Assignment 7 Due: 6:00PM, Friday 22 May.

Name:

Student ID:

Explainer: Question 1 should be completed in **WebWork** by 6:00PM, Friday 22 May. WebWork should be accessed via Assignment 7 WebWork in the Assignments panel of the MAST10005 LMS Site.

You should upload a scan of neatly presented solutions to Question 2 in **Gradescope**, which should be accessed via Assignment 7 Written Part in the Assignments section of the MAST10005 LMS Site. Please do not include your answers to Question 1 in your solutions to the written part.

1. You should complete this question in WebWork by 6:00PM, Friday 22 May. It will test your knowledge of derivatives of parametric curves. Completing Question 1 *before* you attempt Question 2 will make Question 2 easier because you will have already checked that your calculation of the derivative of $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by

$$\mathbf{r}(t) = t(t-2)^3\mathbf{i} + t(t-2)^2\mathbf{j}$$

at various points.

2. A particle is moving according to the parametric curve $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by

$$\mathbf{r}(t) = t(t-2)^3\mathbf{i} + t(t-2)^2\mathbf{j}.$$

(a) Use Definition 3.44 from the lecture slides to find the t values for which $\mathbf{r}$ has a cusp. Calculate $\mathbf{r}(t)$ at this point.

Solution: Using our calculations from WebWork:

$$\mathbf{r}'(t) = (t-2)^2(4t-2)\mathbf{i} + (t-2)(3t-2)\mathbf{j} = \mathbf{0}$$

1M

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$$\Rightarrow t \in \{2, \tfrac{1}{2}\} \text{ and} t \in \{2, \tfrac{2}{3}\}$$

$$\Rightarrow t = 2$$

The function $y'(t) = (t-2)(3t-2)$ changes sign at $t = 2$

1M

Check for sign change

so there is a cusp when $t = 2$

1A

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at position $\mathbf{r}(2) = \mathbf{0}$.

1A

(b) In WebWork Problem 3, you calculated $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ for each t in the set

$$T = \{0, \tfrac{1}{2}, 1, 3\}.$$

For each $t \in T$, calculate the unit vector $\widehat{\mathbf{r}'(t)}$ in the direction of $\mathbf{r}'(t)$.

Solution:

[Not marked]

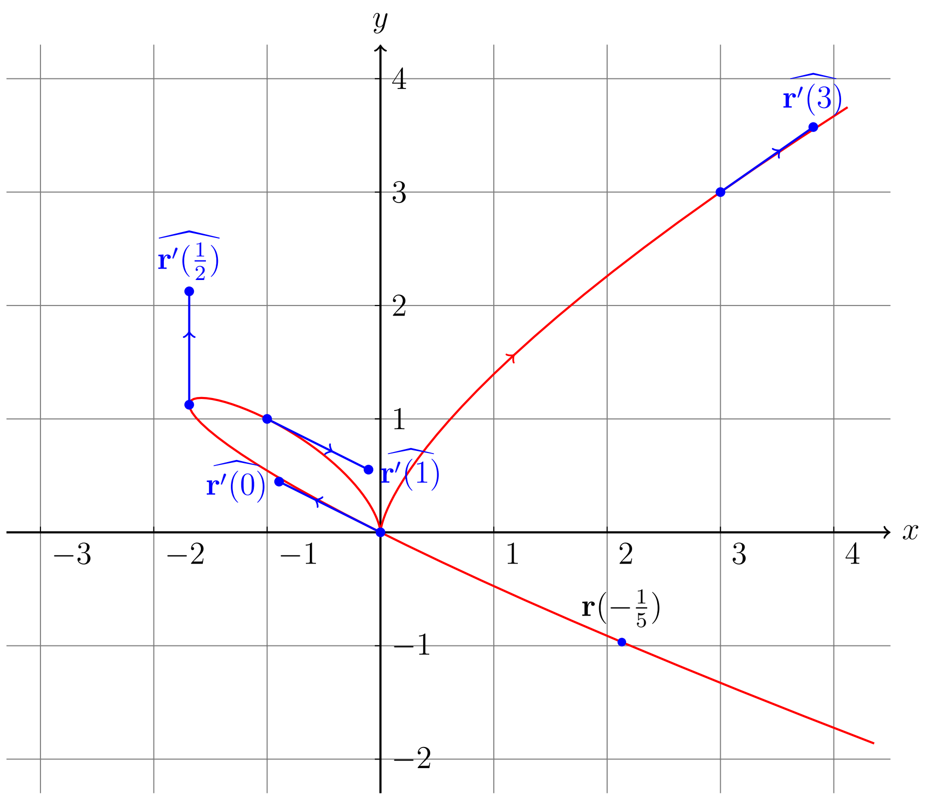
$$\mathbf{r}'(0) = -8\mathbf{i} + 4\mathbf{j} \Rightarrow \widehat{\mathbf{r}'(0)} = \frac{1}{4\sqrt{5}}\mathbf{r}'(0) = \frac{1}{\sqrt{5}}(-2\mathbf{i} + \mathbf{j})$$

