School of Mathematics and Statistics
MAST10007 Linear Algebra, Semester 1 2020
Written assignment 3 Solutions

1. (a) W is **not** a subspace of $V = \mathcal{P}_2$. For example, it does not contain the zero vector: $\mathbf{0}(x) = 0 + 0x + 0x^2$. It is also not closed under addition and not closed under scalar multiplication.

   (b) W **is** a subspace of $V = M_{2,2}$. We prove this using the subspace theorem:

   (0) W is non-empty since the zero matrix $\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is in W.

   (In fact, any matrix of the form $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ is in W.)

   (1) Assume that $A, B \in W$. This means that A, B are 2×2 matrices with $A^T = A$ and $B^T = B$. Then $A + B \in M_{2,2}$ and

   $$(A+B)^T = A^T + B^T = A + B$$

   by properties of transpose. Hence $A + B \in W$ and W is closed under addition.

   (2) If $A \in W$ and $\alpha \in \mathbb{R}$ then $\alpha A \in M_{2,2}$ and

   $$(\alpha A)^T = \alpha A^T = \alpha A$$

   by properties of transpose. So $\alpha A \in W$ and W is closed under scalar multiplication.

   Hence, by the subspace theorem, W is a subspace of V.

   (c) This is **not** a subspace of $V = \mathbb{R}^3$. We prove this by giving an explicit example showing that the set is not closed under vector addition. Let $\mathbf{u} = (1, 0, 1)$ and $\mathbf{v} = (1, 0, -1)$. Then $\mathbf{u}$ and $\mathbf{v}$ are in U. However, $\mathbf{u} + \mathbf{v} = (2, 0, 0)$ is not in U since $0^2 \neq 2^2 + 0^2$.

2. We use the isomorphism between $\mathcal{P}_3$ and $\mathbb{R}^4$ given by $a_0 + a_1x + a_2x^2 + a_3x^3 \leftrightarrow (a_0, a_1, a_2, a_3)$. So we first convert the polynomials to vectors in $\mathbb{R}^4$:

   $$(1, 2, -1, 3), (2, 3, 1, 4), (1, 0, 5, -1).$$

   Writing the vectors as columns and reducing to reduced row echelon form gives:

   $$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 0 \\ -1 & 1 & 5 \\ 3 & 4 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -2 \\ 0 & 3 & 6 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = B$$

   Since $\operatorname{rank}(A) = 2 <$ number of vectors $= 3$, the vectors in $\mathbb{R}^4$ are linearly dependent. Hence the polynomials are also linearly **dependent**.

   In matrix B we can see that column 3 $= -3$(column 1)$+2$(column 2), so the same relation holds for the columns of A. Hence

   $$(1, 0, 5, -1) = -3(1, 2, -1, 3) + 2(2, 3, 1, 4).$$

   Converting back to polynomials gives:

   $$1 + 5x^2 - x^3 = -3(1 + 2x - x^2 + 3x^3) + 2(2 + 3x + x^2 + 4x^3).$$

   (Note: you can check your answer by expanding out the right hand side.)