MAST20004 Probability
Semester 1, 2021
Assignment 3: Solutions

Due 3 pm, Friday 7 May 2021

Name:
Student ID:

Important instructions:

(1) This assignment contains 4 questions, **two** of which will be randomly selected to be marked. Each marked question is worth 10 points and each unmarked question with substantial working is worth 1 point.

(2) To complete this assignment, you need to write your solutions into the blank answer spaces following each question in this assignment PDF.

- If you have a printer (or can access one), then you must print out the assignment template and handwrite your solutions into the answer spaces.
- If you do not have a printer but you can figure out how to annotate a PDF using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly onto the assignment PDF and save a copy for submission.

Failing both of these methods, you may handwrite your answers as normal on blank paper and then scan for submission (but note that you will thereby miss valuable practice for the exam process). In that case, however, your document should have the same length as the assignment template otherwise Gradescope will reject your submission. So you will need to add as many blank pages as necessary to reach that criterion.

Scan your assignment to a PDF file using your mobile phone (we recommend Cam - Scanner App), then upload by going to the Assignments menu on Canvas and submit the PDF to the **GradeScope tool** by first selecting your PDF file and then clicking on 'Upload PDF'.

Note that here you do not need to submit any Matlab code with your assignment.

(3) A poor presentation penalty of 10% of the total available marks will apply unless your submitted assignment meets all of the following requirements:

- it is a single pdf with all pages in correct template order and the correct way up, and with any blank pages with additional working added only at the end of the template pages;
- has all pages clearly readable;

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- has all pages cropped to the A4 borders of the original page and is imaged from directly above to avoid excessive 'keystoning'.

These requirements are easy to meet if you use a scanning app on your phone and take some care with your submission - please review it before submitting to double check you have satisfied all of the above requirements.

(4) Late submission within 20 hours after the deadline will be penalised by 5% of the total available marks for every hour or part thereof after the deadline. After that, the Gradescope submission channel will be closed, and your submission will no longer be accepted. You are strongly encouraged to submit the assignment a few days before the deadline just in case of unexpected technical issues. If you are facing a rather exceptional/extreme situation that prevents you from submitting on time, please contact the tutor coordinator **Robert Maillardet** with formal proofs such as medical certificate.

(5) Working and reasoning must be given to obtain full credit. Clarity, neatness, and style count.

Problem 1. Let $X \stackrel{d}{=} R(-2,1)$, $Y_1 = |X|$ and $Y_2 = \sqrt{|X|}$.

(i) Derive the cdf and pdf of Y_1.

Sol Since $S_X = (-2,1)$, we have $S_{Y_1} = (0,2)$. The pdf of X is $f_X(x) = \frac{1}{3}$ for $-2 < x < 1$ and 0 elsewhere, so the cdf F_{Y_1} is

$$F_{Y_1}(x) = \mathbb{P}(|X| \le x)$$
$$= \begin{cases} 0, & \text{if } x < 0, \\ \int_{-x}^{x} \frac{1}{3} dt = \frac{2}{3}x, & \text{if } 0 \le x \le 1, \\ \int_{-1}^{1} \frac{1}{3} dt + \int_{1}^{x} \frac{1}{3} dt = \frac{1}{3} + \frac{1}{3}x, & \text{if } 1 < x \le 2, \\ 1, & \text{if } x > 2. \end{cases}$$

For the pdf, we differentiate the cdf to get

$$f_{Y_1}(x) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x > 2, \\ \frac{2}{3}, & \text{if } 0 \le x \le 1, \\ \frac{1}{3}, & \text{if } 1 < x \le 2. \end{cases}$$

(ii) Compute $\mathbb{E}(Y_2)$ using the pdf of X.

Sol Since $f_X(x) = \frac{1}{3}$ for $-2 < x < 1$ and 0 elsewhere, f_X is symmetric on $(-1,1)$, we have

$$\mathbb{E}(Y_2) = \int_{-2}^{-1} \sqrt{-x}\frac{1}{3}dx + 2\int_0^1 \sqrt{x}\frac{1}{3}dx$$
$$= \int_1^2 \sqrt{x}\frac{1}{3}dx + 2\int_0^1 \sqrt{x}\frac{1}{3}dx$$
$$= \left[\frac{2}{9}x^{3/2}\right]\Big|_1^2 + \left[\frac{4}{9}x^{3/2}\right]\Big|_0^1$$
$$= \frac{2}{9}\left(1 + 2^{3/2}\right).$$

(iii) Derive the cdf and pdf of Y_2.

