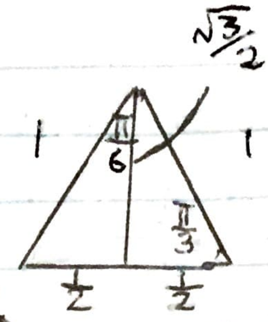
(2.) $A=\{z\in\mathbb{C} \mid z^3=\frac{\sqrt{3}+i}{2}\}$

(a) $B=\{z\in\mathbb{C} \mid z^3=\frac{\sqrt{3}-i}{2}\}$



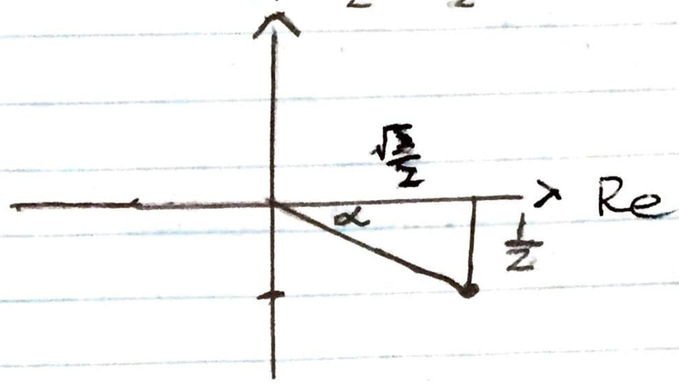
$z^3=\frac{\sqrt{3}}{2}-\frac{1}{2}i$ $\qquad \tan\alpha=\frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$

$z^3=S e^{i\alpha}$

$S=\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2+\left(-\frac{1}{2}\right)^2}$ $\qquad \alpha=-\frac{\pi}{6}$

$S=\sqrt{3/4+1/4}=\sqrt{1}$

$S=1$



Solving $z^3=e^{-i\pi/6}$

If $z=re^{i\theta}$, then

$r^3=1$ $\qquad 3\theta=-\pi/6+2k\pi,\quad k=0,1,2$

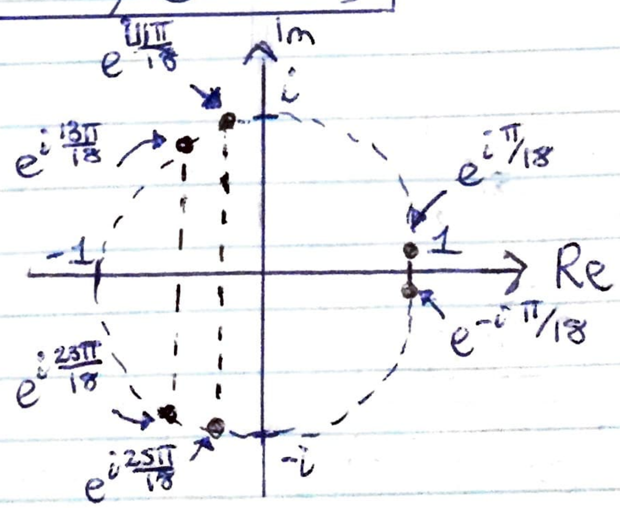
$r=\sqrt[3]{1}$ $\qquad \theta=\frac{1}{3}\left(-\frac{\pi}{6}+2k\pi\right)$

$r=1$ $\qquad \theta=-\pi/18+\frac{2k\pi}{3}=-\frac{\pi}{18}+\frac{12k\pi}{18}$

$\theta=-\frac{\pi}{18},\ \frac{11\pi}{18},\ \frac{23\pi}{18}$

$$\boxed{z\in\{e^{-i\pi/18},\ e^{i\,11\pi/18},\ e^{i\,23\pi/18}\}}$$

(b)



