THE UNIVERSITY OF
MELBOURNE

Student number	1079860

Semester 1 Assessment, 2021

School of Mathematics and Statistics

MAST20004 Probability

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 28 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader and blank loose-leaf paper.
- One double sided A4 page of notes (handwritten only).
- No calculators are permitted.

Instructions to Students

- If you have a printer, print the exam one-sided. If you cannot print, download the exam to a second device and disconnect that device from the internet.
- Ask the supervisor if you want to use the device running Zoom.
- Check your scanned PDF before submitting.
- You must submit while in the Zoom room and no submissions will be accepted after you have left the Zoom room.

Writing

- There are 10 questions with marks as shown. The total number of marks available is 105.
- Working and/or reasoning must be given to obtain full credit. Clarity, neatness, and style count.
- Write your answers in the boxes provided on the exam that you have printed. If you need more space, you can use blank paper. Note this in the answer box, so the marker knows. The extra pages can be added to the end of the exam to scan.
- If you have been unable to print the exam write your answers on A4 paper. The first page should contain only your student number, the subject code and the subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

Scanning

- Put the pages in number order and the correct way up. Add any extra pages to the end. Use a scanning app to scan all pages to PDF. Scan directly from above. Crop pages to A4. Check PDF is readable.

Submitting

- **You must submit while in the Zoom room.** No submissions will be accepted after you have left the Zoom room.
- Go to the Gradescope window. Choose the Canvas assignment for this exam. Submit your file. Get Gradescope confirmation on email. Tell your supervisor it is submitted.

Question 1 (7 marks)

Events A and B satisfy $\mathbb{P}(A) = \frac{1}{3}$, $\mathbb{P}(A \cap B) = \frac{1}{4}$ and $\mathbb{P}(A^c \cap B) = \frac{1}{4}$.

(a) Compute

(i) $\mathbb{P}(A^c)$.

$$P(A^c) = 1 - P(A) = 1 - \frac{1}{3} = \frac{2}{3}$$

(ii) $\mathbb{P}(B|A)$.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

(iii) $\mathbb{P}(B)$.

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

$$\frac{1}{4} = P(B) - \frac{1}{4} \Rightarrow P(B) = \frac{1}{2}$$

(iv) $\mathbb{P}(A^c|B^c)$.

$$P(A^c|B^c) = \frac{P(A^c \cap B^c)}{P(B^c)} = \frac{P((A \cup B)^c)}{1 - P(B)}$$

$$= \frac{1 - (P(A) + P(B) - P(A \cap B))}{1 - P(B)} = \frac{1 - \frac{7}{12}}{1 - \frac{6}{12}} = \frac{\frac{5}{12}}{\frac{6}{12}} = \frac{5}{6}$$

(b) Is there a positive or negative relationship between the two events A and B? Explain.

$P(B|A) > P(B)$

$\frac{3}{4} > \frac{1}{2}$ $\Rightarrow$ the occurrence of A increases the chance of the occurrence of B and so there is a positive relationship

Question 2 (9 marks)

A student takes a multiple choice exam in which each question has 5 choices, exactly one of which is correct. If the student knows the correct answer, then they select that answer. Otherwise, they select an answer equally likely from the 5 choices. Suppose the student knows the answer to 50% of the questions.

(a) What is the probability that the student gets a correct answer to a given question?

$m=5$

$$P(\text{correct}) = P(\text{correct}|\text{student knew}) \cdot P(\text{student knew}) + P(\text{correct}|\text{student guss}) \cdot P(\text{student guess})$$

$$= 1 \cdot \frac{1}{2} + \frac{1}{5} \cdot \frac{1}{2}$$

$$= \frac{1}{2} + \frac{1}{10} = \frac{5}{10} + \frac{1}{10} = \frac{6}{10}$$

$$= \frac{3}{5}$$

(b) If the student gets the correct answer to a given question, what is the probability they knew the correct answer?

$$P(\text{student knew} \mid \text{correct}) = \frac{P(\text{correct} \mid \text{student knew}) \cdot P(\text{student knew})}{P(\text{correct} \mid \text{student knew}) \cdot P(\text{student knew}) + P(\text{correct} \mid \text{student guess}) \cdot P(\text{student guess})}$$

$$= \frac{1 \cdot \frac{1}{2}}{\frac{3}{5}} = \frac{1}{2} \cdot \frac{5}{3} = \frac{5}{6}$$

(c) For an exam with 20 questions, state the assumption(s) and derive the probability that they get correct answers to at least 12 questions. (No need to compute the final numerical answer.)