Sol We have $S_{Y_2} = (0, \sqrt{2})$. The pdf of X is $f_X(x) = \frac{1}{3}$ for $-2 < x < 1$, so the cdf F_{Y_2} is

$$F_{Y_2}(x) = \mathbb{P}(|X| \le x^2)$$
$$= \begin{cases} 0, & \text{if } x < 0, \\ \int_{-x^2}^{x^2} \frac{1}{3} dt = \frac{2}{3}x^2, & \text{if } 0 \le x \le 1, \\ \int_{-1}^{1} \frac{1}{3} dt + \int_{1}^{x^2} \frac{1}{3} dt = \frac{1}{3} + \frac{1}{3}x^2, & \text{if } 1 < x \le \sqrt{2}, \\ 1, & \text{if } x > \sqrt{2}. \end{cases}$$

One can also use f_{Y_1} in (i) to obtain the same answer. For the pdf, we differentiate the cdf to get

$$f_{Y_2}(x) = \begin{cases} 0, & \text{if } x < 0 \text{ or } x > \sqrt{2}, \\ \frac{4}{3}x, & \text{if } 0 \le x \le 1, \\ \frac{2}{3}x, & \text{if } 1 < x \le \sqrt{2}. \end{cases}$$

(iv) Compute $\mathbb{E}(Y_2)$ using the pdf of Y_2.

Sol Using the pdf in (iii), we have

$$\mathbb{E}(Y_2) = \int_0^1 x\frac{4}{3}x dx + \int_1^{\sqrt{2}} x\frac{2}{3}x dx$$
$$= \left[\frac{4}{9}x^3\right]\Big|_0^1 + \left[\frac{2}{9}x^3\right]\Big|_1^{\sqrt{2}}$$
$$= \frac{2}{9}\left(1 + 2^{3/2}\right).$$

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Problem 2. For each of the following statements, determine whether it is true or false. If it is true, give a proof; if it is false, give a counterexample.

(i) If f and g are two pdfs, then $pf + (1-p)g$ is also a pdf for all $0 \le p \le 1$.

Sol True. *Proof:* Since $f \ge 0$, $g \ge 0$ and $0 \le p \le 1$, we have $pf + (1-p)g \ge 0$. Also,

$$\begin{aligned}&\int_{-\infty}^{\infty} (pf(x) + (1-p)g(x))dx \\ &= p\int_{-\infty}^{\infty} f(x)dx + (1-p)\int_{-\infty}^{\infty} g(x)dx \\ &= p + (1-p) = 1,\end{aligned}$$

both conditions of a pdf are satisfied, so $pf + (1-p)g$ is a pdf.

(ii) If X and Y are two random variables defined on the same sample space and $\mathbb{P}(X \ne Y) = 0$, then $X \overset{d}{=} Y$.

Sol True. *Proof:* Since $\mathbb{P}(X \ne Y) = 0$, for all $x \in \mathbb{R}$,

$$0 \le \mathbb{P}(X \le x, X \ne Y) \le \mathbb{P}(X \ne Y) = 0,$$

hence $\mathbb{P}(X \le x, X \ne Y) = 0$. Likewise, for all $x \in \mathbb{R}$, $\mathbb{P}(Y \le x, X \ne Y) = 0$, therefore

$$\begin{aligned}F_X(x) &= \mathbb{P}(X \le x, X \ne Y) + \mathbb{P}(X \le x, X = Y) \\ &= 0 + \mathbb{P}(Y \le x, X = Y) \\ &= \mathbb{P}(Y \le x) - \mathbb{P}(Y \le x, X \ne Y) \\ &= \mathbb{P}(Y \le x) - 0 = F_Y(x).\end{aligned}$$

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(iii) If X and Y are two random variables defined on the same sample space and $X \stackrel{d}{=} Y$, then $\mathbb{P}(X \neq Y) = 0$.

Sol False. For a counterexample, we toss a fair coin once, let X be the number of heads and Y be the number of tails, then $\Omega = \{H, T\}$ with $\mathbb{P}(\{H\}) = \mathbb{P}(\{T\}) = 0.5$, $X(H) = 1$, $X(T) = 0$, $Y(H) = 0$, $Y(T) = 1$,

$$\mathbb{P}(X = 0) = \mathbb{P}(X = 1) = \frac{1}{2}$$

and

$$\mathbb{P}(Y = 0) = \mathbb{P}(Y = 1) = \frac{1}{2}$$

so $X \stackrel{d}{=} Y$ but $\mathbb{P}(X \neq Y) = \mathbb{P}(\Omega) = 1$.

(iv) If X is a nonnegative random variable such that $\lim_{x\to\infty}\{x\mathbb{P}(X > x)\} = 0$, then $\mathbb{E}(X^2)$ exists.

