MAST10006 Calculus 2, Semester 1, 2020

Assignment 8

School of Mathematics and Statistics, The University of Melbourne

This assignment is due at **6pm Monday 19 October 2020**.

Name: James La Fontaine	Student ID: 1079860

The format of this assignment is different to previous assignments. The questions are on the following pages, along with spaces for you to write your answers. This is the same format that we will use for the exam this semester so this assignment is a chance to get familiar with the format and submission process.

To complete this assignment, you have two options. In both cases, you should write your answer on paper, rather than annotating a pdf or typing your solutions. Here are the two options.

1. Print a hard-copy of this assignment. Write your name and student ID on this page, and write your answers in the spaces provided on the following pages.
2. If you don't have access to a printer, then write your answers on blank sheets of A4 paper, with
   - Your name and student ID (only) on page 1
   - Your answer to question 1(a) on page 2
   - Your answer to question 1(b) on page 3
   - Your answer to question 1(c) on page 4
   - Your answer to question 1(d) on page 5

In both cases, if you need additional space, you can use extra blank pages, attached to the end (page 5 and onwards). To submit, scan your work into a single PDF file with all pages matching the order described above. Upload your PDF into Gradescope via Canvas.

Further instructions:

- Submit your solutions as a single PDF file with correctly oriented pages in the order as explained above
- This assignment is worth 2.22% of your final MAST10006 mark.
- There is only one question for this assignment.
- Marks may be awarded for:
  - Correct use of appropriate mathematical techniques
  - Accuracy and validity of any calculations or algebraic manipulations
  - Clear justification or explanation of techniques and rules used
  - Clear communication of mathematical ideas through diagrams
  - Use of correct mathematical notation and terminology
- You must use methods taught in MAST10006 Calculus 2 to solve the assignment questions.
- It is **your responsiblity to ensure your assignment is successfully uploaded before the deadline**. Delays due to minor technical issues eg slow internet are not grounds for special consideration.

Questions are on the following pages

Question 1

In this assignment, we will investigate a new car suspension system, and how it travels on a road surface with different roughness properties. To do this, we isolate the suspension system from the rest of the car. We simplify it to a *mass-spring-damper* model as shown (Figure 1). The mass represents the car, while the damper is a shock absorber that provides damping to the system.

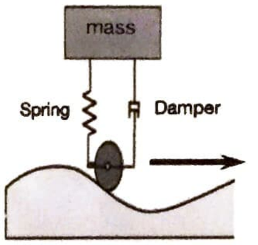


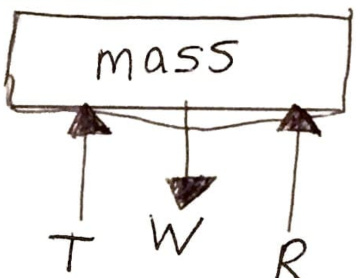
Figure 1

Throughout, we assume the mass is $m = 300kg$, the spring constant is $k = 1500N/cm$, and the damping constant is $\beta = 600N/cm$.

(a) Firstly we assume that the only forces acting on the car are gravity, the spring force and damping force. Draw a force diagram indicating the vertical forces exerted on the car. Indicate the direction of motion. Use Newton's Law of motion to show that the system response, $y(t)$, satisfies the differential equation

$$\ddot{y} + 2\dot{y} + 5y = 0$$

where the *system response* is the vertical displacement of the car from equilibrium.
Hint: While any choice of signs will work, since moving up corresponds to an extension of the spring, it might be suitable to have up correspond to positive displacement.



Equation of motion from Newton's Second Law:

$m\ddot{y} = W + T + R$

$\Rightarrow m\ddot{y} = mg - k(s+y) - \beta\dot{y}$

$\Rightarrow m\ddot{y} = mg - ks - ky - \beta\dot{y}$

$\Rightarrow m\ddot{y} = -ky - \beta\dot{y}$, $mg = ks$

$\Rightarrow m\ddot{y} + \beta\dot{y} + ky = 0$

$\Rightarrow 300\ddot{y} + 600\dot{y} + 1500y = 0$

$\Rightarrow \ddot{y} + 2\dot{y} + 5y = 0$

(b) We first examine the free vibrations of our suspension system in response to being driven across a single bump on the road, which initially stretches the spring by an amount of 1cm. Assuming that the vertical velocity is initially 0 cm/s, and that no other forces are acting on the car, find the system response $y(t)$ after driving across the single bump on the road.

