Question 2 marked, total 13

MAST10006 Calculus 2, Semester 2, 2020

Assignment 4

School of Mathematics and Statistics, The University of Melbourne

- Submit your assignment solutions online in Canvas before **6pm, Monday 14 September 2020**
- Submit your solutions as a single PDF file with the pages in the right order and correct orientation. You may be penalised a mark if you do not.
- This assignment is worth 2.22% of your final MAST10006 mark.
- Answer all questions below. Of these questions, one will be chosen for marking.
- Marks may be awarded for:
  - Correct use of appropriate mathematical techniques
  - Accuracy and validity of any calculations or algebraic manipulations
  - Clear justification or explanation of techniques and rules used
  - Use of correct mathematical notation and terminology
- You must use methods taught in MAST10006 Calculus 2 to solve the assignment questions.

1. Mathematical models are an important tool in understanding and managing infectious disease epidemics. According to the SIR model (first studied by Kermack & McKendrick in 1927), the expected incidence of a disease in the early stages of an epidemic is approximated by

$$f(t) = k\,\mathrm{sech}^2\left(\alpha t - \phi\right), \quad t \geq 0$$

where k, α and ϕ are positive constants and t is the time in days since the start of the disease outbreak.

The *incidence* of a disease is the rate of new cases of the disease per unit time, typically measured as the number of new cases in a single day. $f(t)$ thus represents the expected number of new cases at day t.

For simplicity, assume the unrealistic but simple values $k = 20000, \alpha = 2, \phi = 30$, which gives

$$f(t) = 20000\,\mathrm{sech}^2\left(2t - 30\right), \quad t \geq 0.$$

(a) Sketch by hand the graph of $y = f(t)$. The graph should have the correct shape, exact value labels for any intercepts, labels for asymptotes and stationary points.

(b) Suppose that 1% of the new disease cases on any given day require hospitalisation, and that the hospital system's capacity can handle at most 10 new hospitalisations per day. Find the exact value of t for which the hospital system is first expected to be over capacity. Then use any device invented after the 19th century to find the day for which this occurs.

(c) The expected total reported cases at time t days is given by

$$T(t) = \int_0^t f(u)\,du.$$

Find an expression for $T(t)$ in terms of t.

(d) Suppose that the population consists of 1 million people. According to this model, how many people are expected to remain uninfected by the disease in the long term?

Solution.

(a) We can approach this problem by taking the graph of $y = \mathrm{sech}(x)$ and performing transformations to it, keeping track of important features of the graph. Alternatively, we can find the stationary points and intercepts algebraically first. We take the second approach in these solutions.

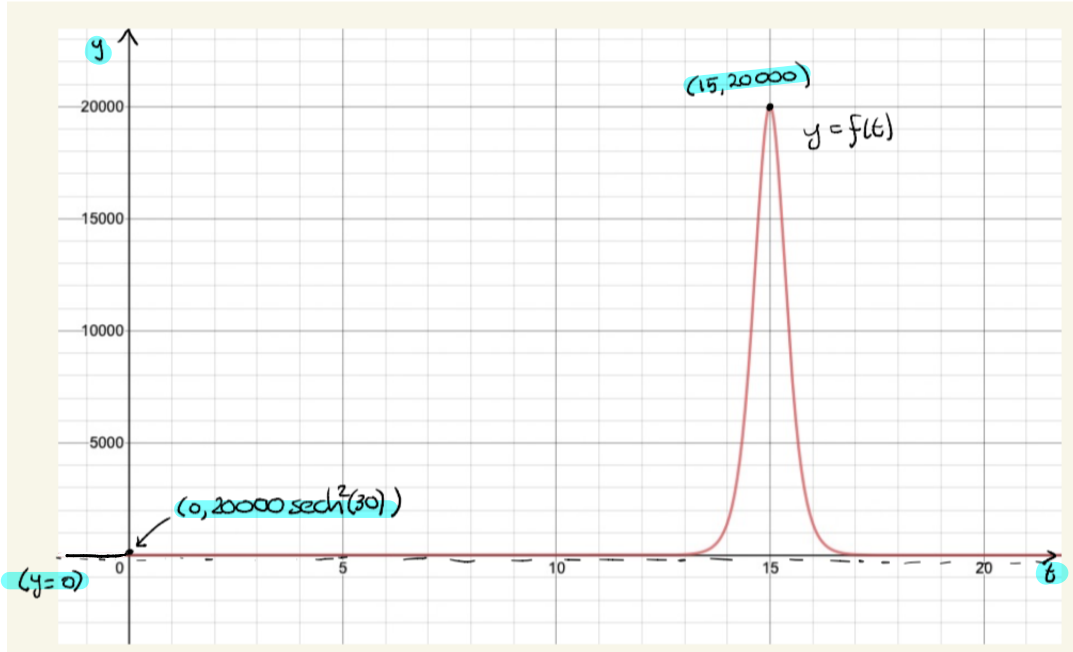
Find y-intercept: $f(0) = 20000\,\text{sech}^2(-30) = 20000\,\text{sech}^2(30) \approx 7.05 \times 20^{-22}$.
There are no x-intercepts since $f(t) > 0$.
Find stationary points:

