Question 3

a) $F_X(x) = \int_1^x \theta x^{-(\theta+1)}\,dx, \quad x \geq 1,\ \theta > 0$

$= \theta \int_1^x x^{-\theta-1}\,dx = \theta\left[-\frac{x^{-\theta}}{\theta}\right]_1^x$

$= \theta\left(-\frac{x^{-\theta}}{\theta} + \frac{1}{\theta}\right) = 1 - x^{-\theta}$

$F_{X_{(1)}}(x) = P(X_{(1)} \leq x) = 1 - P(X_{(1)} > x)$

$= 1 - [P(X > x)]^n = 1 - [1 - F(x)]^n$

$= 1 - (1 - (1 - x^{-\theta}))^n = 1 - (1 - 1 + x^{-\theta})^n$

$= 1 - (x^{-\theta})^n = 1 - x^{-n\theta}, \quad x \geq 1,\ \theta > 0$

$0 \quad$ otherwise

b) $\pi_p = F^{-1}(p)$

$x = 1 - y^{-n\theta}$

$y^{-n\theta} = 1 - x$

$y = (1-x)^{-\frac{1}{n\theta}} = \frac{1}{(1-x)^{\frac{1}{n\theta}}} = F^{-1}(x)$

$\Rightarrow F^{-1}(p) = \frac{1}{(1-p)^{\frac{1}{n\theta}}}, \quad 0 \leq p < 1, \quad \theta > 0$

c) Asymptotic variance of $\hat{M} \approx \frac{1}{4nf(m)^2}$

$\approx \frac{1}{4n(\theta m^{-(\theta+1)})^2}$ $\qquad M = \pi_{0.5} = F^{-1}(0.5)$

$\approx \frac{1}{4n\theta^2 m^{-2(\theta+1)}}$ $\qquad = \frac{1}{0.5^{\frac{1}{n\theta}}} = 2^{\frac{1}{n\theta}}$

$\approx$ [illegible] $\approx \frac{1}{4n\theta^2 2^{-\frac{2(\theta+1)}{n\theta}}}$