Written Assignment 6 (Gradescope)

Graded

Student

James La Fontaine

Total Points

21 / 21 pts

Question 1

Q1 orthogonal projection — **8** / 8 pts

1.1 **1a** — **1** / 1 pt

- ✔ **+ 1 pt** 1 mark for correct explanations (geometric or otherwise)
- **+ 0 pts** No marks

1.2 **1b** — **1** / 1 pt

- ✔ **+ 1 pt** 1 mark for correct answer
- **+ 0 pts** No marks

1.3 **1c** — **1** / 1 pt

- ✔ **+ 1 pt** 1 mark for correct answer
- **+ 0 pts** No marks

1.4 **1d** — **2** / 2 pts

- ✔ **+ 1 pt** 1 method mark for finding $P_{S,B}^{-1}$
- ✔ **+ 1 pt** 1 mark for correct answer
- **+ 0 pts** No marks

1.5 **1e** — **3** / 3 pts

- ✔ **+ 1 pt** 1 method mark for a correct method, e.g. using $[T]_S = P_{S,B}[T]_B P_{B,S}$
- ✔ **+ 1 pt** 1 answer mark for correct $[T]_S$
- ✔ **+ 1 pt** 1 method mark for going from matrix $[T]_S$ to formula for $T(x, y)$
- **+ 0 pts** No marks

Question 2

Q2 eigenvalues and eigenvectors **8** / 8 pts

2.1 2a **5** / 5 pts

- ✔ **+ 1 pt** 1 method mark for looking at $\det(A - \lambda I) = 0$
- ✔ **+ 1 pt** 1 answer mark for two correct eigenvalues
- ✔ **+ 1 pt** 1 method mark for solving $(A - \lambda I)\mathbf{x} = \mathbf{0}$ (for at least one eigenvalue)
- ✔ **+ 1 pt** 1 answer mark for a correct eigenvector for $\lambda = 2$
- ✔ **+ 1 pt** 1 answer mark for a correct eigenvector for $\lambda = 3$
- **+ 0 pts** No marks

2.2 2b **3** / 3 pts

- ✔ **+ 1 pt** 1 answer mark for a correct D
- ✔ **+ 1 pt** 1 answer mark for a correct P
- ✔ **+ 1 pt** 1 answer mark for a correct P^{-1}
- **+ 0 pts** No marks

Question 3

Q3 eigenspaces **5** / 5 pts

3.1 3a **4** / 4 pts

- ✔ **+ 1 pt** 1 method mark for solving $(B - 2I)\mathbf{x} = \mathbf{0}$
- ✔ **+ 1 pt** 1 answer mark for a correctly written basis for $\lambda = 2$ eigenspace
- ✔ **+ 1 pt** 1 method mark for solving $(B + 2I)\mathbf{x} = \mathbf{0}$
- ✔ **+ 1 pt** 1 answer mark for a correctly written basis for $\lambda = -2$ eigenspace
- **+ 0 pts** No marks

3.2 3b **1** / 1 pt

- ✔ **+ 1 pt** 1 mark for correct answer and correct explanation
- **+ 0 pts** No marks

School of Mathematics and Statistics
MAST10007 Linear Algebra, Semester 1 2020
Written assignment 6

Submit your assignment online in Canvas before 12 noon on Monday 1st June.

Name: James La Fontaine

Student ID: 1079860

- This assignment is worth $1\frac{1}{9}\%$ of your final MAST10007 mark.
- Your solutions should be neatly handwritten in blue or black pen, then uploaded as a single PDF file in **GradeScope**.
- Full explanations and working must be shown in your solutions.
- Marks may be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- You must use methods taught in MAST10007 Linear Algebra to solve the assignment questions.