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(\frac{20000}{\cosh^2(2t-3)}\right) \\ &= -\frac{80000\sinh(2t-3)}{\cosh^3(2t-3)} \\ &= -80000\,\text{sech}^2(2t-30)\tanh(2t-30) \end{aligned}$$

Since $\text{sech}\,\theta > 0$, the only stationary point occurs when $\tanh(2t-30) = 0 \implies t = 15$.
Since $f(15) = 20000\,\text{sech}^2(0) = 20000$, the stationary point is therefore

$$(15, 20000).$$

Here is the graph:



Should really be hand drawn

(b) The hospital's capacity becomes exceeded ~~when~~ just after 10 hospitalisations. We therefore solve $0.01 f(t) = 10$:

$$\begin{aligned} 0.01 \times 20000\,\text{sech}^2(2t-30) &= 10 \\ \text{sech}^2(2t-30) &= \frac{1}{20} \\ 2t - 30 &= \pm\text{arcsech}\sqrt{\frac{1}{20}} \\ t &= \pm\frac{1}{2}\text{arcsech}\sqrt{\frac{1}{20}} + 15 \end{aligned}$$

So $t \approx 13.9$ or $t \approx 16.08$.
The exact value of t for which the capicity will be exceeded is $t = -\frac{1}{2}\text{arcsech}\sqrt{\frac{1}{20}} + 15$.
This first occurs on the 14th day since the start of the outbreak.
Here we have $t=0$ to $t=1$ as the first day.
It's reasonable to have $t=0$ to $t=1$ be the zeroth day, in which case "the 13th day" is a reasonable answer.

(c)

$$
\begin{aligned}
T(t) &= \int_0^t 20000\,\mathrm{sech}^2\,(2u-30)\;\mathrm{d}u \\
&= 20000\left[\frac{1}{2}\tanh(2u-30)\right]_0^t \\
&= 10000\,(\tanh(2t-30)-\tanh(-30)) \\
&= 10000\,(\tanh(2t-30)+\tanh(30))
\end{aligned}
$$

(d) In the long run, the number that are infected is

$$
\begin{aligned}
\lim_{t\to\infty} T(t) &= 10000\left(\lim_{t\to\infty}\left(\frac{e^{2t-30}-e^{-2t+30}}{e^{2t+30}+e^{-2t+30}}\right)+\tanh(30)\right) && \text{limit laws} \\
&= 10000\left(\lim_{t\to\infty}\left(\frac{1-e^{60}e^{-4t}}{1+e^{60}e^{-4t}}\right)+\tanh(30)\right) \\
&= 10000\,(1+\tanh(30)) && \text{limit laws and } \lim_{x\to\infty}\frac{1}{a^x}=0 \\
&\approx 20000
\end{aligned}
$$

So the population expected to not be infected is $1000000 - 20000 = 980000$ people.

Notes for self reflection:
1a) Did you include all labels on your graph?

1b) Did you include the "0.01" factor, accounting for only 1% of new cases needing hospitalisation?

1b) It's reasonable to have instead solved 0.01f(t) = 11 if you explained that we need 11 people to be hospitalised before the hospital is over capacity. Note that a clarification was given to calculate 0.01f(t) = 10.

1b) In this case the "negative" square root still gave a positive answer and was the answer required.

