MAST30027 Modern Applied Statistics Assignment 1

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Tutorial: Wed 1-2PM, Yidi Deng

Question 1

(a)

```
> library(faraway)
> data(orings)
>
> logL = function(betas, orings) {
+     eta = cbind(1, orings$temp) %*% betas
+     return (sum(orings$damage * log(pnorm(eta)) + (6 - orings$damage)
                                    * log(1 - pnorm(eta)) ))
+ }
>
> (betahat = optim(c(10, -.1), logL, orings=orings,
                    control=list(fnscale=-1))$par)
[1]  5.5917242 -0.1058008
```

$\hat{\beta}_0 = 5.5917$

$\hat{\beta}_1 = -0.1058$

(b)

$$\eta_i = \beta_0 + \beta_1 t_i$$

$$l(\beta_0, \beta_1) = c + \sum_i \left[(y_i \log F(\eta_i) + (6-y_i)\log(1-F(\eta_i)) \right] \quad \text{where } F(\eta_i) = \Phi(\eta_i)$$

$$\frac{\partial \eta_i}{\partial \beta_0} = 1 \qquad \frac{\partial \eta_i}{\partial \beta_1} = t_i$$

$$\frac{\partial l(\beta_0,\beta_1)}{\partial \beta_0} = \sum_i \left[y_i \frac{1}{F(\eta_i)} f(\eta_i)\times 1 + (6-y_i)\frac{1}{1-F(\eta_i)} \times -f(\eta_i)\times 1 \right]$$

$$= \sum_i \left[y_i \frac{f(\eta_i)}{F(\eta_i)} + (y_i - 6) \frac{f(\eta_i)}{1-F(\eta_i)} \right]$$

$$\frac{\partial^2 l(\beta_0,\beta_1)}{\partial^2 \beta_0} = \sum_i \left[y_i \times \frac{F(\eta_i)\times f'(\eta_i)\times 1 - f(\eta_i) f(\eta_i)\times 1}{(F(\eta_i))^2} + (y_i-6)\frac{(1-F(\eta_i)) f'(\eta_i)\times 1 - f(\eta_i) \times -f(\eta_i)\times 1}{(1-F(\eta_i))^2} \right]$$

$$= \sum_i \left[y_i \times \frac{F(\eta_i) f'(\eta_i) - f(\eta_i)^2}{F(\eta_i)^2} + (y_i - 6)\frac{f'(\eta_i) - F(\eta_i) f'(\eta_i) + f(\eta_i)^2}{(1-F(\eta_i))^2} \right]$$

$$= \sum_i \left[y_i \times \frac{p_i f'(\eta_i) - f(\eta_i)^2}{p_i^2} + (y_i - 6)\frac{f'(\eta_i) - p_i f'(\eta_i) + f(\eta_i)^2}{(1-p_i)^2} \right]$$

$$E\left(\frac{\partial^2 L(\beta_0,\beta_1)}{\partial^2 \beta_0}\right) = \sum_i \left[6p_i \times \frac{p_i f'(\eta_i) - f(\eta_i)^2}{p_i^2} - 6(1-p_i)\frac{f'(\eta_i) - p_i f'(\eta_i) + f(\eta_i)^2}{(1-p_i)^2} \right]$$

$$= \sum_i \left[6 \times \frac{p_i f'(\eta_i) - f(\eta_i)^2}{p_i} - 6 \times \frac{f'(\eta_i) - p_i f'(\eta_i) + f(\eta_i)^2}{1-p_i} \right]$$

$$= 6\sum_i \left[\frac{(p_i f'(\eta_i) - f(\eta_i)^2)(1-p_i)}{p_i(1-p_i)} - \frac{(f'(\eta_i) - p_i f'(\eta_i) + f(\eta_i)^2) p_i}{p_i(1-p_i)} \right]$$

$$= 6\sum_i \left[\frac{p_i f'(\eta_i) - p_i^2 f'(\eta_i) - f(\eta_i)^2 + p_i f(\eta_i)^2}{p_i(1-p_i)} - \frac{p_i f'(\eta_i) - p_i^2 f'(\eta_i) + p_i f(\eta_i)^2}{p_i(1-p_i)} \right]$$

$$= 6\sum_i \frac{-f(\eta_i)^2}{p_i(1-p_i)} = \boxed{-\frac{3}{\pi}\sum_i \frac{e^{-\eta_i^2}}{p_i(1-p_i)}}$$

