THE UNIVERSITY OF
MELBOURNE

Student Number 1079860

Semester 1 Assessment, 2020

School of Mathematics and Statistics

MAST10007 Linear Algebra

This exam consists of 21 pages (including this page)

Authorised materials: printed one-sided copy of the Exam or the Masked Exam made available earlier (or an offline electronic PDF reader), 2 double-sided A4 sheets of notes (handwritten or printed), and blank A4 paper

Instructions to Students

- During exam writing time you may only interact with the device running the Zoom session with supervisor permission. The screen of any other device must be visible in Zoom from the start of the session.
- If you have a printer, print out the exam single-sided and hand write your solutions into the answer spaces.
- If you do not have a printer, or if your printer fails on the day of the exam,

  (a) download the exam paper to a second device (not running Zoom), disconnect it from the internet as soon as the paper is downloaded and read the paper on the second device;

  (b) write your answers on the Masked Exam PDF if you were able to print it single-sided before the exam day.

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- If you are unable to answer the whole question in the answer space provided then you can append additional handwritten solutions to the end of your exam submission. If you do this you MUST make a note in the correct answer space or page for the question, warning the marker that you have appended additional remarks at the end.
- Assemble all the exam pages (or template pages) in correct page number order and the correct way up, and add any extra pages with additional working at the end.
- Scan your exam submission to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
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- Confirm with your Zoom supervisor that you have GradeScope confirmation of submission before leaving Zoom supervision.
- You should attempt all questions.
- Marks may be awarded for
  - Using appropriate mathematical techniques.
  - Accuracy of the solution.
  - Showing full working, including results used.
  - Using correct mathematical notation.
- There are 11 questions with marks as shown. The total number of marks available is 100.

Question 1 (8 marks)

2×1 2×2 1×2

(a) Let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $C = \begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$.

Calculate the following, if they exist:

(i) AC

(ii) BA

(iii) $A^T C$

(i) $\begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 3+8 & 1+4 \end{bmatrix} = \begin{bmatrix} 11 & 5 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 4 & 2 \end{bmatrix}$ is not defined

(b) Let $M = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 0 & 1 & 3 \end{bmatrix}$ and $N = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 1 & 1 & 2 & -1 \\ 1 & 0 & 1 & 3 \\ a & 0 & 0 & 0 \end{bmatrix}$ where $a \in \mathbb{R}$.

(i) Use row operations to calculate $\det(M)$.

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 0 & 1 & 3 \end{vmatrix} \underset{R_1 \leftrightarrow R_2}{=} -\begin{vmatrix} 1 & 2 & -1 \\ 2 & -1 & 1 \\ 0 & 1 & 3 \end{vmatrix} \underset{R_2 \to R_2 - 2R_1}{=} -\begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 1 & 3 \end{vmatrix}$$

$$\underset{R_3 \to R_3 + \frac{1}{5}R_2}{=} -\begin{vmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & 0 & \frac{18}{5} \end{vmatrix} = -\left(1 \times -5 \times \frac{18}{5}\right)$$

$$= -(-18)$$

$$\det(M) = 18$$

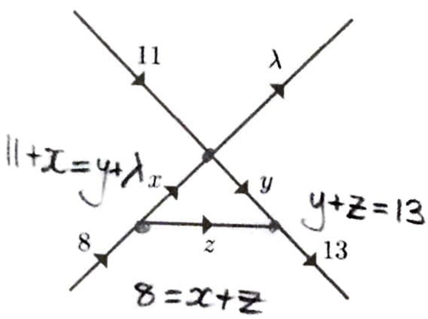
(ii) Write $\det(N)$ in terms of $\det(M)$ and a.

$$\det(N) = \begin{vmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 2 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$-a\det(M) = -18a$$

Question 2 (8 marks)

Consider the following flow diagram where the flow into any vertex must equal the flow out, and $x, y, z, \lambda \in \mathbb{R}$.



(a) Set up a linear system in the unknowns x, y and z using equations corresponding to the flow at each of the vertices.

$x - y = \lambda - 11$

$x + z = 8$

$y + z = 13$

(b) Find the values of λ for which the system in part (a) is (i) consistent and (ii) inconsistent. For (i) find the general solution.