$X = \#$ correct answers $\stackrel{d}{=} Bi\left(20, \frac{3}{5}\right)$

$$P(X \geq 12) = P(X > 11) = 1 - P(X \leq 11)$$

$$= 1 - F_X(11)$$

$$= 1 - \sum_{x=0}^{11} \binom{20}{11} \left(\frac{3}{5}\right)^{11} \left(\frac{2}{5}\right)^{9}$$

Question 3 (10 marks)

Let A, B, C be three events with $\mathbb{P}(A) > 0$, $\mathbb{P}(B) > 0$ and $\mathbb{P}(C) > 0$. For each of the following statements, determine whether it is true or false. If it is true, give a proof; if it is false, give a counterexample.

(a) If $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$, then $A \cap B = \emptyset$.

$$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$P(A\cup B) = P(A)+P(B) \Leftrightarrow P(A\cap B) = 0$$

$$\Rightarrow A\cap B = \phi$$

Therefore this statement is true $\square$

(b) If $\mathbb{P}(A|B \cap C) = \mathbb{P}(A|B)$, then $\mathbb{P}(A \cap C|B) = \mathbb{P}(A|B)\mathbb{P}(C|B)$.

$$P(A|B\cap C) = \frac{P(A\cap B\cap C)}{P(B\cap C)} = \frac{P(A\cap B)}{P(B)} = P(A|B)$$

True.

(c) Assume that A, B and C satisfy two assumptions: (i) A and B are mutually independent, (ii) A and C are mutually independent; then A and $B \cap C$ are mutually independent.

False. Let $B = C$

For A and $B\cap C$ to be mutually independent

$$P(A\cap(B\cap C)) = P(A)\cdot P(B\cap C) = P(A)\cdot P(B)$$

(d) If f and g are two probability density functions, then $\frac{1}{2}(f+g)$ is also a probability density function.

$$\int_{-\infty}^{\infty} f \, dx = 1, \quad \int_{-\infty}^{\infty} g \, dx = 1$$

$$\geq 0 \qquad \geq 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{1}{2}(f+g) \, dx = \frac{1}{2}\left(\int_{-\infty}^{\infty} f \, dx + \int_{-\infty}^{\infty} g \, dx\right)$$

$$= \frac{1}{2}(1+1) = 1 \checkmark$$

and clearly $\frac{1}{2}\left(\int_{-\infty}^{\infty} f \, dx + \int_{-\infty}^{\infty} g \, dx\right) \geq 0$ as $\int_{-\infty}^{\infty} f \, dx$ and $\int_{-\infty}^{\infty} g \, dx \geq 0$ ~~so this~~ statement is true.

(e) For a continuous nonnegative random variable X, if $\mathbb{E}(X^2)$ exists, then $\mathbb{E}(X)$ exists.

$X \geq 0$

$E(X^2)$ exists

$$\Rightarrow \int_0^{\infty} |x^2| f_X(x) \, dx < \infty$$

$$\Rightarrow \int_0^{\infty} x^2 f_X(x) \, dx < \infty$$

$$\Rightarrow \int_0^{\infty} x f_X(x) \, dx < \infty$$

therefore $E(X)$ converges absolutely and exists. This statement is true.

Question 4 (10 marks)

A continuous random variable X has probability density function given by

$$f_X(x) = \begin{cases} c(1-|x|) & \text{for } -1 < x < 1, \\ 0 & \text{otherwise,} \end{cases}$$

where c is a constant.

(a) Find the value of c.

$\int_{-1}^{1} c(1-|x|)\,dx = 1$

$\Rightarrow c \int_{-1}^{1} 1-|x|\,dx = 1$

$\Rightarrow 2c \int_{0}^{1} 1-x\,dx = 1$ (even function)

$\Rightarrow 2c\left[x - \frac{x^2}{2}\right]_0^1 = 1 \Rightarrow 2c - c = 1$

$\Rightarrow c = 1$

(b) Find the cumulative distribution function F_X of X.