$\ddot{y} + 2\dot{y} + 5y = 0$

Try $y = e^{\lambda t}$

$\dot{y} = \lambda e^{\lambda t}$

$\ddot{y} = \lambda^2 e^{\lambda t}$

Sub into ODE:

$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 5e^{\lambda t} = 0$

$(\lambda^2 + 2\lambda + 5)e^{\lambda t} = 0$

$\lambda^2 + 2\lambda + 5 = 0, \quad e^{\lambda t} \neq 0$

$\lambda = \dfrac{-2 \pm \sqrt{4 - 4(1)(5)}}{2}$

$\lambda = -1 \pm 2i$

Therefore:

GS: $y(t) = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t)$

Find A and B:

$\dot{y}(t) = -Ae^{-t}\cos(2t) - 2Ae^{-t}\sin(2t) - Be^{-t}\sin(2t) + 2Be^{-t}\cos(2t)$

Now $y(0) = 1$

$\Rightarrow Ae^{0}\cos(0) + Be^{0}\sin(0) = 1$

$\Rightarrow A = 1$

$\dot{y}(0) = 0$

$\Rightarrow -Ae^{0}\cos(0) - 2Ae^{0}\sin(0) - Be^{0}\sin(0) + 2Be^{0}\cos(0) = 0$

$\Rightarrow -A + 2B = 0$

$\Rightarrow -1 + 2B = 0$

$\Rightarrow 2B = 1$

$\Rightarrow B = 1/2$

$\Rightarrow y(t) = e^{-t}\cos(2t) + \frac{1}{2}e^{-t}\sin(2t)$

(c) When the car is driven over a road surface, the roughness profile of the road exerts a force on the wheels which is transferred up to the car. We can model this as a forced vibration on the car. Assuming the same initial conditions as before, find $y(t)$ if the system is now driven on a road surface with a roughness profile that enforces the following forced vibration $F(t)$ on the system:

$$F(t) = 300e^{-t}\sin t,$$

so that the ODE describing the system becomes

$$\ddot{y} + 2\dot{y} + 5y = e^{-t}\sin t.$$

$\ddot{y} + 2\dot{y} + 5y = e^{-t}\sin(t)$

Find y_p:

Try $y_p = \alpha e^{-t}\cos(t) + \beta e^{-t}\sin(t)$

$\dot{y}_p = -\alpha e^{-t}\cos(t) - \alpha e^{-t}\sin(t) - \beta e^{-t}\sin(t) + \beta e^{-t}\cos(t)$

$\ddot{y}_p = \alpha e^{-t}\cos(t) + \alpha e^{-t}\sin(t) + \alpha e^{-t}\sin(t) - \alpha e^{-t}\cos(t) + \beta e^{-t}\sin(t) - \beta e^{-t}\cos(t) - \beta e^{-t}\cos(t) - \beta e^{-t}\sin(t)$

$= 2\alpha e^{-t}\sin(t) - 2\beta e^{-t}\cos(t)$

Sub into ODE:

$2\alpha e^{-t}\sin(t) - 2\beta e^{-t}\cos(t) + 2(-\alpha e^{-t}\cos(t) - \alpha e^{-t}\sin(t) - \beta e^{-t}\sin(t) + \beta e^{-t}\cos(t)) + 5(\alpha e^{-t}\cos(t) + \beta e^{-t}\sin(t)) = e^{-t}\sin(t)$

$\Rightarrow (-2\beta - 2\alpha + 2\beta + 5\alpha)e^{-t}\cos(t) + (2\alpha - 2\alpha - 2\beta + 5\beta)e^{-t}\sin(t) = e^{-t}\sin(t)$

Equate coefficients:

① $3\alpha = 0 \Rightarrow \alpha = 0$

② $3\beta = 1 \Rightarrow \beta = 1/3$

Therefore, GS (IH):

$y(t) = Ae^{-t}\cos(2t) + Be^{-t}\sin(2t) + \frac{1}{3}e^{-t}\sin(t)$

$\dot{y}(t) = -Ae^{-t}\cos(2t) - 2Ae^{-t}\sin(2t) - Be^{-t}\sin(2t) + 2Be^{-t}\cos(2t) - \frac{1}{3}e^{-t}\sin(t) + \frac{1}{3}e^{-t}\cos(t)$

$y(0) = 1$

$\Rightarrow Ae^{0}\cos(0) + Be^{0}\sin(0) + \frac{1}{3}e^{0}\sin(0) = 1$

$\Rightarrow A = 1$

$\dot{y}(0) = 0$

$\Rightarrow -Ae^{0}\cos(0) - 2Ae^{0}\sin(0) - Be^{0}\sin(0) + 2Be^{0}\cos(0) - \frac{1}{3}e^{0}\sin(0) + \frac{1}{3}e^{0}\cos(0) = 0$