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & \lambda-11 \\ 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & 13 \end{array}\right] \underset{\substack{R_1 \leftrightarrow R_2 \\ R_2 \leftrightarrow R_3}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & 13 \\ 1 & -1 & 0 & \lambda-11 \end{array}\right]$$

$$\underset{R_3 \to R_3 - R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & 13 \\ 0 & -1 & -1 & \lambda-19 \end{array}\right]$$

$$\underset{R_3 \to R_3 + R_2}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & 13 \\ 0 & 0 & 0 & \lambda-6 \end{array}\right]$$

i) the system is consistent only if $\lambda = 6$

let $z = t$, $t \in \mathbb{R}$

$y = 13 - t$

$x = 8 - t$ $\Rightarrow (x, y, z) = (8, 13, 0) + t(-1, -1, 0)$

ii) the system is inconsistent when $\lambda \neq 6$

(c) Find all solutions to the consistent linear system (for correctly chosen λ) such that $x \geq 0$, $y \geq 0$ and $z \geq 0$.

$x \geq 0 \Rightarrow 8 - t \geq 0 \Rightarrow 8 \geq t \Rightarrow t \leq 8$

$y \geq 0 \Rightarrow 13 - t \geq 0 \Rightarrow 13 \geq t \Rightarrow t \leq 13$

$z \geq 0 \Rightarrow t \geq 0$

solutions are $\{(8, 13, 0) + t(-1, -1, 0) \mid 0 \leq t \leq 8\}$

Question 3 (12 marks)

Consider the four points $O(0,0,0)$, $A(1,1,0)$, $B(0,1,1)$ and $C(1,2,4)$ in $\mathbb{R}^3$.

(a) Find a vector equation of the line L containing B and C.

Let $r_0 = B$ $\qquad \vec{BC} = C - B = (1,2,4) - (0,1,1)$

$= (1,1,3)$

$\Rightarrow \underset{\sim}{r} = (0,1,1) + t(1,1,3)$

(b) Find a Cartesian equation for the plane Π_1 containing O, A and B.

$\underset{\sim}{r} = s(1,1,0) + t(0,1,1)$

$\underset{\sim}{n}_1 = A \times B = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{j}$

$+ \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} k = i - j + k$

$\Rightarrow x - y + z = 1 - 1 + 0 \Rightarrow x - y + z = 0$

(c) Find a Cartesian equation for the plane Π_2 containing A, B and C.

let $r_0 = A$ $\qquad \vec{AB} = B - A = (0,1,1) - (1,1,0)$

$\underset{\sim}{n}_2 = \vec{AB} \times \vec{AC}$ $\qquad = (-1,0,1)$

$\vec{AC} = C - A = (1,2,4) - (1,1,0)$

$\begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 4 \end{vmatrix} i - \begin{vmatrix} -1 & 1 \\ 0 & 4 \end{vmatrix} j$ $\qquad = (0,1,4)$

$+ \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} k$

$= -i + 4j - k \Rightarrow -x + 4y - z = -1 + 4(1) - 0$

$-x + 4y - z = 3$

(d) Find the cosine of angle between the planes Π_1 and Π_2.

$$\cos\theta = \frac{\underset{\sim}{n}_1 \cdot \underset{\sim}{n}_2}{\|\underset{\sim}{n}_1\|\|\underset{\sim}{n}_2\|} = \frac{(1,-1,0)\cdot(-1,4,-1)}{\sqrt{1+1}\,\sqrt{1+16+1}}$$

$$= \frac{-1-4-1}{\sqrt{36}} = -\frac{6}{6} = -1$$

$$\arccos(-1) = \pi$$

S A
T C

(e) Find the volume of the parallelepiped having OA, OB and OC as edges.

$\underset{\sim}{u} = \vec{OC}$ $\quad \underset{\sim}{v} = \vec{OA}$ $\quad \underset{\sim}{w} = \vec{OB}$

$$V = |u \cdot (v \times w)|$$

from (b) $v \times w = (1,-1,1)$

$$\Rightarrow V = |(1,2,4)\cdot(1,-1,1)|$$

$$= |1-2+4| = |3|$$

$$= 3$$

Question 4 (10 marks)

In each part of this question, determine whether W is a subspace of the real vector space V. For each part, give a complete proof using the subspace theorem, or a specific counterexample to show that some subspace property fails.