$F_X(x) = \int_{-1}^{x} c(1-|x|)\,dx$

$\left[x + \frac{x^2}{2}\right]_{-1}^{x} = x + \frac{x^2}{2} - (-1 + \frac{1}{2})$

$F_X(x) = \int_{-1}^{x} 1 + x\,dx$ for $= x + \frac{x^2}{2} + \frac{1}{2}$ $-1 < x < 0$

$F_X(x) = \int_{-1}^{0} 1+x\,dx + \int_{0}^{x} 1-x\,dx$ $x - \frac{x^2}{2} - (0)$ for $0 < x < 1$

$= x + \frac{x^2}{2} + \frac{1}{2} + x - \frac{x^2}{2}$

(c) Find the mean and variance of X.

$$E[X] = \int_{-1}^{1} x(1-|x|)\,dx$$

$$=$$

(d) Use the Bienaymé-Chebyshev inequality to give a lower bound for the probability that X takes values within 3 standard deviations of its mean and compare it with the actual probability. Comment on your findings.

$$P(|X-\mu| \leq 3\sigma) \geq 1 - \frac{1}{9} = \frac{8}{9}$$

$$P(|X-\mu| \geq 3\sigma) \leq \frac{1}{9}$$

$$P(X = \frac{1}{9})$$

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(e) Let $Y = \log(X^2)$. Find the probability density function of Y.

$Y = \log(x^2) \Rightarrow e^Y = x^2 \Rightarrow X = \pm\sqrt{e^Y}$

$\Rightarrow P(Y=y) = P(\log(x^2) = y)$

$= P(x^2 = e^y)$

$= P(x = |e^y|)$

$= f_x(|e^y|)$

$f_Y(y) = \begin{cases} 1-|e^y| & -1<|e^y|<1 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} 1-e^y & e^y<1 \\ 0 & \text{otherwise} \end{cases}$

$= \begin{cases} 1-e^y & y<0 \\ 0 & \text{otherwise} \end{cases}$

Question 5 (15 marks)

Accidents in a Poisson town happen according to a Poisson process with rate one accident per week, and the cost of the damage caused by each accident is an exponential random variable with mean \$1000, independent of other accidents.

$\rightarrow \frac{1}{\alpha} = 1000 \Rightarrow \alpha = \frac{1}{1000}$

(a) Find the probability that in a given week there are at least 2 accidents.

$X = \#$ accidents in a week $\stackrel{d}{=} Pn(1)$

$$P(X \geq 2) = 1 - P(X < 2)$$
$$= 1 - (P(X = 0) + P(X = 1))$$
$$= 1 - \left(\frac{e^{-1} 1^0}{0!} + \frac{e^{-1} 1^1}{1!}\right)$$
$$= 1 - \left(\frac{1}{e} + \frac{1}{e}\right)$$
$$= 1 - \frac{2}{e}$$

(b) An accident with a cost of damage less than \$500 is classified as a minor accident. For a week with X accidents in the Poisson town, let M be the number of minor accidents in that week.

(i) Compute the conditional probability mass function of M given $X = n$ for $n = 0, 1, 2, \ldots$.

$C =$ cost of accident damage $\stackrel{d}{=} \exp\left(\frac{1}{1000}\right)$

$P(C < 500) = P(C \leq 500)$ $(P(C = 500) = 0$ as C is continuous$)$

$= F_C(500) = 1 - e^{-\frac{1}{1000} \times 500} = 1 - e^{-\frac{1}{2}} = 1 - \frac{1}{\sqrt{e}}$

$M \stackrel{d}{=} Bi\left(X, 1 - \frac{1}{\sqrt{e}}\right)$

$p_{M|X}(m \mid X = n) = \begin{cases} \binom{n}{m}\left(1 - \frac{1}{\sqrt{e}}\right)^m \left(\frac{1}{\sqrt{e}}\right)^{n-m} & x \geq 0,\ 0 \leq m \leq x \\ 0 & \text{otherwise} \end{cases}$

(ii) Derive the probability mass function of M.

i)

(c) Let T be the total cost of damage in a week caused by the accidents in the Poisson town.

