THE UNIVERSITY OF MELBOURNE

Student number 1079860

Semester 2 Special Assessment, 2021

School of Mathematics and Statistics

MAST20005 Statistics

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 23 pages (including this page) with 7 questions and 80 total marks

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier, blank loose-leaf paper and one of the following calculators: Casio FX-82 (any suffix), Casio FX-95 (any suffix), Casio FX-570 (any suffix).
- One double sided A4 page of notes (handwritten or printed).
- No headphones or earphones are permitted.

Instructions to Students

- Wave your hand right in front of your webcam if you wish to communicate with the supervisor at any time (before, during or after the exam).
- You must not be out of webcam view at any time without supervisor permission.
- You must not write your answers on an iPad or other electronic device.
- Off-line PDF readers (i) must have the screen visible in Zoom; (ii) must only be used to read exam questions (do not access other software or files); (iii) must be set in flight mode or have both internet and Bluetooth disabled as soon as the exam paper is downloaded.

Writing

- You should attempt all questions.
- Full reasoning must be shown and penalties may be imposed for poorly presented, unclear, untidy and badly worded solutions.
- If you are writing answers on the exam or masked exam and need more space, use blank paper. Note this in the answer box, so the marker knows.
- If you are only writing on blank A4 paper, the first page must contain only your student number, subject code and subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.

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Question 1 (9 marks)

The following regression model was fitted: $Y_i = \alpha + \beta x_i + e_i$, where $e_i \sim \mathrm{N}(0, \sigma^2)$, for $i = 1, \dots, 10$. Some R output from this analysis is shown below (with some of the details removed).

```
> summary(y)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  2.453   5.321   5.943   6.371   7.049  10.517
> sd(y)
[1] 2.27713
> sort(y)
 2.453  4.947  5.181  5.742  5.867  6.019  6.304  7.297  9.380  10.516525
> cor(y, x)
[1] 0.975
> m <- lm(y ~ x)
> summary(m)

Call:
lm(formula = y ~ x)

Residuals:
     Min       1Q   Median       3Q      Max
-0.79982 -0.38653 -0.01469  0.29758  0.84883

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.008833   0.544065  -0.016    0.987
x            1.551464   0.125646  12.348 1.72e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.5393 on 8 degrees of freedom
Multiple R-squared: (??),   Adjusted R-squared: (??)
F-statistic: 152.5 on 1 and 8 DF,  p-value: 1.724e-06
```

(a) For each of the following quantities, state or calculate its value if possible, or otherwise explain why it is not possible.

(i) $y_{(8.5)}$

(ii) β

(iii) $\sum_i (y_i - \bar{y})^2$

(iv) r^2

i) $y_{(8.5)} = \frac{y_{(8)} + y_{(9)}}{2} = \frac{7.297 + 9.380}{2} = 8.3385$

ii) $\hat{\beta} = 1.551464$ but we cannot calculate β exactly

iii) $\sum_i (y_i - \bar{y})^2 = 46.6696$

iv) $r^2 = 0.450625$

(b) For each of the following hypotheses, carry out the test if possible, using a 5% significance level, or explain what further information you need in order to carry out the test.

(i) $H_0: \beta = 0$ versus $H_1: \beta \neq 0$

(ii) $H_0: \beta = 1.5$ versus $H_1: \beta \neq 1.5$

i) $\frac{\hat{\beta}}{se(\hat{\beta})} \sim t_8$

0.975 quantile from t_8

$\frac{1.551464}{0.125646} = 12.3479 > 2.306$

$\Rightarrow$ reject H_0

ii) $\hat{\beta} \pm c\, se(\hat{\beta})$

0.975 quantile from t_8

$1.551464 \pm 2.306 \times 0.125646$

$(1.2617, 1.8412)$

$\Rightarrow$ do not reject H_0

(c) Calculate a 90% confidence interval for β.

