MAST30025 Linear Statistical Models
Assignment 1

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Question 1

Since A is symmetric, there exists an orthogonal matrix P ($\implies P^T = P^{-1}$) which diagonalises A such that

$$
\begin{aligned}
& P^TAP = D \\
\implies\ & PP^TAP = PD \\
\implies\ & AP = PD \\
\implies\ & APP^T = PDP^T \\
\implies\ & A = PDP^T \\
\implies\ & A^2 = PDP^TPDP^T = PDDP^T = PD^2P^T
\end{aligned}
$$

D is a diagonal matrix with A's eigenvalues of 0s and 1s on its diagonal. Therefore D^2 can be represented like so

$$D^2 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} = D$$

$$
\begin{aligned}
\implies\ & D^2 = D \\
\implies\ & PD^2P^T = PDP^T \\
\implies\ & A^2 = A \implies A \textit{ is idempotent}
\end{aligned}
$$

Question 2

$A = A^2, \;\; B = B^2, \;\; A + B = (A+B)^2$

(a)

$A + B = (A+B)^2 = A^2 + AB + BA + B^2$
$\implies A = A^2 + AB + BA + B^2 - B$
$\implies A = A^2 + AB + BA + B - B$
$\implies A - A^2 = AB + BA$
$\implies A - A = AB + BA$
$\implies AB + BA = 0$

(b)

$A = PDP^T, \;\; B = P\Lambda P^T$

$so \;\; from \;\; (a) \;\; we \;\; have$

$PDP^T P\Lambda P^T + P\Lambda P^T PDP^T = 0$
$\implies PD\Lambda P^T + P\Lambda DP^T = 0$
$\implies PD\Lambda P^T = -P\Lambda DP^T$
$\implies D\Lambda = -\Lambda D$
$\implies \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} = \begin{bmatrix} -\Lambda_{11} & -\Lambda_{12} \\ -\Lambda_{21} & -\Lambda_{22} \end{bmatrix} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$
$\implies \begin{bmatrix} I_r\Lambda_{11} + 0 & I_r\Lambda_{12} + 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\Lambda_{11}I_r + 0 & 0 \\ -\Lambda_{21}I_r + 0 & 0 \end{bmatrix}$
$\implies \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\Lambda_{11} & 0 \\ -\Lambda_{21} & 0 \end{bmatrix}$

$Therefore \;\; we \;\; have$

$\Lambda_{11} = -\Lambda_{11} \implies \Lambda_{11} = 0$
$\Lambda_{12} = 0$
$-\Lambda_{21} = 0 \implies \Lambda_{21} = 0$

(c)

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = \begin{bmatrix}I_r & 0\\0 & 0\end{bmatrix}\begin{bmatrix}\Lambda_{11} & \Lambda_{12}\\\Lambda_{21} & \Lambda_{22}\end{bmatrix} = \begin{bmatrix}I_r & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & \Lambda_{22}\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & \Lambda_{22}\end{bmatrix}\begin{bmatrix}I_r & 0\\0 & 0\end{bmatrix}$$

$\implies 0 = D\Lambda = \Lambda D$
$\implies 0 = D\Lambda P^T = \Lambda D P^T$
$\implies 0 = PD\Lambda P^T = P\Lambda D P^T$
$\implies 0 = PDP^T P\Lambda P^T = P\Lambda P^T PDP^T$

$from\ \ (b)\ \ A = PDP^T\ \ and\ \ B = P\Lambda P^T$

$\implies AB = BA = 0$

Question 3

$var\ A\vec{y} = E[(A\vec{y} - A\vec{\mu})(A\vec{y} - A\vec{\mu})^T]$
$= E[A(\vec{y} - \vec{\mu})(\vec{y} - \vec{\mu})^T A^T]$
$= A\, E[(\vec{y} - \vec{\mu})(\vec{y} - \vec{\mu})^T]\, A^T$
$= A\,(var\ \vec{y})A^T$
$\implies var\ A\vec{y} = A\,(var\ \vec{y})A^T$

