MAST20004 Probability
Semester 1, 2021
Assignment One: Questions

Due 3 pm, Friday 26 March 2021

Name: James La Fontaine	
Student ID: 1079860	

Important instructions:

(1) This assignment contains 4 questions, **two** of which will be randomly selected to be marked. Each marked question is worth 10 points and each unmarked question with substantial working is worth 1 point.

(2) To complete this assignment, you need to write your solutions into the blank answer spaces following each question in this assignment PDF.

- If you have a printer (or can access one), then you must print out the assignment template and handwrite your solutions into the answer spaces.
- If you do not have a printer but you can figure out how to annotate a PDF using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly onto the assignment PDF and save a copy for submission.

Failing both of these methods, you may handwrite your answers as normal on blank paper and then scan for submission (but note that you will thereby miss valuable practice for the exam process). In that case, however, your document should have the same length as the assignment template otherwise Gradescope will reject your submission. So you will need to add as many blank pages as necessary to reach that criterion.

Scan your assignment to a PDF file using your mobile phone (we recommend Cam - Scanner App), then upload by going to the Assignments menu on Canvas and submit the PDF to the GradeScope tool by first selecting your PDF file and then clicking on 'Upload PDF'.

(3) A poor presentation penalty of 10% of the total available marks will apply unless your submitted assignment meets all of the following requirements:

- it is a single pdf with all pages in correct template order and the correct way up, and with any blank pages with additional working added only at the end of the template pages;
- has all pages clearly readable;
- has all pages cropped to the A4 borders of the original page and is imaged from directly above to avoid excessive 'keystoning'.

These requirements are easy to meet if you use a scanning app on your phone and take some care with your submission - please review it before submitting to double check you have satisfied all of the above requirements.

(4) Late submission within 20 hours after the deadline will be penalised by 5% of the total available marks for every hour or part thereof after the deadline. After that, the Gradescope submission channel will be closed, and your submission will no longer be accepted. You are strongly encouraged to submit the assignment a few days before the deadline just in case of unexpected technical issues. If you are facing a rather exceptional/extreme situation that prevents you from submitting on time, please contact the tutor coordinator **Robert Maillardet** with formal proofs such as medical certificate.

(5) Working and reasoning must be given to obtain full credit. Clarity, neatness, and style count.

Problem 1. Let $A_1, \ldots, A_n$ be a finite collection of sets.

(i) Formulate a version of De Morgan's laws for the collection of sets.

① $(A_1 \cup A_2 \cup \ldots \cup A_n)^c = A_1^c \cap A_2^c \cap \ldots \cap A_n^c$

② $(A_1 \cap A_2 \cap \ldots \cap A_n)^c = A_1^c \cup A_2^c \cup \ldots \cup A_n^c$

① $\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$ ② $\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$

(ii) Prove the De Morgan's laws in (i) by the elementwise method.

① let $x \in \left(\bigcup_{i=1}^{n} A_i\right)^c$

$\Rightarrow x \notin A_1 \cup A_2 \cup \ldots \cup A_n$

$\Rightarrow x \notin A_1$ and $x \notin A_2$ and ... and $x \notin A_n$

$\Rightarrow x \in A_1^c$ and $x \in A_2^c$ and... and $x \in A_n^c$

$\Rightarrow x \in A_1^c \cap A_2^c \cap \ldots \cap A_n^c$

$\Rightarrow \left(\bigcup_{i=1}^{n} A_i\right)^c \subset \bigcap_{i=1}^{n} A_i^c$

let $x \in \bigcap_{i=1}^{n} A_i^c$

$\Rightarrow x \in A_1^c$ and $x \in A_2^c$ and ... and $x \in A_n^c$

$\Rightarrow x \notin A_1$ and $x \notin A_2$ and ... and $x \notin A_n$

$\Rightarrow x \notin A_1 \cup A_2 \cup \ldots \cup A_n$

$\Rightarrow x \in (A_1 \cup A_2 \cup \ldots \cup A_n)^c$

$\Rightarrow \bigcap_{i=1}^{n} A_i^c \subset \left(\bigcup_{i=1}^{n} A_i\right)^c$