$\hat{\beta} \pm c\ \mathrm{se}(\hat{\beta})$

↳ 0.95 quantile from t_8

$1.551464 \pm 1.860 \times 0.125646$

$(1.3178,\ 1.7852)$

Question 2 (8 marks)

According to the WorldAtlas website, about 75% of the world's population have brown eyes. The next most common eye color is blue, at 10%. The remaining 15% of the population have hazel, amber, grey, green or even red-violet colors.
A survey was conducted at a university among the 1st year students. A total of 100 students responded to this survey. The following table gives a summary of the responses:

	Brown	Blue	Other
Number of students	64	17	19

(a) Test whether the survey responses are consistent with the proportions given by the WorldAtlas website. Use a 1% significance level.

$$Q_{k-1} = \sum_{i=1}^{3} \frac{(O_i - E_i)^2}{E_i} = \frac{(64-75)^2}{75} + \frac{(17-10)^2}{10} + \frac{(19-15)^2}{15}$$

$= 7.58 < 9.21034 \rightarrow 0.99$ quantile from χ^2_2

$\Rightarrow$ not enough evidence to reject the null hypothesis that the survey responses are consistent with the World Atlas proportions at 1% significance level

H_0: survey responses consistent with World Atlas

H_1: survey responses not consistent with WorldAtlas

(b) Among the students that took the survey, five students indicated that they have green eyes. Calculate a 90% confidence interval for the proportion of green-eyed people in the population.

$\hat{p} = \frac{5}{100} = 0.05$

$\frac{\hat{p}-p}{\sqrt{\hat{p}(1-\hat{p})/n}} \approx N(0,1)$

$c \to$ 0.95 quantile from standard normal

$\Rightarrow$ 90% CI is $\hat{p} \pm c\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$0.05 \pm 1.645\sqrt{\frac{0.05(0.95)}{100}}$

$(0.0141; 0.0859)$

(c) We wish to estimate the proportion of grey-eyed people in the world, p. In our preliminary survey at the university, four students indicated that they have grey eyes. How large a sample do we need if we want to estimate p with a margin of error at most 0.01, using a 95% confidence level?

$\hat{p} = \frac{4}{100} = 0.04$

$c \to$ 0.975 quantile from standard normal

$n = \frac{c^2\hat{p}(1-\hat{p})}{\epsilon^2}$

$n = \frac{1.96^2 * 0.04(0.96)}{0.01^2}$

≈ 1476

or worst case scenario $\hat{p} = 0.5$

$\Rightarrow n = 9604$

Question 3 (12 marks)

A random sample from an exponential distribution with rate parameter λ (see details in the appendix) is given as follows.

0.19 0.34 0.47 0.78 0.98 1.19 2.32 2.40 5.16 5.92

(a) Let X_i be the i-th observation. Show that the maximum likelihood estimator of λ is the reciprocal of the sample mean, $1/\bar{X}$.

$$f(x) = \lambda e^{-\lambda x}$$

$$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda x}$$

$$= \lambda^n e^{-\lambda \sum^n x_i}$$

$$l(\lambda) = n\ln(\lambda) - \lambda \sum x_i$$

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} - \sum x_i$$

$$\frac{\partial l}{\partial \lambda} = 0 \Rightarrow \frac{n}{\lambda} - \sum x_i = 0$$

$$\frac{n}{\lambda} = \sum x_i$$

$$\frac{1}{\lambda} = \frac{\sum x_i}{n}$$

$$\hat{\lambda} = \frac{n}{\sum x_i} = \frac{1}{\bar{x}}$$

(b) Using the sample, give an estimate of λ.

$$\hat{\lambda} = \frac{10}{0.19+0.34+0.47+0.78+0.98+1.19+2.32+2.40+5.16+5.92}$$

$$= 0.5063$$

(c) Derive the 30th percentile, $\pi_{0.3}$, as a function of λ.

$$\pi_{0.3} = F^{-1}(0.3)$$

$$F^{-1}(x) = \int_0^x \lambda e^{-\lambda x} \, dx$$

$$= \lambda \left[-\frac{1}{\lambda} e^{-\lambda x} \right]_0^x$$

$$= \lambda \left[-\frac{1}{\lambda} e^{-\lambda x} + \frac{1}{\lambda} e^{0} \right]$$

$$= 1 - e^{-\lambda x}$$

$$F^{-1}(0.3) = 1 - e^{-0.3\lambda} = \pi_{0.3}$$

(d) Calculate a 95% confidence interval for $\pi_{0.3}$. Use a method that incorporates your previous estimate of λ, rather than a completely distribution-free method.