$\Rightarrow -A + 2B + \frac{1}{3} = 0$

$\Rightarrow -1 + 2B + \frac{1}{3} = 0$

$\Rightarrow B = 1/3$

$\Rightarrow y(t) = e^{-t}\cos(2t) + \frac{1}{3}e^{-t}\sin(2t) + \frac{1}{3}e^{-t}\sin(t)$

(d) Now let us test a different road profile with

$$G(t) = 300e^{-t}\sin 2t.$$

In this case, you can take as given that

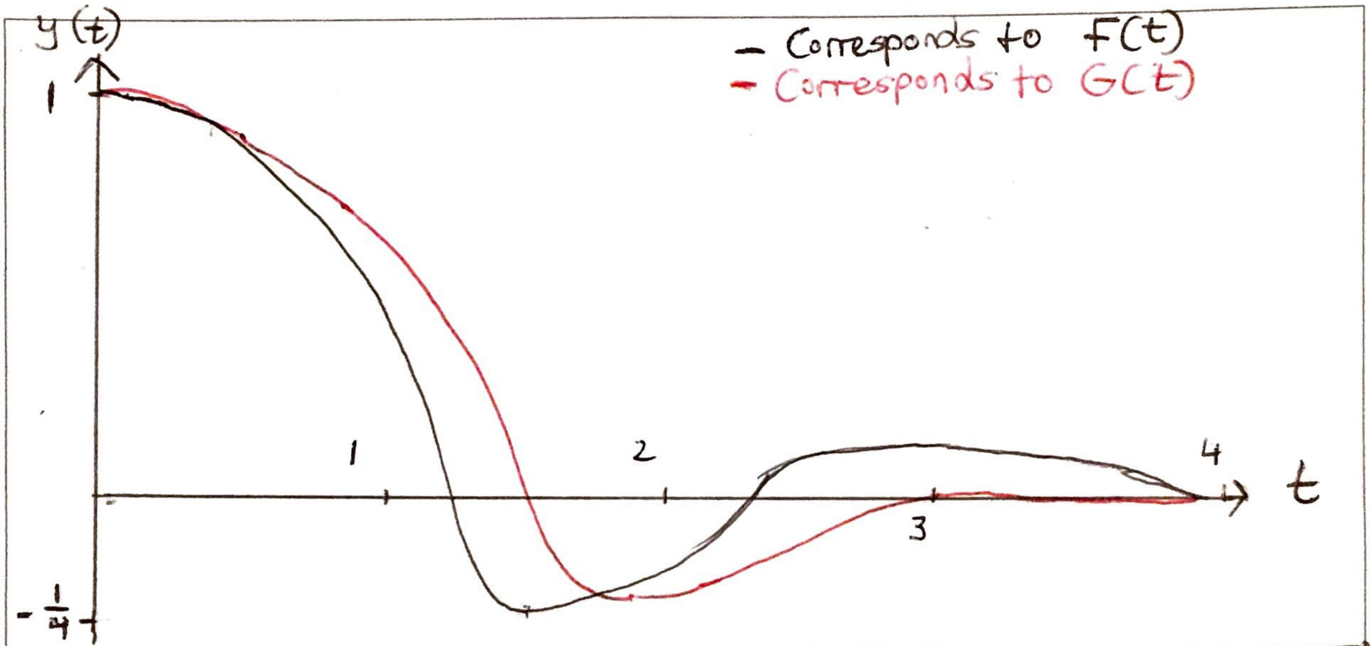
$$y(t) = e^{-t}\left(\frac{5}{8}\sin 2t + \cos 2t\right) - \frac{1}{4}te^{-t}\cos 2t$$

is the solution to the system response, given the same initial conditions.

i. Plot the response $y(t)$ to both road profiles on the same axes, clearly labelling which graph corresponds to which road profile.

ii. Which road profile do you think gives a smoother ride? Give a reason for your answer.

For this question, you can use a calculator or other app for your plot. You do not need to label intercepts or stationary points, but your graph should be of a suitable scale that allows you to compare the responses of the two road profiles.



I think road profile G(t) gives a smoother ride as it doesn't cause any noticeable stretching of the suspension after it compresses unlike road profile F(t) which induces slightly more oscillation and demonstrates under damping to a greater extent.

For your interest. *Generally, in suspension design, one aims for critical damping. If there is not enough damping, one gets residual vibrations after hitting a bump. This can be dangerous in terms of car control and steering. Also, this can make passengers sick especially those who suffer from car or motion sickness. On the other hand, if the car is over-damped, the suspension will be sluggish and slow to respond to the bump. This too is dangerous as the tire has less grip on the road.*

End of assignment