Assignment 3

Graded

Student

James La Fontaine

Total Points

18.5 / 22 pts

Question 1

transform **7.5** / 10 pts

1.1 **(no title)** **3** / 3 pts

- ✔ **+ 0.5 pts** $S_{Y_1} = (0, 2)$, give the credit if they deal with $x < 0$ and $x > 2$ in the next two parts
- ✔ **+ 0.5 pts** The method for F_{Y_1}
- ✔ **+ 0.5 pts** $F_{Y_1}(x) = \frac{2}{3}x,\ if\ 0 \leq x \leq 1$
- ✔ **+ 0.5 pts** $F_{Y_1}(x) = \frac{1}{3} + \frac{1}{3}x,\ if\ 1 < x \leq 2$
- ✔ **+ 0.5 pts** $f_{Y_1}(x) = \frac{2}{3},\ if\ 0 \leq x \leq 1$
- ✔ **+ 0.5 pts** $f_{Y_1}(x) = \frac{1}{3},\ if\ 1 < x \leq 2$
- **+ 0 pts** totally wrong or no answer

1.2 **(no title)** **0** / 2 pts

- **+ 0.5 pts** $E(Y_2) = \int_{-2}^{1} \sqrt{|x|} f_X(x) dx$
- **+ 0.5 pts** $= \int_1^2 \sqrt{x}\frac{1}{3}dx + 2\int_0^1 \sqrt{x}\frac{1}{3}dx$
- **+ 0.5 pts** Correct computation
- **+ 0.5 pts** correct answer
- ✔ **+ 0 pts** no credit

1.3 **(no title)** **3** / 3 pts

- ✔ **+ 0.5 pts** $S_{Y_2} = (0, \sqrt{2})$, give the credit if they address the cdf and pdf outside $(0, \sqrt{2})$
- ✔ **+ 0.5 pts** reasoning for F_{Y_2}
- ✔ **+ 0.5 pts** $F_{Y_2}(x) = \frac{2}{3}x^2\ if\ 0 \leq x \leq 1$
- ✔ **+ 0.5 pts** $F_{Y_2}(x) = \frac{1}{3} + \frac{1}{3}x^2\ if\ 1 < x \leq \sqrt{2}$
- ✔ **+ 0.5 pts** $f_{Y_2}(x) = \frac{4}{3}x\ if\ 0 \leq x \leq 1$
- ✔ **+ 0.5 pts** $f_{Y_2}(x) = \frac{2}{3}x\ if\ 1 < x \leq \sqrt{2}$
- **+ 0 pts** no credit

1.4 **(no title)** **1.5** / 2 pts

- ✔ **+ 1 pt** $\int_0^1 x\frac{4}{3}xdx + \int_1^{\sqrt{2}} x\frac{2}{3}xdx$
- ✔ **+ 0.5 pts** Correct derivation
- **+ 0.5 pts** $\frac{2}{9}(1 + 2^{1.5})$
- **+ 0 pts** no credit

Question 2

conceptual **1** / 1 pt

✔ **+ 1 pt** Substantial work

+ 0 pts non-substantial work

Question 3

discrete multivariate **9** / 10 pts

3.1 **(no title)** **2** / 2 pts

✔ **+ 1 pt** half of the cdf statements are correct

✔ **+ 1 pt** the other half of the statements are correct

+ 0 pts no credit

3.2 **(no title)** **1** / 2 pts

+ 0.5 pts The formula for computing p_X or correct details of derivation

✔ **+ 0.5 pts** $p_X(x) = \frac{1}{6}$ for $x = \pm 1$, $\frac{2}{3}$ for $x = 0$.

+ 0.5 pts The formula for computing p_Y or correct details of derivation

✔ **+ 0.5 pts** $p_Y(y) = \frac{1}{3}$ for $y = 1$, $\frac{2}{3}$ for $y = 0$.

+ 0 pts no credit

1 Justify.

3.3 **(no title)** **2** / 2 pts

✔ **+ 1 pt** Correctly identifying the points

✔ **+ 1 pt** correct computation and answer

+ 0 pts no credit

3.4 **(no title)** **2** / 2 pts

✔ **+ 0.5 pts** Correct computation for EX and answer

✔ **+ 0.5 pts** Correct computation for EY and answer

✔ **+ 1 pt** Correct computation for $E(XY)$ and answer

+ 0 pts no credit

3.5 **(no title)** **2** / 2 pts

✔ **+ 1 pt** No

✔ **+ 1 pt** reasoning

+ 0 pts no credit

Question 4

continuous multivariate

1 / 1 pt

- ✔ **+ 1 pt** Substantial work
- **+ 0 pts** non-substantial work

MAST20004 Probability
Semester 1, 2021
Assignment 3

Due 3 pm, Friday 7 May 2021

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Student ID:	1079860

Important instructions:

(1) This assignment contains 4 questions, **two** of which will be randomly selected to be marked. Each marked question is worth 10 points and each unmarked question with substantial working is worth 1 point.