(i) Represent T as a suitable sum and state the assumptions.

$T = \sum_{i=1}^{x} C_i$ where C_i = the cost of accident i

(ii) Compute $E(T|X)$ and $\mathrm{Var}(T|X)$.

(iii) Compute the mean and variance of T using the formulas $E(T) = E\{E(T|X)\}$ and $\mathrm{Var}(T) = E\{\mathrm{Var}(T|X)\} + \mathrm{Var}\{E(T|X)\}$.

(iv) Four students wrote Matlab programs for simulating the values of $\mathbb{E}(T)$ and $\mathrm{Var}(T)$, only one of them is correct. Which of them is correct?

(I)

```
nreps=10000 ;
for i=1:nreps
N=randpois(1,1);
T(i)=sum(-log(1-rand(N,1))/1000);
end
Tmean=mean(T)
VarT=std(T)^2
```

(II)

```
nreps=10000 ;
for i=1:nreps
N=randpois(1,1);
T(i)=sum(-(1-randn(N,1))/1000);
end
Tmean=mean(T)
VarT=std(T)^2
```

(III)

```
nreps=10000 ;
for i=1:nreps
N=randpois(1,1);
T(i)=sum(-1000*(1-randn(N,1)));
end
Tmean=mean(T)
VarT=std(T)^2
```

(IV)

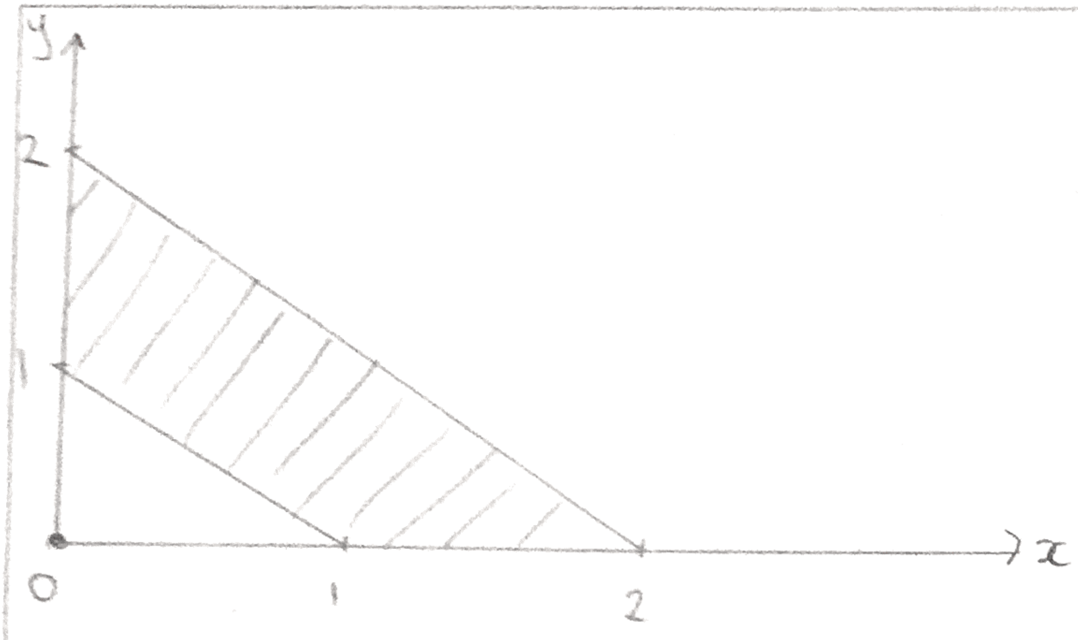
```
nreps=10000 ;
for i=1:nreps
N=randpois(1,1);
T(i)=sum(-1000*log(1-rand(N,1)));
end
Tmean=mean(T)
VarT=std(T)^2
```

Question 6 (13 marks)

The joint probability density function of the random variables X and Y is given by

$$f(x,y) = \begin{cases} cx, & x > 0,\ y > 0,\ 1 \le x + y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Draw the support of (X, Y).



