TASK 5A

$I_D = \{1, 2, 3\}$ $F: (\forall x (Q(x))) \vee \exists x ((\forall y R(y,x) \vee Q(x)) \Rightarrow \exists z \forall y \ P(z,y))$

Predicates:

$Q(x)$: x is an odd number

$P(z,y): z=y$

$P(z,y)$	1	2	3
1	t	f	f
2	f	t	f
3	f	f	t

$Q(x)$	
1	t
2	f
3	t

$R(y,x): y > x$

$R(y,x)$	1	2	3
1	f	f	f
2	t	f	f
3	t	t	f

Valuation: $\sigma(x) = 1$, $\sigma(z) = 2$

$Q(2) = f$ so $\forall x \ Q(x)$ evaluates to f

$\sigma(x) = 1$ so $\exists x ((\forall y \ R(y,x) \vee Q(x))$ evaluates to t as $Q(1) = t$

$\exists z \ \forall y \ (P(z,y))$ evaluates to f as there is no number z in I_D which is equal to all other numbers

$f \vee (t \Rightarrow f)$ evaluates to f so F is non-valid.

TASK 5B

Show that $F \vee \neg G$ is valid $\equiv$ show that $\neg F \wedge G$ is unsatisfiable

$\neg((\forall x\, Q(x)) \vee \exists x\, ((\forall y\, R(y,x) \vee Q(x)) \Rightarrow \exists z \forall y\, P(z,y)))$
$\wedge (\exists x\, \forall y\, (P(x,y) \vee (\exists z\, R(y,z) \Rightarrow \forall w\, Q(w))))$

① Remove $\Rightarrow$

$\neg((\forall x\, Q(x)) \vee \exists x\, (\neg(\forall y\, R(y,x) \vee Q(x)) \vee \exists z \forall y\, P(z,y)))$
$\wedge (\exists x\, \forall y\, (P(x,y) \vee (\neg(\exists z\, R(y,z)) \vee \forall w\, Q(w))))$

② Push negations in

$((\exists x\, \neg Q(x)) \wedge \forall x\, ((\forall y\, R(y,x) \vee Q(x)) \wedge \forall z\, \exists y\, \neg P(z,y)))$
$\wedge (\exists x\, \forall y\, (P(x,y) \vee (\forall z\, \neg R(y,z)) \vee \forall w\, Q(w)))$

③ Standardize bound variables apart

$((\exists x\, \neg Q(x)) \wedge \forall u\, ((\forall y\, R(y,u) \vee Q(u)) \wedge \forall z\, \exists v\, \neg P(z,v)))$
$\wedge (\exists r\, \forall t\, (P(r,t) \vee (\forall q\, \neg R(t,q)) \vee \forall w\, Q(w)))$

④ Remove $\exists$ $\quad x \mapsto a,\ v \mapsto f(z,u),\ r \mapsto b$

$((\neg Q(a)) \wedge \forall u\, ((\forall y\, R(y,u) \vee Q(u)) \wedge \forall z\, \neg P(z, f(z,u)))$
$\wedge (\forall t\, (P(b,t) \vee (\forall q\, \neg R(t,q)) \vee \forall w\, Q(w)))$

⑤ + ⑥ Remove $\forall$ and convert to CNF

$\neg Q(a) \wedge (R(y,u) \vee Q(u)) \wedge \neg P(z, f(z,u))$
$\wedge (P(b,t) \vee \neg R(t,q) \vee Q(w))$

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$\{\neg Q(a)\}, \{R(y,u), Q(u)\}, \{\neg P(z, f(z,w))\},$
$\{P(b,t), \neg R(t,q), Q(w)\}$