(2) To complete this assignment, you need to write your solutions into the blank answer spaces following each question in this assignment PDF.

- If you have a printer (or can access one), then you must print out the assignment template and handwrite your solutions into the answer spaces.
- If you do not have a printer but you can figure out how to annotate a PDF using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly onto the assignment PDF and save a copy for submission.

Failing both of these methods, you may handwrite your answers as normal on blank paper and then scan for submission (but note that you will thereby miss valuable practice for the exam process). In that case, however, your document should have the same length as the assignment template otherwise Gradescope will reject your submission. So you will need to add as many blank pages as necessary to reach that criterion.

Scan your assignment to a PDF file using your mobile phone (we recommend Cam - Scanner App), then upload by going to the Assignments menu on Canvas and submit the PDF to the **GradeScope tool** by first selecting your PDF file and then clicking on 'Upload PDF'.

Note that here you do not need to submit any Matlab code with your assignment.

(3) A poor presentation penalty of 10% of the total available marks will apply unless your submitted assignment meets all of the following requirements:

- it is a single pdf with all pages in correct template order and the correct way up, and with any blank pages with additional working added only at the end of the template pages;

- has all pages clearly readable;

- has all pages cropped to the A4 borders of the original page and is imaged from directly above to avoid excessive 'keystoning'.

These requirements are easy to meet if you use a scanning app on your phone and take some care with your submission - please review it before submitting to double check you have satisfied all of the above requirements.

(4) Late submission within 20 hours after the deadline will be penalised by 5% of the total available marks for every hour or part thereof after the deadline. After that, the Gradescope submission channel will be closed, and your submission will no longer be accepted. You are strongly encouraged to submit the assignment a few days before the deadline just in case of unexpected technical issues. If you are facing a rather exceptional/extreme situation that prevents you from submitting on time, please contact the tutor coordinator **Robert Maillardet** with formal proofs such as medical certificate.

(5) Working and reasoning must be given to obtain full credit. Clarity, neatness, and style count.

Problem 1. Let $X \stackrel{d}{=} R(-2,1)$, $Y_1 = |X|$ and $Y_2 = \sqrt{|X|}$.

(i) Derive the cdf and pdf of Y_1.

$X \stackrel{d}{=} R(-2,1) \Rightarrow f_X(x) = \begin{cases} 1/3, & -2 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

$F_X(x) = \begin{cases} 0 & x \leq -2 \\ \frac{x+2}{3}, & -2 < x < 1 \\ 1, & x \geq 1 \end{cases}$

$S_{Y_1} = (0,2)$

$F_{Y_1}(y) = P(Y \leq y) = P(|X| \leq y) = P(-y \leq X \leq y)$

$= F_X(y) - F_X(-y)$

$F_{Y_1}(y) = \begin{cases} 0, & y \leq 0 \\ \frac{2y}{3}, & 0 < y < 1 \\ \cancel{y/3} + \frac{2}{3}, & 1 < y < 2 \\ 1, & y \geq 2 \end{cases}$

$f_{Y_1}(y) = \begin{cases} 2/3, & 0 < y < 1 \\ 1/3, & 1 < y < 2 \\ 0, & \text{otherwise} \end{cases}$

(ii) Compute $E(Y_2)$ using the pdf of X.

$S_{Y_2} = (0, \sqrt{2})$ but $x < 1$ for $f_X(x)$

$E(Y_2) = \int_0^1 x \cdot f_X(x)\, dx$

$= 1/3$

(iii) Derive the cdf and pdf of Y_2.

$$F_{Y_2}(y) = P(Y_2 \leq y) = P(\sqrt{|X|} \leq y)$$
$$= P(-y^2 \leq X \leq y^2)$$
$$= \begin{cases} 0, & y \leq 0 \\ \frac{2y^2}{3}, & 0 < y < 1 \\ \frac{y^2-1}{3} + \frac{2}{3}, & 1 < y < \sqrt{2} \\ 1, & y \geq \sqrt{2} \end{cases}$$

$$f_{Y_2}(y) = \begin{cases} \frac{4y}{3}, & 0 < y < 1 \\ \frac{2y}{3}, & 1 < y < \sqrt{2} \\ 0, & \text{otherwise} \end{cases}$$

(iv) Compute $\mathbb{E}(Y_2)$ using the pdf of Y_2.

$$E[Y_2] = \int_0^\infty y \cdot f_{Y_2}(y)\, dy$$
$$= \int_0^1 y \cdot \frac{4y}{3}\, dy + \int_1^{\sqrt{2}} y \cdot \frac{2y}{3}\, dy$$
$$= \left[\frac{4y^3}{9}\right]_0^1 + \left[\frac{2y^3}{9}\right]_1^{\sqrt{2}}$$
$$= \frac{4}{9} + \frac{4\sqrt{2}}{9} - \frac{2}{9}$$
$$= \frac{4\sqrt{2}}{9} - \frac{2}{9} \simeq 0.8508$$

Problem 2. For each of the following statements, determine whether it is true or false. If it is true, give a proof; if it is false, give a counterexample.