$$\frac{\partial^2 l(\beta_0,\beta_1)}{\partial \beta_0 \partial \beta_1} = \sum_i \left[y_i \times \frac{F(\eta_i)\times f'(\eta_i)\times t_i - f(\eta_i) f(\eta_i)\times t_i}{(F(\eta_i))^2} + (y_i-6)\frac{(1-F(\eta_i)) f'(\eta_i)\times t_i - f(\eta_i) \times -f(\eta_i)\times t_i}{(1-F(\eta_i))^2} \right]$$

$$= \sum_i \left[y_i \times \frac{F(\eta_i) f'(\eta_i) t_i - f(\eta_i)^2 t_i}{F(\eta_i)^2} + (y_i - 6)\frac{(1-F(\eta_i)) f'(\eta_i) t_i + f(\eta_i)^2 t_i}{(1-F(\eta_i))^2} \right]$$

$$E\left(\frac{\partial^2 l(\beta_0,\beta_1)}{\partial \beta_0 \partial \beta_1}\right) = \sum_i \left[6p_i \times \frac{p_i f'(\eta_i) t_i - f(\eta_i)^2 t_i}{p_i^2} - 6(1-p_i)\frac{f'(\eta_i) t_i - p_i f'(\eta_i) t_i + f(\eta_i)^2 t_i}{(1-p_i)^2} \right]$$

$$= 6\sum_i \left[\frac{(p_i f'(\eta_i) t_i - f(\eta_i)^2 t_i)(1-p_i)}{p_i(1-p_i)} - 6(1-p_i)\frac{(f'(\eta_i) t_i - p_i f'(\eta_i) t_i + f(\eta_i)^2 t_i) p_i}{(1-p_i) p_i} \right]$$

$$= 6\sum_i \left[\frac{p_i f'(\eta_i) t_i - p_i^2 f'(\eta_i) t_i - f(\eta_i)^2 t_i + p_i f(\eta_i)^2 t_i}{p_i(1-p_i)} - \frac{p_i f'(\eta_i) t_i - p_i^2 f'(\eta_i) t_i + p_i f(\eta_i)^2 t_i}{(1-p_i) p_i} \right]$$

$$= 6\sum_i \left[\frac{-f(\eta_i)^2 t_i}{p_i(1-p_i)} \right] = \boxed{-\frac{3}{\pi}\sum_i \frac{e^{-\eta_i^2} t_i}{p_i(1-p_i)}}$$

$$= E\left(\frac{\partial^2 l(\beta_0,\beta_1)}{\partial \beta_1 \partial \beta_0}\right) \text{ as both partial derivatives are continuous}$$

$$\frac{\partial l(\beta_0,\beta_1)}{\partial \beta_1} = \sum_i \left[y_i \frac{1}{F(\eta_i)} f(\eta_i) \times t_i + (6-y_i) \frac{1}{1-F(\eta_i)} \times -f(\eta_i) \times t_i \right]$$

$$= \sum_i \left[y_i \frac{f(\eta_i) t_i}{F(\eta_i)} + (y_i - 6) \frac{f(\eta_i) t_i}{1-F(\eta_i)} \right]$$

$$\frac{\partial^2 l(\beta_0,\beta_1)}{\partial^2 \beta_1} = \sum_i \left[y_i \times \frac{F(\eta_i) \times f'(\eta_i) t_i \times t_i - f(\eta_i) t_i f(\eta_i) \times t_i}{(F(\eta_i))^2} + (y_i - 6) \frac{(1-F(\eta_i)) f'(\eta_i) t_i \times t_i - f(\eta_i) t_i \times -f(\eta_i) \times t_i}{(1-F(\eta_i))^2} \right]$$

$$= \sum_i \left[y_i \times \frac{F(\eta_i) f'(\eta_i) t_i^2 - f(\eta_i)^2 t_i^2}{F(\eta_i)^2} + (y_i - 6) \frac{f'(\eta_i) t_i^2 - F(\eta_i) f'(\eta_i) t_i^2 + f(\eta_i)^2 t_i^2}{(1-F(\eta_i))^2} \right]$$

$$= \sum_i \left[y_i \times \frac{p_i f'(\eta_i) t_i^2 - f(\eta_i)^2 t_i^2}{p_i^2} + (y_i - 6) \frac{f'(\eta_i) t_i^2 - p_i f'(\eta_i) t_i^2 + f(\eta_i)^2 t_i^2}{(1-p_i)^2} \right]$$