$\neg Q(a) \qquad P(b,t), \neg R(t,q), Q(w)$

$w \mapsto a$

$P(b,t), \neg R(t,q) \qquad R(y,u), Q(u)$

$t \mapsto y, q \mapsto u$

$P(b,y), Q(u) \qquad \neg P(z, f(z,u))$

$z \mapsto b, y \mapsto f(b,u)$

$Q(u)$

$u \mapsto a$

$\perp$

Therefore $\neg(F \vee \neg G)$ is unsatisfiable
and so $F \vee \neg G$ is valid

TASK 6A

Show $(\forall x\, P(a,x,x)) \wedge (\forall x \forall y \forall z (\neg P(x,y,z) \vee P(s(x),y,s(z))))$
$\wedge (\forall x \forall y \forall z (\neg P(x,y,z) \vee P(y,x,z)))$
$\wedge (\forall x \exists y (\neg E(x) \vee P(y,y,x)))$
$\wedge (\forall x \forall y (\neg P(y,y,x) \vee E(x)))$
$\wedge \neg(\forall x (\neg E(x) \vee E(s(s(x)))))$ is unsatisfiable

$\forall x\, P(a,x,x) = \{P(a,x,x)\}$ (6.1)

$\forall x \forall y \forall z (\neg P(x,y,z) \vee P(s(x),y,s(z)))$

$= \{\neg P(w,y,z), P(s(w),y,s(z))\}$ (6.2)

$\forall x \forall y \forall z (\neg P(x,y,z) \vee P(y,x,z))$

$= \{\neg P(q,r,t), P(r,q,t)\}$ (6.3)

$\forall x \exists y (\neg E(x) \vee P(y,y,x))$ $\quad y \mapsto f(u)$

$= \{\neg E(u), P(f(u),f(u),u)\}$ (6.4)

$\forall x \forall y (\neg P(y,y,x) \vee E(x))$

$= \{\neg P(v,v,p), E(p)\}$ (6.5)

$\neg(\forall x (\neg E(x) \vee E(s(s(x)))))$ $\quad x \mapsto b$

$= \{E(b)\}, \{\neg E(s(s(b)))\}$ (6.6)

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$$E(b) \qquad \neg E(u),\ P(fu, fu, u)$$

$$u \mapsto b$$

$$P(fb, fb, b)$$

?

$$\neg E(s(s(b))) \qquad \neg P(v, v, p),\ E(p)$$

$$p \mapsto s(s(b))$$

$$\neg P(v, v, s(s(b)))$$

?

TASK 6B

clauses from 6A

$\{P(a,x,x)\}$ (6.1)

$\{\neg P(w,y,z), P(s(w), y, s(z))\}$ (6.2)

$\{\neg P(q,r,t), P(r,q,t)\}$ (6.3)

$\{\neg E(u), P(f(u), f(u), u)\}$ (6.4)

$\{\neg P(v,v,p), E(p)\}$ (6.5)

$\{\neg E(0), E(s(s(0)))\}$ (6.6)

$\{\neg E(s(s(s(n))))\}$ $\neg$ (6.7)

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$\neg E(o), E(s(s(o)))$ $\qquad$ $\neg P(v, v, p), E(p)$

$p \mapsto o$

$E(s(s(o))), \neg P(v, v, o)$

$\neg E(s(s(s(n))))$

$o \mapsto s(n)$

$\neg P(v, v, s(n))$ $\qquad$ $\neg P(w, y, z), P(s(w), y, s(z))$

$z \mapsto n,$ $v \mapsto s(w),$ $y \mapsto s(w)$

$P(a, x, x)$ $\qquad$ $\neg P(w, s(w), n)$

$w \mapsto a,\ x \mapsto s(w),$ $n \mapsto s(w)$

$\perp$

Therefore 6.7 is a logical consequence of axioms (6.1)–(6.5) and theorem (6.6)

$p \mapsto o \mapsto s(n)$

p is any natural number such that it is the addition of 2 equal natural numbers and is therefore an even number under (6.5). P can be mapped to o where o is any natural even number implying s(s(o)) (or o+2) is also an even number under (6.6). o can be mapped to s(n) where n is any number which satisfies E(s(s(s(n)))) (i.e. any odd number). (6.7) is satisfied as s(s(s(n))) can be rewritten as s(s(o)) which we know to be an even number from (6.6). It can be concluded that all successors of odd numbers are even numbers in $\mathbb{N}$