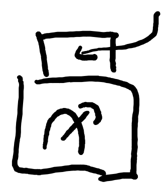
Lecture 5 - Multi-qubit systems

(As we know, possible to have a....)

Dim.-4 quantum system (e.g. "spin-3/2 baryon", photon in one of 4 places...)

$$\begin{bmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\end{bmatrix}, \quad \xrightarrow{U}, \quad$$



$\begin{cases} |1\rangle \text{ w.prob } |\alpha_1|^2 \\ \vdots \\ |4\rangle \text{ w.prob } |\alpha_4|^2 \end{cases}$

(state transforms via...)

4x4 unitary matrix

(general state is a vector...) $\in \mathbb{C}^4$

(measuring device)

(That's all fine. But the most typical dim-4 object is....) 2 qubits.

E.g. polarization of 2 photons.

↳ If "0" = horz., "1" = vert.:

state is $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$$|00\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \qquad |11\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

(Similarly, 3 qubits = 8-dim state, etc.)

Today: Say Alice has a qubit $|\psi\rangle \in \mathbb{C}^2$
Bob ~~~ $|\phi\rangle \in \mathbb{C}^2$.

(Their qubits are going to have a playdate: get together, undergo some joint transformation, get measured. This is exactly Q.C., by the way — tho you usually imagine $\gg 2$ qubits.)

Q1 What is the joint 4-dim. state?

Q2 If Bob applies unitary $U \in \mathbb{C}^{2\times 2}$ to his qubit, what is new 4-dim state?

(e.g. $R_{45°}$ or H)

(tension: how to apply a 2x2 matrix to a 4-d vec.?)

Q3 If we measure only Alice's qubit, what happens? (Note: We already know what happens if we "collectively measure" both.)

next time

Answers: What you'd guess is correct.
(by analogy with probabilistic computing)

cor: (After today) you'll know all rules of quantum computing! (Assuming, as you may WLOG, that you always end playdates with collective measurement.)

Alice's photon: $\alpha_0|0\rangle + \alpha_1|1\rangle$.

Bob's photon: $\beta_0|0\rangle + \beta_1|1\rangle$.

(Perhaps prepped in different labs. But now they bring them together...)

Can be viewed as a joint system of 4 basic states:

$$\gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle$$

Alice's, Bob's

(Must be viewed this way if the 2 photons are going to interact.

So what are $\gamma_{00}, \ldots, \gamma_{11}$?)

(If you think for 2 minutes, there's only one plausible option. And ... it's correct! ☺)

(Imagine the analogue with probabilities. Like Alice flips a 0/1 coin that comes up with probs. α_0, α_1 & Bob flips a β_0/β_1 coin too.)

γ_{01}?

"The ~~probability~~ amplitude of $|01\rangle$ is the ~~probability~~ amplitude that Alice's qubit is "0" – namely α_0 – times the ~~probability~~ amplitude that Bob's qubit is "1" – namely β_1. So $\gamma_{01} = \alpha_0\beta_1$."

✓

Indeed: <u>QM Law 4:</u>

<u>Alice's qubit</u>, <u>Bob's</u> $\mapsto$ <u>Joint state</u>

$$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} \qquad \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} \qquad \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}\begin{bmatrix}\alpha_0\beta_0\\ \alpha_0\beta_1\\ \alpha_1\beta_0\\ \alpha_1\beta_1\end{bmatrix}$$

Check: $|\alpha_0\beta_0|^2+|\alpha_0\beta_1|^2+|\alpha_1\beta_0|^2+|\alpha_1\beta_1|^2$

$$=(|\alpha_0|^2+|\alpha_1|^2)(|\beta_0|^2+|\beta_1|^2) = 1\cdot 1 = 1. \checkmark$$

<u>More generally:</u>

Alice's d-level <u>system</u>, <u>Bob's e-level system</u> $\mapsto$ <u>Joint state is d·e -dimensional</u>

$$\begin{array}{c} |1\rangle \\ |2\rangle \\ \vdots \\ |d\rangle \end{array}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_d\end{bmatrix} \qquad \begin{array}{c} |1\rangle \\ \vdots \\ |e\rangle \end{array}\begin{bmatrix}\beta_1\\ \vdots\\ \beta_e\end{bmatrix} \qquad \begin{array}{c} |11\rangle \\ |12\rangle \\ \vdots \\ |1e\rangle \\ |21\rangle \\ \vdots \\ |de\rangle \end{array}\begin{bmatrix}\alpha_1\beta_1\\ \alpha_1\beta_2\\ \vdots\\ \alpha_1\beta_e\\ \alpha_2\beta_1\\ \vdots\\ \alpha_d\beta_e\end{bmatrix}$$

