A Survey of Applications of Boost Invariance - Project Background

Jennifer L. James[1]

Vanderbilt University, Department of Physics and Astronomy, 37211 TN
Nashville, U.S.A

1. Introduction

Systems take advantage of Lorentz transformations across many disciplines of physics. Of interest is boost invariance — where one frame is Lorentz boosted to another frame within a system. The author wishes to highlight general understandings of boost invariance in many areas of physics while targeting systems using fluid-dynamics, especially in that of heavy-ion collisions.

2. Boost Invariance

A Lorentz boost is made along the longitudinal direction parallel to the beam axis $\hat{z}$. A Lorentz boost is significant as it allows for quantities to remain intact while being in a frame where velocity does not need to be accounted for. At ultra-relativistic speeds where $v = c = 1$, the rapidity, y, of the system is no longer velocity dependent and only relies on the angle, θ, of emission for example. This is the foundation for boost

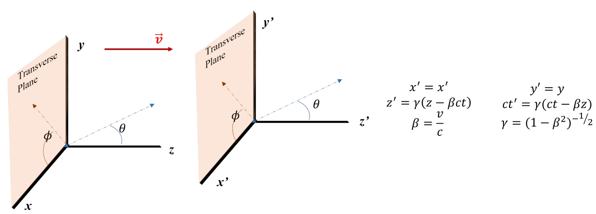

Figure 1: Figure 1 demonstrates a boost along the longitudinal axis where $\vec{v}$ is approximately the speed of light. To the right is the relativistic Lorentz transformation equations in the z direction.

invariance across many systems. We can now evaluate boost invariance within hydrodynamics itself.

3. Relativistic Fluid-Dynamics

Ideal hydrodynamics is an approximation used for many "perfect fluid" systems, such as quark gluon plasma (QGP). Ideal hydrodynamics is invariant when scaling given that the equation of state, too, is scale invariant. Therefore, a boost in the z direction should not affect the analysis given there are no dissipative processes such as viscosity. For example, a quantity given at mid-rapidity ($\eta_s = 0$) may be given at all other rapidity's along the longitudinal direction.

3.1. Iterations from Landau-Fermi Hydrodynamics

Landau and Lifshitz treatment of relativistic fluid dynamics requires that relativistic effects in fluid dynamics are not solely due to a large velocity but the fact that the microscopic motion of the fluid of the particles are large as well [1]. In ideal-relativistic hydrodynamics, we define the stress-energy tensor $T^{\mu\nu}$ and its conservation law as:

$$T^{\mu\nu} = (\epsilon + P)\mathbf{u}^{\mu}\mathbf{u}^{\nu} + Pg^{\mu\nu} \quad \text{and} \quad \partial_{\mu}T^{\mu\nu} = 0 \tag{1}$$

Where ϵ is energy density and P is pressure. In which the equations of motion are defined for the physical system pertaining to $T^{\mu\nu}$. Landau found an analytic asymptotic solution using Fermi's statistical model for relativistic boson and fermion gasses [2]. Fermi's model used blackbody (or ideal-relativistic Bose gas)

URL: `jennifer.l.james@vanderbilt.edu` (Jennifer L. James)

relations which only correctly describes the cooling and pion condensation process. Landau's model had two phases: state of matter produced in the collision and the cooling stage. In short, Landau's model was not boost invariant and proved somewhat problematic in describing ultra-relativistic fluids [3].

3.2. Bjorken boost invariant hydrodynamics

A new model was developed by J.D. Bjorken, which was mainly used to describe high energy collisions. He was able to base his model off of that of Landau's, however, the boundary conditions were adjusted. He first assumed a "central-plateau" structure for the particle production as a function of the rapidity variable is to occur only at high energies. This was done to assure that the space-time evolution looks the same in all center-of-mass like frames whether the collision occurred between two nucleon, two nuclei, or nucleon-nucleus. This assumption also implies a symmetry property of the system to which the hydrodynamic equations respect as well providing simple solutions [4]. Using the initial conditions set by these assumptions, a space-time diagram of the quark gluon plasma could be made (Fig. 2a).

