Calculus 2 Assignment 4

1. $f(t) = 20000\,\text{sech}^2(2t-30)$, $t \geq 0$

a) <u>y-intercept</u>

$y = 20000\,\text{sech}^2(2(0)-30)$

$y = 20000\,\text{sech}^2(-30)$

<u>x-intercepts</u>

$20000\,\text{sech}^2(2t-30) = 0$

$\text{Sech}^2(2t-30) = 0$

$\text{Sech}(2t-30) = 0$ $\quad$ $\text{Sech} > 0$

$2t-30 = \text{arcSech}(0)$

$=$ undefined over $\mathbb{R}$ as $\text{dom}(\text{arcsech}) = (0,1]$

<u>horizontal asymptotes</u>

$y = \dfrac{20000}{\cosh^2(2t-30)}$ $\quad$ degree $n <$ degree d

$\Rightarrow y = 0$ is the horizontal asymptote

<u>vertical asymptote</u>

occurs when $\cosh^2(2t-30) = 0$

$\cosh(2t-30) = 0$, $\cosh > 0$

$2t-30 = \text{arccosh}(0)$

$=$ undefined over $\mathbb{R}$ as $\text{dom}(\text{arccosh}) = [1,\infty)$

Stationary point

occur when $f'(t) = 0$

$$f'(t) = \frac{d}{dt}\left(20000 \operatorname{sech}^2(2t-30)\right)$$

$$= 20000 \frac{d}{dt}\left(\operatorname{sech}^2(2t-30)\right)$$

$$= 20000 \cdot 2\operatorname{sech}(2t-30) \cdot \frac{d}{dt}\left(\operatorname{sech}(2t-30)\right)$$

$$= 40000 \operatorname{sech}(2t-30) \cdot -\operatorname{sech}(2t-30)\tanh(2t-30) \cdot 2$$

