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Wednesday 9AM
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1 $V = P_2, \quad W = \{ \underset{\sim}{p} \in P_2 : \underset{\sim}{p}(0) + \underset{\sim}{p}(2) = 1 \}$

(a)

Let $\underset{\sim}{p} = \frac{1}{2} + 0x + 0x^2 \in W$ and $\alpha = 2 \in \mathbb{R}$

Then $\alpha \underset{\sim}{p} = 1 + 0x + 0x^2 \notin W$

as $\underset{\sim}{p}(0) + \underset{\sim}{p}(2) = 1 + 0(0) + 0(0)^2 + 1 + 0(2) + 0(4)$

$= 2 \neq 1$

$\Rightarrow W$ is not closed under scalar multiplication

$\Rightarrow W$ is not a subspace of V.

b) $V = M_{2,2}, \quad W = \{ A \in M_{2,2} : A^T = A \}$

Define two vectors in W and a scalar in $\mathbb{R}$

Let $\underset{\sim}{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in W, \quad \underset{\sim}{v} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in W, \quad \alpha \in \mathbb{R}$

Then $\underset{\sim}{u}^T = \underset{\sim}{u}$ $\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$\Rightarrow b = c$

and $\underset{\sim}{v}^T = \underset{\sim}{v}$

$\Rightarrow f = g$

⓪ W is not empty

$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in W$ since $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$b = c \Rightarrow 1 = 1$ ✓

$\Rightarrow W$ is not empty

① Closure under vector addition

For $\underset{\sim}{u}, \underset{\sim}{v} \in W$ as above

$\underset{\sim}{u} + \underset{\sim}{v} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$

$\Rightarrow b+f = c+g$

$\Rightarrow b+f = b+f$ as $b=c$ and $f=g$ ✓

$\Rightarrow \underset{\sim}{u} + \underset{\sim}{v} \in W$

② Closure under scalar multiplication

For $\underset{\sim}{u} \in W$ as above, $\alpha \in \mathbb{R}$

$\alpha \underset{\sim}{u} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$

$\Rightarrow \alpha b = \alpha c$

$\Rightarrow \alpha b = \alpha b$ (as $b=c$) ✓

$\Rightarrow \alpha \underset{\sim}{u} \in W$

Conclusion: W is a subspace of V by the Subspace Theorem as it is not empty and is closed under vector addition and scalar multiplication.

$\Rightarrow$ W is a real vector space using the operations from $M_{2,2}$

(c) $V = \mathbb{R}^3$, $W = \{(x, y, z) \in \mathbb{R} : z^2 = x^2 + y^2\}$

Let $\underset{\sim}{u} = (3, 4, 5) \in W$ and $\underset{\sim}{v} = (0, 1, 1) \in W$

Then $\underset{\sim}{u} + \underset{\sim}{v} = (3, 5, 6) \notin W$

as $3^2 + 5^2 \neq 6^2$

$34 \neq 36$

$\Rightarrow$ W is not closed under vector addition and so is not a subspace of V.

(2) $\underset{\sim}{p_1} = 1 + 2x - x^2 + 3x^3$ $\quad$ $\underset{\sim}{p_3} = 1 + 0x + 5x^2 - x^3$

$\underset{\sim}{p_2} = 2 + 3x + x^2 + 4x^3$

If this set of polynomials is linearly dependent then $\alpha_1 \underset{\sim}{p_1} + \alpha_2 \underset{\sim}{p_2} + \alpha_3 \underset{\sim}{p_3} = \underset{\sim}{0}$ for $\alpha \in \mathbb{R}$, $\alpha \neq 0$

$$\Rightarrow \alpha_1 (1 + 2x - x^2 + 3x^3) + \alpha_2 (2 + 3x + x^2 + 4x^3) + \alpha_3 (1 + 0x + 5x^2 - x^3) = 0 + 0x + 0x^2 + 0x^3$$

Two polynomials are equal $\Leftrightarrow$ their coefficients agree.

Equating coefficients of $1, x, x^2, x^3$:

$$\alpha_1 + 2\alpha_2 + \alpha_3 = 0$$
$$2\alpha_1 + 3\alpha_2 + 0\alpha_3 = 0$$
$$-\alpha_1 + \alpha_2 + 5\alpha_3 = 0$$
$$3\alpha_1 + 4\alpha_2 - \alpha_3 = 0$$

Solving by row reduction

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 3 & 0 & 0 \\ -1 & 1 & 5 & 0 \\ 3 & 4 & -1 & 0 \end{array}\right] \underset{\substack{R_3 \to R_3 + R_1 \\ R_4 \to R_4 - 3R_1}}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right]$$

parameter

$$\overset{R_2 \to -R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right] \underset{R_4 \to R_4 + 2R_2}{\overset{R_3 \to R_3 - 3R_2}{\sim}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$(\text{rank} = 2) < (\text{number of vectors}) = 3$

$\Rightarrow$ 1 parameter in solution

$\boxed{\Rightarrow \text{The polynomials are linearly dependent in } P_3}$

Linear combination

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \overset{R_1 \to R_1 - 2R_2}{\sim} \begin{array}{c} \begin{array}{cccc} \underset{\sim}{P_1} & \underset{\sim}{P_2} & \underset{\sim}{P_3} & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$\underset{\sim}{P_3} = -3(\underset{\sim}{P_1}) + 2(\underset{\sim}{P_2})$

$$\boxed{1 + 0x + 5x^2 - x^3 = -3(1 + 2x - x^2 + 3x^3) + 2(2 + 3x + x^2 + 4x^3)}$$