Question 4

(a)

$$\left|V - \lambda I\right| = 0$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0 \implies (1-\lambda)((1-\lambda)^2 - 1) = 0$$

$$\implies (1-\lambda)(1 - 2\lambda + \lambda^2 - 1) = 0 \implies (1-\lambda)(\lambda^2 - 2\lambda) = 0$$

so we have

$$1 - \lambda = 0 \implies \lambda = 1$$

or

$$\lambda^2 - 2\lambda = 0 \implies \lambda(\lambda - 2) = 0 \implies \lambda = 0, 2$$

$$(V - \lambda I)\vec{x} = 0$$

for $\lambda = 0$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = 0$

$x_2 - x_3 = 0 \implies x_2 = x_3$

$$\implies \vec{x} = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

where $t \in \mathbb{R} \setminus \{0\}$

for $\lambda = 1$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_2 = 0$

$-x_3 = 0$

$\implies x_1 \neq 0$ *as* $\vec{x} \neq 0$

$$\implies \vec{x} = t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$where \;\; t \in \mathbb{R} \setminus \{0\}$

$for \;\; \lambda = 2$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 = 0$
$-x_2 - x_3 = 0 \implies -x_2 = x_3$

$$\implies \vec{x} = t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$where \;\; t \in \mathbb{R} \setminus \{0\}$

(b)

$z_1 = 3y_1 + 2y_2 + y_3$

$$= \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = A\vec{y} \;\; where \;\; A = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$$

$\implies z_1 \sim MVN(A\vec{\mu},\ AVA^T)$

$$A\vec{\mu} = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = 3 + 4 + 4 = 11$$

[R code for calculating AVA^T]

```
A = matrix(c(3,2,1), 1, 3)
V = matrix(c(1, 0, 0, 0, 1, -1, 0, -1, 1), 3, 3)
varZ1 = A %*% V %*% t(A)
> varZ1 = 10
```

$\implies AVA^T = 10$
$\implies z_1 \sim MVN(11,\ 10)$
$\implies z_1 \sim N(11,\ 10)$

(c)

$z_2 = y_1^2 + (\frac{y_2+y_3}{2})^2 + (\frac{y_2-y_3}{2})^2$
$= y_1^2 + \frac{1}{4}y_2^2 + \frac{1}{2}y_2y_3 + \frac{1}{4}y_3^2 + \frac{1}{4}y_2^2 - \frac{1}{2}y_2y_3 + \frac{1}{4}y_3^2$
$= y_1^2 + \frac{1}{2}y_2^2 + \frac{1}{2}y_3^2$

$$\implies z_2 = \vec{y}^T B \vec{y} \ \ where \ \ B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$BV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

[R code for calculating** BV**]

```
B = matrix(c(1, 0, 0, 0, 1/2, 0, 0, 0, 1/2), 3, 3)
V = matrix(c(1, 0, 0, 0, 1, -1, 0, -1, 1), 3, 3)
B %*% V
(B %*% V) %*% (B %*% V)
```

$$BV = (BV)^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$\implies BV \ \ is \ \ idempotent \ \ and \ \ rank(BV) = tr(BV) = 2 \ \ as \ \ BV \ \ is \ \ idempotent \ \ and \ \ symmetric$
$\implies By \ \ Theorem \ \ 3.8, \ \ z_2 \sim \chi^2_{2,\frac{11}{2}} \ \ with \ \ \frac{11}{2} = \frac{1}{2}\vec{\mu}^T B \vec{\mu} = \lambda$

[R code for calculating** λ**]

```
B = matrix(c(1, 0, 0, 0, 1/2, 0, 0, 0, 1/2), 3, 3)
muVec = matrix(c(1,2,4), 3, 1)
ncp = 1/2 * t(muVec) %*% B %*% muVec
```

Question 5

(a)