$$= -80000 \operatorname{sech}^2(2t-30)\tanh(2t-30)$$

$$\Rightarrow -80000 \operatorname{sech}^2(2t-30)\tanh(2t-30) = 0$$

$$\operatorname{sech}^2(2t-30)\tanh(2t-30) = 0$$

$$\tanh(2t-30) = 0$$

$$2t-30 = \operatorname{arctanh}(0)$$

$$2t-30 = 0$$

$$2t = 30$$

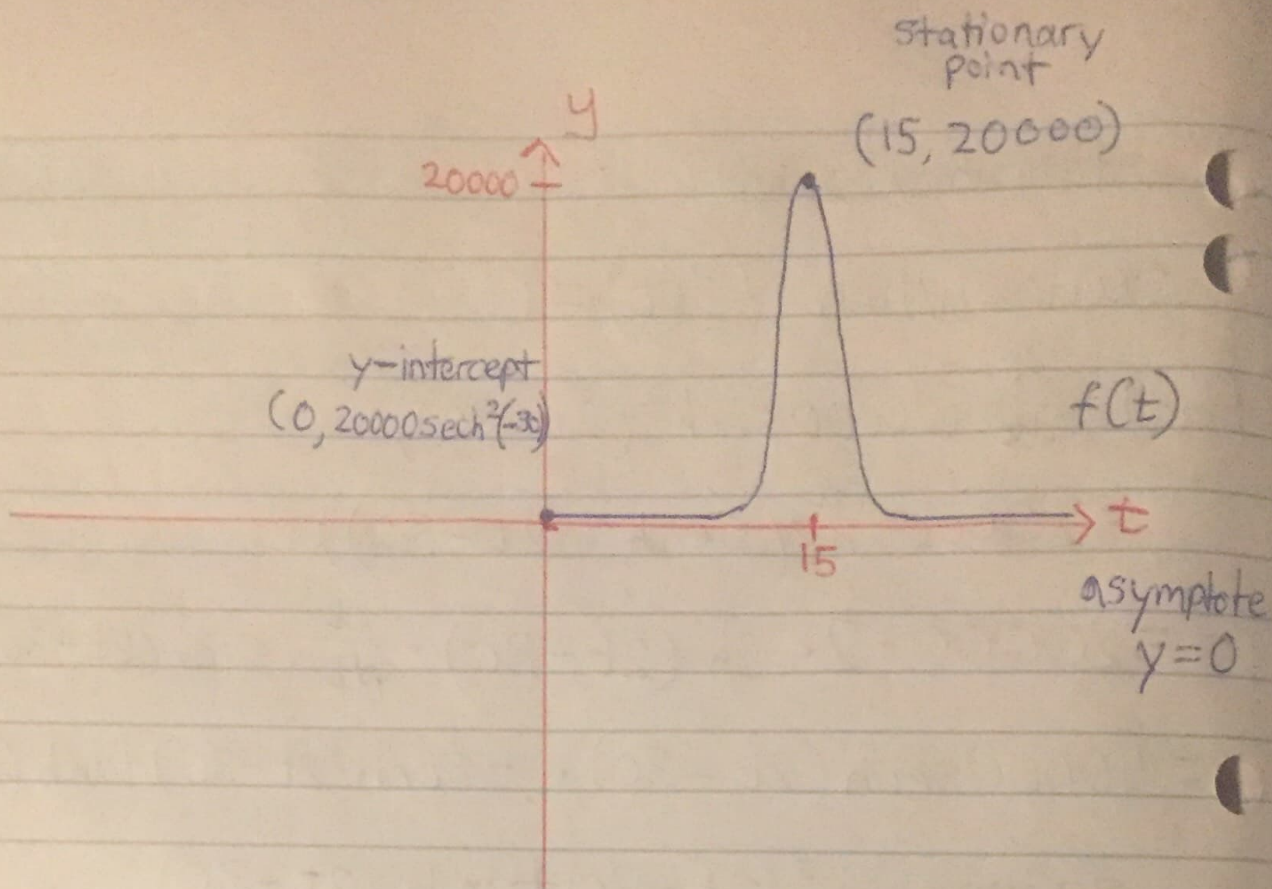
$$t = 15$$

sub $t = 15$ into $f(t)$

$$f(15) = 20000 \operatorname{sech}^2(0)$$

$$= 20000$$

turning point at $(15, 20000)$



b) $0.01 \cdot f(t)$ cases require hospitalisation

$\Rightarrow$ the hospital will first be over capacity when $0.01 \cdot f(t)$ is first ~~equal to~~ greater than 10.

$\Rightarrow 200\,\text{sech}^2(2t-30) = 10$

$\text{sech}^2(2t-30) = \frac{1}{20}$

$|\text{sech}(2t-30)| = \pm\frac{1}{\sqrt{20}}$

$\text{sech}(2t-30) = \frac{1}{\sqrt{20}}$, range(sech) > 0

$\text{sech}(2t-30) = \text{sech}(-2t+30)$ (as sech is an even function)

$\Rightarrow \text{sech}(2t-30) = \frac{1}{\sqrt{20}}$ and $\Rightarrow \text{sech}(-2t+30) = \frac{1}{\sqrt{20}}$

$2t - 30 = \text{arcsech}\left(\frac{1}{\sqrt{20}}\right)$ $\qquad -2t+30 = \text{arcsech}\left(\frac{1}{\sqrt{20}}\right)$

$t = \dfrac{\text{arcsech}\left(\frac{1}{\sqrt{20}}\right) + 30}{2}$ $\qquad t = -\dfrac{\text{arcsech}\left(\frac{1}{\sqrt{20}}\right) - 30}{2}$

$t \approx 16$ $\qquad \boxed{t \approx 14}$

($t < 14$ so day 14 will be the first day)

Therefore, the hospital system is first expected to be over capacity 14 days after the start of the disease outbreak.