Therefore

$\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} A_i^c$

② Let $x \in \left(\bigcap_{i=1}^{n} A_i\right)^c$

$\Rightarrow x \notin A_1 \cap A_2 \cap \ldots \cap A_n$

$\Rightarrow x \notin A_1$ or $x \notin A_2$ or ... or $x \notin A_n$

$\Rightarrow x \in A_1^c$ or $x \in A_2^c$ or ... or $x \in A_n^c$

$\Rightarrow x \in A_1^c \cup A_2^c \cup \ldots \cup A_n^c$

$\Rightarrow \left(\bigcap_{i=1}^{n} A_i\right)^c \subset \bigcup_{i=1}^{n} A_i^c$

let $x \in \bigcup_{i=1}^{n} A_i^c$

$\Rightarrow x \in A_1^c \cup A_2^c \cup \ldots \cup A_n^c$

$\Rightarrow x \in A_1^c$ or $x \in A_2^c$ or ... or $x \in A_n^c$

$\Rightarrow x \notin A_1$ or $x \notin A_2$ or ... or $x \notin A_n$

$\Rightarrow x \notin A_1 \cap A_2 \cap \ldots \cap A_n$

$\Rightarrow x \in \left(\bigcap_{i=1}^{n} A_i\right)^c$

$\Rightarrow \bigcup_{i=1}^{n} A_i^c \subset \left(\bigcap_{i=1}^{n} A_i\right)^c$

$\Rightarrow$ Therefore

$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} A_i^c$

$\blacksquare$

Problem 2. Let A, B, C be three events with $\mathbb{P}(B) > 0$ and $\mathbb{P}(B^c) > 0$. For each of the following statements, determine whether it is true or false. If it is true, give a proof; if it is false, give a counterexample.

(i) If A and B are independent, then $\mathbb{P}(A|B) = \mathbb{P}(A|B^c)$.

A and B are independent $\Rightarrow P(A\cap B) = P(A)\cdot P(B)$

$A = (A\cap B^c) \cup (A\cap B)$ and $(A\cap B^c)\cap(A\cap B) = \emptyset$

$\Rightarrow$ By axiom 3, $P(A) = P(A\cap B^c) + P(A\cap B)$

$= P(A\cap B^c) + P(A)\cdot P(B)$ (independence)

$\Rightarrow P(A\cap B^c) = P(A)\cdot(1-P(B))$

$= P(A)\cdot P(B^c)$

$\Rightarrow$ A and B^c are independent

$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{P(A)\cdot P(B)}{P(B)} = P(A)$

$P(A|B^c) = \frac{P(A\cap B^c)}{P(B^c)} = \frac{P(A)\cdot P(B^c)}{P(B^c)} = P(A)$

$\Rightarrow P(A|B) = P(A|B^c)$

Therefore this statement is true. $\blacksquare$

(ii) If $\mathbb{P}(A|B) = \mathbb{P}(A|B^c)$, then A and B are independent.

$P(A|B) = P(A|B^c)$

B and B^c are disjoint and exhaustive events

so by the Law of Total Probability,

$P(A) = P(A|B)\cdot P(B) + P(A|B^c)\cdot P(B^c)$

$= P(A|B)\cdot P(B) + P(A|B)\cdot(1-P(B))$

$= P(A|B)(P(B)+1-P(B))$

$= P(A|B)$

$\Rightarrow P(A) = P(A|B)$

$\Rightarrow P(A\cap B) = P(A)\cdot P(B)$

$\Rightarrow$ A and B are independent

Therefore this statement is true. $\blacksquare$

(iii) If A, B, C are independent, then A and $B^c \cap C$ are independent.

A, B, C are mutually independent:

$\Rightarrow P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$ $\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$\Rightarrow P(A \cap C) = P(A) \cdot P(C)$ $\Rightarrow P(B \cap C) = P(B) \cdot P(C)$

$C = (B^c \cap C) \cup (B \cap C)$ and $(B^c \cap C) \cap (B \cap C) = \emptyset$

$\Rightarrow$ By axiom 3, $P(C) = P(B^c \cap C) + P(B \cap C)$
$= P(B^c \cap C) + P(B) \cdot P(C)$

$\Rightarrow P(B^c \cap C) = (1 - P(B)) \cdot P(C)$
$= P(B^c) \cdot P(C)$

Now, $P(A \cap (B^c \cap C)) = P(A) \cdot P(B^c \cap C)$ if A and $B^c \cap C$ are independent

$P(A \cap (B^c \cap C)) = P(A \cap C) - P(A \cap B \cap C)$
$= P(A) \cdot P(C) - P(A) \cdot P(B) \cdot P(C)$
$= P(A) \cdot P(C)(1 - P(B)) = P(A) \cdot P(C) \cdot P(B^c)$
$= P(A) \cdot P(B^c \cap C)$

There fore this statement is true. ☒

(iv) If A and B are independent, A and C are independent, and A and $B \cap C$ are independent, then A, B, C are independent.

