MAST10006 Calculus 2, Semester 1, 2020

Assignment 7

School of Mathematics and Statistics, The University of Melbourne

This assignment is due at **6pm Monday 12 October 2020**.

Name: Solutions, Question 1 marked, Total 12 Student ID: ____________________

The format of this assignment is different to previous assignments. The questions are on the following pages, along with spaces for you to write your answers. This is the same format that we will use for the exam this semester so this assignment is a chance to get familiar with the format and submission process. **There will be a mark for correctly following the upload procedure**.

To complete this assignment, you have two options. In both cases, you should write your answer on paper, rather than annotating a pdf or typing your solutions. Here are the two options.

1. Print a hard-copy of this assignment. Write your name and student ID on this page, and write your answers in the spaces provided on the following pages. If you need additional space, you can use extra blank pages, but make a note at the end of the relevant answer box that you have done so. To submit, scan your work into a single PDF file **with all pages (including this one) in order** and **any additional pages at the end**. Upload your PDF into Gradescope via Canvas.

2. If you don't have access to a printer, then write your answers on blank sheets of A4 paper, with

   - Your name and student ID (only) on page 1
   - Your answer to question 1(a),(b) on page 2
   - Your answer to question 1(c) on page 3
   - Your answer to question 2(a),(b) on page 4
   - Your answer to question 2(c) on page 5

   Scan your work into a single PDF file **with all pages in order**. If you can't fit your answers on the pages as described above, then you can use extra pages but put those pages **at the end** end of the document (page 6 onwards). Upload your PDF into Gradescope via Canvas.

Further instructions:

- Submit your solutions as a single PDF file with correctly oriented pages in the order as explained above
- This assignment is worth 2.22% of your final MAST10006 mark.
- Answer all questions below. One of the two questions will be chosen for marking.
- Marks may be awarded for:
  - Correct use of appropriate mathematical techniques
  - Accuracy and validity of any calculations or algebraic manipulations
  - Clear justification or explanation of techniques and rules used
  - Clear communication of mathematical ideas through diagrams
  - Use of correct mathematical notation and terminology
- You must use methods taught in MAST10006 Calculus 2 to solve the assignment questions.
- It is **your responsiblity to ensure your assignment is successfully uploaded before the deadline**. Delays due to minor technical issues eg slow internet are not grounds for special consideration.

Questions are on the following pages

Question 1 - Whey to go! An Australian entrepreneur has just been awarded a "World Vodka Award" four times in a row. See if you can find it: `http://www.worldvodkaawards.com`. Here's what some people had to say about the vodka:

"It's the only known vodka in the world to incorporate sheep's whey - whey to go!".

"The vodka can be described as a creamy sweet nose, velvet body with a smooth attitude to finish. The nose has a delicate brown sugar sweetness with a delightful floral undertone. Therefore, upon tasting you are welcomed with fresh pear and golden apple. Rounding out with hints of wild spice, leather and mineral freshness to finish. Super smooth with little heat and plenty of character. Designed to drink straight at room temperature. Serve alone in a large aromatic glass or as a vodka martini with a pair of quality green olives."

Vodka consists mainly of water and ethanol. Before bottling, the ethanol is diluted with water to the standard bottle concentration of 40%. Suppose a 2500 gallon tank initially contains 600 gallons of pure ethanol. An ethanol-water mix of $\left(\frac{1+\cos t}{6}\right)$ % ethanol is run at 6 gallons/min and the mixture stirred so that it mixes uniformly. Simultaneously the mixture is withdrawn at 3 gallons/min.

(a) Show that $x(t)$ satisfies a differential equation of the form

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$$\frac{dx}{dt} + \frac{1}{200+t}x = \frac{1+\cos(t)}{100}$$

where $x(t)$ is the amount of ethanol in the tank at time t. **Solution.**

Let $x(t)$ be the amount of ethanol in the tank at time t in gallons.

Inflow $= 6$ (gallons/min) $\times (0.01)\left(\frac{1+\cos t}{6}\right)$ (gallons / gallon) $= \frac{1+\cos t}{100}$ (gallons / min) .

Outflow $= \frac{x}{600+3t}$ (gallons / gallon) $\times$ 3 (gallons / min) $= \frac{x}{200+t}$ (gallons / min) .

1J: Derive ODE from $\frac{dx}{dt}$ = inflow - outflow

$$\begin{aligned}
\frac{dx}{dt} &= \text{inflow} - \text{outflow} \\
&= \frac{1+\cos t}{100} - \frac{x}{200+t} \\
\implies \frac{dx}{dt} + \frac{1}{200+t}x &= \frac{1+\cos(t)}{100}
\end{aligned}$$

(b) When will the tank overflow? **Solution.**

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The volume in the tank at time t is $V(t) = 600 + 3t$.