(a) $V = \mathbb{R}^4$, $W = \{(a, b, c, d) \in \mathbb{R}^4 : ab = cd\}$

Let $\underset{\sim}{u} = (0,1,0,0) \in W$, $\underset{\sim}{v} = (1,1,1,1) \in W$

$\underset{\sim}{u} + \underset{\sim}{v} = (1,2,1,1) \notin W$

as $1 \times 2 \neq 1 \times 1$

$\Rightarrow W$ is not a subspace of V

(b) $V = M_{3,3}$, $W = \{A \in M_{3,3} : A^2 = 3A\}$

Let $A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \in W$, $\alpha = 2$.

$\alpha A = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix} \notin W$

as $\begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix}^2 \neq \begin{bmatrix} 18 & 18 & 18 \\ 18 & 18 & 18 \\ 18 & 18 & 18 \end{bmatrix}$

$\Rightarrow W$ is not a subspace of V

(e) $V = \mathcal{P}_3$, $W = \{p(x) \in \mathcal{P}_3 : p(20) = p(19)\}$

Question 5 (7 marks)

You are given the following matrices related by a sequence of elementary row operations:

$$A = \begin{bmatrix} 1 & 0 & -2 & 2 \\ -2 & 1 & 7 & 0 \\ 1 & 2 & 4 & 11 \\ 1 & 1 & 1 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Using this information (or otherwise), answer the following questions:

(a) Show that

$$\{\underbrace{1 - 2x + x^2 + x^3}_{\underset{\sim}{p_1}}, \underbrace{x + 2x^2 + x^3}_{\underset{\sim}{p_2}}, \underbrace{-2 + 7x + 4x^2 + x^3}_{\underset{\sim}{p_3}}, \underbrace{2 + 11x^2 + 6x^3}_{\underset{\sim}{p_4}}\}$$

is not a linearly independent subset of $\mathcal{P}_3$ by writing one of the polynomials as a linear combination of the others.

$\underset{\sim}{p_3} = -2\underset{\sim}{p_1} + 3\underset{\sim}{p_2}$ from matrix B

$-2 + 7x + 4x^2 + x^3 = -2(1 - 2x + x^2 + x^3) + 3(x + 2x^2 + x^3)$

$\Rightarrow -2\underset{\sim}{p_1} + 3\underset{\sim}{p_2} - \underset{\sim}{p_3} + 0\underset{\sim}{p_4} = \underset{\sim}{0}$

$\Rightarrow$ the subset is linearly dependent

(b) Is

$$\{(1, -2, 1, 1), (0, 1, 2, 1), (-2, 7, 4, 1), (2, 0, 11, 6)\}$$

a spanning set for $\mathbb{R}^4$? Explain your answer.

This is not a spanning set as the rank$(B) = 3 \neq 4$ and therefore the augmented matrix $[A|X]$ does not have a solution for all $X \in \mathbb{R}^4$

(c) Find a basis for the subspace of $M_{2,2}$ defined by

$$W = \text{Span}\left\{\begin{bmatrix}1 & 0\\-2 & 2\end{bmatrix}, \begin{bmatrix}-2 & 1\\7 & 0\end{bmatrix}, \begin{bmatrix}1 & 2\\4 & 11\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 6\end{bmatrix}\right\}.$$

What is the dimension of W?

$$A = \begin{bmatrix}1 & -2 & 1 & 1\\0 & 1 & 2 & 1\\-2 & 7 & 4 & 1\\2 & 0 & 11 & 6\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 1 & 1\\0 & 1 & 2 & 1\\0 & 3 & 6 & 3\\0 & 4 & 9 & 4\end{bmatrix}$$

$R_3 \to R_3 + 2R_1$
$R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix}1 & -2 & 1 & 1\\0 & 1 & 2 & 1\\0 & 0 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}$$

$R_3 \to R_3 - 3R_2$
$R_4 \to R_4 - 4R_2$

$R_3 \leftrightarrow R_4$

$$\begin{bmatrix}1 & -2 & 1 & 1\\0 & 1 & 2 & 1\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\end{bmatrix}$$

$$\text{basis} = \left\{\begin{bmatrix}1 & 0\\-2 & 2\end{bmatrix}, \begin{bmatrix}-2 & 1\\7 & 0\end{bmatrix} \begin{bmatrix}1 & 2\\4 & 11\end{bmatrix}\right\}$$

$\dim(W) = 3$

Question 6 (10 marks)