(first index is Alice's "particle", second is Bob's)

(What is this operation? Has a name in linear algebra.)

$$\vec{a} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix} \mapsto \vec{a} \otimes \vec{b} = \begin{bmatrix} \alpha_1 \vec{\beta} \\ \alpha_2 \vec{\beta} \\ \vdots \\ \alpha_d \vec{\beta} \end{bmatrix}$$

("block vector")

tensor / Kronecker product

(Technically there's a difference. Kron. product is an operation on matrices, tensor prod. is a "formal", coordinate-free operation. Unlike linear algebra nerds, we'll ignore the distinction.)

More generally, if $A = \begin{pmatrix} a_{11} & a_{12} & \cdots \\ \vdots & & \\ & & a_{mn} \end{pmatrix}$, B are matrices,

$m \times n$ $\quad$ $p \times q$

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & \ddots & \cdots & \vdots \\ \vdots & & \ddots & \\ & & & a_{mn}B \end{bmatrix}$$

$mp \times nq$ block matrix

Examples

- $|0\rangle \otimes |0\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} =$ "$|00\rangle$"

formally, an <u>abbreviation</u> of $|0\rangle \otimes |0\rangle$

(Recall: meaning is that Alice prepares qubit in state $|0\rangle$, Bob preps one in state $|+\rangle$, then we view as joint system.)

- $|0\rangle \otimes |+\rangle = \begin{bmatrix}1\\0\end{bmatrix} \otimes \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\\0\end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$

$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ (looks like you just stuck a 0 in first slot)

- $|+\rangle \otimes |0\rangle = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \otimes \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\\0\end{bmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$

(Note: not commutative!)

- $|+\rangle \otimes |-\rangle = \begin{bmatrix}\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}\end{bmatrix} \otimes \begin{bmatrix}\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}$

$= \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right)$

$= \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$

Properties of $\otimes$

- Acts just like "noncommutative" multiplication:
  - $(A+B)\otimes C = A\otimes C + B\otimes C$
  - $A\otimes(B+C) = A\otimes B + A\otimes C$

$$A \otimes (B\otimes C) = (A\otimes B)\otimes C = \text{"}A\otimes B\otimes C\text{"}$$

(E.g., if Alice, Bob, Charlie each have a qubit...)

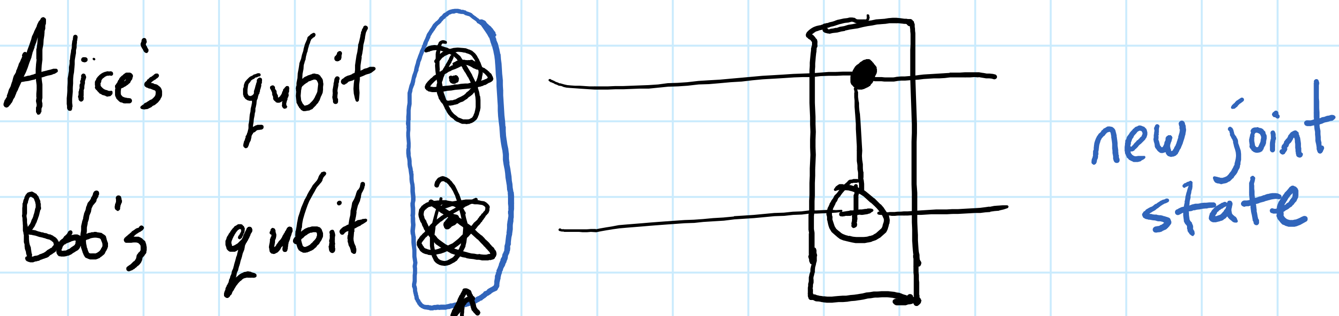
$$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} \otimes \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} \otimes \begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix} = \begin{bmatrix}\alpha_0\beta_0\gamma_0\\ \alpha_0\beta_0\gamma_1\\ \alpha_0\beta_1\gamma_0\\ \vdots\\ \alpha_1\beta_1\gamma_1\end{bmatrix}$$

(In Quantum Computing, should imagine 100s or 1000s of qubits!)

- $(A\otimes B)^\dagger = A^\dagger \otimes B^\dagger$ (Doesn't reverse order. Easy to prove this one.)

- $(A\otimes B)\cdot(C\otimes D) = (AC)\otimes(BD)$
(This one's subtler. We'll come back to it...)