Sol False. For a counterexample, let $X \stackrel{d}{=}$Pareto$(1, 2)$ with pdf

$$f_X(t) = \begin{cases} \frac{2}{t^3}, & \text{for } t \geq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Then for $x \geq 1$, $x\mathbb{P}(X > x) = x\int_x^\infty \frac{2}{t^3}dt = x \times \frac{1}{x^2} = \frac{1}{x} \to 0$ as $x \to \infty$, but

$$\int_1^x t^2 f_X(t)dt = \int_1^x \frac{2}{t}dt = 2\ln x \to \infty \text{ as } x \to \infty,$$

that is, $\int_1^\infty t^2 f_X(t)dt$ diverges, hence $\mathbb{E}(X^2)$ does not exist.

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(v) If X is a nonnegative continuous random variable such that $\mathbb{E}(X^2)$ exists, then

$$\lim_{x\to\infty}\{\mathbb{P}(X > x)x^2\} = 0.$$

Sol True. *Proof* Since $\mathbb{E}(X^2)$ exists, we have

$$\int_0^\infty t^2 f_X(t)dt < \infty,$$

which implies $\lim_{x\to\infty}\int_x^\infty t^2 f_X(t)dt = 0$. However,

$$\int_x^\infty t^2 f_X(t)dt \geq \int_x^\infty x^2 f_X(t)dt = x^2\mathbb{P}(X > x) \geq 0,$$

by the sandwitch theorem, $\lim_{x\to\infty}\left\{x^2\mathbb{P}(X > x)\right\} = 0$.

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Problem 3. Let (X, Y) be a bivariate random variable with joint pmf

$$\mathbb{P}((X,Y) = (-1,0)) = \mathbb{P}((X,Y) = (1,0)) = \frac{1}{6}, \ \mathbb{P}((X,Y) = (0,1)) = \mathbb{P}((X,Y) = (0,0)) = \frac{1}{3}.$$

(i) Derive the cdf of (X, Y). (Hint: you may wish to draw a graph and consider a number of different regions where the cdf is a constant.)

Sol Since $S_X = \{-1, 0, 1\}$, we seperate $X \le x$ for x values into four cases $(-\infty, -1)$, $[-1, 0)$, $[0, 1)$ and $[1, \infty)$, then work on the corresponding values of y, giving

$$F_{(X,Y)}(x,y) = \begin{cases} 0, & \text{if } x < -1, y \in \mathbb{R}, \\ 0, & \text{if } -1 \le x < 0, y < 0, \\ \frac{1}{6}, & \text{if } -1 \le x < 0, y \ge 0, \\ 0, & \text{if } 0 \le x < 1, y < 0, \\ \frac{1}{2}, & \text{if } 0 \le x < 1, 0 \le y < 1, \\ \frac{5}{6}, & \text{if } 0 \le x < 1, y \ge 1, \\ 0, & \text{if } x \ge 1, y < 0, \\ \frac{2}{3}, & \text{if } x \ge 1, 0 \le y < 1, \\ 1, & \text{if } x \ge 1, y \ge 1. \end{cases}$$

(ii) Find the marginal pmfs of X and Y.

Sol Using the formula $p_X(x) = \sum_{y \in S_Y} \mathbb{P}((X,Y) = (x,y))$ for $x \in S_X = \{-1, 0, 1\}$, we have

$$p_X(x) = \begin{cases} \frac{1}{6}, & \text{if } x = \pm 1, \\ \frac{2}{3}, & \text{if } x = 0. \end{cases}$$

Similarly, using $p_Y(y) = \sum_{x \in S_X} \mathbb{P}((X,Y) = (x,y))$ for $y \in S_Y = \{0, 1\}$, we have

$$p_Y(y) = \begin{cases} \frac{1}{3}, & \text{if } y = 1, \\ \frac{2}{3}, & \text{if } y = 0. \end{cases}$$

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(iii) Calculate $\mathbb{P}(X \leq Y)$.

Sol We have $\{X > Y\} = \{(X,Y) = (1,0)\}$ so

$$\begin{aligned}&\mathbb{P}(X \leq Y)\\&= 1 - \mathbb{P}(\{(X,Y) = (1,0)\})\\&= \frac{5}{6}.\end{aligned}$$

(iv) Compute $\mathbb{E}(X)$, $\mathbb{E}(Y)$ and $\mathbb{E}(XY)$.

Sol Using the marginal pmf in (ii), we have

$$\mathbb{E}(X) = (-1) \times \frac{1}{6} + 0 \times \frac{2}{3} + 1 \times \frac{1}{6} = 0,$$

$$\mathbb{E}(Y) = 0 \times \frac{2}{3} + 1 \times \frac{1}{3} = \frac{1}{3}.$$

Next, we have

$$\mathbb{E}(XY) = (-1)\times 0 \times \frac{1}{6} + 0 \times 1 \times \frac{1}{3} + 1 \times 0 \times \frac{1}{6} + 0 \times 0 \times \frac{1}{3} = 0.$$

(v) Are X and Y independent? Explain.