Let $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ be an ordered basis for a real vector space V, and suppose that

$$\mathbf{c}_1 = \mathbf{b}_1, \quad \mathbf{c}_2 = \mathbf{b}_1 - \mathbf{b}_2, \quad \mathbf{c}_3 = \mathbf{b}_1 - 3\mathbf{b}_2 + \mathbf{b}_3$$

(a) Write down the coordinate vectors $[\mathbf{c}_1]_B$, $[\mathbf{c}_2]_B$ and $[\mathbf{c}_3]_B$.

$$[c_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad [c_2]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$[c_3]_B = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

(b) Show that $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is a basis for V.

$$\begin{array}{ccc} \tilde{c}_1 & \tilde{c}_2 & \tilde{c}_3 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & - & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ basis $\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_3\}$ is a basis for V using the column method

(c) Find the transition matrix $P_{\mathcal{B},\mathcal{C}}$ (that converts coordinates with respect to $\mathcal{C}$ to coordinates with respect to $\mathcal{B}$).

$$P_{B,C} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) Find the transition matrix $P_{\mathcal{C},\mathcal{B}}$ (that converts coordinates with respect to $\mathcal{B}$ to coordinates with respect to $\mathcal{C}$)

$$P_{C,B} = (P_{B,C})^{-1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + 3R_3 \\ R_1 \to R_1 - R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad R_2 \to -R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & -1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \quad = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

(e) Using your answer to part (d), find $[2\mathbf{b}_1 + 3\mathbf{b}_2 - \mathbf{b}_3]_{\mathcal{C}}$.

$$[2b_1 + 3b_2 - b_3]_C = P_{C,B} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & -1 \end{bmatrix}$$

Question 7 (10 marks)

Consider the function $T : \mathcal{P}_2 \to \mathcal{P}_2$ given by

$$T(p(x)) = p''(x) + xp'(x) - p(x),$$

where $p'(x)$ and $p''(x)$ are the first and second derivatives of $p(x)$.
Let $\mathcal{B} = \{1, x, x^2\}$ be the standard ordered basis of $\mathcal{P}_2$.

(a) Prove that T is a linear transformation.

Let $\underset{\sim}{u} = x^2$ $\quad \underset{\sim}{v} = x^2+1$ $\quad \alpha \in \mathbb{R}$

1. $T(\underset{\sim}{u}+\underset{\sim}{v}) = T(2x^2+1)$
$= 4 + 4x^2 - 2x^2 - 1$
$= 2x^2 + 3$

$T(\underset{\sim}{u}) = 2 + 2x^2 - x^2$ $\quad T(\underset{\sim}{u}) + T(\underset{\sim}{v}) = 2x^2+3$
$T(\underset{\sim}{v}) = 2 + 2x^2 - x^2 - 1$
$\Rightarrow T(\underset{\sim}{u}+\underset{\sim}{v}) = T(\underset{\sim}{u}) + T(\underset{\sim}{v})$

2. $T(\alpha \underset{\sim}{u}) = \alpha 2 + \alpha x^2$
$= \alpha(2 + \alpha x^2)$
$= \alpha T(\underset{\sim}{u})$

$\Rightarrow T$ preserves vector addition and scalar multiplication and is a linear transformation

(b) Find the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$.

$T(1) = -1$
$T(x) = 0$
$T(x^2) = 2 + x^2$

$$[T]_B = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Find a basis for the image of T.

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underset{R_2 \leftrightarrow R_3}{\sim} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

basis for $\text{Im}(T) = \{(-1+0x+0x^2), (2+0x+x^2)\}$

(d) Find a basis for the kernel of T.

$$\left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \underset{R_2 \leftrightarrow R_3}{\sim} \left[\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

let $x_2 = t$
$x_3 = 0$
$x_1 = 0$

$(x_1, x_2, x_3) = (0,0,0) + t(0,1,0)$

basis for $\text{Ker}(T) = \{(0,1,0)\}$

(e) Determine whether T is injective, surjective or neither.

T is not injective as $\text{Ker}(T) \neq \{\underset{\sim}{0}\}$

T is not surjective as $\text{Im}(T) \neq P_2$

you cannot obtain polynomials with x (degree 1) through this linear transformation

Question 8 (7 marks)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by

$$T(x, y) = (x + y, 2x + y).$$

Consider the following two ordered bases of $\mathbb{R}^2$

$$\mathcal{S} = \{(1, 0), (0, 1)\} \quad \text{and} \quad \mathcal{B} = \{(1, 2), (3, 4)\}.$$

(a) Find the matrix $[T]_{\mathcal{S}}$ of T with respect to the basis $\mathcal{S}$.

$$[T(x,y)] = \begin{bmatrix} x+y \\ 2x+y \end{bmatrix}$$

$$\Rightarrow [T]_S = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$$

(b) Is T invertible? (You should explain your reasoning.)