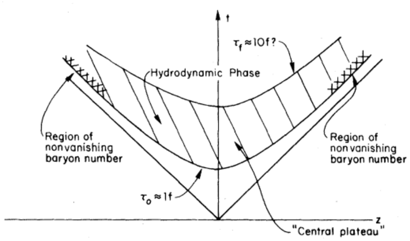

Figure 2: Fig 2a

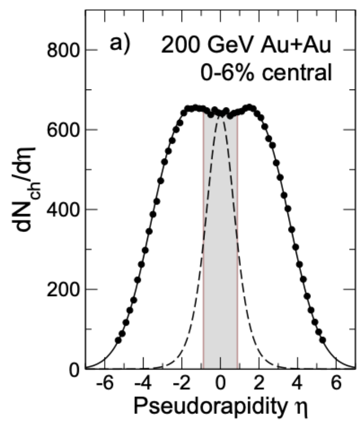

Figure 3: Fig 2b

Figure 2b demonstrates pseudo rapidity ($dN_{ch}/d\eta$) distributions of charged particles emitted in Au+Au collisions shown for different centrality bins at a collision energy of 200 GeV [5].

Bjorken was able to create a relatively simple model of relativistic fluid mechanics for that of longitudinal expansion in heavy-ion collisions. He discovered that kinematic quantities (p_z and E) were approximately that of quantities in the definite region of space-time

$$v^2 = \frac{p^z}{E} \simeq \frac{z}{t} \tag{2}$$

Wherefore the kinematics may best be analyzed using proper time τ and space-time rapidity η_s

$$\tau \equiv \sqrt{t^2 - z^2} \quad \text{and} \quad \eta_s \equiv \frac{1}{2} log\left(\frac{t+z}{t-z}\right) \tag{3}$$

by the following relationship between rapidity and space-time rapidity given they are evaluated at the proper time

$$y \equiv \frac{1}{2} log\left(\frac{p_z + E}{E - p_z}\right) \simeq \frac{1}{2} log\left(\frac{t+z}{t-z}\right) \equiv \eta_s \tag{4}$$

The relationship between rapidity and space-time rapidity can be approximated to be associated with a definite angle θ in the detector[6]

$$\eta_s \simeq y \simeq \frac{1}{2} log\left(\frac{p + p_z}{p - p_z}\right) = \frac{1}{2} log\left(\frac{1 + cos\theta}{1 - cos\theta}\right) = \ln cot(\theta/2) \equiv \eta_{pseudo} \tag{5}$$

Since Bjorken assumed energy density to be uniform in space-time rapidity, the relationship between fluid rapidity and space-time rapidity remains fixed as the fluid flows into the forward light cone (Fig. 2a). All of these assumptions can be collectively seen in the following equations

$$e(t, \mathbf{x}) = e(\tau), \qquad u^{\mu}(t, \mathbf{x}) = (u^{\tau}, u^{x}, u^{y}, u^{\eta}) = (cosh(\eta_s), 0, 0, sinh(\eta_s)). \tag{6}$$

Which can be changed to curvilinear coordinates

$$x^{\mu} = (t, \mathbf{x}_{\perp}, \eta_s), \qquad g_{\mu\nu} = diag(-1, 1, 1, \tau^2), \quad \text{and} \quad (u^{\tau}, u^{x}, u^{y}, u^{\eta}) = (1, 0, 0, 0) \tag{7}$$

where being boost invariant in this coordinate system implies that the space-time rapidity η_s isn't depended upon by anything else[6]. To briefly summarize Bjorken boost invariant hydrodynamics, we have found that space-time rapidity is analogous to rapidity when evaluated at proper time and that the ansatz of equation 6 can be used to define an equation of state for ideal fluids in local equilibrium[4].