(b) Show that $c = \frac{6}{7}$.

fix y, $1-y < x < 2-y$

$$\int_{1-x}^{2-x}\int_{1-y}^{2-y} cx \, dx \, dy = 1$$

$$c\int_{1-x}^{2-x}\left[\frac{x^2}{2}\right]_{1-y}^{2-y} dy = 1$$

$$c\int_{1-x}^{2-x} \frac{(2-y)^2}{2} - \frac{(1-y)^2}{2} dy = 1$$

$$\frac{1}{2}c\int_{1-x}^{2-x} 4-4y+y^2-(1-2y+y^2)\, dy = 1$$

$$\frac{1}{2}c\int_{1-x}^{2-x} 3-2y \, dy = 1$$

$$\frac{1}{2}c\left[3y-y^2\right]_{1-x}^{2-x} = 1 \Rightarrow \frac{1}{2}c\left[3(2-x)-(2-x)^2 - (3(1-x)-(1-x)^2)\right] = 1$$

$$\Rightarrow \frac{1}{2}c[(6-3x-4+4x-x^2-(3-3x-1+2x-x^2) = 1$$

$$\Rightarrow \frac{1}{2}c(2x) = 1$$

$\Rightarrow$

$c = 6/7$

(c) Determine the marginal probability density function of X.

$$f_X(x) = \int_{1-x}^{2-x} f_{(X,Y)}(x,y)\,dy$$

$$= \int_{1-x}^{2-x} \frac{6}{7}x\,dy$$

$$= \frac{6}{7}\int_{1-x}^{2-x} x\,dy$$

$$= \frac{6}{7}\left[xy\right]_{1-x}^{2-x}$$

$$= \frac{6}{7}\left[x(2-x) - x(1-x)\right]$$

$$= \frac{6}{7}\left[2x - x^2 - x + x^2\right]$$

$$= \frac{6}{7}x$$

(d) Determine the conditional probability density function of Y given $X = x$.

$$f_{Y|X=x}(y \mid X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$= \begin{cases} 1 & 1-y \le x < 2-y \\ 0 & \text{otherwise} \end{cases}$$

(e) Are X and Y independent? Explain.

X and Y are not independent as

(f) Compute the probability $\mathbb{P}(Y > 1 | X = \frac{1}{2})$.

$P(Y>1|X=\frac{1}{2}) = 1$

(g) Compute the probability $\mathbb{P}(Y > 1)$.

$P(Y>1) =$

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Question 7 (10 marks)

Let P be a random point uniformly distributed inside the unit circle, and let (X, Y) be the Cartesian coordinates of P. The joint probability density function of (X, Y) is thus given by

$$f_{(X,Y)}(x, y) = \begin{cases} \frac{1}{\pi}, & 0 \le x^2 + y^2 \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Derive the joint and marginal probability density functions for the *polar coordinates* (R, Θ) of P. Check that your marginal probability density functions integrate to 1 over the appropriate domain.

Polar coordinates (r, θ) where

$r = \sqrt{x^2+y^2}$

$\tan\theta = \frac{y}{x} \Rightarrow \theta = \arctan\left(\frac{y}{x}\right)$

$R, \Theta = \left(\sqrt{x^2+y^2}, \arctan\left(\frac{y}{x}\right)\right)$

$f_{(R,\Theta)}(r, \theta) = \begin{cases} \frac{1}{\pi}, & 0 \le r^2 \le 1 \\ 0, & \text{elsewhere} \end{cases}$

$f_R(r) = \int_0^{\pi} f_{(R,\Theta)}(r, \theta)\, d\theta$

$= \left[\frac{1}{\pi}\theta\right]_0^{\pi}$

$= 1$

$f_R(r) = \begin{cases} \frac{1}{\pi} & 0 \le r \le 1 \\ 0 & \text{otherwise} \end{cases}$

$f_\Theta(\theta) = \begin{cases} \frac{1}{\pi} & 0 \le \theta < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$

(b) Are the polar coordinates independent?

Yes the polar coordinates are independent as the radius r does not affect the angle with the x-axis θ

(c) Derive the cumulative distribution function F_R for R.

$$F_R = \int_0^1 1 \, dr = P(R \le r)$$
$$= P(\sqrt{x^2+y^2} \le r)$$
$$= P(x^2+y^2 \le r^2)$$

(d) Explain how you would use F_R to generate an observation of R from an observation of $U \stackrel{d}{=} R(0,1)$.

