MAST10006 Calculus 2, Semester 2, 2020

Assignment 1

School of Mathematics and Statistics, The University of Melbourne

- Submit your assignment solutions online in Canvas before **6pm, Monday 24 August 2020**
- Submit your solutions as a single PDF file with the pages in the right order and correct orientation. You may be penalised a mark if you do not.
- This assignment is worth 2.22% of your final MAST10006 mark.
- Answer all questions below. Of these questions, one will be chosen for marking.
- Marks may be awarded for:
  - Correct use of appropriate mathematical techniques
  - Accuracy and validity of any calculations or algebraic manipulations
  - Clear justification or explanation of techniques and rules used
  - Use of correct mathematical notation and terminology
- You must explicitly state if you use the Sandwich Theorem, l'Hôpital's Rule, limit laws, continuity or standard limits in your answers when evaluating limits.
- You must use methods taught in MAST10006 Calculus 2 to solve the assignment questions.

1. (a) Evaluate the limit

$$\lim_{x\to\pi} \frac{e^x - e^\pi}{\cos\left(\frac{x}{2}\right)}$$

or explain why it does not exist.

Solution.

$$\begin{aligned}
\lim_{x\to\pi} \frac{e^x - e^\pi}{\cos\left(\frac{x}{2}\right)} &= \lim_{x\to\pi} \frac{e^x - 0}{-\frac{1}{2}\sin\left(\frac{x}{2}\right)} && \text{L'Hôpital's Rule, type } \left(\frac{0}{0}\right) \\
&= \frac{\lim\limits_{x\to\pi} e^x}{-\frac{1}{2}\lim\limits_{x\to\pi}\sin(\frac{x}{2})} && \text{Limit laws} \\
&= \frac{e^\pi}{-\frac{1}{2}\sin(\lim\limits_{x\to\pi}\frac{x}{2})} && \text{since } e^z \text{ and } \sin z \text{ are continuous} \\
&= \frac{e^\pi}{-\frac{1}{2}} && \text{Limit laws} \\
&= -2e^\pi
\end{aligned}$$

(b) For which value of a is the function defined by the rule

$$f(x) = \begin{cases} \frac{e^x - e^\pi}{\cos\left(\frac{x}{2}\right)} & x < \pi \\ a\sin\left(\frac{\pi^2}{2x}\right) & x \geq \pi \end{cases}$$

continuous? Explain why the function is continuous for your answer with reference to the definition of continuity.

Solution.

For f to be continuous at $x = \pi$, we require:

$$\lim_{x\to\pi} f(x) = f(\pi)$$

Calculate $f(\pi)$:

$$f(\pi) = a\sin\left(\frac{\pi^2}{2\pi}\right) = a$$

Calculate the left and right limits:

$$\begin{aligned}
\lim_{x\to\pi+} f(x) &= \lim_{x\to\pi} a\sin\left(\frac{\pi^2}{2x}\right) \\
&= a\sin\left(\lim_{x\to\pi}\frac{\pi^2}{2x}\right) && \text{since } \sin z \text{ is continuous for } z\in\mathbb{R} \\
&= a\sin\left(\frac{\pi^2}{2\pi}\right) && \text{limit laws} \\
&= a
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to\pi-} f(x) &= \lim_{x\to\pi} \frac{e^x - e^\pi}{\cos\left(\frac{x}{2}\right)} \\
&= \lim_{x\to\pi} \frac{e^x}{-\frac{1}{2}\sin\left(\frac{x}{2}\right)} && \text{L'Hôpital's Rule, type } \left(\frac{0}{0}\right) \\
&= \frac{e^\pi}{-\frac{1}{2}} && \text{continuity of } e^x, \sin\left(\frac{x}{2}\right), \text{ and limit laws} \\
&= -2e^\pi
\end{aligned}$$

For the $\lim_{x\to\pi} f(x)$ to be defined, we need $\lim_{x\to\pi+} f(x) = \lim_{x\to\pi-} f(x)$.
From the above, this occurs when $a = -2e^\pi$.
Since, when $a = -2e^\pi$, we have

$$f(\pi) = -2e^\pi = \lim_{x\to\pi} f(x),$$

the function f is continuous at $x = \pi$.

Now, for f to be continuous, we need f to be continuous for all values in its domain. Since the rule for f is defined only when

$$\cos\left(\frac{x}{2}\right) \neq 0 \iff x \neq \pi + 2k\pi \text{ where } k < 0 \text{ and } k\in\mathbb{Z},$$

the implied domain of the function is

$$\{x\in\mathbb{R} \mid x\neq \pi+2k\pi, k<0, k\in\mathbb{Z}\} \cup [\pi, \infty).$$

Check that f is continuous on this domain for $x < \pi$:
On the specified domain,

- e^x is an exponential so is continuous.
- $-e^\pi$, and $\frac{x}{2}$ are polynomials, so are continuous.
- $\cos z$ is a trigonometric functions so is continuous.

Therefore,

- $e^x - e^\pi$ is a sum of continuous functions so is continuous.
- $\cos\left(\frac{x}{2}\right)$ is a composition of continuous functions so is continuous.