$P(A \cap B) = P(A) \cdot P(B)$ $P(A \cap C) = P(A) \cdot P(C)$

$P(A \cap (B \cap C)) = P(A) \cdot P(B \cap C)$

For A, B, C to be mutually independent we must also have

$P(B \cap C) = P(B) \cdot P(C)$

Consider the rolling of a fair die:

$A = \{3, 4, 5, 6\}$ $P(A) = 2/3$

$B = \{2, 3, 4\}$ $P(B) = 1/2$

$C = \{2, 4, 6\}$ $P(C) = 1/2$

$P(A \cap B) = 1/3 = 2/3 \cdot 1/2$ ✓

$P(A \cap C) = 1/3 = 2/3 \cdot 1/2$ ✓

$P(A \cap (B \cap C)) = 1/6 = 2/3 \cdot 1/2 \cdot 1/2$ ✓

$P(B \cap C) = 1/3 \neq 1/2 \cdot 1/2 = 1/4$

Therefore A, B, C are not mutually independent and the statement is false. ☒

Problem 3. There are three urns labelled as u_1, u_2 and u_3: initially u_1 contains 2 black balls and 3 white balls, u_2 has 1 black ball and 4 white balls, and u_3 holds 3 black balls and 2 white balls. We first mix the balls in u_1, randomly draw a ball from u_1, place it in u_2; next, mix the balls in u_2, randomly draw a ball from u_2 and place it in u_3; finally, mix the balls in u_3 and randomly draw a ball from u_3.

(i) What is the probability that the ball drawn from u_2 is white?

W_2 = ball drawn from u_2 is white

W_1 = ball drawn from u_1 is white $\quad P(W_1) = 3/5$

B_1 = ball drawn from u_1 is black $\quad P(B_1) = 2/5$

W_1 and B_1 partition the sample space

Therefore by using the Law of Total Probability,

$$P(W_2) = P(W_2|W_1)\cdot P(W_1) + P(W_2|B_1)\cdot P(B_1)$$
$$= \frac{5}{6}\cdot\frac{3}{5} + \frac{2}{3}\cdot\frac{2}{5}$$
$$= \frac{23}{30}$$

(ii) What is the probability that the ball drawn from u_3 is white?

W_3 = ball drawn from u_3 is white

B_2 = ball drawn from u_2 is black

$$P(B_2) = 1 - P(W_2) = 7/30$$

W_2 and B_2 partition the sample space so by the Law of Total Probability,

$$P(W_3) = P(W_3|W_2)\cdot P(W_2) + P(W_3|B_2)\cdot P(B_2)$$
$$= \frac{1}{2}\cdot\frac{23}{30} + \frac{1}{3}\cdot\frac{7}{30} = \frac{83}{180}$$

(iii) Given that the ball drawn from u_3 is white, what is the probability that the ball drawn from u_2 is white?

$$P(W_2|W_3) = \frac{P(W_3|W_2)\cdot P(W_2)}{P(W_3|W_2)\cdot P(W_2) + P(W_3|B_2)\cdot P(B_2)}$$

(Bayes' formula)

$$= \frac{\frac{1}{2}\cdot\frac{23}{30}}{\frac{1}{2}\cdot\frac{23}{30} + \frac{1}{3}\cdot\frac{7}{30}} = \frac{69}{83}$$

(iv) Given that the ball drawn from u_3 is white, what is the probability that the ball drawn from u_1 is white?