1c) Leave your answer as a function of t.

1d) Did you calculate the limit? It's good to give both the exact value, and then answer the question asked.

2. Evaluate the following integrals. State which method you use, document each step carefully and keep track of any assumptions that you make.

   (a) $\int 6x\sin(x^2)\,\mathrm{d}x$.

   (b) $\int \frac{x^2}{\sqrt{9x^2-1}}\,\mathrm{d}x$.

   (c) $\int e^{-x}\cos(x)\,\mathrm{d}x$. Use integration by parts.

   (d) $\int e^{-x}\cos(x)\,\mathrm{d}x$. Use the complex exponential.

   (e) Comment briefly on which of (c) and (d) you found simpler, giving a brief reason.

Solution.

1M: use derivative substitution

2

(a) Use a derivative substitution. $\boxed{\text{Let } u = x^2,\ \frac{du}{dx} = 2x.}$ Then

$$\begin{aligned}\int 6x\sin(x^2)\,\mathrm{d}x &= 3\int\left(\frac{d}{dx}(x^2)\right)\sin(x^2)\,\mathrm{d}x\\ &= 3\int \sin(u)\,\mathrm{d}u\\ &= -3\cos(u)+c\\ &= \underline{-3\cos(x^2)+c}\end{aligned}$$

1A

5

(b) Use a hyperbolic substitution. Let

$$\left.\begin{aligned}x &= \frac{1}{3}\cosh(\theta)\\ \theta &= \operatorname{arccosh}(3x)\\ \frac{dx}{d\theta} &= \frac{1}{3}\sinh(\theta)\end{aligned}\right]$$

1M: use a valid hyperbolic substitution

Since we also need $9x^2-1\neq 0 \Rightarrow x\neq \pm 1/3$

This substitution is valid for $3x > 1$ and $\theta > 0$. Therefore

$$\begin{aligned}\frac{x^2}{\sqrt{9x^2-1}} &= \frac{\frac{1}{9}\cosh^2(\theta)}{\sqrt{\cosh^2(\theta)-1}}\\ &= \frac{1}{9}\frac{\cosh^2(\theta)}{|\sinh(\theta)|}\\ &= \frac{1}{9}\frac{\cosh^2(\theta)}{\sinh(\theta)} \qquad \underline{\text{since } \sinh(\theta)\geq 0 \text{ for } \theta\geq 0}\end{aligned}$$

1J: use $\sinh\theta \geq 0$ to remove $|\ |$ signs.

Therefore

$$\begin{aligned}\int\frac{x^2}{\sqrt{9x^2-1}}\,dx &= \int\frac{1}{9}\frac{\cosh^2(\theta)}{\sinh(\theta)}\frac{dx}{d\theta}\,d\theta\\ &= \frac{1}{27}\int\boxed{\cosh^2(\theta)}\,d\theta\\ &= \frac{1}{27}\int\frac{\cosh(2\theta)+1}{2}\,d\theta\\ &= \frac{1}{54}\left(\frac{1}{2}\sinh(2\theta)+\theta\right)+c\\ &= \frac{1}{54}(\sinh(\theta)\cosh(\theta)+\theta)+c\\ &= \underline{\frac{1}{54}\left(3x\sqrt{9x^2-1}+\operatorname{arccosh}(3x)\right)+c}\end{aligned}$$

1A: correct integrand after substitution, up to a constant factor

1M: use a valid method such as double angle formula to calculate integral.

Must simplify compositions of hyperbolic functions and their inverses.

1A

Alternative (non-recommended) solution to Assignment 4, 2b

Let $I = \int \frac{x^2}{\sqrt{9x^2-1}}\,dx$

Let $x = \frac{1}{3}\sec\theta$

$\frac{dx}{d\theta} = \frac{1}{3}\sec\theta\tan\theta$ 1M

$\theta = \operatorname{arcsec}(3x)$

Also need $9x^2 - 1 \neq 0$

We make this substitution for $\theta \in (0, \frac{\pi}{2})$, $3x \in (1, \infty)$

Then $I = \int \frac{\frac{1}{9}\sec^2\theta}{\sqrt{\sec^2\theta - 1}} \cdot \frac{1}{3}\sec\theta\tan\theta\,d\theta$