(i) If f and g are two pdfs, then $pf + (1-p)g$ is also a pdf for all $0 \leq p \leq 1$.

$f \geq 0$ & $g \geq 0$

and

$\int_{-\infty}^{\infty} f(t)\, dt = 1$ & $\int_{-\infty}^{\infty} g(t)\, dt = 1$

$pf + (1-p)g \geq 0$ for $0 \leq p \leq 1$, $f \geq 0$, $g \geq 0$ ✓

$\int_{-\infty}^{\infty} pf(t) + (1-p)g(t)\, dt$

$= p\int_{-\infty}^{\infty} f(t)\, dt + (1-p)\int_{-\infty}^{\infty} g(t)\, dt = p + 1 - p = 1$ ✓

Therefore this statement is true. $\square$

(ii) If X and Y are two random variables defined on the same sample space and $\mathbb{P}(X \neq Y) = 0$, then $X \stackrel{d}{=} Y$.

$P(X = Y) = 1$

$\Rightarrow P(X \leq x) = P(Y \leq y) \quad \forall\, x, y$ in S_X, S_Y

$\Rightarrow F_X(x) = F_Y(y)$

$\Rightarrow X \stackrel{d}{=} Y$

Therefore the statement is true.

$\square$

(iii) If X and Y are two random variables defined on the same sample space and $X \overset{d}{=} Y$, then $\mathbb{P}(X \neq Y) = 0$.

Consider the toss of a fair coin,

X = number of heads, $X \overset{d}{=} \text{Ber}(\frac{1}{2})$

Y = number of tails, $Y \overset{d}{=} \text{Ber}(\frac{1}{2})$

$X \overset{d}{=} Y$

However,

$P(X \neq Y) = 1 \neq 0$ As only a head or tail can be tossed.

Therefore this statement is false. $\square$

(iv) If X is a nonnegative random variable such that $\lim_{x\to\infty}\{x\mathbb{P}(X > x)\} = 0$, then $\mathbb{E}(X^2)$ exists.

$X \geq 0$

$\lim_{x\to\infty} (x \cdot (1 - F_X(x)) = 0$

$E[X^n] = \int_0^\infty x^n f_X(x)\, dx$

$= \int_0^\infty (n \int_0^x y^{n-1} dy) f_X(x)\, dx$

$= n \int_0^\infty y^{n-1} dy \int_y^\infty f_X(x)\, dx$

$= n \int_0^\infty y^{n-1} P(X > y)\, dy$

$\Rightarrow E[X^2] = 2\int_0^\infty x P[X > x]\, dx < \infty$

Therefore this statement is true. $\square$

(v) If X is a nonnegative continuous random variable such that $E(X^2)$ exists, then

$$\lim_{x \to \infty} \{P(X > x)x^2\} = 0.$$

This statement is false.

Problem 3. Let (X, Y) be a bivariate random variable with joint pmf

$$\mathbb{P}((X,Y) = (-1,0)) = \mathbb{P}((X,Y) = (1,0)) = \frac{1}{6}, \quad \mathbb{P}((X,Y) = (0,1)) = \mathbb{P}((X,Y) = (0,0)) = \frac{1}{3}.$$

(i) Derive the cdf of (X, Y). (Hint: you may wish to draw a graph and consider a number of different regions where the cdf is a constant.)

$p_{(X,Y)}(x,y)$	$x=-1$	$x=0$	$x=1$
$y=0$	$1/6$	$1/3$	$1/6$
$y=1$	0	$1/3$	0

$F_{(X,Y)}(x,y)$	$x=-1$	$x=0$	$x=1$
$y=0$	$1/6$	$1/2$	$2/3$
$y=1$	$1/6$	$5/6$	1

(ii) Find the marginal pmfs of X and Y.

y	0	1
$p_Y(y)$	$2/3$	$1/3$

x	-1	0	1
$p_X(x)$	$1/6$	$2/3$	$1/6$

(iii) Calculate $\mathbb{P}(X \leq Y)$.

$$1 - P(X > Y) = 1 - P_{(X,Y)}(1,0)$$
$$= 1 - 1/6 = 5/6$$

(iv) Compute $\mathbb{E}(X)$, $\mathbb{E}(Y)$ and $\mathbb{E}(XY)$.

$$E[X] = \sum_{x=-1}^{1} x \, P_X(x) = -1 \cdot \frac{1}{6} + 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{6}$$
$$\boxed{= 0}$$

$$E[Y] = \sum_{y=0}^{1} y \, P_Y(y) = 0 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3} \boxed{= \frac{1}{3}}$$

$$E[XY] = \sum_{(x,y) \in S_{(X,Y)}} xy \, P_{(X,Y)}(x,y)$$
$$= 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{3} + 0 \cdot \frac{1}{6} + -1 \cdot 0 + 0 \, \frac{1}{3} + 1 \cdot 0$$
$$= 0$$

(v) Are X and Y independent? Explain.