$$\mathbf{r}'(\tfrac{1}{2}) = \tfrac{3}{4}\mathbf{j} \Rightarrow \widehat{\mathbf{r}'(\tfrac{1}{2})} = \mathbf{j}$$

$$\mathbf{r}'(1) = 2\mathbf{i} - \mathbf{j} \Rightarrow \widehat{\mathbf{r}'(1)} = \frac{1}{\sqrt{52}}\mathbf{r}'(1) = \frac{1}{\sqrt{5}}(2\mathbf{i} - \mathbf{j})$$

$$\mathbf{r}'(3) = 10\mathbf{i} + 7\mathbf{j} \Rightarrow \widehat{\mathbf{r}'(3)} = \frac{1}{\sqrt{149}}\mathbf{r}'(3) = \frac{1}{\sqrt{149}}(10\mathbf{i} + 7\mathbf{j})$$

(c) For each $t \in T$, plot the unit vector in the direction of $\widehat{\mathbf{r}'(t)}$ on the following diagram. **In each case, base this unit vector at the point** $\mathbf{r}(t)$, not at the origin. To plot the unit vectors accurately your calculator to find decimal approximations.



1A

Plotting unit vectors

(d) Using the vectors you plotted on the above diagram and your answer to (a), to plot the path range($\mathbf{r}$) indicating the direction of motion with arrows. It may help to plot the point $\mathbf{r}(-\frac{1}{5})$ on the diagram.

1A

Plausible shape.

(e) Prove that the path of $\mathbf{r}$ (also called range($\mathbf{r}$)) is a subset of the Cartesian curve

$$C = \{(x, y) \in \mathbb{R}^2 \mid y^4 = x^3 + 2x^2 y\}.$$

As always for a subset proof, you should to start with:

"Let $(x, y) \in \text{range}(\mathbf{r})$. This means ..."

and work towards showing that $(x, y) \in C$. This means you need to show that

$$y^4 = x^3 + 2x^2 y$$

Solution:

Let $(x, y) \in \text{range}(\mathbf{r})$.

1M

State hypothesis

This means $(x, y) = \mathbf{r}(t) = t(t-2)^3\mathbf{i} + t(t-2)^2\mathbf{j}$ for some $t \in \mathbb{R}$, so $x = t(t-2)^3$ and $y = t(t-2)^2$. Hence

$$\begin{aligned} y^4 &= t^4(t-2)^8 \\ x^3 + 2x^2 y &= t^3(t-2)^9 + 2(t^2(t-2)^6)(t(t-2)^2) \\ &= t^3(t-2)^9 + 2t^3(t-2)^8 \\ &= t^3(t-2)^8(t-2+2) \\ &= t^4(t-2)^8 \end{aligned}$$

Hence $(x, y) \in C$.

2M

Overall proof

1L

Whole written part: clear structure, and ALL mathematical notation is correct.

Assignment Information

This assignment is worth $\frac{20}{9}$% of your final MAST10005 mark.

Full working should be shown in your solutions to Question 2. There will be 1 mark overall for correct mathematical notation.
Full solutions to the assignment will be uploaded to the LMS site approximately 3 days after the assignment is due.