c) $T(t) = \int_0^t f(u)\, du$

$$= \int_0^t 20000 \operatorname{sech}^2(2u-30)\, du$$

Let $v = 2u-30$

$\frac{dv}{du} = 2$ sub $u=t \Rightarrow v = 2t-30$

$\frac{1}{2}\frac{dv}{du} = 1$ sub $u=0 \Rightarrow v = -30$

$$\Rightarrow T(t) = 20000 \int_0^t \operatorname{sech}^2(v) \frac{1}{2} \frac{dv}{du}\, du$$

$$= 20000 \cdot \frac{1}{2} \int_{-30}^{2t-30} \operatorname{sech}^2(v)\, dv$$

$$= 10000 \left[\tanh(v)\right]_{-30}^{2t-30}$$

$T(t) = 10000(\tanh(2t-30) - \tanh(-30))$

or

$T(t) = 10000\left(\tanh(2t-30) - \frac{e^{-30}-e^{30}}{e^{-30}+e^{30}}\right)$

d) Total number of expected cases $\approx 20,000$

$\frac{20000}{1000000} = 0.02 \Rightarrow 2\%$ of the population are expected to be infected

This model says that roughly 98% of the population will remain uninfected which is roughly 980,000 people

2.

a) $\int 6x\sin(x^2)\,dx$

Let $u = x^2$

$\frac{du}{dx} = 2x \Rightarrow 3\frac{du}{dx} = 6x$

Derivative Substitution

$\int 6x\sin(x^2)\,dx$

$= \int 3\frac{du}{dx}\sin(u)\,dx$

$= 3\int \sin(u)\,du$

$= -3\cos(u) + C$

$= -3\cos(x^2) + C$

b) $\int \frac{x^2}{\sqrt{9x^2-1}}\,dx$

$= \int \frac{x^2}{\sqrt{9(x^2-\frac{1}{9})}}\,dx$

$= \frac{1}{3}\int \frac{x^2}{\sqrt{x^2-\frac{1}{9}}}\,dx$

Let $x = \frac{1}{3}\cosh\theta$, Hyperbolic Substitution

$\theta = \operatorname{arccosh}(3x)$

This is valid when

$3x \in \operatorname{dom}(\operatorname{arccosh})$ and $\theta \in \operatorname{range}(\operatorname{arccosh})$

$\Rightarrow 3x \geq 1 \Rightarrow x \geq \frac{1}{3}$ and $\theta \geq 0$

· Also need $\sqrt{9x^2-1} \neq 0$

$|9x^2 - 1| \neq 0$

$|9x^2 - 1| \neq 0$, ~~$9x^2 - 1 \neq 0$ as~~ $|0| = 0$

$9x^2 \neq 1$

$x^2 \neq 1/9$

$x \neq 1/3 \Rightarrow \theta \neq 0$ as $\theta = 0$ when $x = 1/3$

$\Rightarrow x > \frac{1}{3}$ and $\theta > 0$

- $x = \frac{1}{3}\cosh\theta$
- $\frac{dx}{d\theta} = \frac{1}{3}\sinh\theta$
- $\frac{x^2}{\sqrt{x^2 - \frac{1}{9}}} = \frac{1/9 \cosh^2\theta}{\sqrt{1/9\cosh^2\theta - \frac{1}{9}}}$

$$= \frac{1/9 \cosh^2\theta}{\sqrt{1/9(\cosh^2\theta - 1)}}$$

$$= \frac{1/9 \cosh^2\theta}{\sqrt{1/9 \sinh^2\theta}}$$

$$= \frac{\frac{1}{9}\cosh^2\theta}{\frac{1}{3}|\sinh\theta|}$$

$$= \frac{\frac{1}{9}\cosh^2\theta}{\frac{1}{3}\sinh\theta}, \quad \sinh > 0 \text{ when } \theta > 0$$

