MAST20004 Probability
Semester 1, 2021
Assignment One: Solutions

Due 3 pm, Friday 26 March 2021

Name:
Student ID:

Important instructions:

(1) This assignment contains 4 questions, **two** of which will be randomly selected to be marked. Each marked question is worth 10 points and each unmarked question with substantial working is worth 1 point.

(2) To complete this assignment, you need to write your solutions into the blank answer spaces following each question in this assignment PDF.

- If you have a printer (or can access one), then you must print out the assignment template and handwrite your solutions into the answer spaces.
- If you do not have a printer but you can figure out how to annotate a PDF using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly onto the assignment PDF and save a copy for submission.

Failing both of these methods, you may handwrite your answers as normal on blank paper and then scan for submission (but note that you will thereby miss valuable practice for the exam process). In that case, however, your document should have the same length as the assignment template otherwise Gradescope will reject your submission. So you will need to add as many blank pages as necessary to reach that criterion.

Scan your assignment to a PDF file using your mobile phone (we recommend Cam - Scanner App), then upload by going to the Assignments menu on Canvas and submit the PDF to the GradeScope tool by first selecting your PDF file and then clicking on 'Upload PDF'.

(3) A poor presentation penalty of 10% of the total available marks will apply unless your submitted assignment meets all of the following requirements:

- it is a single pdf with all pages in correct template order and the correct way up, and with any blank pages with additional working added only at the end of the template pages;
- has all pages clearly readable;
- has all pages cropped to the A4 borders of the original page and is imaged from directly above to avoid excessive 'keystoning'.

These requirements are easy to meet if you use a scanning app on your phone and take some care with your submission - please review it before submitting to double check you have satisfied all of the above requirements.

(4) Late submission within 20 hours after the deadline will be penalised by 5% of the total available marks for every hour or part thereof after the deadline. After that, the Gradescope submission channel will be closed, and your submission will no longer be accepted. You are strongly encouraged to submit the assignment a few days before the deadline just in case of unexpected technical issues. If you are facing a rather exceptional/extreme situation that prevents you from submitting on time, please contact the tutor coordinator **Robert Maillardet** with formal proofs such as medical certificate.

(5) Working and reasoning must be given to obtain full credit. Clarity, neatness, and style count.

Problem 1. Let $A_1, \ldots, A_n$ be a finite collection of sets.

(i) Formulate a version of De Morgan's laws for the collection of sets.

(a) For the union, De Morgan's law is

$$(\cup_{i=1}^{n} A_i)^c = \cap_{i=1}^{n} A_i^c.$$

(b) For the intersection, De Morgan's law is

$$(\cap_{i=1}^{n} A_i)^c = \cup_{i=1}^{n} A_i^c.$$

(ii) Prove the De Morgan's laws in (i) by the elementwise method.

(a) Proof: $(\cup_{i=1}^{n} A_i)^c \subseteq \cap_{i=1}^{n} A_i^c$: for each $\omega \in (\cup_{i=1}^{n} A_i)^c$, we have $\omega \notin \cup_{i=1}^{n} A_i$, so $\omega \notin A_i$ for all $1 \leq i \leq n$, which means $\omega \in A_i^c$ for all $1 \leq i \leq n$, that is, $\omega \in \cap_{i=1}^{n} A_i^c$, as expected.

$\cap_{i=1}^{n} A_i^c \subseteq (\cup_{i=1}^{n} A_i)^c$: for each $\omega \in \cap_{i=1}^{n} A_i^c$, it follows that $\omega \in A_i^c$ for all $1 \leq i \leq n$, hence $\omega \notin A_i$ for all $1 \leq i \leq n$, which is the same as $\omega \notin \cup_{i=1}^{n} A_i$. This ensures that $\omega \in (\cup_{i=1}^{n} A_i)^c$.

(b) Proof: $(\cap_{i=1}^{n} A_i)^c \subseteq \cup_{i=1}^{n} A_i^c$: for each $\omega \in (\cap_{i=1}^{n} A_i)^c$, we have $\omega \notin \cap_{i=1}^{n} A_i$, which in turn ensures $\omega \notin A_i$ for some $1 \leq i \leq n$, hence $\omega \in A_i^c$ for some $1 \leq i \leq n$, that is, $\omega \in \cup_{i=1}^{n} A_i^c$, as claimed.

$\cup_{i=1}^{n} A_i^c \subseteq (\cap_{i=1}^{n} A_i)^c$: for each $\omega \in \cup_{i=1}^{n} A_i^c$, we can see that $\omega \in A_i^c$ for some $1 \leq i \leq n$, hence $\omega \notin A_i$ for some $1 \leq i \leq n$, which is the same as $\omega \notin \cap_{i=1}^{n} A_i$. This implies $\omega \in (\cap_{i=1}^{n} A_i)^c$.

