Assignment 4

Graded

Student

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Total Points

11 / 13 pts

Question 1

Question 2

11 / 13 pts

- ✔ **+ 1 pt** 2a) M: use derivative substitution
- ✔ **+ 1 pt** 2a) A: $-3\cos(x^2)+c$
- ✔ **+ 1 pt** 2b) M: Use a valid hyperbolic substitution. Expecting $x=\frac{1}{3}\cosh\theta$.
- ✔ **+ 1 pt** 2b) J: Stated use of $\sinh\theta > 0$ to remove absolute value signs. ($\sinh\theta \geq 0$) is okay too.
- ✔ **+ 1 pt** 2b) A: $\int \cosh^2(\theta)d\theta$. correct integrand after substitution up to a constant factor.
- ✔ **+ 1 pt** 2b) M: Use a valid method such as a double angle formula to calculate the integral.
- **+ 1 pt** 2b) A: $\frac{1}{54}\left(3x\sqrt{9x^2-1}+\operatorname{arccosh}(3x)\right)+c$. Must simplify compositions of hyperbolic functions and their inverses.
- ✔ **+ 1 pt** 2c) M: Use integration by parts twice to obtain an equation for $\int e^{-x}\cos(x)\ dx$.
- ✔ **+ 1 pt** 2c) A: $\frac{1}{2}e^{-x}\left(\sin(x)-\cos(x)\right)+c$
- ✔ **+ 1 pt** 2d) M: Use complex exponential
- ✔ **+ 1 pt** 2d) A: $\frac{1}{2}e^{-x}\left(\sin(x)-\cos(x)\right)+c$
- ✔ **+ 1 pt** 2e) A: For a reason given that contains no mathematical errors.
- **+ 1 pt** N: Notation correction throughout question 2 and no +c's missing in non-answer steps.
- **+ 0 pts** Scored zero or not attempt

1 good

2 something wrong, missing a factor

3 dx

4 would be good to put big brackets around it

5 refer to solutions

6 integrate gives you 1/2 outside?

No questions assigned to the following page.

Calculus 2 Assignment 4

1. $f(t) = 20000\,\text{sech}^2(2t-30)\ ,\quad t \geq 0$

a) <u>y-intercept</u>

$y = 20000\,\text{sech}^2(2(0)-30)$

$y = 20000\,\text{sech}^2(-30)$

<u>x-intercepts</u>

$20000\,\text{sech}^2(2t-30) = 0$

$\text{sech}^2(2t-30) = 0$

$\text{sech}(2t-30) = 0 \qquad \text{sech} > 0$

$2t - 30 = \text{arcsech}(0)$

$= \text{undefined over } \mathbb{R}$ as $\text{dom}(\text{arcsech}) = (0, 1]$

<u>horizontal asymptotes</u>

$y = \dfrac{20000}{\cosh^2(2t-30)}$ degree $n <$ degree d

$\Rightarrow y = 0$ is the horizontal asymptote

<u>vertical asymptote</u>

occurs when $\cosh^2(2t-30) = 0$

$\cosh(2t-30) = 0\ , \quad \cosh > 0$

$2t - 30 = \text{arccosh}(0)$

$= \text{undefined over } \mathbb{R}$

as $\text{dom}(\text{arccosh}) = [1, \infty)$

No questions assigned to the following page.

Stationary point

occur when $f'(t)=0$

$$f'(t)=\frac{d}{dt}\left(20000\operatorname{sech}^2(2t-30)\right)$$

$$=20000\frac{d}{dt}\left(\operatorname{sech}^2(2t-30)\right)$$

$$=20000\cdot 2\operatorname{sech}(2t-30)\cdot\frac{d}{dt}\left(\operatorname{sech}(2t-30)\right)$$

$$=40000\operatorname{sech}(2t-30)\cdot -\operatorname{sech}(2t-30)\tanh(2t-30)\cdot 2$$