$\vec{y} = X\vec{\beta} + \vec{\epsilon}$

$$\begin{bmatrix} 85 \\ 97 \\ 76 \\ 79 \\ 76 \\ 99 \\ 49 \\ 72 \\ 83 \end{bmatrix} = \begin{bmatrix} 1 & 86 \\ 1 & 85 \\ 1 & 89 \\ 1 & 82 \\ 1 & 84 \\ 1 & 86 \\ 1 & 84 \\ 1 & 78 \\ 1 & 92 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \\ \epsilon_7 \\ \epsilon_8 \\ \epsilon_9 \end{bmatrix}$$

(b)

$\vec{b} = (X^T X)^{-1} X^T \vec{y}$

[R code for calculating $\vec{b}$]

```
y = c(85,97,76,79,76,99,49,72,83)
X = matrix(c(rep(1,9),86,85,89,82,84,86,84,78,92), 9, 2)
b = solve(t(X) %*% X, t(X) %*% y)
```

$\vec{b} = \begin{bmatrix} -3.245 \\ 0.973 \end{bmatrix}$

(c)

$s^2 = \frac{(\vec{y}-X\vec{b})^T(\vec{y}-X\vec{b})}{n-(k+1)}$

[R code for calculating s^2]

```
y = c(85,97,76,79,76,99,49,72,83)
X = matrix(c(rep(1,9),86,85,89,82,84,86,84,78,92), 9, 2)
b = solve(t(X) %*% X, t(X) %*% y)
e = y-X %*% b
SSRes = sum(e^2)
s2 = SSRes/(length(y)-length(b))
> s2 = 231.447
```

$s^2 = 231.447$

(d)

$z_i = \frac{e_i}{\sqrt{s^2(1-H_{ii})}}$

[R code for calculating $\vec{z}$]

```
y = c(85,97,76,79,76,99,49,72,83)
X = matrix(c(rep(1,9),86,85,89,82,84,86,84,78,92), 9, 2)
b = solve(t(X) %*% X, t(X) %*% y)
e = y-X %*% b
SSRes = sum(e^2)
s2 = SSRes/(length(y)-length(b))
H = X %*% solve(t(X) %*% X) %*% t(X)
z = e / sqrt(sampleVar*(1-diag(H)))
```

$$\vec{z} = \begin{bmatrix} 0.320 \\ 1.224 \\ -0.550 \\ 0.180 \\ -0.173 \\ 1.300 \\ -2.066 \\ -0.060 \\ -0.298 \end{bmatrix}$$

(e)

$D_i = \frac{1}{k+1} z_i^2 (\frac{H_{ii}}{1-H_{ii}})$

[R code for calculating $\vec{D}$]

```
y = c(85,97,76,79,76,99,49,72,83)
X = matrix(c(rep(1,9),86,85,89,82,84,86,84,78,92), 9, 2)
b = solve(t(X) %*% X, t(X) %*% y)
e = y-X %*% b
SSRes = sum(e^2)
s2 = SSRes/(length(y)-length(b))
H = X %*% solve(t(X) %*% X) %*% t(X)
z = e / sqrt(sampleVar*(1-diag(H)))
cooksDist = (1/length(b))*(z^2)*(diag(H)/(1-diag(H)))
```

$$\vec{D} = \begin{bmatrix} 0.007 \\ 0.094 \\ 0.045 \\ 0.004 \\ 0.002 \\ 0.112 \\ 0.293 \\ 0.002 \\ 0.042 \end{bmatrix}$$

(f)

$Let\ \ \hat{y} = the\ \ point\ \ estimate\ \ for\ \ x_1 = 90$

$Then\ \ \hat{y} = \vec{t}^T \vec{b}$

$where\ \ \vec{t} = \begin{bmatrix} 1 \\ 90 \end{bmatrix}\ \ and\ \ \vec{b} = \begin{bmatrix} -3.245 \\ 0.973 \end{bmatrix}$

[R code for calculating $\hat{y}$]

```
y = c(85,97,76,79,76,99,49,72,83)
X = matrix(c(rep(1,9),86,85,89,82,84,86,84,78,92), 9, 2)
b = solve(t(X) %*% X, t(X) %*% y)
> c(1,90) %*% b = 84.312
```

$\hat{y} = 84.312$