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Problem 2. Let A, B, C be three events with $\mathbb{P}(B) > 0$ and $\mathbb{P}(B^c) > 0$. For each of the following statements, determine whether it is true or false. If it is true, give a proof; if it is false, give a counterexample.

(i) If A and B are independent, then $\mathbb{P}(A|B) = \mathbb{P}(A|B^c)$.

True. Proof: By the definition of independence,

$$\mathbb{P}(A|B) = \mathbb{P}(A) \tag{1}$$

or equivalently,

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B). \tag{2}$$

On the other hand, $A \cap B$ and $A \cap B^c$ form a partition of A, so by the finite additivity, we have $\mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c) = \mathbb{P}(A)$, giving

$$\begin{aligned} \mathbb{P}(A \cap B^c) &= \mathbb{P}(A) - \mathbb{P}(A \cap B) \\ &\overset{(2)}{=} \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) \\ &= \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A)\mathbb{P}(B^c), \end{aligned}$$

which implies $\mathbb{P}(A|B^c) = \mathbb{P}(A) \overset{(1)}{=} \mathbb{P}(A|B)$.

(ii) If $\mathbb{P}(A|B) = \mathbb{P}(A|B^c)$, then A and B are independent.

True. Proof: It follows from $\mathbb{P}(A|B) = \mathbb{P}(A|B^c)$ that

$$\mathbb{P}(A \cap B)/\mathbb{P}(B) = \mathbb{P}(A \cap B^c)/\mathbb{P}(B^c),$$

hence

$$\mathbb{P}(A \cap B)\mathbb{P}(B^c) = \mathbb{P}(A \cap B^c)\mathbb{P}(B), \tag{3}$$

which gives

$$\begin{aligned} &\mathbb{P}(A \cap B)(1 - \mathbb{P}(B)) \\ &= \mathbb{P}(A \cap B)\mathbb{P}(B^c) \\ &\overset{(3)}{=} \mathbb{P}(A \cap B^c)\mathbb{P}(B) \\ &= (\mathbb{P}(A) - \mathbb{P}(A \cap B))\mathbb{P}(B). \end{aligned}$$

Now, we expand both sides to get $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$ and the independence between A and B follows.

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(iii) If A, B, C are mutually independent, then A and $B^c \cap C$ are independent.

True. Proof: Since $\{(A \cap C) \cap B, (A \cap C) \cap B^c\}$ forms a partition of $A \cap C$, by the finite additivity,

$$\mathbb{P}((A \cap C) \cap B) + \mathbb{P}((A \cap C) \cap B^c) = \mathbb{P}(A \cap C),$$

hence

$$\mathbb{P}((A \cap C) \cap B^c) = \mathbb{P}(A \cap C) - \mathbb{P}((A \cap C) \cap B). \tag{4}$$

Similarly, we can obtain

$$\mathbb{P}(C \cap B^c) = \mathbb{P}(C) - \mathbb{P}(C \cap B). \tag{5}$$

Now, by the mutual independence of A, B, C, we obtain

$$\begin{aligned}
&\mathbb{P}(A \cap (B^c \cap C)) \overset{(4)}{=} \mathbb{P}(A \cap C) - \mathbb{P}((A \cap C) \cap B) \\
&= \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \\
&= \mathbb{P}(A)(\mathbb{P}(C) - \mathbb{P}(C \cap B)) \overset{(5)}{=} \mathbb{P}(A)\mathbb{P}(B^c \cap C).
\end{aligned}$$

(iv) If A and B are independent, A and C are independent, and A and $B \cap C$ are independent, then A, B, C are mutually independent.

False. Counterexample: Let $\Omega = \{w_1, w_2, w_3, w_4\}$ with $\mathbb{P}(\{w_i\}) = 1/4$ for $1 \leq i \leq 4$. Set $A = \{w_1, w_2\}$, $B = C = \{w_2, w_3\}$, then

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{w_2\}) = 1/4 = 1/2 \times 1/2 = \mathbb{P}(A)\mathbb{P}(B),$$

so A and B are independent. Since $B = C$, A and C are independent and A and $B \cap C = B$ are independent but

$$\mathbb{P}(A \cap B \cap C) = 1/4 \neq 1/8 = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C),$$

that is, A, B, C are NOT mutually independent.

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Problem 3. There are three urns labelled as u_1, u_2 and u_3; initially u_1 contains 2 black balls and 3 white balls, u_2 has 1 black ball and 4 white balls, and u_3 holds 3 black balls and 2 white balls. We first mix the balls in u_1, randomly draw a ball from u_1, place it in u_2; next, mix the balls in u_2, randomly draw a ball from u_2 and place it in u_3; finally, mix the balls in u_3 and randomly draw a ball from u_3.

