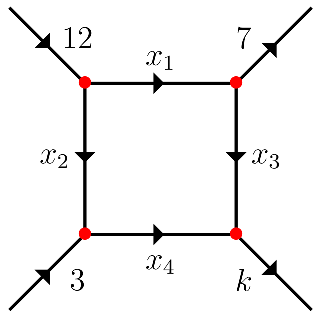
School of Mathematics and Statistics

MAST10007 Linear Algebra, Semester 1 2020

Solutions to Written Assignment 1

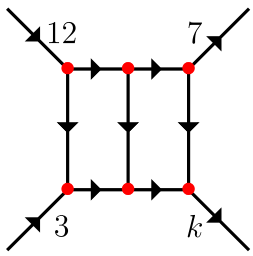
1. Consider the following flow diagram where the flow into any vertex must equal the flow out.



(a) Write down the equations corresponding to the flow at each of the vertices.

(b) Find the values of $k \in \mathbb{R}$ for which the system in part (a) is (i) consistent and (ii) inconsistent, by reducing the augmented matrix to row-echelon form. For (i) find the general solution.

(c) Add an extra path to the the flow diagram as shown below. For the value(s) of k in (b)(i) show that the new system is consistent. (Hint: You can answer this question without setting up the linear system.)



Solution:

(a)

$$12 = x_1 + x_2, \quad 3 + x_2 = x_4, \quad x_1 = 7 + x_3, \quad x_3 + x_4 = k$$

[1 mark]

(b)

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -1 & -3 \\ 1 & 0 & -1 & 0 & 7 \\ 0 & 0 & 1 & 1 & k \end{array}\right] \underset{\sim}{R3 \to R3 - R1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & -1 & -1 & 0 & -5 \\ 0 & 0 & 1 & 1 & k \end{array}\right] \underset{\sim}{R3 \to R3 + R2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & -1 & -1 & -8 \\ 0 & 0 & 1 & 1 & k \end{array}\right]$$

$$\underset{\sim}{R3 \to -R3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 1 & 1 & k \end{array}\right] \underset{\sim}{R4 \to R4 - R3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & k-8 \end{array}\right]$$

(Note that multiple row echelon forms are possible, but can easily check the answer by working out the reduced row echelon form, which is unique.) [2 marks]

The solution is consistent when $k = 8$ and inconsistent for $k \neq 8$. [1 mark]

When $k = 8$, reduce to reduced row-echelon form:

$$\underset{\sim}{R1 \to R1 - R2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 15 \\ 0 & 1 & 0 & -1 & -3 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Set $x_4 = t \in \mathbb{R}$, then

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 15-t \\ -3+t \\ 8-t \\ t \end{bmatrix} = \begin{bmatrix} 15 \\ -3 \\ 8 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

[2 marks]

(c) When $k = 8$ the system is consistent since every solution of (a) is a solution of the new system by setting the new edge flow to be zero. [1 mark]

Completely correct notation, e.g. $R1 \to R1 + R3$ and $\sim$ (not $=$) [2 marks]

[TOTAL 9 marks]

2. (a) Use row operations to find the inverse of

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

(b) In part (a), your implementation of the algorithm should have produced

$$[A|I] \sim [A_1|B_1] \sim \ldots \sim [A_m|B_m] \sim [I|B].$$

Choose any intermediate augmented matrix $[A_k|B_k]$ (not $[A|I]$ or $[I|B]$.) Calculate $B_kA - A_k$.

Solution:

(a)

$$[A|I] = \left[\begin{array}{rrr|rrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\overset{R2\to R2-R1}{R3\to R3+R1}} \left[\begin{array}{rrr|rrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\overset{R2\to -\frac{1}{2}R2}{\sim} \left[\begin{array}{rrr|rrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right] \overset{R3\to R3-2R2}{\sim} \left[\begin{array}{rrr|rrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$\overset{R3\to \frac{1}{2}R3}{\sim} \left[\begin{array}{rrr|rrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \underset{\sim}{\overset{R1\to R1+R3}{R2\to R2+R3}} \left[\begin{array}{rrr|rrr} 1 & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\overset{R1\to R1-R2}{\sim} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

[2 marks]

Hence

$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

[1 mark]

(b) For example, taking $k = 3$,

$$[A_3|B_3] = \left[\begin{array}{rrr|rrr} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$B_3A - A_3 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} = \text{zero matrix}$$

[1 mark]

(In fact: $B_kA - A_k$ is always the zero matrix – can you prove this?)

Completely correct notation, e.g. $R1 \to R1 + R3$ and $\sim$ (not $=$) [2 marks]

[TOTAL 6 marks]