(e) Explain how to modify the joint probability density functions $f_{(X,Y)}(x,y)$ and $f_{(R,\Theta)}(r,\theta)$ in order for P to be a random point uniformly distributed inside the first quarter of the unit circle.

Does that modify the marginal distributions of the polar coordinates R and Θ? Give a simple argument without computation.

alter the range for which $f_{X,Y}$ is defined to $0 \leq x+y \leq 1$ and for $f_{R,\Theta}$ $0 \leq \theta < \frac{\pi}{2}$

It does not modify the marginal distributions of the polar coordinates as $0 < r < 1$ still holds across the whole quarter of the unit circle as well as $0 \leq \Theta < \frac{\pi}{2}$

Question 8 (6 marks)

Let (X, Y) be a general bivariate normal random variable.

(a) If $\mathrm{Cov}(X, Y) = 0$, show that X, Y are independent.

$\mathrm{Cov}(X, Y) = 0 \Rightarrow E((X - \mu_X)(Y - \mu_Y)) = 0$

$X \overset{d}{=} N(0,1)$

$Y \overset{d}{=} N(0,1)$

$\Rightarrow E(XY) - E(X) - E(Y) = 0$

$\Rightarrow E(XY) = E(X) \cdot E(Y)$

$\Rightarrow X$ and Y are independent

(b) If $\mathrm{Var}(X) = \mathrm{Var}(Y)$, show that $\mathrm{Cov}(X + Y, X - Y) = 0$.

$\mathrm{Var}(X) = \mathrm{Var}(Y)$

$\mathrm{Cov}(X+Y, X-Y) = 0$

$\Rightarrow (1 \cdot 1)\mathrm{Var}(X) + (1 \cdot -1 + 1 \cdot 1)\mathrm{Cov}(X, Y) + (1 \cdot -1)\mathrm{Var}(Y) = 0$

$\Rightarrow \mathrm{Var}(X) - \mathrm{Var}(Y) = 0$

$\Rightarrow \mathrm{Var}(X) - \mathrm{Var}(X) = 0$ ✓

(c) Assume that $\mu_X = 0$, $\sigma_X^2 = 1$, $\mu_Y = -1$, $\sigma_Y^2 = 4$, and $\rho = 1/2$. We can show that $X + Y$ is a normal random variable with mean $\mu_{X+Y} = \mathbb{E}(X+Y)$ and variance $\sigma_{X+Y}^2 = \mathrm{Var}(X+Y)$.
Compute μ_{X+Y} and σ_{X+Y}^2, and then write an expression for $\mathbb{P}(X + Y > 0)$ in terms of the cumulative distribution function of the standard normal distribution. *No numerical answer is required.*

$X \stackrel{d}{=} N(0,1)$, $Y \stackrel{d}{=} N(-1,4)$, $\rho(X,Y) = \frac{1}{2}$

$\mu_{X+Y} = E(X+Y)$

$= E(X) + E(Y) = 0 - 1 = -1$

Question 9 (15 marks)

Let X be a geometric random variable with parameter (success probability) $p = 3/4$, and let $X_1, X_2, \ldots, X_n$ be independent and identically distributed copies of X. Let $S_n = X_1 + X_2 + \cdots + X_n$.

(a) Prove that the mgf (moment generating function) of X is $M_X(t) = \frac{3}{4 - e^t}$, and determine the domain of values of t for which it is well defined.

$$M_x(t) = E(e^{tx}) \qquad X \stackrel{d}{=} G(3/4)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{1}{4^x} \cdot \frac{3}{4}$$

$$= \frac{3}{4} \sum_{x=0}^{\infty} \frac{e^{tx}}{4^x}$$

defined for

$$0 \le t < \log_e(4)$$

(b) Find the mgf $M_{S_n}(t)$ of S_n. Then state the distribution of S_n.

