School of Mathematics and Statistics

MAST10007 Linear Algebra, Semester 1 2020

Written assignment 4

Submit your assignment online in Canvas before 12 noon on Monday 11th May.

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- This assignment is worth $1\frac{1}{9}\%$ of your final MAST10007 mark.
- Your solutions should be neatly handwritten in blue or black pen, then uploaded as a single PDF file in **GradeScope**.
- Full explanations and working must be shown in your solutions.
- Marks may be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- You must use methods taught in MAST10007 Linear Algebra to solve the assignment questions.

New submission guidelines:

- This assignment is being handled using a similar process to that planned for the final exam so you can start to become familiar with it.
- If you have access to a printer, then you should print out this assignment sheet and handwrite your solutions into the answer boxes.
- If you do not have access to a printer, but you can annotate a PDF file using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly in the boxes on the assignment PDF and save a copy for submission.
- Otherwise, you may handwrite your answers as normal on blank paper and then scan for submission.
- The answer boxes should typically provide sufficient space for your solution, but if you find you need extra space please take a blank sheet of paper and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- Scan your assignment to a PDF file using your mobile phone or scanner, then upload by going to the Assignments menu on Canvas and submit the PDF to the **GradeScope** tool by first selecting your PDF file and then clicking on 'Upload pdf'.

1. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ and $M_{2,2}$ be the vector space of 2×2 matrices with real entries. Define

$$S = \{A \in M_{2,2} \mid \text{there exists } r \in \mathbb{R} \text{ such that } A\mathbf{v} = r\mathbf{v}\}.$$

(a) Write down an element of S that is not a scalar multiple of the identity matrix I.

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \in S$, since $\begin{bmatrix} 1+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$\left(\begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} r \\ 2r \end{bmatrix}\right)$ $\quad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \neq \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) Prove that S is a subspace of $M_{2,2}$.

Let $\underset{\sim}{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S$, $\underset{\sim}{v} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in S$, $\alpha \in \mathbb{R}$

Then there exists a $r \in \mathbb{R}$ such that

$\underset{\sim}{u}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = r\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} r \\ 2r \end{bmatrix} \Rightarrow \begin{matrix} a+2b=r \\ c+2d=2r \end{matrix}$

$\underset{\sim}{v}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = r\begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a'+2b' \\ c'+2d' \end{bmatrix} = \begin{bmatrix} r \\ 2r \end{bmatrix} \Rightarrow \begin{matrix} a'+2b'=r \\ c'+2d'=2r \end{matrix}$

0 S is not empty:

$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \in S$ since $\begin{bmatrix} 1+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $3 \in \mathbb{R}$

$\Rightarrow$ S is not empty

1 Closure under vector addition:

$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$

$(a+a') + 2(b+b') = r \in \mathbb{R}$ as $a, a', b, b' \in \mathbb{R}$

$(c+c') + 2(d+d') = 2r \in \mathbb{R}$ as $c, c', d, d' \in \mathbb{R}$

$\Rightarrow \underset{\sim}{u} + \underset{\sim}{v} \in S$

2 Closure under scalar multiplication:

$\alpha \underset{\sim}{u} = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$

$\Rightarrow \alpha a + \alpha 2b = r$

$\alpha(a+2b) = r \in \mathbb{R}$

$\Rightarrow \alpha c + \alpha 2d = 2r$

$\alpha(c+2d) = 2r \in \mathbb{R}$

$\Rightarrow \alpha \underset{\sim}{u} \in S$

$\Rightarrow$ S is a subspace of $M_{2,2}$

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(c) Find a basis for S and calculate the dimension of S.

Hint. there exists $r \in \mathbb{R}$ such that $A\mathbf{v} = r\mathbf{v} \Leftrightarrow \begin{bmatrix} -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0.$

2. Let $\mathcal{P}_2$ be the real vector space of polynomials of degree at most 2. Define

$$p_0(x) = \tfrac{1}{2}(x-1)(x-2), \quad p_1(x) = -x(x-2), \quad p_2(x) = \tfrac{1}{2}x(x-1).$$

(a) Prove that every polynomial $f(x) \in \mathcal{P}_2$ satisfies

$$f(x) = f(0)p_0(x) + f(1)p_1(x) + f(2)p_2(x).$$

$P_0(x) = 1 - \frac{3}{2}x + \frac{1}{2}x^2$, $P_1(x) = 0 + 2x - x^2$, $P_2(x) = 0 - \frac{1}{2}x + \frac{1}{2}x^2$

$f(0) = \alpha_1$ $f(1) = \alpha_2$ $f(2) = \alpha_3$

$a_0 + a_1 x + a_2 x^2 = \alpha_1 P_0(x) + \alpha_2 P_1(x) + \alpha_3 P_2(x)$

$\Rightarrow (\alpha_1 + 0\alpha_2 + 0\alpha_3, -\frac{3}{2}\alpha_1 + 2\alpha_2 - \frac{1}{2}\alpha_3, \frac{1}{2}\alpha_1 - \alpha_2 + \frac{1}{2}\alpha_3)$

$A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{matrix} R_2 \to R_2 + \frac{3}{2}R_1 \\ R_3 \to R_3 - \frac{1}{2}R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix}$

$R_3 \to R_3 + \frac{1}{2}R_2 \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

rank(A) = no. of vectors
So there is a unique solution for all
$f(x) = a_0 + a_1 x + a_2 x^2$

$\Rightarrow$ Equation is satisfied for all $f(x) \in P_2$ //

(b) Use (a), or otherwise, to prove that $\{p_0(x), p_1(x), p_2(x)\}$ is a basis of $\mathcal{P}_2$.

From (a), $P_0(x)$, $P_1(x)$, $P_2(x)$ span P_2.

$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$

$(1, -\frac{3}{2}, \frac{1}{2}) \neq \alpha(0, 2, -1)$ for $\alpha \in \mathbb{R}$

$(1, -\frac{3}{2}, \frac{1}{2}) \neq \alpha(0, -\frac{1}{2}, \frac{1}{2})$ for $\alpha \in \mathbb{R}$

$(0, 2, -1) \neq \alpha(0, -\frac{1}{2}, \frac{1}{2})$ for $\alpha \in \mathbb{R}$

$\Rightarrow$ They are linearly independent.

$\Rightarrow \{P_0(x), P_1(x), P_2(x)\}$ is a basis of P_2. //