4. Expanding on The Foundational Models

4.1. Heavy-ion collisions

The Bjorken model is a foundation many physicists have used to investigate fluid dynamics in heavy-ion collisions. However, his assumption of translational invariance in the transverse plane may mislead the calculation of subsequent hydrodynamic flow. Collective flow of matter may reveal the properties of an expanding matter such as the quark gluon plasma, thus it is important to consider radial expansion in the transverse plane. Various relativistic viscous hydrodynamic studies have been done to describe the fluid as well. Largely known is Gubser flow, which assumes a 3+1D expansion such that the system goes from early time free-streaming regime to intermediate thermalization and back to free-streaming in the late time regime [7]. Viscous hydrodynamics takes advantage of kinetic theory — or rather the *continuum Boltzmann equation* — which aim to reveal non-equilibrium effects[8].

5. Tasks and Project Goals

Further studies of related hydrodynamic models will be made, especially in that of viscosity information studies. For example, Paul and Ulrike Romatschke investigate what information should be gained when including viscous dynamics to the foundations Bjorken built [9]. Plots they've produced along with figure 2 above will be replicated. Another topic to investigate is how kinetic theory merges into the viscous hydrodynamic limits[10][11].

References

[1] L. D. Landau, E. M. Lifshitz, Fluid Dynamics Vol. 6, Butterworth-Heinemann, 1987.

[2] J. Nowakowski, F. Cooper, Multiparticle production in high-energy collisions: Validity of statistical and hydrodynamical approximations, Phys. Rev. D 9 (1974) 771–775. `doi:10.1103/PhysRevD.9.771`. URL `https://link.aps.org/doi/10.1103/PhysRevD.9.771`

[3] A. Bialas, R. A. Janik, R. B. Peschanski, Unified description of Bjorken and Landau 1+1 hydrodynamics, Phys. Rev. C 76 (2007) 054901. `arXiv:0706.2108`, `doi:10.1103/PhysRevC.76.054901`.

[4] J. D. Bjorken, Highly Relativistic Nucleus-Nucleus Collisions: The Central Rapidity Region, Phys. Rev. D 27 (1983) 140–151. `doi:10.1103/PhysRevD.27.140`.

[5] B. B. Back, the PHOBOS Collaboration, Recent results from PHOBOS, Acta Physica Hungarica A) Heavy Ion Physics 25 (2006) 1588–2675. `doi:10.1556/APH.25.2006.2-4.41`.

[6] D. A. Teaney, Viscous Hydrodynamics and the Quark Gluon Plasma, 2010, pp. 207–266. `arXiv:0905.2433`, `doi:10.1142/9789814293297_0004`.

[7] A. Dash, V. Roy, Hydrodynamic attractors for Gubser flow, Phys. Lett. B 806 (2020) 135481.

arXiv:2001.10756, doi:10.1016/j.physletb.2020.135481.

[8] X. SHAN, X.-F. YUAN, H. CHEN, Kinetic theory representation of hydrodynamics: a way beyond the navier–stokes equation, Journal of Fluid Mechanics 550 (2006) 413–441. doi:10.1017/S0022112005008153.

[9] P. Romatschke, U. Romatschke, Viscosity information from relativistic nuclear collisions: How perfect is the fluid observed at rhic?, Phys. Rev. Lett. 99 (2007) 172301. doi:10.1103/PhysRevLett.99.172301.
URL https://link.aps.org/doi/10.1103/PhysRevLett.99.172301

[10] G. Baym, Thermal equilibration in ultrarelativistic heavy-ion collisions, Physics Letters B 138 (1) (1984) 18–22. doi:https://doi.org/10.1016/0370-2693(84)91863-X.
URL https://www.sciencedirect.com/science/article/pii/037026938491863X

[11] P. F. Kolb, U. Heinz, Hydrodynamic description of ultrarelativistic heavy-ion collisionsdoi:10.48550/ARXIV.NUCL-TH/0305084.
URL https://arxiv.org/abs/nucl-th/0305084