let $N \stackrel{d}{=} N(\frac{n}{3},$

$$M_{S_n}(t) = E(e^{t(X_1 + X_2 + \ldots + X_n)})$$

$$= E(e^{tX_1} \cdot e^{tX_2} \cdot \ldots \cdot e^{tX_n})$$

$$= E(e^{tX_1}) \cdot E(e^{tX_2}) \cdot \ldots \cdot E(e^{tX_n})$$

$$= \left(\frac{3}{4 - e^t}\right)^n = \left(\frac{3}{4}\left(1 - \frac{1}{e^t}\right)\right)^n = \left(\frac{3}{4}\left(1 - e^{-t}\right)\right)^n$$

$$= \left(\frac{3}{4}\left(1 - \cancel{1} - t + \frac{t^2}{2} + o(2)\right)\right)^n$$

$$= \left(-\frac{3}{4}t + \frac{3t^2}{4}\right)^n$$

$$S_n \stackrel{d}{=} N\left(\frac{n}{3}, \quad\right)$$

(c) Find the mgf $M_{\bar{X}_n}(t)$ of the sample mean $\bar{X}_n = \frac{S_n}{n}$.

$$M_{\bar{X}_n}(t) = E(e^{t\frac{S_n}{n}}) = E(e^{\frac{t}{n}S_n})$$

$$=$$

(d) Find the limit $\lim_{n\to\infty} M_{\bar{X}_n}(t)$ using the result of (c). What distribution does the limiting mgf correspond to? Interpret your result.

(e) Let

$$Z_n = \frac{3}{2}\sqrt{n}\left(\bar{X}_n - \frac{1}{3}\right)$$

Find $M_{Z_n}(t)$, the mgf of Z_n. Then find the limiting mgf $\lim_{n\to\infty} M_{Z_n}(t)$. What is the limiting distribution of Z_n? Interpret your result.

Question 10 (10 marks)

Suppose brands A and B have consumer loyalties of 0.7 and 0.8, meaning that a customer who buys A one week will with probability 0.7 buy it again the next week, or try the other brand with probability 0.3. Let X_n be the brand bought by a specific customer on Week n, $n \geq 1$.

(a) Model $\{X_n\}$ as a Markov chain by determining the state space S and the transition matrix P.

$S = \{A, B\}$

$P(X_1 = A \mid X_0 = A) = 0.7$

$P(X_1 = A \mid X_0 = B) = 0.2$

$P(X_1 = B \mid X_0 = A) = 0.3$

$P(X_1 = B \mid X_0 = B) = 0.8$

$$P = \begin{array}{c|cc} & A & B \\ \hline A & 0.7 & 0.3 \\ B & 0.2 & 0.8 \end{array}$$

(b) If Peter buys brand A on Week 1, what is the probability that he is loyal and keeps buying brand A for the next two weeks?

$= P(X_3 = A \cap X_2 = A \mid X_1 = A)$

$= P(X_3 = A \mid X_2 = A \cap X_1 = A)$

$= P(X_3 = A \mid X_2 = A)$

$= 0.7$

(c) If Peter buys brand A on Week 1, what is the probability that he buys brand A on Week 3? *Note that this is a different question from (b).*

$P(X_3 = A \mid X_1 = A)$

$= P_{AA}^{2}$

$= \left(\frac{7}{10}\right)^2 + \frac{3}{10} \cdot \frac{2}{10} = \frac{49}{100} + \frac{6}{100}$

$= \frac{55}{100} = 0.55$

(d) What is the limiting market share for each of these brands?

(e) Suppose now that there is a third brand with loyalty 0.9, that the third brand does not influence the loyalties of the first two, and that a consumer who changes any brand picks one of the other two with equal probability. If X_n still denotes the brand bought by a specific customer on Week n ($n \geq 1$), what is the new state space S, and transition matrix P corresponding to the Markov chain?

(f) What conclusion can be drawn from the following Matlab output with P in (e):

```
>> P^10

ans =

    0.1958    0.2921    0.5121
    0.1947    0.3172    0.4881
    0.1707    0.2441    0.5852

>> P^50

ans =

    0.1818    0.2727    0.5455
    0.1818    0.2727    0.5455
    0.1818    0.2727    0.5455

>> P^100

ans =

    0.1818    0.2727    0.5455
    0.1818    0.2727    0.5455
    0.1818    0.2727    0.5455
```

The limiting distribution can be observed by P^{50}

$\vec{\pi} = (0.1818, 0.2727, 0.5455)$

End of Exam — Total Available Marks = 105