$= \frac{1}{27}\int \frac{\sec^3\theta}{|\tan\theta|} \cdot \tan\theta\,d\theta$

$= \frac{1}{27}\int \sec^3\theta\,d\theta$, $\tan\theta > 0$ for $\theta \in (0, \frac{\pi}{2})$ 1A 1J

$= \frac{1}{54}\sec\theta\tan\theta + \frac{1}{54}\log|\sec\theta + \tan\theta| + c$, see ☆

$= \frac{1}{54}\sec\theta\tan\theta + \frac{1}{54}\log(\sec\theta + \tan\theta) + c$, $\sec\theta > 1$, $\tan\theta > 0$ for $\theta \in (0, \frac{\pi}{2})$

$= \frac{1}{54}(3x)\sqrt{9x^2-1} + \frac{1}{54}\log(3x + \sqrt{9x^2-1}) + c$ 1A

☆ 1M

$\int \sec^3\theta\,d\theta = \int \sec\theta \frac{d}{d\theta}[\tan\theta]\,d\theta$

$= \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\,d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta\,d\theta$

$= \sec\theta\tan\theta - \int \sec^3\theta\,d\theta + \log|\sec\theta + \tan\theta|$

This gives an equation for $\int \sec^3\theta\,d\theta$ that can be solved

$\Rightarrow \int \sec^3\theta\,d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log|\sec\theta + \tan\theta| + c$

Remark.
We should also consider separately $\theta \in (\frac{\pi}{2}, \pi)$ where tan and sec are negative.
But okay to let this go for marking.

2 (c) Use integration by parts. Let

$$u_1 = e^{-x}, \frac{dv_1}{dx} = \cos(x).$$

$$\frac{du_1}{dx} = -e^{-x}, v_1 = \sin(x).$$

Therefore

$$\int e^{-x}\cos(x)\,\mathrm{d}x = e^{-x}\sin(x) + \int e^{-x}\sin(x)\,\mathrm{d}x. \tag{1}$$

To calculate $\int e^{-x}\sin(x)$, we use integration by parts again. Let

$$u_2 = e^{-x}, \frac{dv_2}{dx} = \sin(x).$$

$$\frac{du_2}{dx} = -e^{-x}, v_2 = -\cos(x).$$

Therefore

$$\int e^{-x}\sin(x)\,\mathrm{d}x = -e^{-x}\cos(x) - \int e^{-x}\cos(x)\,\mathrm{d}x \tag{2}$$

1M: use integration by parts twice to obtain an equation for $\int e^{-x}\cos(x)\,dx$

Substitute (2) into (1):

$$\int e^{-x}\cos(x)\,\mathrm{d}x = e^{-x}\sin(x) - e^{-x}\cos(x) - \int e^{-x}\cos(x)\,\mathrm{d}x$$

$$\implies 2\int e^{-x}\cos(x)\,\mathrm{d}x = e^{-x}(\sin(x) - \cos(x)) + c$$

$$\implies \int e^{-x}\cos(x)\,\mathrm{d}x = \underbrace{\frac{1}{2}e^{-x}(\sin(x) - \cos(x)) + c}_{1A}$$

2 (d)

$$\int e^{-x}\cos(x)\,\mathrm{d}x = Re\left(\int e^{(-1+i)x}\,\mathrm{d}x\right)$$

1M: Use complex exponential [can also use $\cos(x) = \frac{1}{2}e^{ix} + \frac{1}{2}e^{-ix}$]

$$= Re\left(\frac{1}{-1+i}e^{(-1+i)x} + c + di\right)$$

$$= Re\left(\frac{-1-i}{2}e^{-x}(\cos(x) + i\sin(x)) + c + di\right)$$

$$= Re\left(\frac{e^{-x}}{2}((-\cos(x) + \sin(x)) - i(\cos(x) + \sin(x))) + c + di\right)$$

No deduction for just writing $+c$

$$= \underbrace{\frac{1}{2}e^{-x}(\sin(x) - \cos(x)) + c}_{1A}$$

1 (e) Any reasonable not containing incorrect mathematics is okay here.
1A: for a reason given, that contains no mathematical errors

End of assignment

1 N: All notation correct including no "+c's" missing in non-answer steps.