$$E\left(\frac{\partial^2 l(\beta_0,\beta_1)}{\partial^2 \beta_1}\right) = \sum_i \left[6 p_i \frac{p_i f'(\eta_i) t_i^2 - f(\eta_i)^2 t_i^2}{p_i^2} - 6(1-p_i) \frac{f'(\eta_i) t_i^2 - p_i f'(\eta_i) t_i^2 + f(\eta_i)^2 t_i^2}{(1-p_i)^2} \right]$$

$$= 6 \sum_i \left[\frac{(p_i f'(\eta_i) t_i^2 - f(\eta_i)^2 t_i^2)(1-p_i)}{p_i (1-p_i)} - \frac{(f'(\eta_i) t_i^2 - p_i f'(\eta_i) t_i^2 + f(\eta_i)^2 t_i^2) p_i}{(1-p_i) p_i} \right]$$

$$= 6 \sum_i \left[\frac{p_i f'(\eta_i) t_i^2 - p_i^2 f'(\eta_i) t_i^2 - f(\eta_i)^2 t_i^2 + p_i f(\eta_i)^2 t_i^2}{p_i (1-p_i)} - \frac{p_i f'(\eta_i) t_i^2 - p_i^2 f'(\eta_i) t_i^2 + p_i f(\eta_i)^2 t_i^2}{(1-p_i) p_i} \right]$$

$$= \boxed{-\frac{3}{\pi} \sum_i \frac{e^{-\eta_i^2} t_i^2}{p_i (1-p_i)}}$$

$$\Rightarrow \mathcal{I}(\beta) = \frac{3}{\pi} \begin{bmatrix} \sum_i \frac{e^{-(\Phi^{-1}(p_i))^2}}{p_i(1-p_i)} & \sum_i \frac{e^{-(\Phi^{-1}(p_i))^2} t_i}{p_i(1-p_i)} \\ \sum_i \frac{e^{-(\Phi^{-1}(p_i))^2} t_i}{p_i(1-p_i)} & \sum_i \frac{e^{-(\Phi^{-1}(p_i))^2} t_i^2}{p_i(1-p_i)} \end{bmatrix}$$

```
> iprobit = function(x) pnorm(x)
> phat = iprobit(betahat[1] + orings$temp*betahat[2])
> I11 = 3/pi * sum(dexp(qnorm(phat)^2) / (phat*(1-phat)))
> I12 = 3/pi * sum(orings$temp * dexp(qnorm(phat)^2) / (phat*(1-phat)))
> I22 = 3/pi * sum(orings$temp^2 * dexp(qnorm(phat)^2) / (phat*(1-phat)))
>
> Iinv = solve(matrix(c(I11, I12, I12, I22), 2, 2))
>
> betahat[1] + c(-1,1)*qnorm(0.975)*sqrt(Iinv[1,1])
 [1] 2.239762 8.943686
>
> betahat[2] + c(-1,1)*qnorm(0.975)*sqrt(Iinv[2,2])
 [1] -0.15784670 -0.05375481
```

95% Confidence Interval for $\hat{\beta}_0$: $(2.2398, \ 8.9437)$

95% Confidence Interval for $\hat{\beta}_1$: $(-0.1578, \ -0.0538)$

(c)

```
> logL.F = function(betas, orings) {
+     eta = cbind(1, orings$temp) %*% betas
+     return (sum(orings$damage * log(pnorm(eta)) + (6 - orings$damage)
                                                    * log(1 - pnorm(eta)) ))
+ }
>
> logL.R = function(beta0, orings) {
+     eta = beta0
+     return (sum(orings$damage * log(pnorm(eta)) + (6 - orings$damage)
                                                    * log(1 - pnorm(eta)) ))
+ }
>
>
> (betahat.F = optim(c(10, -.1), logL.F, orings=orings,
                               control=list(fnscale=-1))$par)
[1]  5.5917242 -0.1058008
>
> (betahat.R = optim(c(5), logL.R, orings=orings, control=list(fnscale=-1))$par)
[1] -1.40625
>
> (LR = -2*(logL.R(betahat.R, orings) - logL.F(betahat.F, orings)))
[1] 20.76711
>
> pchisq(LR, df=1, lower=FALSE)
[1] 5.186617e-06
```

p-value < 0.05, so we reject $H_0 : \beta_1 = 0$

(d)