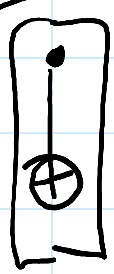
(Perhaps a lot to take in. Let's go back to just a 2-qubit system as we move on to talk about <u>unitary transformations</u>)



joint state $\alpha_{00}|00\rangle + \alpha_{01} \cdots$ etc.

<u>def</u>:

(Like all multi-qubit gates, rather tough to implement in practice. First done with "ion trap" qubits in '95: post-Shor! Use some complicated magnetic field)



= "CNOT" = 4x4 unitary defined as follows:

"If the 'control qubit •' is 0,
do nothing;
else if the control qubit is 1,
apply a NOT to the 'target qubit ⊕'."

Formally:

$$\begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \downarrow & \downarrow & \downarrow & \downarrow \\ |00\rangle & |01\rangle & |11\rangle & |10\rangle \end{array}$$

(We've said how CNOT operates on the classical basis states. That defines how it acts on any 2-qubit state in $\mathbb{C}^4$, by linearity.)

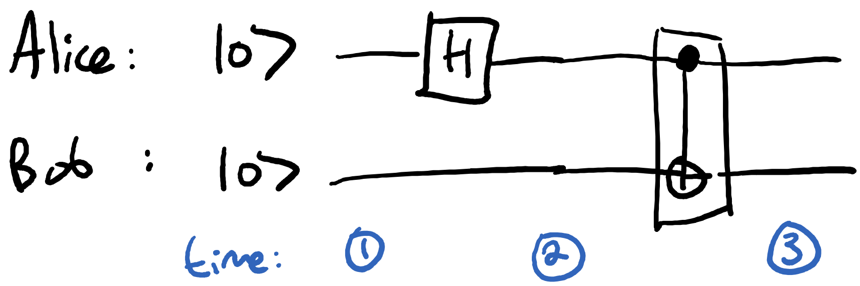
$$CNOT = \begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline |00\rangle & 1 & 0 & 0 & 0 \\ |01\rangle & 0 & 1 & 0 & 0 \\ |10\rangle & 0 & 0 & 0 & 1 \\ |11\rangle & 0 & 0 & 1 & 0 \end{array} \qquad \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \mapsto \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{11} \\ \alpha_{10} \end{bmatrix}$$

(A permutation matrix, hence unitary, as we saw.)

$$CNOT\left(\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \right)$$
$$= \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{11}|11\rangle + \alpha_{10}|10\rangle$$

(Say in words:
With ~~probability~~ amplitude α_{00}, the state is $|00\rangle$, which when CNOTted gives $|00\rangle$;
with ~~probability~~ amplitude α_{01}, the state is $|01\rangle$, which when CNOTted gives $|01\rangle$;
etc....)

(Here's our first interesting "quantum circuit"; a tiny but important one in quantum computing...)



State at time ①: $|00\rangle$

② : $|+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$

(Up until this point, A & B could have been in sep. labs. But now their qubits must physically come together to enter the "CNOT gate".)

③ : ("with ampl. $\frac{1}{\sqrt{2}}$, state is $|00\rangle \xmapsto{\text{CNOT}} |00\rangle$,
" " " $|10\rangle \longmapsto |11\rangle$.")

(" $|\Phi^+\rangle$ ") $\quad \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ (A very famous & important state in Q.M. and Q.C.)

"Bell state" aka "EPR pair" (Einstein-Podolsky-Rosen, who used it to describe a "paradox" in Q.M. ... which is actually perfectly fine. More on it later.)

$$= \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$$

thm: Bell state is not of the form $|\psi\rangle \otimes |\phi\rangle$

for any $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$.

(I.e., it cannot be achieved by taking two "separate" qubits and viewing them as joint state)

proof: In any $\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} \otimes \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$,

product of blue amplitudes

=

~~ green ~~ $= \alpha_0\alpha_1\beta_0\beta_1$.

But for $\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $0 \cdot 0 \neq \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}$. ∎

def: Entangled := not unentangled

:= of the form $|\psi\rangle \otimes |\phi\rangle$.

(So Bell state is entangled. Not always easy to tell.)

Q: Is $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$ entangled?

A: No, it's $|+\rangle \otimes |+\rangle$.

Transforms on subsets of qubits

(Always confused me when I was first learning.)

Say Alice & Bob make an EPR pair,

$$\frac{1}{\sqrt{2}}|\underset{A}{0}\underset{B}{0}\rangle + \frac{1}{\sqrt{2}}|\underset{A}{1}\underset{B}{1}\rangle .$$

Then Bob puts his photon into a NOT gate.