(c) $z^6-\sqrt{3}z^3+1=0$

Set $w=z^3$ so $P(z)=z^6-\sqrt{3}z^3+1=0$ has a root z if and only if $\underset{②}{w=z^3}$ and $\underset{①}{w^2-\sqrt{3}w+1=0}$

① The roots of $w^2 - \sqrt{3}w + 1 = 0$, $a=1$, $b=-\sqrt{3}$, $c=1$

$$w = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{\sqrt{3} \pm \sqrt{3-4(1)(1)}}{2(1)} = \frac{\sqrt{3} \pm \sqrt{-1}}{2}$$

$$= \frac{\sqrt{3} \pm i}{2} = \frac{\sqrt{3}}{2} \pm \frac{1}{2} i$$

② (From previous solutions)

$z^3 = e^{i\pi/6}$ and $z^3 = e^{-i\pi/6}$

$r = 1$

$\theta = \pi/18, \frac{13\pi}{18}, \frac{25\pi}{18}$

$r = 1$

$\theta = -\frac{\pi}{18}, \frac{11\pi}{18}, \frac{23\pi}{18}$

$$\boxed{z \in \{ e^{-i\pi/18}, e^{i\pi/18}, e^{i11\pi/18}, e^{i13\pi/18}, e^{i23\pi/18}, e^{i25\pi/18} \}}$$

d) Yes, the answers in (c) do come in conjugate pairs as predicted by Theorem 1.81.

[E.G. 1.52: $e^{-i\theta} = \overline{e^{i\theta}}$]

$$\boxed{\begin{aligned} e^{-i\pi/18} &= \overline{e^{i\pi/18}} \\ e^{i11\pi/18} &= e^{-i25\pi/18} = \overline{e^{i25\pi/18}} \\ e^{i13\pi/18} &= e^{-i23\pi/18} = \overline{e^{i23\pi/18}} \end{aligned}}$$

e) $$\boxed{\begin{aligned} P(z) = &(z - e^{-i\pi/18})(z - e^{i\pi/18})(z - e^{i11\pi/18})(z - e^{i13\pi/18}) \\ &(z - e^{i23\pi/18})(z - e^{i25\pi/18}) \end{aligned}}$$

(f) $(z-e^{i\theta})(z-e^{-i\theta}), \quad \theta \in \mathbb{R}$

$= z^2 - e^{-i\theta}z - e^{i\theta}z + (-e^{i\theta})(-e^{-i\theta})$

1.53

$= z^2 - e^{-i\theta}z - e^{i\theta}z + 1$

1.52

$= z^2 - \overline{e^{i\theta}}z - e^{i\theta}z + 1$

1.39

$= z^2 - 2\cos\theta z + 1$

$$\boxed{f(z) = z^2 - 2\cos(\theta)z + 1}$$

Theorem 1.53

$(-e^{i\theta})(-e^{-i\theta})$

$= e^{i(\theta + (-\theta))}$

$= e^{i(0)} = 1$

Example 1.52

$e^{-i\theta} = \overline{e^{i\theta}}$

Theorem 1.39

$z + \bar{z} = 2\mathrm{Re}(z)$

$-(z+\bar{z}) = -2\mathrm{Re}(z)$

$-(\overline{e^{i\theta}} + e^{i\theta}) = -2\mathrm{Re}(e^{i\theta}$

$= -2\cos\theta$

(g) $P(z) = (z^2 - 2\cos(\theta)z + 1)(z - e^{i11\pi/18})(z - e^{i13\pi/18})(z - e^{i23\pi/18})(z - e^{i25\pi/18})$

(from (d))

$= (z^2 - 2\cos(\theta)z + 1)(z - e^{-i23\pi/18})(z - e^{i23\pi/18})(z - e^{-i25\pi/18})(z - e^{i25\pi/18})$

(from (f))

$\therefore \boxed{P(z) = (z^2 - 2\cos(\theta)z + 1)(z^2 - 2\cos(\theta)z + 1)(z^2 - 2\cos(\theta)z + 1}$