$$P(W_1|W_3) = \frac{P(W_3|W_1)\cdot P(W_1)}{P(W_3|W_1)\cdot P(W_1) + P(W_3|B_1)\cdot P(B_1)}$$

$$P(W_3|W_1) = P(W_2|W_1)\cdot P(W_3|W_2) + P(B_2|W_1)\cdot P(W_3|B_2)$$

$$= \frac{5}{6}\cdot\frac{1}{2} + \frac{1}{6}\cdot\frac{1}{3} = \frac{17}{36}$$

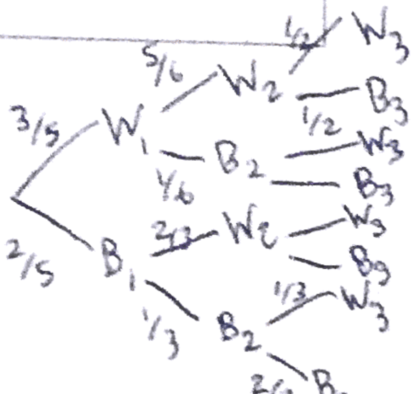
$$P(W_3|B_1) = P(W_2|B_1)\cdot P(W_3|W_2) + P(B_2|B_1)\cdot P(W_3|B_2)$$

$$= \frac{2}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{3} = \frac{4}{9}$$

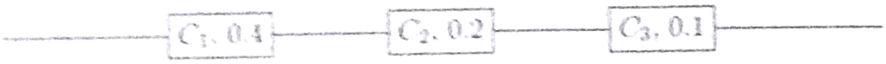
$$\Rightarrow P(W_1|W_3) = \frac{\frac{17}{36}\cdot\frac{3}{5}}{\frac{17}{36}\cdot\frac{3}{5} + \frac{4}{9}\cdot\frac{2}{5}}$$

(Bayes' formula)

$$= \frac{51}{83}$$



Problem 4. A circuit contains three mutually independent components C_1, C_2, C_3 in series as shown in the figure below.



The probability of failure for each component is indicated in the figure respectively.

(i) What is the probability that the circuit will fail?

F_i = Component C_i fails

$$
\begin{aligned}
P(\text{system fails}) &= 1 - P(\text{system doesn't fail}) \\
&= 1 - P(F_1^c \cap F_2^c \cap F_3^c) \\
&= 1 - 0.6 \times 0.8 \times 0.9 \\
&= 0.568 \\
&= \frac{71}{125}
\end{aligned}
$$

(ii) What is the probability that exactly one component fails?

let A = one component fails

$$
A = (F_1 \cap F_2^c \cap F_3^c) \cup (F_2 \cap F_1^c \cap F_3^c) \cup (F_3 \cap F_1^c \cap F_2^c)
$$

These events are disjoint so by axiom 3 and by independence,

$$
P(A) = P(F_1) \cdot P(F_2^c) \cdot P(F_3^c) + P(F_2) \cdot P(F_1^c) \cdot P(F_3^c) + P(F_3) \cdot P(F_1^c) \cdot P(F_2^c)
$$

$$
= 0.536 = \frac{67}{125}
$$

(iii) Given that exactly one of the components fails, what are the respective probabilities that the failed component is C_1, C_2, C_3?

$$P(F_1|A) = \frac{P(A|F_1)\cdot P(F_1)}{P(A|F_1)\cdot P(F_1)+P(A|F_2)\cdot P(F_2)+P(A|F_3)\cdot P(F_3)}$$

$$= \frac{P(F_2^c \cap F_3^c)\cdot P(F_1)}{P(F_2^c \cap F_3^c)\cdot P(F_1)+P(F_1^c \cap F_3^c)\cdot P(F_2)+P(F_1^c \cap F_2^c)\cdot P(F_3)}$$

$$= 0.6486 = \frac{24}{37}$$

(Bayes' Formula)

$P(F_2|A) = 0.2432 = 9/37$

$P(F_3|A) = 0.1081 = 4/37$

(each component failing is disjoint in this case)

(iv) Assume that exactly one of the components failed, an engineer is assigned to detect the failed component. To minimise the number X of tests, what is the order of the components they should check? Explain your answer.

They should check C_1 first, then C_2 and C_3 last (although it must be C_3 if it isn't C_1 or C_2). This gives the highest probability of finding the failed components earlier by checking the most likely culprits first and thus minimising the number of tests.

(v) Derive the probability mass function of X in (iv).

x	1	2	3
$P_X(x)$	$\frac{24}{37}$	$\frac{9}{37}$	$\frac{4}{37}$

$$P_X(x) = \begin{cases} \frac{24}{37} & \text{if } x=1 \\ \frac{9}{37} & \text{if } x=2 \\ \frac{4}{37} & \text{if } x=3 \end{cases}$$