T is invertible as T is surjective ($\mathrm{Im}(T) = \mathbb{R}^2$) and T is injective ($\ker(T) = \{0\}$)

(c) Find the matrix $[T]_{\mathcal{B}}$ of T with respect to the basis $\mathcal{B}$.

$$[T]_B = P_{B,S}[T]_S P_{S,B}$$

$$P_{S,B} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$P_{B,S} = \frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$\Rightarrow [T]_B = -\frac{1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 7 \\ 4 & 10 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 0 & -2 \\ -2 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

Question 9 (12 marks)

A student gets some exercise by going for a walk each day, either to the local park or to the nearby river.

- If the student goes to the park one day, then the next day she goes to the park with probability $\frac{1}{4}$.
- If she goes to the river one day, then the next day she goes to the river with probability $\frac{1}{2}$.

Let p_n be the probability that the student goes to the park on day n, and r_n be the probability that the student goes to the river on day n.

(a) Explain briefly why

$$\begin{bmatrix} p_{n+1} \\ r_{n+1} \end{bmatrix} = A \begin{bmatrix} p_n \\ r_n \end{bmatrix},$$

where

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} \end{bmatrix}.$$

(b) Find the eigenvalues and corresponding eigenvectors for A.

$\det(A-\lambda I)=0$

$\begin{vmatrix} \frac{1}{4}-\lambda & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{2}-\lambda \end{vmatrix} = 0$

$(\frac{1}{4}-\lambda)(\frac{1}{2}-\lambda) - \frac{3}{8} = 0$

$\frac{1}{8} - \frac{1}{4}\lambda - \frac{1}{2}\lambda + \lambda^2 - \frac{3}{8} = 0$

$\lambda^2 - \frac{3}{4}\lambda - \frac{1}{4} = 0$

$(\lambda-1)(\lambda+\frac{1}{4}) = 0 \Rightarrow \lambda = 1, -\frac{1}{4}$

For $\lambda = 1$:

$(A-1I)\underset{\sim}{v} = \underset{\sim}{0}$

$\left[\begin{array}{cc|c} -\frac{3}{4} & \frac{1}{2} & 0 \\ \frac{3}{4} & -\frac{1}{2} & 0 \end{array}\right] \underset{R_2 \to R_2 + R_1}{\sim} \left[\begin{array}{cc|c} -\frac{3}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

let $\underset{\sim}{v_2} = t,\ t \in \mathbb{R}$

$-\frac{3}{4}\underset{\sim}{v_1} = -\frac{1}{2}t \Rightarrow \underset{\sim}{v_1} = \frac{2}{3}t \therefore \begin{bmatrix} \underset{\sim}{v_1} \\ \underset{\sim}{v_2} \end{bmatrix} = t\begin{bmatrix} \frac{2}{3} \\ 1 \end{bmatrix}$

For $\lambda = -\frac{1}{4}$:

$\left[\begin{array}{cc|c} \frac{2}{4} & \frac{1}{2} & 0 \\ \frac{3}{4} & \frac{3}{4} & 0 \end{array}\right] \underset{R_2 \to R_2 - \frac{3}{2}R_1}{\sim} \left[\begin{array}{cc|c} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$

let $\underset{\sim}{v_2} = t,\ t \in \mathbb{R}$

$\frac{1}{2}\underset{\sim}{v_1} = -\frac{1}{2}t \therefore \begin{bmatrix} \underset{\sim}{v_1} \\ \underset{\sim}{v_2} \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\underset{\sim}{v_1} = -t$

(c) Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

$$P = \begin{bmatrix} \frac{2}{3} & -1 \\ 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}$$

(d) Describe the limiting behaviour of A^n as $n \to \infty$.

(e) What is the long term probability that the student visits the river?

Question 10 (10 marks)

Consider $\mathcal{P}_1$ with the inner product given by

$$\langle p(x), q(x)\rangle = p(1)q(1) + 2p(2)q(2).$$

(a) Prove that the formula for $\langle p(x), q(x)\rangle$ defines an inner product on $\mathcal{P}_1$.

