15-251: Great Theoretical Ideas In Computer Science

Homework 5: DragonBall Graph Theory (due 23:59 Monday, February 17)

Directions: *Write up carefully argued solutions to the following problems. The first task is to be complete and correct. The more subtle task is to keep it simple and succinct. Your solution should be clear enough that it should explain to someone who does not already understand the answer why it works. You may use any results proven in lecture without proof. Anything else must be argued rigorously. Unless otherwise specified, all answers are expected to be given in closed form.*

Are we there yet?

Nappa is a very impatient person and wanted to make sure that he wouldn't get too bored during his trip from H.F.I.L. to Earth. Nappa thought back to his days in Saiyan University and remembered how he didn't like math when he was there. Still solving math problems was a decent way for him to pass the time.

The 2^k worlds that Nappa could possibly pass through on his way to Earth can all be cool in some ways or not. It turns out that there are k different dimensions of coolness and a world's awesomeness factor is how many different dimensions they are cool in. It turns out that every world is different so no two worlds are cool in exactly the same way even if their awesomeness factors are the same. Two worlds are only connected to each other if they are cool in all the same ways except for one.

Help Nappa prove that for every perfect matching of the worlds using their connections, the number of connections in the matching that connect worlds of awesomeness factor i to worlds of awesomeness factor $i+1$ is $\binom{k-1}{i}$, for $0 \leq i \leq k-1$

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Converting your ki into a blast of energy isn't actually very easy. Just because your power level is over 8000 doesn't mean you can actually use all that power properly. Goku was never very good at learning hard things but is still worried about making some ki blasts.

Every ki blast actually has special points that need to be charged with ki. The problem is if adjacent points are charged with the same type of ki, the entire blast could blow up in your face. The good news is that there is a way to configure all blasts so that all of the points touch the outside of the blast and don't have any connections between neighboring points cross. This means that the graphs are *outerplanar* — they are planar graphs that can be drawn with all vertices touching the outside face without their edges crossing.

There are 3 types of ki: Personal, Earth, and Heaven. Prove to Goku that it is possible to make every ki blast without it blowing up in your face.

Incidentally, this shows why almost all ki blasts are circular or close to it; it's just so much easier to make them when they are already configured properly.

Note: Goku doesn't understand the 4-color theorem since it's too confusing. He likes to keep things nice and simple.

Lasers

Goku thinks that firing lasers is a little bit harder than making simple ki blasts. All he knows about lasers is that the graph of their ki points doesn't have a $K_{2,3}$ or a K_4 minor. Show Goku that he's completely wrong and that lasers and ki-blasts are basically the same thing.

Refer to the previous problem if you don't know about ki-blasts.

Going Super Saiyan

While most people in Dragonball can usually transform, the graph of their transformations pretty much determines how strong they can be. The graph of transformations is simply made of the various tranformations (or states) that a person can be in. There is an edge between two states if you can transform between the two states without changing into another state first.

Every transformation graph G with n transformations can be described by the sequence of the transformation's power levels. A transformations power level is just the number of other states that you can transform into from the one you are in. Let $P(G)$ be the list of power levels sorted from strongest to weakest. An arbitrary sequence of power levels S is *attainable* if there is some transformation graph G such that $P(G) = S$. For example, the sequences $3, 2, 2, 1$ and $2, 2, 2, 2$ is attainable, but the sequence $3, 3, 3, 1$ is not.

After intense training, some people can powerup and make it easier or harder to achieve certain transformations. A powerup is an operation that lets someone change their transformation graph. Pretend that originally, you could go between states x and y and between states z and w but not between x and w or between y and z. After a powerup, you will now be able to transform between x and w and between y and z but not between x and y or w and z.

One might wonder how many power levels in an attainable power level sequence can be distinct. We call an attainable power level sequence *powerful* if it has as many different power levels as possible. Likewise, a graph is powerful if its power level sequence is.

(a) Show that for any transformation graph of size $n \geq 2$ there must be two transformations with the same power level.

(b) Find a family of powerful graphs for all sizes n.

(c) Characterize all powerful graphs.

(d) Show that graphs with the same power level sequence can be changed into one another by a series of power ups.

For part (B) it is a good idea to enumerate a few small examples. Then ponder deeply and do not forget to show that your answer captures all the proper graphs.