Therefore, $\frac{e^x-e^\pi}{\cos\left(\frac{x}{2}\right)}$ is a quotient of continuous functions so is continuous as long as $\cos\left(\frac{x}{2}\right) \neq 0$.
So for $x < \pi$ and $x \neq \pi + 2k\pi$ for $k\in\mathbb{Z}$, $f(x) = \frac{e^x-e^\pi}{\cos\left(\frac{x}{2}\right)}$ is continuous.

Check that f is continuous on this domain for $x > \pi$: On the domain specified above,

- $\frac{\pi^2}{2x}$ is a quotient of polynomials so is continuous.
- $\sin z$ is a trigonometric function so is continuous.
- a is a constant function so is continuous. (constant functions are polynomials)

Therefore $\sin\left(\frac{\pi^2}{2x}\right)$ is a composition of continuous functions so is continuous.

Therefore $a\sin\left(\frac{\pi^2}{2x}\right)$ is continuous since it is a product of continuous functions.

So for $x > \pi$ $f(x) = a\sin\left(\frac{\pi^2}{2x}\right)$ is continuous.

Therefore, f is continuous for $a = -2e^{\pi}$.

Notes for self reflection:

1. a) • Did you state L'Hôpital's Rule?

   • Did you state that $\sin z$ and e^z are continuous?

   b). Did you justify that f was continuous for $x < \pi$ and $x > \pi$?

   • Was $\lim_{x \to \pi} f(x) = f(\pi)$ written on your assignment?

   This shows that you are checking the definition of continuity

2. Evaluate the following limits of sequences, or explain why they do not exist:

5 (a) $\lim_{n\to\infty} \dfrac{n\log(\cos^2(n)+3)}{2020^n}$

Solution.
First find lower and upper bounds for the sequence.

$$
\begin{aligned}
& -1 \le \cos(n) \le 1 \\
\implies & 0 \le \cos^2(n) \le 1 \\
\implies & 3 \le \cos^2(n)+3 \le 4 \\
\implies & \log(3) \le \log(\cos^2(n)+3) \le \log(4) \qquad \text{since log is an increasing function} \\
\implies & \frac{n\log(3)}{2020^n} \le \frac{n\log(\cos^2(n)+3)}{2020^n} \le \frac{n\log 4}{2020^n}
\end{aligned}
$$

1M: use bounds for cos to find bounds for sequence

1A: correctly derived bounds

Calculate the limits of the lower and upper bound:

$$\lim_{n\to\infty}\frac{n\log(3)}{2020^n}=0$$

$$\lim_{n\to\infty}\frac{n\log(4)}{2020^n}=0$$

1J: use standard limits and explicitly calculate limits of bounds

Using standard limit $\frac{n^p}{a^n}\to 0$, and limit laws.
By the Sandwich Theorem,

1J

$$\lim_{n\to\infty}\frac{n\log(\cos^2(n)+3)}{2020^n}=0$$ 1A

2 (b) $\lim_{n\to\infty} \sin\left(\dfrac{(2n-1)\pi}{4}\right)$

Solution.
If $n=4k+2$ and $n=4k+3$ for $k\in\mathbb{N}$, then $\sin\left(\frac{(2n-1)\pi}{4}\right)=\frac{1}{\sqrt{2}}$.
If $n=4k$ and $n=4k+1$, for $k\in\mathbb{N}$, then $\sin\left(\frac{(2n-1)\pi}{4}\right)=-\frac{1}{\sqrt{2}}$.
Therefore the sequence does not approach any single value, so diverges.

1J: a valid reason "oscillates from $-\frac{1}{\sqrt{2}}$ to $\frac{1}{\sqrt{2}}$" is okay
Note: cannot apply limit laws or continuity

1A

3 (c) $\lim_{n\to\infty}\tan\left((2020n)^{\frac{1}{n}}\right)$

Solution.

1J: change to $x\in(0,\infty)$

$$
\begin{aligned}
\lim_{n\to\infty}\tan\left((2020n)^{\frac{1}{n}}\right) &= \lim_{x\to\infty}\tan\left((2020x)^{\frac{1}{x}}\right) && \text{change to a continuous function, } x\in(0,\infty) \\
&= \tan\left(\lim_{x\to\infty}(2020x)^{\frac{1}{x}}\right) && \tan z \text{ is continuous} \\
&= \tan\left(\lim_{x\to\infty}(2020^{\frac{1}{x}}).\lim_{x\to\infty}(x^{\frac{1}{x}})\right) && \text{limit laws} \\
&= \tan(1.1) && \lim_{x\to\infty}x^{\frac{1}{x}}=1 \text{ and } \lim_{x\to\infty}a^{\frac{1}{x}}=1 \\
&= \tan(1)
\end{aligned}
$$ 1A

1J: use continuity of tan and standard limits

Note: we only need $\tan z$ to be continuous at $z=1$ for this step

1N: notation correct and unambiguous throughout question 2.

End of assignment