$$=-80000\operatorname{sech}^2(2t-30)\tanh(2t-30)$$

$$\Rightarrow -80000\operatorname{sech}^2(2t-30)\tanh(2t-30)=0$$

$$\operatorname{sech}^2(2t-30)\tanh(2t-30)=0$$

$$\tanh(2t-30)=0$$

$$2t-30=\operatorname{arctanh}(0)$$

$$2t-30=0$$

$$2t=30$$

$$t=15$$

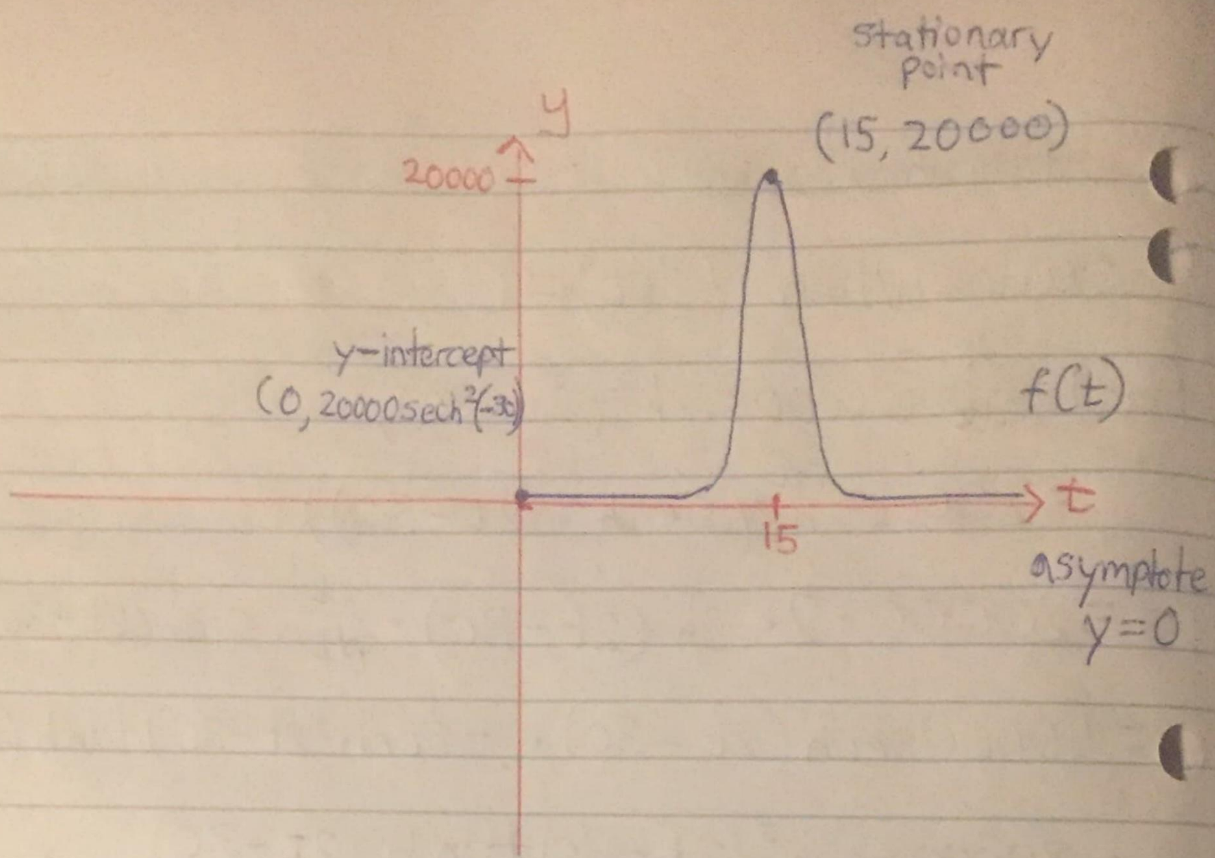
sub $t=15$ into $f(t)$

$$f(15)=20000\operatorname{sech}^2(0)$$

$$=20000$$

turning point at $(15, 20000)$

No questions assigned to the following page.



b) $0.01 \cdot f(t)$ cases require hospitalisation

$\Rightarrow$ the hospital will first be over capacity when $0.01 \cdot f(t)$ is first ~~equal to~~ greater than 10.

$$\Rightarrow 200\,\text{sech}^2(2t-30) = 10$$

$$\text{sech}^2(2t-30) = \frac{1}{20}$$

$$|\text{sech}(2t-30)| = \pm\frac{1}{\sqrt{20}}$$

$$\text{sech}(2t-30) = \frac{1}{\sqrt{20}}, \quad \text{range}(\text{sech}) > 0$$

$$\text{sech}(2t-30) = \text{sech}(-2t+30)$$ (as sech is an even function)

$\Rightarrow \text{sech}(2t-30) = \frac{1}{\sqrt{20}}$ and $\Rightarrow \text{sech}(-2t+30) = \frac{1}{\sqrt{20}}$

$2t - 30 = \text{arcsech}\left(\frac{1}{\sqrt{20}}\right)$ $\qquad -2t+30 = \text{arcsech}\left(\frac{1}{\sqrt{20}}\right)$

$t = \frac{\text{arcsech}\left(\frac{1}{\sqrt{20}}\right)+30}{2}$ $\qquad t = -\frac{\text{arcsech}\left(\frac{1}{\sqrt{20}}\right)-30}{2}$

$t \approx 16$ $\qquad \boxed{t \approx 14}$

($t < 14$ so day 14 will

be the first day)

No questions assigned to the following page.