Sol No, because

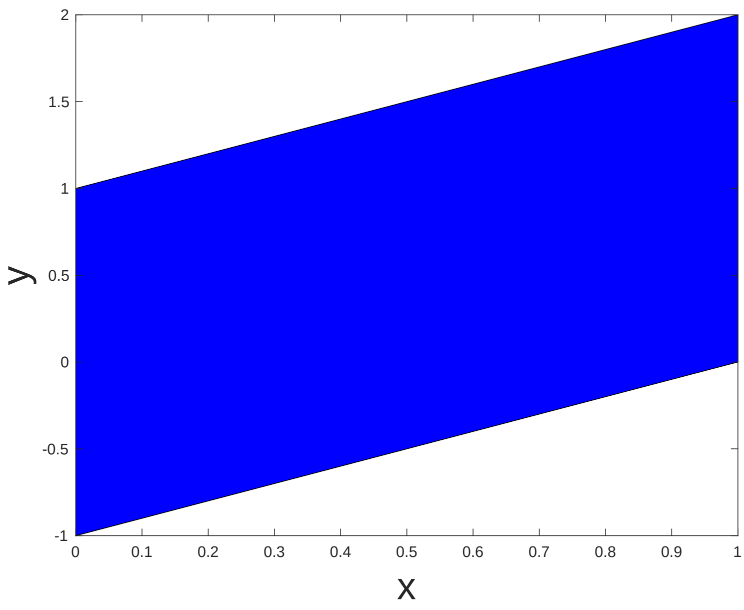
$$\mathbb{P}((X,Y)=(1,0)) = \frac{1}{6} \neq \frac{1}{6} \times \frac{2}{3} = \mathbb{P}(X=1)\mathbb{P}(Y=0).$$

Problem 4. Let

$$f(x,y) = \begin{cases} c, & \text{if } 0 \le x \le 1, x-1 \le y \le x+1, \\ 0, & \text{elsewhere,} \end{cases}$$

be the joint pdf of (X,Y).

(i) Sketch the region for which $f(x,y) > 0$.



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(ii) Derive the value of c.

Sol As a pdf satisfies the property that the total volume under the pdf is 1, we have

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = \int_{0}^{1} \left(\int_{x-1}^{x+1} dy \right) dx = 2c,$$

giving $c = 1/2$.

(iii) Find the marginal pdf of Y. Check that it is a pdf.

Sol We use $f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$ to obtain

$$f_Y(y) = \begin{cases} \int_0^{y+1} \frac{1}{2} dx = \frac{y+1}{2}, & \text{if } -1 < y < 0; \\ \int_0^1 \frac{1}{2} dx = \frac{1}{2}, & \text{if } 0 \leq y < 1; \\ \int_{y-1}^1 \frac{1}{2} dx = \frac{2-y}{2}, & \text{if } 1 \leq y < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

To verify that f_Y is indeed a pdf, we can see that f_Y is non-negative and

$$\begin{aligned} &\int_{-\infty}^{\infty} f_Y(y) dy \\ &= \int_{-1}^{0} \frac{y+1}{2} dy + \int_{0}^{1} \frac{1}{2} dy + \int_{1}^{2} \frac{2-y}{2} dy \\ &= \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1. \end{aligned}$$

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(iv) Determine the conditional pdf of X given $Y = y$. Check that it is indeed a pdf.

Sol For $-1 < y < 0$, X takes values in $[0, y+1]$ and

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{y+1}, & \text{if } 0 \le x \le y+1, \\ 0, & \text{elsewhere.} \end{cases}$$

$f_{X|Y}(x|y)$ is a pdf because it is nonnegative and

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y) = \int_0^{y+1} \frac{1}{y+1} dx = 1.$$

For $0 \le y < 1$, X takes values in $[0, 1]$ and

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} 1, & \text{if } 0 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$f_{X|Y}(x|y)$ is a pdf because it is nonnegative and

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y) = \int_0^{1} 1 dx = 1.$$

For $1 \le y < 2$, X takes values in $[y-1, 1]$ and

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{2-y}, & \text{if } y-1 \le x \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$f_{X|Y}(x|y)$ is a pdf because it is nonnegative and

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y) = \int_{y-1}^{1} \frac{1}{2-y} dx = 1.$$

(v) Are X and Y independent? Explain.

Sol No, because $f_X(x) = \int_{x-1}^{x+1} \frac{1}{2} dy = 1$ for $0 \le x \le 1$ and

$$f(0.5, -0.6) = 0 \ne 1 \times 0.2 = f_X(0.5) f_Y(-0.6).$$