(i) What is the probability that the ball drawn from u_2 is white?

Sol: Let A_i be the event that the ball drawn from u_i is white, $1 \le i \le 3$, then $\mathbb{P}(A_2|A_1) = 5/6$, $\mathbb{P}(A_2|A_1^c) = 4/6$, $\mathbb{P}(A_1) = 1 - \mathbb{P}(A^c) = 3/5$. Since $\{A_1, A_1^c\}$ forms a partition of the sample space, by the total probability formula,

$$\begin{aligned} \mathbb{P}(A_2) &= \mathbb{P}(A_2|A_1)\mathbb{P}(A_1) + \mathbb{P}(A_2|A_1^c)\mathbb{P}(A_1^c) \\ &= 5/6 \times 3/5 + 4/6 \times 2/5 = 23/30. \end{aligned}$$

(ii) What is the probability that the ball drawn from u_3 is white?

Sol: Using the notations in (i), we have from (i) that $\mathbb{P}(A_3|A_2) = 3/6$, $\mathbb{P}(A_3|A_2^c) = 2/6$, $\mathbb{P}(A_2) = 1 - \mathbb{P}(A^c) = 23/30$. Since $\{A_2, A_2^c\}$ forms a partition of the sample space, by the total probability formula,

$$\begin{aligned} \mathbb{P}(A_3) &= \mathbb{P}(A_3|A_2)\mathbb{P}(A_2) + \mathbb{P}(A_3|A_2^c)\mathbb{P}(A_2^c) \\ &= 3/6 \times 23/30 + 2/6 \times 7/30 = 83/180. \end{aligned}$$

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(iii) Given that the ball drawn from u_3 is white, what is the probability that the ball drawn from u_2 is white?

Sol: $\mathbb{P}(A_3|A_2) = 3/6$, (i) ensures $\mathbb{P}(A_2) = 23/30$ and (ii) gives $\mathbb{P}(A_3) = 83/180$, so applying Bayes' formula gives

$$\mathbb{P}(A_2|A_3) = \frac{\mathbb{P}(A_3|A_2)\mathbb{P}(A_2)}{\mathbb{P}(A_3)} = \frac{\frac{3}{6}\cdot\frac{23}{30}}{\frac{83}{180}} = \frac{69}{83}.$$

(iv) Given that the ball drawn from u_3 is white, what is the probability that the ball drawn from u_1 is white?

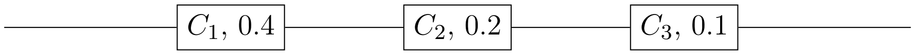
Sol: We start with computing $\mathbb{P}(A_3 \cap A_1)$: $\{A_3 \cap A_1 \cap A_2, A_3 \cap A_1 \cap A_2^c\}$ forms a partition of $A_3 \cap A_1$, using finite additivity,

$$\begin{aligned}
\mathbb{P}(A_3 \cap A_1) &= \mathbb{P}(A_3 \cap A_1 \cap A_2) + \mathbb{P}(A_3 \cap A_1 \cap A_2^c) \\
&= \mathbb{P}(A_3|A_1 \cap A_2)\mathbb{P}(A_1 \cap A_2) \\
&\quad + \mathbb{P}(A_3|A_1 \cap A_2^c)\mathbb{P}(A_1 \cap A_2^c) \\
&= \mathbb{P}(A_3|A_1 \cap A_2)\mathbb{P}(A_2|A_1)\mathbb{P}(A_1) \\
&\quad + \mathbb{P}(A_3|A_1 \cap A_2^c)\mathbb{P}(A_2^c|A_1)\mathbb{P}(A_1) \\
&= \frac{3}{6}\cdot\frac{5}{6}\cdot\frac{3}{5} + \frac{2}{6}\cdot\frac{1}{6}\cdot\frac{3}{5} \\
&= \frac{51}{180} = \frac{17}{60}.
\end{aligned}$$

Now,

$$\mathbb{P}(A_1|A_3) = \frac{\mathbb{P}(A_3 \cap A_1)}{\mathbb{P}(A_3)} = \frac{\frac{51}{180}}{\frac{83}{180}} = \frac{51}{83}.$$

Problem 4. A circuit contains three mutually independent components C_1, C_2, C_3 in series as shown in the figure below.



The probability of failure for each component is indicated in the figure respectively.

(i) What is the probability that the circuit will fail?

Sol: Let A_i be the event that C_i fails, $1 \le i \le 3$, and H be the event that the circuit fails. Noting that the circuit functions if all components function, we obtain

$$\begin{aligned}
\mathbb{P}(H^c) &= \mathbb{P}(A_1^c \cap A_2^c \cap A_3^c) \\
&= \mathbb{P}(A_1^c)\mathbb{P}(A_2^c)\mathbb{P}(A_3^c) \\
&= 0.6 \cdot 0.8 \cdot 0.9 = 0.432,
\end{aligned}$$

hence $\mathbb{P}(H) = 1 - \mathbb{P}(H^c) = 0.568$.