The tank overflows when

$$\begin{aligned}
V &= 2500 \\
\implies 600 + 3t &= 2500 \\
\implies t &= \frac{1900}{3}
\end{aligned}$$

1M: Solve V = 2500 for t

This is $\frac{1900}{3}$ minutes (or 633 minutes and 20 seconds) after the process has started.

1A: Answer with units.

Note $t = \frac{1900}{3}$ is okay if units for t were defined earlier

(c) Find the concentration of ethanol in the tank at time t.

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Will a concentration of 40% be obtained before the tank overflows? **Solution.**

The ODE is linear so we us the integrating factor method. An integrating factor is

$$\begin{aligned} I &= e^{\int (200+t)^{-1}\,dt} \\ &= e^{\log(200+t)} \\ &= 200+t \end{aligned}$$

1A: Any correct integrating factor

Therefore

$$\begin{aligned} \frac{d}{dt}\left(x(200+t)\right) &= \frac{(1+\cos t)(200+t)}{100} \\ \implies x(200+t) &= \frac{1}{100}\int (1+\cos t)(200+t)\,dt \\ &= \frac{1}{100}\int \left(\frac{d}{dt}(t+\sin t)\right)(200+t)\,dt \\ &= \frac{1}{100}\left((t+\sin t)(200+t) - \int t+\sin t\,dt\right) && \text{Integration by parts} \\ &= \frac{1}{100}\left((t+\sin t)(200+t) - \left(\frac{t^2}{2}-\cos t + d\right)\right) && d\in\mathbb{R} \\ \implies x(t) &= \frac{1}{100}\left(t+\sin t - \frac{\frac{t^2}{2}-\cos t - d}{200+t}\right) \\ &= \frac{1}{100}\left(t+\sin t + \frac{-\frac{t^2}{2}+\cos t + d}{200+t}\right) \\ &= \frac{1}{100}\left(\sin t + \frac{200t+\frac{t^2}{2}+\cos t + d}{200+t}\right) \\ &= \frac{1}{100}\left(\frac{200\sin t + t\sin t + 200t + \frac{t^2}{2}+\cos t + d}{200+t}\right) \end{aligned}$$

Note: Can expand at this step, but will need to integrate tcost using integration by parts

1M: use integration by parts. Note: This can be written seperately

1M: Divide by integrating factor to obtain x

Note: These are some possible forms that $x(t)$ may take depending on how the integration and algebra were done. All are acceptable.

Since $x(0) = 600$, we have

$$\begin{aligned} 600 &= \frac{1}{100}\left(0+0-\frac{0-1-d}{200+0}\right) \\ \implies d &= 12000000 - 1 \\ &= 11999999 \end{aligned}$$

1M: Use $x(0)=600$ to find constant

The concentration of ethanol is therefore

$$c(t) = \frac{x}{600+3t} = \frac{1}{100(600+3t)}\left(t+\sin t - \frac{\frac{t^2}{2}-\cos t - 11999999}{200+t}\right)$$

1A: Any correct form for $c(t)$ in terms of t. eg. use any form of $x(t)$ above

The tank overflows when $t = \frac{1900}{3}$. If the mixing dilution process had continue to this time, then the concentration in the tank would have gone down from 1 to

$$c\left(\frac{1900}{3}\right) \approx 0.059$$

Can multiply $c(t)$ by 100 if conc. given as %

Therefore, the concentration drops from 100% to approximately 6% so does reach 40% at some point during the process.

1A: 40% ethanol is obtained

One way to get an idea of when this occurs is to draw a graph.

1J: A brief reason is given. eg. A graph, a calculation that $c(\frac{1900}{3}) < 0.4$, solving $c(t)=0.4$ for t.

Aside: for a more rigorous explanation of why 40% concentration is obtained, one could use the intermediate value theorem, though using this theorem is not required in Calculus 2

1N: notation correct in question 1

1 File submitted in correct format. pg1-name, pg2-1a)b), pg3-1c), pg4-2a)b), pg5-2c), extra pages at end.

Question 2 - Body snatching.
Parasitic wasps lay their eggs on caterpillars, in order to provide their newly hatched babies with a fresh supply of "caterpillar steak tartare". The caterpillar that is incubating the young wasps spends its dying moments protecting the wasps that are eating it alive. The caterpillar does so by banging its head when it detects other predators nearby, an action that is entirely caused by the wasp ... so truely body snatching!
Find out more here (but maybe don't click if you're squeamish!): `https://www.nhm.ac.uk/discover/body-snatchers-eaten-alive.html`

The population x of parasitic wasps living on an army of caterpillars at time t can be modelled by the logistic equation

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{a}\right), \quad x \geq 0$$

where $k > 0$ and $a > 0$ are constants.
Research shows that the wasps can lay as many as 80 eggs on one hapless caterpillar. Suppose we have 100 caterpillars in a given area. In this scenario, the carrying capacity is $a = 8000$.