Let $\underset{\sim}{u} = u_1 x + u_0$, $\underset{\sim}{v} = v_1 x + v_0$, $\underset{\sim}{w} = w_1 x + w_0$, $\alpha \in \mathbb{R}$

1. $\langle u, v\rangle = (u_1 + u_0)(v_1 + v_0) + 2(2u_1 + u_0)(2v_1 + v_0)$
$= (v_1 + v_0)(u_1 + u_0) + 2(2v_1 + v_0)(2u_1 + u_0)$
$= \langle v, u\rangle$

2. $\langle \alpha u, v\rangle = (\alpha u_1 + \alpha u_0)(v_1 + v_0) + 2(2\alpha u_1 + \alpha u_0)(2v_1 + v_0)$
$= \alpha(u_1 + u_0)(v_1 + v_0) + 2\alpha(2u_1 + u_0)(2v_1 + v_0)$
$= \alpha((u_1 + u_0)(v_1 + v_0) + 2(2u_1 + u_0)(2v_1 + v_0))$
$= \alpha\langle u, v\rangle$

3. $\langle u + v, w\rangle = ((u_1 + v_1) + (u_0 + v_0))(w_1 + w_0)$
$+ 2(2(u_1 + v_1) + (u_0 + v_0))(2w_1 + w_0)$
$= (u_1 + u_0)(w_1 + w_0) + 2(2u_1 + u_0)(2w_1 + w_0)$
$+ (v_1 + v_0)(w_1 + w_0) + 2(2v_1 + v_0)(2w_1 + w_0)$
$= \langle u, w\rangle + \langle v, w\rangle$

4. $\langle u, u\rangle = (u_1 + u_0)(u_1 + u_0) + 2(2u_1 + u_0)(2u_1 + u_0)$
$= (u_1 + u_0)^2 + 2(2u_1 + u_0)^2 \geq 0$

$\langle u, u\rangle = 0$

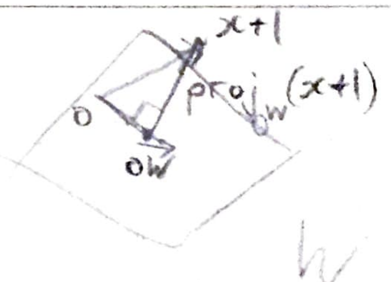
$(u_1 + u_0)^2 + 2(2u_1 + u_0)^2 = 0 \iff u = 0$

$\therefore \langle p(x), q(x)\rangle$ defines an inner product on P_1.

(b) Calculate the length of x with respect to this inner product.

$$\|x\| = \sqrt{\langle x, x \rangle}$$
$$= \sqrt{(1 \times 1) + 2(2 \times 2)}$$
$$= \sqrt{1+8} = \sqrt{9} = 3$$

(c) Let $W \subset \mathcal{P}_1$ be the subspace spanned by x. Find the point of W that is closest to $x+1$ using this inner product.



$$\mathrm{proj}_{\vec{ow}}(x+1)$$
$$= \langle x+1, x \rangle x$$
$$= ((1+1)(1+0) + 2(2+1)(2+0))x$$
$$= (2 + 2(3)(2))x$$
$$= (14)x = 14x$$

$\underset{\sim}{p} = 14x$ or $(0, 14)$

Question 11 (6 marks)

Let A be an arbitrary $n \times n$ matrix with complex entries.

(a) Prove that if λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2.

$$A^2 = PD^2P^{-1}$$

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ 0 & \cdots & \lambda_n \end{bmatrix}$$

$$D^2 = \begin{bmatrix} \lambda_1^2 & \cdots & 0 \\ 0 & \cdots & \lambda_n^2 \end{bmatrix}$$

$\Rightarrow$ If λ is an eigenvalue of A then λ^2 is an eigenvalue of A^2

(b) Is it always true that every eigenvector of A^2 is also an eigenvector of A? Justify your answer by either giving a general proof, or by giving an example of a matrix A where this does not hold.

$A^2 = PD^2P^{-1}$

In general,

$A^k = PD^kP^{-1}$

P is the matrix with eigenvectors of A as its columns.

$\Rightarrow$ Every eigenvector of A^2 is also an eigenvector of A.

End of Exam—Total Available Marks = 100