New state = $\underline{\quad ? \quad}$ $\qquad \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

(Surely you would guess the correct answer ↑)

(It's again just as if amplitudes were probs. and we were doing probabilistic computing. Can use these words...)

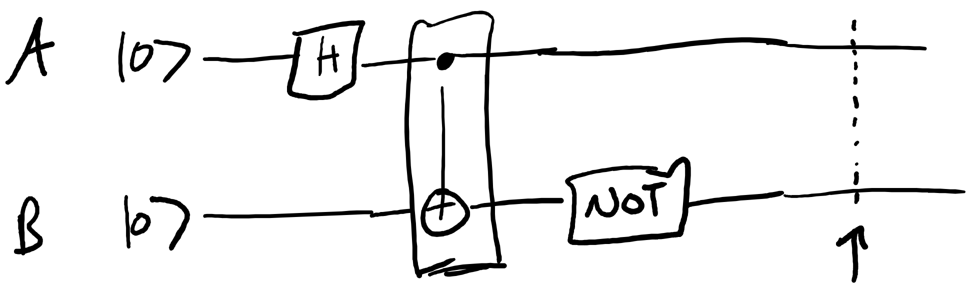
"With ~~probability~~ amplitude $\frac{1}{\sqrt{2}}$, state is $|00\rangle$.

Alice has $|0\rangle$, Bob has $|0\rangle \xrightarrow{\text{NOT}} |1\rangle$.

$\rightsquigarrow |01\rangle$.

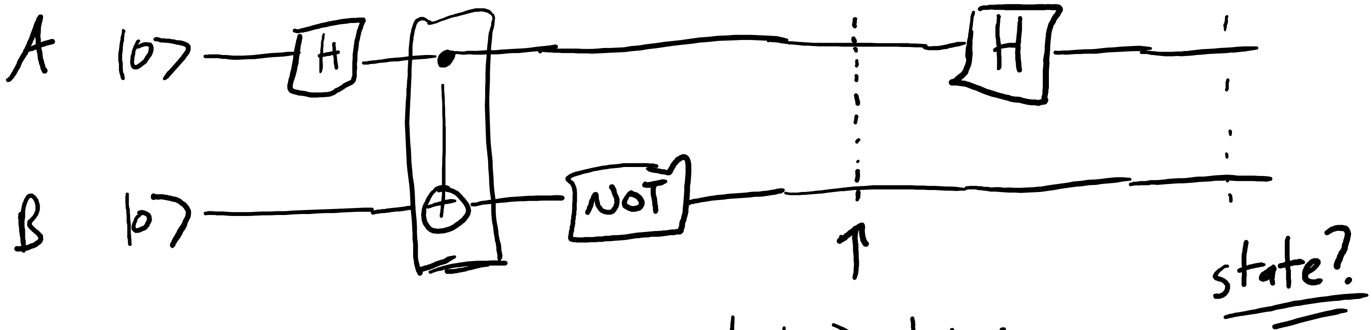
With ~~prob.~~ ampl. $\frac{1}{\sqrt{2}}$, Alice has $|1\rangle$, Bob has $|1\rangle \xrightarrow{\text{NOT}} |0\rangle$.

$\rightsquigarrow |10\rangle$."



$$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

(Say now Alice puts her qubit thru a Hadamard gate.)



$\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

with ampl. $\frac{1}{\sqrt{2}}$: $|01\rangle = |0\rangle \otimes |1\rangle$. Applying H to A's qubit gives $(H|0\rangle) \otimes |1\rangle$

$= |+\rangle \otimes |1\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |1\rangle$

$= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

with ampl. $\frac{1}{\sqrt{2}}$: $|10\rangle = |1\rangle \otimes |0\rangle \mapsto (H|1\rangle) \otimes |0\rangle$

$= |-\rangle \otimes |0\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle$

$= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|10\rangle$

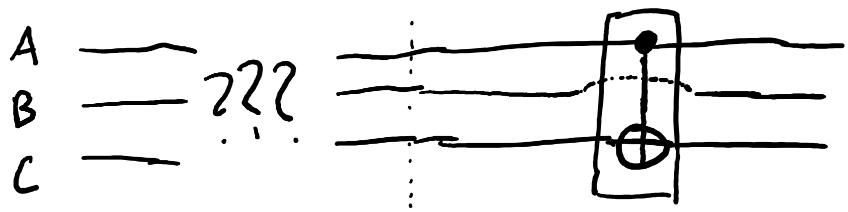
(adding blue-circled stuff.)