New submission guidelines:

- This assignment is being handled using a similar process to that planned for the final exam so you can start to become familiar with it.
- If you have access to a printer, then you should print out this assignment sheet and handwrite your solutions into the answer boxes.
- If you do not have access to a printer, but you can annotate a PDF file using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly in the boxes on the assignment PDF and save a copy for submission.
- Otherwise, you may handwrite your answers on blank paper to **produce a document that mirrors the layout of the assignment template** and then scan for submission. So: put your name and student ID on page 1, your answers to Q1a-d on page 2, your answer to Q1e on page 3, your answers to Q2 on page 4, and your answers to Q3 on page 5.
- The answer boxes should typically provide sufficient space for your solution, but if you find you need extra space please add a blank sheet of paper at the end and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- Scan your assignment to a PDF file using your mobile phone or scanner, then upload by going to the Assignments menu on Canvas and submit the PDF to the **GradeScope** tool by first selecting your PDF file and then clicking on 'Upload pdf'.

1. Orthogonal projection onto the line $y = 2x$ gives a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

(a) Explain briefly, geometrically or otherwise, why $T(1,2) = (1,2)$ and $T(-2,1) = (0,0)$.

The projection of a vector onto itself/ a parallel line will give the same vector

i.e. $\frac{(1,2)\cdot(1,2)}{\sqrt{1^2+2^2}^2}(1,2) = (1,2)$

As $(-2,1)$ is perpendicular to the line, projection onto the line will give the zero vector,

i.e. $\frac{0}{\sqrt{(-2)^2+1^2}^2}(1,2) = (0,0)$

(b) Write down the matrix of T with respect to the ordered basis $\mathcal{B} = \{(1,2),(-2,1)\}$.

$\underset{\sim}{b}_1 = (1,2)$ $\quad$ $\underset{\sim}{b}_2 = (-2,1)$

$T(\underset{\sim}{b}_1) = \underset{\sim}{b}_1 + 0\underset{\sim}{b}_2 = (1,2)$ $\quad$ $T(\underset{\sim}{b}_2) = 0\underset{\sim}{b}_1 + 0\underset{\sim}{b}_2 = (0,0)$

$\Rightarrow [T(\underset{\sim}{b}_1)]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad$ $\Rightarrow [T(\underset{\sim}{b}_2)]_B = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ matrix of T with respect to B is $[T]_B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(c) Write down the transition matrix from $\mathcal{B}$ to the standard basis $\mathcal{S} = \{(1,0),(0,1)\}$.

$P_{S,B} = \left[[\underset{\sim}{b}_1]_S \ [\underset{\sim}{b}_2]_S \right] = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

(d) Find the transition matrix from $\mathcal{S}$ to $\mathcal{B}$.

$P_{B,S} = (P_{S,B})^{-1} = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{1-(-2\cdot 2)} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

(e) Hence, find the matrix of T with respect to the standard basis S and use this to find a general formula for $T(x, y)$.

$$[T]_S = P_{S,B}\,[T]_B\,P_{B,S}$$

$$= \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \right)$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\therefore [T(x,y)] = [T]_S \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Rightarrow T(x,y) = \left(\frac{x+2y}{5}, \frac{2x+4y}{5} \right)$$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix A.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix}$$

$$= (1-\lambda)(4-\lambda) - (1\cdot -2)$$

$$= (1-\lambda)(4-\lambda) + 2$$

$$= \lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2)$$

$\boxed{\therefore \text{ eigenvalues are } \lambda = 2, 3}$

$\lambda = 2$: $(A-2I)\underset{\sim}{v} = \underset{\sim}{0}$ where $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$(A-2I) = \begin{bmatrix} 1-2 & 1 \\ -2 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \underset{R_2 \to R_2 - 2R_1}{\sim} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $\underset{\sim}{v_2} = t$, $t \in \mathbb{R}$. Then $-\underset{\sim}{v_1} = -v_2 = -t$

