School of Mathematics and Statistics

MAST10007 Linear Algebra, Semester 1 2020

Solutions for Written assignment 4

Submit your assignment online in Canvas before 12 noon on Monday 11th May.

Name:

Student ID:

- This assignment is worth $1\frac{1}{9}$% of your final MAST10007 mark.
- Assignments should be neatly handwritten in blue or black pen, then upload a scan of your solutions in **Gradescope**. (See instructions on the Written Assignments page under Modules on Canvas.)
- Full explanations and working must be shown in your solutions.
- Marks may be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- You must use methods taught in MAST10007 Linear Algebra to solve the assignment questions.

1. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ and $M_{2,2}$ be the vector space of 2×2 matrices with real entries. Define

$$S = \{A \in M_{2,2} \mid \text{there exists } r \in \mathbb{R} \text{ such that } A\mathbf{v} = r\mathbf{v}\}.$$

(a) Write down an element of S that is not a scalar multiple of the identity matrix I.

For example, $\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \in S$ since $\begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
(Many other examples are possible.) [1 mark]

(b) Prove that S is a subspace of $M_{2,2}$.

We verify the conditions of the subspace theorem.

(0) $I \in S$ hence S is non-empty. [1 mark]

(1) Let $A, B \in S$ so there exists $r, s \in \mathbb{R}$ such that $Av = rv$ and $Bv = sv$. Then

$$(A + B)v = Av + Bv = rv + sv = (r + s)v$$

hence $A + B \in S$. [2 marks]

(2) Let $\lambda \in \mathbb{R}$. Then $\lambda Av = \lambda rv = (\lambda r)v$ hence $\lambda A \in S$. [1 mark]

By the subspace theorem, S is a subspace of $M_{2,2}$. [1 mark]

(c) Find a basis for S and calculate the dimension of S.

Hint. Use: there exists $r \in \mathbb{R}$ such that $A\mathbf{v} = r\mathbf{v} \Leftrightarrow \begin{bmatrix} -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0.$

For $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the condition $Av = rv$ produces the equations:

$$\begin{aligned} a_{11} + 2a_{12} &= r \\ a_{21} + 2a_{22} &= 2r. \end{aligned}$$

Eliminate r:

$$2(a_{11} + 2a_{12}) = a_{21} + 2a_{22}.$$

This system has matrix $[2\ 4\ -1\ -2]$ which is in row echelon form.
Set $a_{12} = s$, $a_{21} = t$, $a_{22} = u \in \mathbb{R}$. Then $a_{11} = -2s + \frac{1}{2}t + u$ and

$$\begin{aligned} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} &= \begin{bmatrix} -2s + \frac{1}{2}t + u & s \\ t & u \end{bmatrix} \\ &= s \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{bmatrix} + u \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

[1 mark]

so a basis for S is

$$\left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

since this spans S and is linearly independent. [1 mark]

Hence $\dim S = 3$. [1 mark]

Using the hint:

$$(-2 \quad 1) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -2a_{11} + a_{21} - 4a_{12} + 2a_{22} = 0$$

which gives the same linear system as above.

Correct notation throughout Q1: [1 mark]

[TOTAL 10 marks]

2. Let $\mathcal{P}_2$ be the real vector space of polynomials of degree at most 2. Define

$$p_0(x) = \tfrac{1}{2}(x-1)(x-2), \quad p_1(x) = -x(x-2), \quad p_2(x) = \tfrac{1}{2}x(x-1).$$

(a) Prove that any polynomial $f(x) \in \mathcal{P}_2$ satisfies

$$f(x) = f(0)p_0(x) + f(1)p_1(x) + f(2)p_2(x).$$

Method 1: Given any polynomial $f(x) = a + bx + cx^2 \in \mathcal{P}_2$, we have

$$\begin{aligned} &f(0)p_0(x) + f(1)p_1(x) + f(2)p_2(x) \\ &= ap_0(x) + (a+b+c)p_1(x) + (a+2b+4c)p_2(x) \\ &= \frac{1}{2}a(x^2-3x+2) + (a+b+c)(-x^2+2x) + \frac{1}{2}(a+2b+4c)(x^2-x) \\ &= a + bx + cx^2 = f(x). \end{aligned}$$

Method 2: Given any $f \in \mathcal{P}_2$, the polynomial

$$F = f - (f(0)p_0(x) + f(1)p_1(x) + f(2)p_2(x)) \in \mathcal{P}_2$$

satisfies

$$F(0) = 0 = F(1) = F(2).$$

We will prove that $F \equiv 0$. Set $F = a + bx + cx^2$, and solve for a, b, c given $F(0) = 0 = F(1) = F(2)$. This is the linear system

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

and hence has a unique solution $(a, b, c) = (0, 0, 0)$.
Thus $F \equiv 0$ and $f = f(0)p_0(x) + f(1)p_1(x) + f(2)p_2(x)$.

[4 marks]

(b) Use (a), or otherwise, to prove that $\{p_0(x), p_1(x), p_2(x)\}$ is a basis of $\mathcal{P}_2$.

By (a), any $f(x) \in \mathcal{P}_2$ is a linear combination of $\{p_0(x), p_1(x), p_2(x)\}$, hence this is a *spanning set.*
If

$$\mathbf{0} = \alpha_0 p_0(x) + \alpha_1 p_1(x) + \alpha_2 p_2(x) \text{ with } \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R},$$

then evaluate at $x = 0$ to get

$$0 = \alpha_0 \cdot 1 + \alpha_1 \cdot 0 + \alpha_2 \cdot 0 \Rightarrow \alpha_0 = 0.$$

Similarly, evaluate at $x = 1$ to get $\alpha_1 = 0$ and evaluate at $x = 2$ to get $\alpha_2 = 0$. Hence $\{p_0(x), p_1(x), p_2(x)\}$ is *linearly independent.*
Hence $\{p_0(x), p_1(x), p_2(x)\}$ is a basis of $\mathcal{P}_2$.

Alternatively: $\{p_0(x), p_1(x), p_2(x)\}$ spans $\mathcal{P}_2$ by (a), and has number of vectors equal to $\dim \mathcal{P}_2 = 3$, so is a basis for $\mathcal{P}_2$ by a result from lectures (second theorem on slide 211).

[4 marks]

Correct notation throughout Q2: [1 mark]

[TOTAL 9 marks]