Final state: $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$.

(Check: $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = 1$. Q: entangled? A: yes.)

another e.g. Suppose Alice, Bob, & Charlie jointly cook up this 3-qubit state:

(Yes, these 3 ampl.^{2} add to 1.) $\rightarrow$ $$.36|100\rangle + .48|001\rangle - .8|111\rangle$$



final state?

Now A&C put their bits thru a CNOT gate.

ampl. .36: $\underset{ABC}{|100\rangle}$. B stays 0. $\underset{AC}{|10\rangle} \xmapsto{\text{CNOT}} \underset{AC}{|11\rangle}$.

$\rightsquigarrow \underset{(ABC)}{|101\rangle}$.

ampl. .48: $|001\rangle \rightsquigarrow |001\rangle$

ampl -.8: $|111\rangle \mapsto |110\rangle$

Final state $= .36|101\rangle + .48|001\rangle - .8|110\rangle$.

(Aside: if they now put all photons into a big measuring device, readout is "$|101\rangle$" with prob. $(.36)^2$, "$|001\rangle$" w. prob. $(.48)^2$, "$|110\rangle$" with prob. $(-.8)^2$.)

(You now know — by example — Q.M. Law 5: how composite states transform when a unitary is applied to a subsystem. As with conjoining two states, there's a slick linear algebra description using tensor products...)

Suppose we have a 2-qubit state,

$$\begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \in \mathbb{C}^4.$$

Suppose we do unitary $U = \begin{bmatrix} p & r \\ q & s \end{bmatrix}$ to 2nd qubit.

New state $= ?$

Type-check problem?
How to apply 2×2 matrix to 4-dim vector?!

(Well, you know the rule, so in principle can just compute and find out!)

WLOG can assume state is $|00\rangle, |01\rangle, |10\rangle$, or $|11\rangle$.

(Once we know where these 4 states go, linearity determines the rest.)

$$|00\rangle = |0\rangle \otimes |0\rangle \longmapsto |0\rangle \otimes (U|0\rangle)$$

(applying U to 2nd qubit)

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} p \\ q \end{bmatrix}$$

($\because$ $U|0\rangle$ = first col. of U.)

$$= \begin{bmatrix} p \\ q \\ 0 \\ 0 \end{bmatrix}.$$

$$|01\rangle = |0\rangle \otimes |1\rangle \longmapsto |0\rangle \otimes (U|1\rangle) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} r \\ s \\ 0 \\ 0 \end{bmatrix}.$$

$$|10\rangle = |1\rangle \otimes |0\rangle \longmapsto |1\rangle \otimes (U|0\rangle) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \\ q \end{bmatrix}$$

$$|11\rangle \longmapsto \cdots \begin{bmatrix} 0 \\ 0 \\ r \\ s \end{bmatrix}.$$

$\therefore$ overall transform is

$$\begin{bmatrix} p & r & 0 & 0 \\ q & s & 0 & 0 \\ 0 & 0 & p & r \\ 0 & 0 & q & s \end{bmatrix} = \underbrace{I}_{2\times 2} \otimes U.$$

$\llcorner \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(Remember how to read a matrix: first col. is where $|00\rangle$ goes, second col. is where $|01\rangle$ goes, etc.)

Regarding the I in $I \otimes U$:

that's the operation we did to the <u>1st</u> qubit: i.e, the do-nothing unitary $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

<u>ex</u>: Had we applied U to 1st bit and left 2nd qubit alone, we'd have gotten

$$\begin{bmatrix} p & 0 & r & 0 \\ 0 & p & 0 & r \\ q & 0 & s & 0 \\ 0 & q & 0 & s \end{bmatrix} = \left[\begin{array}{c|c} pI & rI \\ \hline qI & sI \end{array}\right] = U \otimes I.$$

fact: If we do U to 1st qubit, V to 2nd qubit, the overall 4x4 unitary is

$$(U \otimes I) \cdot (I \otimes V) = (I \otimes V) \cdot (U \otimes I)$$
$$= U \otimes V.$$

(Note: doesn't matter the temporal order in which we apply U, V.)

(A final remark...)

thm: <u>Every</u> unitary operating on ($2^n \times 2^n$ dimensional) n qubits can be realized by a "circuit" of

- 1-qubit unitaries
- CNOTs (on 2 qubits)

(Since you know the rules of how these work, and you know the rule of what happens when you do a big collective measurement at the end...

You now know all the rules of how a Q.C. works.

Go forth and build Shor's factoring alg! :))