X and Y are not independent as

$P_{(X,Y)}(x,y) \neq P_X(x) \cdot P_Y(y) \ \forall\ x,y$

for example,

$P_{(X,Y)}(-1,0) \neq P_X(-1) \cdot P_Y(0)$

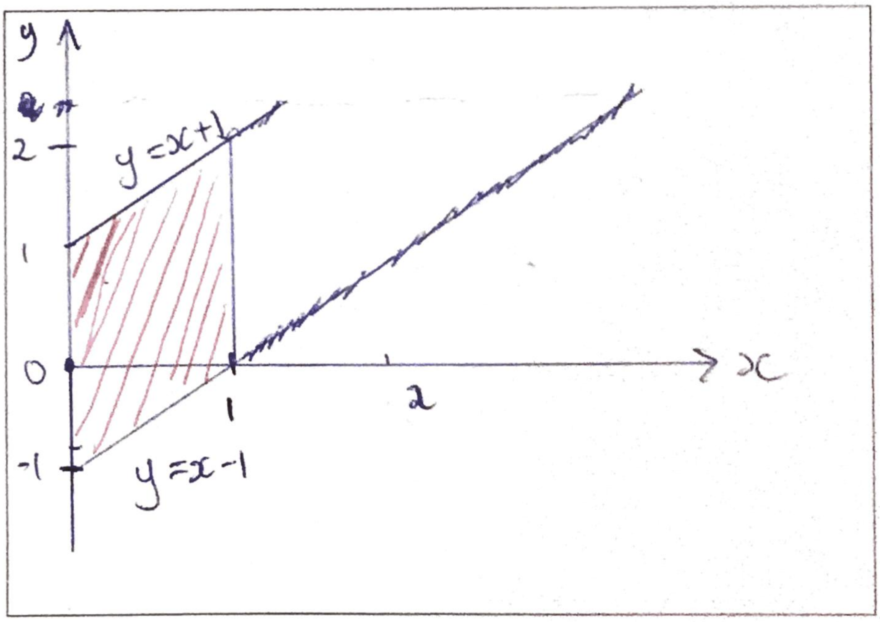
$1/6 \neq 1/6 \cdot 2/3$

Problem 4. Let

$$f(x,y) = \begin{cases} c, & \text{if } 0 \le x \le 1, x-1 \le y \le x+1, \\ 0, & \text{elsewhere,} \end{cases}$$

be the joint pdf of (X,Y).

(i) Sketch the region for which $f(x,y) > 0$.



(ii) Derive the value of c.

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{(X,Y)}(x,y)\,dx\,dy = 1$$

$$\underset{(x)}{\int_0^1}\ \underset{(y)}{\int_0^2} c\ dy\ dx$$ (can reorient into a rectangle)

$$= 2c\int_0^1 1\,dx = 2c$$

$$\Rightarrow c = 1/2$$

(iii) Find the marginal pdf of Y. Check that it is a pdf.

$$f_Y(y) = \int_0^1 \frac{1}{2}\,dx$$

$$= \left[\frac{x}{2}\right]_0^1 = \frac{1}{2}$$

$$= \begin{cases} \frac{1}{2} & 0<y<1 \\ 0 & \text{otherwise} \end{cases}$$

1) $\frac{1}{2} \geq 0$ & $0 \geq 0$ ✓

2) $\int_0^2 \frac{1}{2}\,dx = 1$

$\left[\frac{x}{2}\right]_0^2 = 1$

$1 = 1$ ✓

(iv) Determine the conditional pdf of X given $Y = y$. Check that it is indeed a pdf.

$$f_{(X|Y)}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$\begin{cases} 1, & 0 < x < 1 \quad \cancel{\text{if } 0 \leq x \leq 1,\ x+1 \leq y \leq x+1} \\ 0, & \text{elsewhere} \end{cases}$$

1) $1 \geq 0$ and $0 \geq 0$ ✓

2) $\int_0^1 1 \, dx = 1$

$[x]_0^1 = 1$

$1 = 1$ ✓

(v) Are X and Y independent? Explain.

$$f_{(X|Y)}(x|y) = f_X(x)$$

$$= \int_0^2 \frac{1}{2} \, dx$$

$$= 1 \quad ✓$$

$\Rightarrow X$ and Y are independent.