(ii) What is the probability that exactly one component fails?

Sol: Let H_1 be the event that exactly one component fails. With the notations in (i), we have

$$H_1 = (A_1 \cap A_2^c \cap A_3^c) \cup (A_1^c \cap A_2 \cap A_3^c) \cup (A_1^c \cap A_2^c \cap A_3).$$

As $(A_1 \cap A_2^c \cap A_3^c)$, $(A_1^c \cap A_2 \cap A_3^c)$, $(A_1^c \cap A_2^c \cap A_3)$ are disjoint, we obtain from the finite additivity that

$$\begin{aligned}
\mathbb{P}(H_1) &= \mathbb{P}(A_1 \cap A_2^c \cap A_3^c) + \mathbb{P}(A_1^c \cap A_2 \cap A_3^c) \\
&\quad + \mathbb{P}(A_1^c \cap A_2^c \cap A_3) \\
&= \mathbb{P}(A_1)\mathbb{P}(A_2^c)\mathbb{P}(A_3^c) + \mathbb{P}(A_1^c)\mathbb{P}(A_2)\mathbb{P}(A_3^c) \\
&\quad + \mathbb{P}(A_1^c)\mathbb{P}(A_2^c)\mathbb{P}(A_3) \\
&= 0.4 \cdot 0.8 \cdot 0.9 + 0.6 \cdot 0.2 \cdot 0.9 + 0.6 \cdot 0.8 \cdot 0.1 \\
&= 0.444.
\end{aligned}$$

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(iii) Given that exactly one of the components fails, what are the respective probabilities that the failed component is C_1, C_2, C_3?

Sol: With the notations H_1 and A_i's in (i) and (ii),

$$\begin{aligned}
\mathbb{P}(A_1|H_1) &= \frac{\mathbb{P}(A_1 \cap H_1)}{\mathbb{P}(H_1)} = \frac{\mathbb{P}(A_1 \cap A_2^c \cap A_3^c)}{\mathbb{P}(H_1)} \\
&= \frac{\mathbb{P}(A_1)\mathbb{P}(A_2^c)\mathbb{P}(A_3^c)}{\mathbb{P}(H_1)} = \frac{0.4 \cdot 0.8 \cdot 0.9}{0.444} = \frac{24}{37}, \\
\mathbb{P}(A_2|H_1) &= \frac{\mathbb{P}(A_2 \cap H_1)}{\mathbb{P}(H_1)} = \frac{\mathbb{P}(A_1^c \cap A_2 \cap A_3^c)}{\mathbb{P}(H_1)} \\
&= \frac{\mathbb{P}(A_1^c)\mathbb{P}(A_2)\mathbb{P}(A_3^c)}{\mathbb{P}(H_1)} = \frac{0.6 \cdot 0.2 \cdot 0.9}{0.444} = \frac{9}{37}
\end{aligned}$$

and $\mathbb{P}(A_3|H_1) = 1 - \mathbb{P}(A_1|H_1) - \mathbb{P}(A_2|H_1) = \frac{4}{37}$.

(iv) Assume that exactly one of the components failed, an engineer is assigned to detect the failed component. To minimise the number X of tests, what is the order of the components they should check? Explain your answer.

Sol: From (iii), the failed component is C_1 with probability $24/37$, so the engineer should check C_1 first. If C_1 is fine, then the failed component is C_2 with probability

$$\begin{aligned}
\mathbb{P}(A_2|A_1^c \cap H_1) &= \frac{\mathbb{P}(A_2 \cap A_1^c \cap H_1)}{\mathbb{P}(A_1^c \cap H_1)} = \frac{\frac{\mathbb{P}(A_1^c \cap A_2 \cap A_3^c)}{\mathbb{P}(H_1)}}{\mathbb{P}(A_1^c|H_1)} \\
&= \frac{0.6 \cdot 0.2 \cdot 0.9/0.444}{1 - 24/37} = \frac{9}{13},
\end{aligned}$$

hence the next for checking is C_2 and the last suspect is C_3.

(v) Derive the probability mass function of X in (iv).

Sol: X may take values $S_X = \{1, 2, 3\}$ with $\{X = i\}$ if C_i is the failed component, hence its pmf is $p(1) = 24/37$, $p(2) = 9/37$ and $p(3) = 4/37$.
It is also acceptable to consider the solution with $S_X = \{1, 2\}$ because if both C_1 and C_2 are fine, then there is no need to check C_3 so its pmf is $p(1) = 24/37$ and $p(2) = 13/37$ (this sounds smart but I don't like engineers behaving in this way).