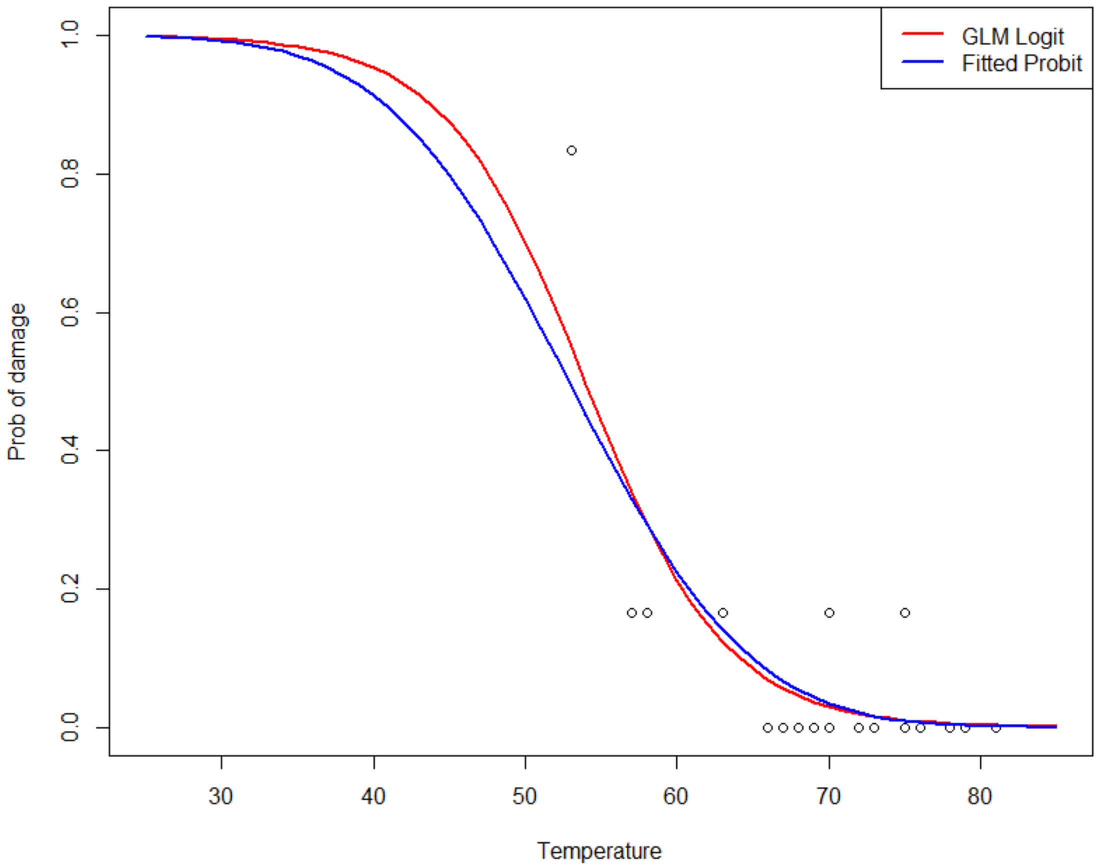
```
> si2 = matrix(c(1, 31), 1, 2) %*% Iinv %*% matrix(c(1, 31), 2, 1)
>
> etahat = betahat[1] + betahat[2]*31
>
> eta_l = etahat - qnorm(0.975)*sqrt(si2)
>
> eta_r = etahat + qnorm(0.975)*sqrt(si2)
>
> c(eta_l, eta_r)
[1] 0.5557902 4.0680114
>
> iprobit(etahat)
[1] 0.9896084
>
> c(iprobit(eta_l), iprobit(eta_r))
[1] 0.7108229 0.9999763
```

$\hat{p} = 0.9896$

95% Confidence Interval for $\hat{p}$: $(0.7108, \ 0.9999)$

(e)

```
> ilogit = function(x) exp(x)/(1+exp(x))
>
> iprobit = function(x) pnorm(x)
>
> log_mod = glm(cbind(damage, 6-damage) ~ temp, family=binomial(link="logit"),
+                   orings)
>
> plot(damage/6 ~ temp, orings, xlim=c(25,85), ylim=c(0,1),
+       xlab="Temperature", ylab="Prob of damage")
> x = seq(25,85,1)
> lines(x, ilogit(log_mod$coefficients[1] + log_mod$coefficients[2]*x), col="red",
                                                                       lwd=2)
> lines(x, iprobit(betahat[1] + betahat[2]*x), col="blue", lwd=2)
>
> legend(x = "topright",
+           legend = c("GLM Logit", "Fitted Probit"),
+           lty = c(1),
+           col = c("red", "blue"),
+           lwd = 2)
```