Therefore, the hospital system is first expected to be over capacity 14 days after the start of the disease outbreak.

c) $T(t) = \int_0^t f(u)\ du$

$= \int_0^t 20000 \operatorname{sech}^2(2u-30)\ du$

Let $v = 2u - 30$

$\frac{dv}{du} = 2$ sub $u = t \Rightarrow v = 2t - 30$

$\frac{1}{2}\frac{dv}{du} = 1$ sub $u = 0 \Rightarrow v = -30$

$\Rightarrow T(t) = 20000 \int_0^t \operatorname{sech}^2(v)\ \frac{1}{2}\frac{dv}{du}\ du$

$= 20000 \cdot \frac{1}{2} \int_{-30}^{2t-30} \operatorname{sech}^2(v)\ dv$

$= 10000 \left[\tanh(v)\right]_{-30}^{2t-30}$

$T(t) = 10000\left(\tanh(2t-30) - \tanh(-30)\right)$

or

$T(t) = 10000\left(\tanh(2t-30) - \frac{e^{-30} - e^{30}}{e^{-30} + e^{30}}\right)$

No questions assigned to the following page.

d) Total number of expected cases $\approx 20,000$

$\frac{20000}{1000000} = 0.02 \Rightarrow 2\%$ of the population are expected to be infected

This model says that roughly 98% of the population will remain uninfected which is roughly 980,000 people

Question assigned to the following page: 1

2.

a) $\int 6x\sin(x^2)\, dx$

Let $u = x^2$

$$\frac{du}{dx} = 2x \Rightarrow 3\frac{du}{dx} = 6x$$

Derivative Substitution

$$\int 6x\sin(x^2)\, dx$$

$$= \int 3\frac{du}{dx}\sin(u)\, dx$$

$$= 3\int \sin(u)\, du$$

$$= -3\cos(u) + C$$

$$= -3\cos(x^2) + C$$

b) $\int \frac{x^2}{\sqrt{9x^2-1}}\, dx$

$$= \int \frac{x^2}{\sqrt{9(x^2-\frac{1}{9})}}\, dx$$

$$= \frac{1}{3}\int \frac{x^2}{\sqrt{x^2-\frac{1}{9}}}\, dx$$

Question assigned to the following page: 1

Let $x = \frac{1}{3}\cosh\theta$, Hyperbolic Substitution

$\theta = \text{arccosh}(3x)$

This is valid when

$3x \in \text{dom}(\text{arccosh})$ and $\theta \in \text{range}(\text{arccosh})$

$\Rightarrow 3x \geq 1 \Rightarrow x \geq \frac{1}{3}$ and $\theta \geq 0$

· Also need $\sqrt{9x^2-1} \neq 0$

$|9x^2-1| \neq 0$

$|9x^2-1| \neq 0$, $\cancel{9x^2-1 > 0 \text{ as}}$ $|0| = 0$

$9x^2 \neq 1$

$x^2 \neq 1/9$

$x \neq 1/3 \Rightarrow \theta \neq 0$ as $\theta = 0$ when $x = 1/3$

$\Rightarrow x > \frac{1}{3}$ and $\theta > 0$

- $x = \frac{1}{3}\cosh\theta$
- $\frac{dx}{d\theta} = \frac{1}{3}\sinh\theta$
- $\frac{x^2}{\sqrt{x^2 - \frac{1}{9}}} = \frac{1/9\cosh^2\theta}{\sqrt{1/9\cosh^2\theta - \frac{1}{9}}}$

$$= \frac{1/9\cosh^2\theta}{\sqrt{1/9(\cosh^2\theta - 1)}}$$

$$= \frac{1/9\cosh^2\theta}{\sqrt{1/9\sinh^2\theta}}$$

Question assigned to the following page: 1

$$= \frac{\frac{1}{9}\cosh^2\theta}{\frac{1}{3}|\sinh\theta|}$$

$$= \frac{\frac{1}{9}\cosh^2\theta}{\frac{1}{3}\sinh\theta}, \quad \sinh > 0 \text{ when } \theta > 0$$

