School of Mathematics and Statistics

MAST10007 Linear Algebra, Semester 1 2020

Solutions to Written assignment 6

Submit your assignment online in Canvas before 12 noon on Monday 1st June.

Name:

Student ID:

- This assignment is worth $1\frac{1}{9}\%$ of your final MAST10007 mark.
- Your solutions should be neatly handwritten in blue or black pen, then uploaded as a single PDF file in **GradeScope**.
- Full explanations and working must be shown in your solutions.
- Marks may be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- You must use methods taught in MAST10007 Linear Algebra to solve the assignment questions.

New submission guidelines:

- This assignment is being handled using a similar process to that planned for the final exam so you can start to become familiar with it.
- If you have access to a printer, then you should print out this assignment sheet and handwrite your solutions into the answer boxes.
- If you do not have access to a printer, but you can annotate a PDF file using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly in the boxes on the assignment PDF and save a copy for submission.
- Otherwise, you may handwrite your answers on blank paper to **produce a document that mirrors the layout of the assignment template** and then scan for submission. So: put your name and student ID on page 1, your answers to Q1a-d on page 2, your answer to Q1e on page 3, your answers to Q2 on page 4, and your answers to Q3 on page 5.
- The answer boxes should typically provide sufficient space for your solution, but if you find you need extra space please add a blank sheet of paper at the end and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- Scan your assignment to a PDF file using your mobile phone or scanner, then upload by going to the Assignments menu on Canvas and submit the PDF to the **GradeScope** tool by first selecting your PDF file and then clicking on 'Upload pdf'.

1. Orthogonal projection onto the line $y = 2x$ gives a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

(a) Explain briefly, geometrically or otherwise, why $T(1, 2) = (1, 2)$ and $T(-2, 1) = (0, 0)$.

$(1, 2)$ lies on the line $y = 2x$ so its orthogonal projection is $(1, 2)$; $(-2, 1)$ is perpendicular to this line so its projection is $(0, 0)$.

(b) Write down the matrix of T with respect to the ordered basis $\mathcal{B} = \{(1, 2), (-2, 1)\}$.

Let $\mathbf{b}_1 = (1, 2), \mathbf{b}_2 = (-2, 1)$ be the vectors in $\mathcal{B}$. Then

$$T(\mathbf{b}_1) = \mathbf{b}_1 = 1\mathbf{b}_1 + 0\mathbf{b}_2 \text{ and } T(\mathbf{b}_2) = \mathbf{0} = 0\mathbf{b}_1 + 0\mathbf{b}_2,$$

hence

$$[T]_{\mathcal{B}} = \left[\ [T(\mathbf{b}_1)]_{\mathcal{B}} \ \ [T(\mathbf{b}_2)]_{\mathcal{B}}\ \right] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

(c) Write down the transition matrix from $\mathcal{B}$ to the standard basis $\mathcal{S} = \{(1, 0), (0, 1)\}$.

$$P_{\mathcal{S},\mathcal{B}} = \left[\ [\mathbf{b}_1]_{\mathcal{S}} \ \ [\mathbf{b}_2]_{\mathcal{S}}\ \right] = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$$

(d) Find the transition matrix from $\mathcal{S}$ to $\mathcal{B}$.

$$P_{\mathcal{B},\mathcal{S}} = (P_{\mathcal{S},\mathcal{B}})^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

(e) Hence, find the matrix of T with respect to the standard basis $\mathcal{S}$ and use this to find a general formula for $T(x, y)$.

$$\begin{aligned}[T]_{\mathcal{S}} &= P_{\mathcal{S},\mathcal{B}}[T]_{\mathcal{B}}P_{\mathcal{B},\mathcal{S}} \\ &= \begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}\frac{1}{5}\begin{bmatrix}1 & 2\\ -2 & 1\end{bmatrix} \\ &= \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\end{aligned}$$

Hence

$$[T(x,y)]_{\mathcal{S}} = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \frac{1}{5}\begin{bmatrix}x+2y\\ 2x+4y\end{bmatrix}$$

and

$$T(x,y) = \frac{1}{5}(x+2y, 2x+4y).$$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix A.

The eigenvalues satisfy

$$0 = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

So the eigenvalues are $\lambda = 2, 3$.

Eigenvectors for $\lambda = 2$ satisfy $(A - 2I)\mathbf{x} = \mathbf{0}$. Now

$$A - 2I = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So $x_1 = x_2$ and an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Eigenvectors for $\lambda = 3$ satisfy $(A - 3I)\mathbf{x} = \mathbf{0}$. Now

$$A - 3I = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$$

So $x_2 = 2x_1$ and an eigenvector is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(b) Find a diagonal matrix D and invertible matrices P, P^{-1} such that $P^{-1}AP = D$.

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}.$$

3. The matrix

$$B = \begin{bmatrix} -1 & 0 & 1 \\ -7 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix},$$

has eigenvalues $-2, 2, 2$.

(a) Find bases for the corresponding eigenspaces.

Eigenspace for $\lambda = 2$: Solving $(B - 2I)\mathbf{x} = \mathbf{0}$ gives

$$B - 2I = \begin{bmatrix} -3 & 0 & 1 \\ -7 & 0 & 5 \\ 3 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_2 = t$, $x_1 = x_3 = 0$, where $t \in \mathbb{R}$,

and the eigenspace has a basis $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Eigenspace for $\lambda = -2$: Solving $(B + 2I)\mathbf{x} = \mathbf{0}$ gives

$$B + 2I = \begin{bmatrix} 1 & 0 & 1 \\ -7 & 4 & 5 \\ 3 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_3 = t$, $x_2 = -3t$, $x_1 = -t$ where $t \in \mathbb{R}$,

and the eigenspace has a basis $\left\{ \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \right\}$.

(b) Is the matrix B diagonalisable? Give a reason for your answer.

No. For $\lambda = 2$, we have
$\dim(\text{eigenspace}) = 1 < 2 = \text{multiplicity of eigenvalue}$.
Hence, the matrix B is not diagonalizable.