Question 2

(a)

```
> library(faraway)
> missing = with(pima, missing <- glucose==0 | diastolic==0 | triceps==0
                                              | bmi==0)
> pima_subset = pima[!missing, c(6,9)]
> str(pima_subset)
'data.frame':   532 obs. of  2 variables:
 $ bmi : num  33.6 26.6 28.1 43.1 31 30.5 30.1 25.8 45.8 43.3 ...
 $ test: int  1 0 0 1 1 1 1 1 1 0 ...
>
> pima_mdl = glm(test ~ bmi, family=binomial(link="logit"),
+                  pima_subset)
>
> phat = ilogit(pima_mdl$coefficients[2]*7)
>
> log_odds = logit(phat)
>
> as.numeric(log_odds)
[1] 0.6980179
```

$logit(\hat{p}) = 0.6980$

(b)

```
> phat_l = ilogit((pima_mdl$coefficients[2]*7 - qnorm(0.975)
                   * summary(pima_mdl)$coefficients[2, 2] * 7))
>
> phat_r = ilogit((pima_mdl$coefficients[2]*7 + qnorm(0.975)
                   * summary(pima_mdl)$coefficients[2, 2] * 7))
>
> CI_logodds = c(logit(phat_l), logit(phat_r))
>
> as.numeric(CI_logodds)
[1] 0.4883237 0.9077121
```

95% Confidence Interval for $logit(\hat{p})$: (0.4883, 0.9077)

Question 3

(a)

$\nu > 0 \qquad \lambda > 0$

$$f(x;\nu,\lambda) = \frac{\lambda^{\nu}}{\Gamma(\nu)} x^{\nu-1} e^{-\lambda x} \qquad x > 0$$

$$f(x) = \frac{\lambda^{\nu}}{\Gamma(\nu)} x^{\nu-1} e^{-\lambda x}$$

$$= \exp\left[-\lambda x + (\nu-1)\log x + \nu\log\lambda - \log\Gamma(\nu)\right]$$

$$= \exp\left[-\lambda x + \nu\log\lambda + (\nu-1)\log x - \log\Gamma(\nu)\right]$$

$$= \exp\left[\frac{x\left(-\frac{\lambda}{\nu}\right) - \log\left(\frac{\nu}{\lambda}\right) - \log(\nu)}{\frac{1}{\nu}} + (\nu-1)\log x - \log\Gamma(\nu)\right]$$

$$= \exp\left[\frac{x\left(-\frac{\lambda}{\nu}\right) - \log\left(\frac{\nu}{\lambda}\right)}{\frac{1}{\nu}} + \frac{-\log(\nu) + \left(1-\frac{1}{\nu}\right)\log x - \frac{1}{\nu}\log\Gamma(\nu)}{\frac{1}{\nu}}\right]$$

let $\theta = -\frac{\lambda}{\nu} \qquad b(\theta) = \log\left(\frac{\nu}{\lambda}\right) = \log\left(-\frac{1}{\theta}\right), \qquad \theta < 0$

$\phi = \frac{1}{\nu} \qquad a(\phi) = \frac{1}{\nu} \qquad \phi > 0$

$$c(x,\phi) = \frac{-\log\left(\frac{1}{\phi}\right) + (1-\phi)\log x - \phi\log\Gamma\left(\frac{1}{\phi}\right)}{\phi}$$

$$\Rightarrow f(x;\nu,\lambda) = \exp\left[\frac{x\theta - b(\theta)}{a(\phi)} + c(x,\phi)\right]$$

$\Rightarrow$ The gamma distribution is an exponential family

(b)

$\theta = -\frac{\lambda}{\nu}$ $\quad b(\theta) = \log(-\frac{1}{\theta})$ $\quad a(\phi) = \frac{1}{\nu}$

variance function $v(\mu) = b''((b')^{-1}(\mu))$

$b'(\theta) = \frac{1}{-\frac{1}{\theta}} \times \frac{1}{\theta^2} = -\frac{1}{\theta}$, $\quad (b')^{-1}(\mu) = -\frac{1}{\mu}$

$b''(\theta) = \frac{1}{\theta^2}$

$\Rightarrow v(\mu) = \frac{1}{(-\frac{1}{\mu})^2}$

$= \frac{1}{\frac{1}{\mu^2}} = \mu^2$

$\Rightarrow$ canonical link

$g(\mu) = -\frac{1}{\mu}$