(a) Find the equilibrium solution(s). **Solution.**

Solve: $\frac{dx}{dt} = 0$

$kx\left(1 - \frac{x}{a}\right) = 0$

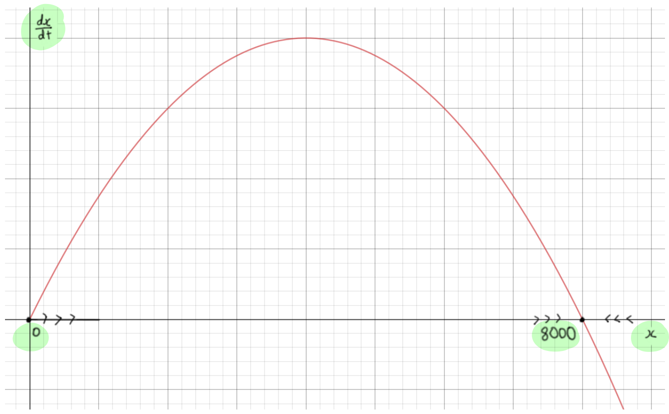
$\implies x = 0$ or $1 - \frac{x}{a} = 0$

$\implies x = 0$ or $x = a$

The equilibrium solutions are $x(t) = 0$ and $x(t) = a = 8000$.

(b) Draw a phase plot for the ODE. Without solving for $x(t)$, comment on the stability of the equilibrium solution(s). **Solution.**

Here is the phase plot:



For $0 < x < 8000$, $\frac{dx}{dt} > 0$, so x is increasing.
Therefore $x(t) = 0$ is unstable.

For $x > 8000$, $\frac{dx}{dt} < 0$, and so x is decreasing.
As before, for $0 < x < 8000$, $\frac{dx}{dt} > 0$, so x is increasing.
Therefore $x(t) = a = 8000$ is stable.

(c) Find the population of the wasps at time t, if the growth rate is $k = 1$ per unit time and there were 80 wasps initially. **Solution.**

The ODE is separable, so we use separation of variables.

$$\begin{aligned} \frac{dx}{dt} &= x\left(1 - \frac{x}{8000}\right) \\ \implies \frac{8000}{x(8000-x)}\frac{dx}{dt} &= 1 \\ \implies \int \frac{8000}{x(8000-x)}\,\mathrm{d}x &= \int 1\,\mathrm{d}t \end{aligned}$$

Use partial fractions:

$$\begin{aligned} \frac{8000}{x(8000-x)} &= \frac{A}{x} + \frac{B}{8000-x} \\ &= \frac{8000A - Ax + Bx}{x(8000-x)} \\ \implies 8000 &= 8000A + (B-A)x \end{aligned}$$

Equating coefficients we have the equations

$$A = 1 \text{ and } B - A = 0.$$

Therefore

$$A = 1, B = 1.$$

Going back to the ODE we have

$$\begin{aligned} & \int \frac{1}{x} + \frac{1}{8000-x}\,\mathrm{d}x = \int 1\,\mathrm{d}t \\ \implies & \log|x| - \log|8000-x| = t + c \qquad c \in \mathbb{R} \\ \implies & \log\left|\frac{x}{8000-x}\right| = t + c \\ \implies & \frac{x}{8000-x} = Ae^t \qquad A = \pm e^c \\ \implies & x = Ae^t(8000-x) \\ \implies & x(1 + Ae^t) = 8000Ae^t \\ \implies & x = \frac{8000Ae^t}{1+Ae^t} \\ & \quad = \frac{8000A}{e^{-t} + A} \end{aligned}$$

Since $x(0) = 80$, we have

$$80 = \frac{8000A}{1+A} \implies A = \frac{1}{99}$$

So the population of wasps at time t is

$$x(t) = \frac{\frac{8000}{99}}{e^{-t} + \frac{1}{99}} = \frac{8000}{99e^{-t} + 1}$$

End of assignment

Notes for question 2.

2b) Axes should be labelled, and the graph should not include negative x values.

2b) Did you indicate or explain how you knew that the equilibrium solutions were stable or unstable?

2c) The logs from integration should have absolute value signs. You should explain how they were removed. Probably the simplest way to do this is to incorporate them into the constant as in the solutions here. You could also have tried considering the domain of x.

2c) Any of the two forms of x(t) at the end would be okay.