$\underset{\sim}{v_1} = t$

$\therefore \underset{\sim}{v} = (t, t)$, $t \in \mathbb{R}$

$\boxed{\text{All eigenvectors of } \lambda = 2 \text{ are } t\begin{bmatrix} 1 \\ 1 \end{bmatrix}, t \in \mathbb{R}, t \neq 0}$

$\lambda = 3$: $(A-3I)\underset{\sim}{v} = \underset{\sim}{0}$ where $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$(A-3I) = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right] \underset{R_2 \to R_2 - R_1}{\sim} \left[\begin{array}{cc|c} -2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \underset{R_1 \to -\frac{1}{2}R_1}{\sim} \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Let $\underset{\sim}{v_2} = t$, $t \in \mathbb{R}$. Then

$\underset{\sim}{v_1} = \frac{1}{2}\underset{\sim}{v_2} = \frac{1}{2}t$

$\therefore \underset{\sim}{v} = (\frac{1}{2}t, t)$, $t \in \mathbb{R}$

$\boxed{\text{eigenvectors of } \lambda = 3 \text{ are } t\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}, t \in \mathbb{R}, t \neq 0}$

(b) Find a diagonal matrix D and invertible matrices P, P^{-1} such that $P^{-1}AP = D$.

$\underset{\sim}{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \underset{\sim}{v_2} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P = [\underset{\sim}{v_1} \,\vdots\, \underset{\sim}{v_2}] = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{1-\frac{1}{2}}\begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix}$$

$$= 2\begin{bmatrix} 1 & -\frac{1}{2} \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

3. The matrix

$$B = \begin{bmatrix} -1 & 0 & 1 \\ -7 & 2 & 5 \\ 3 & 0 & 1 \end{bmatrix},$$

has eigenvalues $-2, 2, 2$.

(a) Find bases for the corresponding eigenspaces.

$\lambda = -2$ $\quad (A+2I)\underset{\sim}{v} = \underset{\sim}{0}$

for $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -7 & 4 & 5 & 0 \\ 3 & 0 & 3 & 0 \end{array}\right] \underset{\substack{R_2 \to R_2 + 7R_1 \\ R_3 \to R_3 - 3R_1}}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 4 & 12 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \underset{R_2 \to \frac{1}{4}R_2}{\sim} \begin{array}{c} \quad\quad\ \ t \\ \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

let $\underset{\sim}{v_3} = t$, $t \in \mathbb{R}$. Then $v_1 = -v_3 = -t$
$v_2 = -3\underset{\sim}{v_3} = -3t$ $\quad \therefore \underset{\sim}{v} = (-t, -3t, t)$

$\therefore$ basis for eigenspace of $\lambda = -2$ is $\boxed{\{(-1, -3, 1)\}}$

$\lambda = 2$ $\quad (A-2I)\underset{\sim}{v} = \underset{\sim}{0}$

for $\underset{\sim}{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ -7 & 0 & 5 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right] \underset{R_1 \to -\frac{1}{3}R_1}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ -7 & 0 & 5 & 0 \\ 3 & 0 & 1 & 0 \end{array}\right] \underset{\substack{R_2 \to R_2 + 7R_1 \\ R_3 \to R_3 - 3R_1}}{\sim} \begin{array}{c} \quad t \quad\quad \\ \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 8/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

let $\underset{\sim}{v_2} = t$, $t \in \mathbb{R}$. Then $\underset{\sim}{v_1} = \frac{1}{3}\underset{\sim}{v_3} = \underset{\sim}{0}$.

$\therefore \underset{\sim}{v} = (0, t, 0)$

basis for eigenspace of $\lambda = 2$ is $\boxed{\{(0, 1, 0)\}}$

(b) Is the matrix B diagonalisable? Give a reason for your answer.

The geometric multiplicity of $\lambda = 2$ is $1 < 2$ (algebraic multiplicity of $\lambda = 2$)
and B is a 3×3 matrix with only 2 linearly independent eigenvectors.

$\Rightarrow$ The matrix B is certainly not diagonalisable.