Therefore $\frac{1}{3}\int \frac{x^2}{\sqrt{x^2 - \frac{1}{9}}}\,dx$

$$= \frac{1}{3}\int \frac{\frac{1}{9}\cosh^2\theta}{\frac{1}{3}\sinh\theta} \cdot \frac{1}{3}\sinh\theta \; d\theta$$

$$= \frac{1}{3}\int \frac{1}{9}\cosh^2\theta \; d\theta$$

$$= \frac{1}{27}\int \cosh^2\theta \; d\theta$$

$$= \frac{1}{27}\int \frac{1}{2}(\cosh(2\theta)+1)\; d\theta, \quad \text{Double Angle formula}$$

$$= \frac{1}{54}\left[\sinh(2\theta) + \theta\right] + C$$

$$= \frac{1}{54}\left[2\sinh\theta\cosh\theta + \theta\right] + C, \quad \text{Double Angle formula}$$

$$= \frac{1}{54}\left[2\sqrt{\cosh^2\theta - 1}\,\cosh\theta + \theta\right] + C$$

$$= \frac{1}{54}\left[2\sqrt{9x^2-1}\cdot 3x + \operatorname{arccosh}(3x)\right] + C$$

$$= \frac{x}{9}\sqrt{9x^2-1} + \frac{1}{54}\operatorname{arccosh}(3x) + C$$

$$= \frac{x}{3}\sqrt{x^2 - \frac{1}{9}} + \frac{1}{54}\operatorname{arccosh}(3x) + C$$

c) $\int e^{-x} \cos(x)\, dx$

Let $u = e^{-x}$ $\quad \frac{dv}{dx} = \cos(x)$

$\frac{du}{dx} = -e^{-x}$ $\quad v = \sin(x)$, Integration by parts

Therefore $\int e^{-x} \cos(x)\, dx$

$$= e^{-x} \sin(x) + \int e^{-x} \sin(x)\, dx$$

First integrate $\int e^{-x} \sin(x)\, dx$:

Let $u = e^{-x}$ $\quad \frac{dv}{dx} = \sin(x)$, Integration by parts

$\frac{du}{dx} = -e^{-x}$ $\quad V = -\cos(x)$

Therefore $\int e^{-x} \sin(x)$

$$= -e^{-x} \cos(x) - \int e^{-x} \cos(x)\, dx$$

Putting everything together:

$\Rightarrow \int e^{-x} \cos(x)\, dx$

$= e^{-x} \sin(x) + (-e^{-x} \cos(x) - \int e^{-x} \cos(x)\, dx)$

$\Rightarrow 2\int e^{-x} \cos(x)\, dx = e^{-x} \sin(x) - e^{-x} \cos(x) + C$

$\Rightarrow \int e^{-x} \cos(x)\, dx = \frac{e^{-x} \sin(x) - e^{-x} \cos(x)}{2} + d$, $d = \frac{1}{2} C$

$$= e^{-x} \left(\frac{1}{2} \sin(x) - \frac{1}{2} \cos(x)\right) + d$$

d) $\int e^{-x}\cos(x)\,dx$, complex exponential

$$e^{-x}\cos(x) = e^{-x}\,\mathrm{Re}(e^{ix})$$
$$= \mathrm{Re}(e^{-x}\cdot e^{ix}), \quad e^{-x} \in \mathbb{R}$$
$$= \mathrm{Re}(e^{ix-x}) = \mathrm{Re}(e^{(-1+i)x})$$

Therefore $\int e^{-x}\cos(x)\,dx = \int \mathrm{Re}(e^{(-1+i)x})\,dx$

$$= \mathrm{Re}\int e^{(-1+i)x}\,dx$$

$$= \mathrm{Re}\left[\frac{1}{-1+i}e^{(-1+i)x} + c + di\right], \quad c, d \in \mathbb{R}$$

$$= \mathrm{Re}\left[\frac{-1-i}{(-1+i)(-1-i)}e^{-x}e^{ix} + c + di\right]$$

$$= \mathrm{Re}\left[\frac{-1-i}{2}e^{-x}(\cos(x) + i\sin(x)) + c + di\right]$$

$$= \mathrm{Re}\left[\frac{e^{-x}}{2}(-\cos(x) + \sin(x) - i\cos(x) - i\sin(x)) + c + di\right]$$

$$= \frac{e^{-x}}{2}(\sin(x) - \cos(x)) + C$$

$$= e^{-x}\left(\frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)\right) + C$$

e) I found (d) simpler as using the complex exponential involves easier integration and simple algebra with complex numbers