Therefore $\frac{1}{3}\int \frac{x^2}{\sqrt{x^2-\frac{1}{9}}}\,dx$

$$= \frac{1}{3}\int \frac{\frac{1}{9}\cosh^2\theta}{\frac{1}{3}\sinh\theta}\cdot\frac{1}{3}\sinh\theta\,d\theta$$

$$= \frac{1}{3}\int \frac{1}{9}\cosh^2\theta\,d\theta$$

$$= \frac{1}{27}\int \cosh^2\theta\,d\theta$$

$$= \frac{1}{27}\int \frac{1}{2}(\cosh(2\theta)+1)\,d\theta, \quad \text{Double Angle formula}$$

$$= \frac{1}{54}[\sinh(2\theta)+\theta]+C$$

$$= \frac{1}{54}[2\sinh\theta\cosh\theta+\theta]+C, \quad \text{Double Angle formula}$$

$$= \frac{1}{54}[2\sqrt{\cosh^2\theta-1}\,\cosh\theta+\theta]+C$$

$$= \frac{1}{54}[2\sqrt{9x^2-1}\cdot 3x+\operatorname{arccosh}(3x)]+C$$

$$= \frac{x}{9}\sqrt{9x^2-1}+\frac{1}{54}\operatorname{arccosh}(3x)+C$$

$$= \frac{x}{3}\sqrt{x^2-\frac{1}{9}}+\frac{1}{54}\operatorname{arccosh}(3x)+C$$

Question assigned to the following page: 1

c) $\int e^{-x} \cos(x)\, dx$

Let $u = e^{-x}$ $\quad \frac{dv}{dx} = \cos(x)$

$\frac{du}{dx} = -e^{-x}$ $\quad v = \sin(x)$, Integration by parts

Therefore $\int e^{-x} \cos(x)\, dx$

$$= e^{-x}\sin(x) + \int e^{-x} \sin(x)\, dx$$

First integrate $\int e^{-x} \sin(x)\, dx$:

Let $u = e^{-x}$ $\quad \frac{dv}{dx} = \sin(x)$, Integration by parts

$\frac{du}{dx} = -e^{-x}$ $\quad v = -\cos(x)$

Therefore $\int e^{-x} \sin(x)$

$$= -e^{-x}\cos(x) - \int e^{-x}\cos(x)\, dx$$

Putting everything together:

$\Rightarrow \int e^{-x} \cos(x)\, dx$

$= e^{-x}\sin(x) + \left(-e^{-x}\cos(x) - \int e^{-x}\cos(x)\, dx\right)$

$\Rightarrow 2\int e^{-x}\cos(x)\, dx = e^{-x}\sin(x) - e^{-x}\cos(x) + C$

$\Rightarrow \int e^{-x}\cos(x)\, dx = \frac{e^{-x}\sin(x) - e^{-x}\cos(x)}{2} + d$, $d = \frac{1}{2}C$

$$= e^{-x}\left(\frac{1}{2}\sin(x) - \frac{1}{2}\cos(x)\right) + d$$

Question assigned to the following page: 1

d) $\int e^{-x}\cos(x)\,dx$, complex exponential

$$e^{-x}\cos(x) = e^{-x}\,\mathrm{Re}(e^{ix})$$
$$= \mathrm{Re}(e^{-x}\cdot e^{ix}), \quad e^{-x}\in\mathbb{R}$$
$$= \mathrm{Re}(e^{ix-x}) = \mathrm{Re}(e^{(-1+i)x})$$

Therefore $\int e^{-x}\cos(x)\,dx = \int \mathrm{Re}(e^{(-1+i)x})\,dx$

$$= \mathrm{Re}\int e^{(-1+i)x}\,dx$$
$$= \mathrm{Re}\left[\frac{1}{-1+i}e^{(-1+i)x} + c + di\right], \quad c,d\in\mathbb{R}$$
$$= \mathrm{Re}\left[\frac{-1-i}{(-1+i)(-1-i)}e^{-x}e^{ix} + c + di\right]$$
$$= \mathrm{Re}\left[\frac{-1-i}{2}e^{-x}(\cos(x)+i\sin(x)) + c + di\right]$$
$$= \mathrm{Re}\left[\frac{e^{-x}}{2}(-\cos(x)+\sin(x)-i\cos(x)-i\sin(x)) + c + di\right]$$
$$= \frac{e^{-x}}{2}(\sin(x)-\cos(x)) + C$$
$$= e^{-x}\left(\tfrac{1}{2}\sin(x) - \tfrac{1}{2}\cos(x)\right) + C$$

e) I found (d) simpler as using the complex exponential involves easier integration and simple algebra with complex numbers