School of Mathematics and Statistics
MAST10007 Linear Algebra, Semester 1 2020
Written assignment 5

Submit your assignment online in Canvas before 12 noon on Monday 25th May.

Name:

Student ID:

- This assignment is worth $1\frac{1}{9}\%$ of your final MAST10007 mark.
- Your solutions should be neatly handwritten in blue or black pen, then uploaded as a single PDF file in **GradeScope**.
- Full explanations and working must be shown in your solutions.
- Marks may be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- You must use methods taught in MAST10007 Linear Algebra to solve the assignment questions.

New submission guidelines:

- This assignment is being handled using a similar process to that planned for the final exam so you can start to become familiar with it.
- If you have access to a printer, then you should print out this assignment sheet and handwrite your solutions into the answer boxes.
- If you do not have access to a printer, but you can annotate a PDF file using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly in the boxes on the assignment PDF and save a copy for submission.
- Otherwise, you may handwrite your answers on blank paper to produce a document that *mirrors the layout of the assignment template* and then scan for submission. So: put your name and student ID on page 1, put your answers to Q1a and Q1b on page 2, put your answers to Q2a and Q2b on page 3, put your answer to Q2c on page 4.
- The answer boxes should typically provide sufficient space for your solution, but if you find you need extra space please take a blank sheet of paper and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- Scan your assignment to a PDF file using your mobile phone or scanner, then upload by going to the Assignments menu on Canvas and submit the PDF to the **GradeScope** tool by first selecting your PDF file and then clicking on 'Upload pdf'.

1. Determine whether the following are linear transformations. Explain your answers by giving an appropriate proof or counterexample.

   (a) $T\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $T(a,b) = \sqrt{a^2+b^2}$.

Since $T(1,0) = \sqrt{1^2+0^2} = 1$ and $T(-(1,0)) = T(-1,0) = \sqrt{(-1)^2+0^2} = 1$, we have $T(-(1,0)) \neq -T(1,0)$. Hence T is not a linear transformation.

   (b) $S\colon M_{2,2} \to M_{2,2}$ defined by $S(A) = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix}$.

For all $A, B \in M_{2,2}$ and $\alpha \in \mathbb{R}$, we have

$$S(A+B) = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} (A+B) \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} B \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix} = S(A) + S(B)$$

and

$$S(\alpha A) = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} (\alpha A) \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix} = \alpha \left(\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} A \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix} \right) = \alpha S(A).$$

Hence S is a linear transformation.

2. Consider the function $T\colon \mathcal{P}_2 \to \mathcal{P}_2$ defined by

$$T(p(x)) = x^2 p''(x),$$

where $p''(x)$ is the second order derivative of $p(x)$.

(a) Verify that T is a linear transformation.

For all $p(x), q(x) \in \mathcal{P}_2$ and $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} T(p(x)+q(x)) &= x^2(p(x)+q(x))'' \\ &= x^2(p''(x)+q''(x)) = x^2p''(x) + x^2q''(x) = T(p(x)) + T(q(x)) \end{aligned}$$

and

$$T(\alpha p(x)) = x^2(\alpha p(x))'' = x^2(\alpha p''(x)) = \alpha\left(x^2 p''(x)\right) = \alpha T(p(x))$$

Hence T is a linear transformation.

(b) Find the kernel of T.

For $p(x) = ax^2 + bx + c \in \mathcal{P}_2$, the vector $p(x)$ lies in $\ker(T)$ if and only if $x^2 p''(x) = 0$, that is, $2ax^2 = 0$, which is equivalent to $a = 0$. Hence the kernel of T is

$$\ker(T) = \{bx + c : b, c \in \mathbb{R}\} = \mathcal{P}_1.$$

(c) Consider the ordered basis

$$\mathcal{L} = \{\tfrac{1}{2}(x-1)(x-2), -x(x-2), \tfrac{1}{2}x(x-1)\}$$

of $\mathcal{P}_2$. (You have already proved in Written Assignment 4 that $\mathcal{L}$ is a basis for $\mathcal{P}_2$.) Find the matrix representation $[T]_{\mathcal{L}}$ of T with respect to $\mathcal{L}$.
(You may use results from Written Assignment 4, if desired.)

Let $p_0(x) = \frac{1}{2}(x-1)(x-2)$, $p_1(x) = -x(x-2)$ and $p_2(x) = \frac{1}{2}x(x-1)$. Then

$$T(p_0(x)) = x^2 p_0(x)'' = x^2, \quad T(p_1(x)) = x^2 p_1(x)'' = -2x^2, \quad T(p_2(x)) = x^2 p_2(x)'' = x^2.$$

By the result in Written Assignment 4 Q2(a), we have

$$f(x) = f(x)p_0(x) + f(1)p_1(x) + f(2)p_2(x) \text{ for all } f(x) \in \mathcal{P}_2,$$

so

$$x^2 = 0p_0(x) + 1p_1(x) + 4p_2(x).$$

Hence

$$[T(p_0(x))]_{\mathcal{L}} = [T(p_2(x))]_{\mathcal{L}} = [x^2]_{\mathcal{L}} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \quad [T(p_1(x))]_{\mathcal{L}} = [-2x^2]_{\mathcal{L}} = -2[x^2]_{\mathcal{L}} = \begin{bmatrix} 0 \\ -2 \\ -8 \end{bmatrix},$$

and so

$$[T]_{\mathcal{L}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 4 & -8 & 4 \end{bmatrix}.$$