Calculus 2 Written Assignment 5

1. $x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$ $\qquad x > 0$

a) $u = y/x$

$\Rightarrow y = ux$

$\Rightarrow \frac{dy}{dx} = x\frac{du}{dx} + u$, product rule

$x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

$\Rightarrow x\left(x\frac{du}{dx} + u\right) = ux + \sqrt{x^2+u^2x^2}$

$\Rightarrow x\frac{du}{dx} + u = u + \frac{\sqrt{x^2+u^2x^2}}{x}$, $x>0$

$\Rightarrow \frac{du}{dx} = \frac{\sqrt{x^2(1+u^2)}}{x^2}$, $x > 0$

$\Rightarrow \frac{du}{dx} = \frac{\sqrt{1+u^2}}{x^2} \cdot x$, $x > 0$

$\Rightarrow \frac{du}{dx} = \frac{\sqrt{1+u^2}}{x}$

b) $\frac{du}{dx} = \sqrt{1+u^2} \cdot \frac{1}{x}$, is separable
- use sep. of variables

$\frac{1}{\sqrt{1+u^2}} \frac{du}{dx} = \frac{1}{x}$, $\sqrt{1+u^2} \geq 1$

$\int \frac{1}{\sqrt{1+u^2}} \, du = \int \frac{1}{x} \, dx$

$\text{arcsinh}(u) = \log(x) + C$, $x > 0$

$u = \sinh(\log(x) + C)$

$u = \frac{1}{2}\left(e^{\log(x)+C} - e^{-(\log(x)+C)}\right)$

$u = \frac{1}{2}\left(e^{\log(x)+C} - \frac{1}{e^{\log(x)+C}}\right)$

$u = \frac{1}{2}\left(e^C x - \frac{1}{e^C x}\right) = \frac{e^C x}{2} - \frac{1}{2e^C x}$

$u(x) = \frac{Ax}{2} - \frac{1}{2Ax}$, $A = e^C$

c) $\frac{y}{x} = \frac{Ax}{2} - \frac{1}{2Ax}$

$$y = \frac{Ax^2}{2} - \frac{1}{2A}$$

$$\lim_{x \to 0^+} y(x) = -\frac{e}{2}$$

$$\lim_{x \to 0^+} \left(\frac{Ax^2}{2} - \frac{1}{2A} \right) = -\frac{e}{2}$$

$$\lim_{x \to 0^+} \left(\frac{A}{2} x^2 \right) - \lim_{x \to 0^+} \left(\frac{1}{2A} \right) = -\frac{e}{2}$$, limit laws

$$\frac{A}{2} \lim_{x \to 0^+} (x^2) - \frac{1}{2A} = -\frac{e}{2}$$, limit laws

$$\frac{A}{2} \left(\lim_{x \to 0^+} x \right)^2 - \frac{1}{2A} = -\frac{e}{2}$$, limit laws

$$\frac{A}{2} (0)^2 - \frac{1}{2A} = -\frac{e}{2}$$, limit laws

$$-\frac{1}{2A} = -\frac{e}{2}$$

$$\frac{1}{A} = e$$

$$A = \frac{1}{e} = e^{-1}$$

So $y(x) = \frac{x^2}{2e} - \frac{e}{2}$

2. $t\log(t)\frac{dr}{dt} + r = \frac{t}{(2t^2-9)^{3/2}}$

Rewrite as:

$$\frac{dr}{dt} + \frac{1}{t\log(t)}r = \frac{t}{t\log(t)(2t^2-9)^{3/2}}, \quad t>0, \ \log(t)\neq 0$$

$$\frac{dr}{dt} + \frac{1}{t\log(t)}r = \frac{1}{\log(t)(2t^2-9)^{3/2}}$$

Is linear with $P(x) = \frac{1}{t\log(t)}$

$Q(x) = \frac{1}{\log(t)(2t^2-9)^{3/2}}$

• Find an integrating factor:

$I(x) = e^{\int \frac{1}{t\log(t)}dt}$

Solve: $\int \frac{1}{t\log(t)} dt$

Let $u = \log(t)$

$\frac{du}{dt} = \frac{1}{t}$, Derivative Substitution

$\int \frac{du}{dt}\frac{1}{u} dt = \int \frac{1}{u} du = \log|u| + C$

$\Rightarrow I(x) = e^{(\log|\log t| + C)}$

$= |\log t| = \log t$, only need 1 integrating factor

• Multiply ODE by I

$$\log(t)\frac{dr}{dt} + \frac{1}{t}r = \frac{1}{(2t^2-9)^{3/2}}$$

$$\Rightarrow \frac{d}{dt}\left(\log(t)\, r\right) = \frac{1}{(2t^2-9)^{3/2}}$$

$$\Rightarrow \log(t)\, r = \int \frac{1}{(2t^2-9)^{3/2}}\, dt$$

Solve: $\int \frac{1}{(2t^2-9)^{3/2}}\, dt$

$$= \int \frac{1}{\left(2\left(t^2-\frac{9}{2}\right)\right)^{3/2}}\, dt$$

$$= \int \frac{1}{2\sqrt{2}} \frac{1}{\left(t^2-\frac{9}{2}\right)^{3/2}}\, dt$$

$$= \frac{1}{2\sqrt{2}} \int \frac{1}{\left(t^2-9/2\right)^{3/2}}\, dt$$

Let $t = \frac{3}{\sqrt{2}}\cosh\theta$

$\theta = \operatorname{arccosh}\left(\frac{\sqrt{2}t}{3}\right)$, Hyperbolic Substitution

This is valid when

$\frac{\sqrt{2}t}{3} \in \operatorname{dom}(\operatorname{arccosh})$ and $\theta \in \operatorname{range}(\operatorname{arccosh})$

$\Rightarrow \frac{\sqrt{2}t}{3} \geq 1$ and $\theta \geq 0$

$\Rightarrow t \geq \frac{3}{\sqrt{2}}$ and $\theta \geq 0$

Also need $\left(\sqrt{t^2-9/2}\right)^3 \neq 0$

$\Rightarrow t > \frac{3}{\sqrt{2}} \Rightarrow \theta > 0$

$$t = \frac{3}{\sqrt{2}} \cosh\theta \qquad \cosh\theta = \frac{\sqrt{2}\,t}{3}$$

- $$\frac{dt}{d\theta} = \frac{3}{\sqrt{2}} \sinh\theta$$

- $$\frac{1}{\left(\sqrt{t^2 - \frac{9}{2}}\right)^3} = \frac{1}{\left(\sqrt{\frac{9}{2}\cosh^2\theta - \frac{9}{2}}\right)^3}$$

$$= \frac{1}{\left(\sqrt{\frac{9}{2}(\cosh^2\theta - 1)}\right)^3} = \frac{1}{\left(\frac{\sqrt{3}}{2}\sqrt{\sinh^2\theta}\right)^3}$$

$$= \frac{1}{\left(\frac{\sqrt{3}}{2}(\sinh\theta)\right)^3}, \quad \sinh\theta > 0 \text{ as } \theta > 0$$

$$= \frac{1}{\frac{3\sqrt{3}}{8}\sinh^3\theta} = \frac{8}{3\sqrt{3}\sinh^3\theta}$$

Therefore $\frac{1}{2\sqrt{2}} \int \frac{1}{(t^2 - 9/2)^{3/2}}\,dt$

$$= \frac{1}{2\sqrt{2}} \int \frac{8}{3\sqrt{3}\sinh^3\theta} \cdot \frac{3}{\sqrt{2}} \sinh\theta \; d\theta$$

$$= \frac{1}{2\sqrt{2}} \int \frac{8}{\sqrt{6}\sinh^2\theta} \; d\theta$$

$$= \frac{1}{2\sqrt{2}} \int \frac{8\sqrt{6}}{6\sinh^2\theta} \; d\theta$$

$$= \frac{8\sqrt{6}}{12\sqrt{2}} \int \frac{1}{\sinh^2\theta} \; d\theta$$

$$= \frac{2\sqrt{3}}{3} \int \operatorname{cosech}^2\theta \; d\theta$$

$$= -\frac{2\sqrt{3}}{3}\left[\coth\theta + C\right]$$

$$= -\frac{2\sqrt{3}}{3}\left[\frac{\cosh\theta}{\sinh\theta} + C\right]$$

$$= -\frac{2\sqrt{3}}{3}\left[\frac{\cosh\left(\operatorname{arccosh}\left(\frac{\sqrt{2}t}{3}\right)\right)}{\sqrt{\cosh^2\theta - 1}} + C\right]$$

$$= -\frac{2\sqrt{3}}{3}\left[\frac{\sqrt{2}t/3}{\sqrt{2/9\,t^2 - 1}} + C\right], \quad t > \frac{3}{\sqrt{2}}$$

$$= -\frac{2\sqrt{6}t}{9} \cdot \frac{1}{\sqrt{2/9\,t^2 - 1}} + d, \quad d = -\frac{2\sqrt{3}}{3} \cdot C$$

$$= -\frac{2\sqrt{6}t}{3\sqrt{2t^2 - 9}} + d$$

Therefore:

$$\log(t)\, r = -\frac{2\sqrt{6}t}{3\sqrt{2t^2 - 9}} + d$$

$$r = -\frac{2\sqrt{6}t}{3\log(t)\sqrt{2t^2 - 9}} + \frac{d}{\log(t)}$$