$\hat{\lambda} = 0.5063$ $\alpha = 0.05$

$$\hat{\pi}_{0.3} \approx N\left(\pi_{0.3}, \frac{0.3(0.7)}{n f(\pi_{0.3})^2}\right)$$

95% CI:

0.975 quantile from standard normal

$$1 - e^{-0.3 \times 0.5063} \pm c \sqrt{\frac{0.3 \times 0.7}{10 \times (0.5063 \times e^{-0.5063 \times \hat{\pi}_{0.3}})^2}}$$

$(0.4616, 0.7434)$

Question 4 (14 marks)

Let $X_1, \ldots, X_n$ be a random sample from the following discrete distribution.

x	0	1	2
$p(x)$	$1-3\theta$	2θ	θ

(a) Determine a sufficient statistic for θ.

$L(\theta) = \prod_{i=1}^{n} p(x|\theta)$ $\qquad F_k = \sum I(x_i = k)$

$= (1-3\theta)^{F_0} (2\theta)^{F_1} (\theta)^{F_2} \times 1$

$\Rightarrow \sum_{i=1}^{n} I(x_i = k)$ is a sufficient statistic for θ

(b) Find the maximum likelihood estimator (MLE) of θ.

$l(\theta) = F_0 \ln(1-3\theta) + F_1 \ln(2\theta) + F_2 \ln(\theta)$

$\frac{\partial l}{\partial \theta} = \frac{-3F_0}{1-3\theta} + \frac{F_1}{2\theta} + \frac{F_2}{\theta}$

$\frac{\partial l}{\partial \theta} = 0 \Rightarrow \frac{-3F_0}{1-3\theta} + \frac{F_1}{2\theta} + \frac{F_2}{\theta} = 0$

$\frac{-3F_0 \times 2\theta^2}{(1-3\theta)2\theta^2} + \frac{F_1(1-3\theta)\theta}{(1-3\theta)2\theta^2} + \frac{F_2(1-3\theta)2\theta}{(1-3\theta)2\theta^2} = 0$

$\frac{-3F_0 \times 2\theta^2 + \theta F_1 - 3\theta^2 F_1 + 2\theta F_2 - 6\theta^2 F_2}{(1-3\theta)2\theta^2} = 0$

(c) Is the MLE unbiased?

Yes the $E(\hat{\theta}) = \theta$ and so the

MLE is always unbiased

(d) Find the Cramér–Rao lower bound for unbiased estimator of θ.

$$C.R.L.B = \frac{1}{I(\theta)} = \frac{1}{E\left(-\frac{\partial^2 l}{\partial \theta}\right)}$$

(e) Does the MLE achieve the Cramér–Rao lower bound?

The MLE achieves the Cramér – Rao lower bound as $n \to \infty$ $(\mathrm{Var}(\hat{\theta}) \to$ C.R.L.B$)$

Question 5 (7 marks)

Martina sets up an exercise to teach her students about Bayesian inference. Inside a box, she places 1 white ball, 1 yellow ball and 1 blue ball. She then opens a new packet of 7 balls, all of which are the same colour, and places them inside the box as well. She didn't pay attention to what colour they were, but she knows they are all either white, yellow or blue.
Martina asks one of her students to select a ball at random. It is white. The student returns the ball into the box, and Martina mixes the balls thoroughly. She asks another student to select a ball at random, and this time it is blue.
Based on these observations, Martina asks her students to infer the colour of the balls that came from the new packet.

(a) Without knowing anything further, one might think the possible colours for the balls in the new packet are equally plausible. Write down a prior that represents this assumption.

$f(\theta) = \frac{1}{3}$

$\theta \sim U(0,3)$

θ	white	yellow	blue
$f(\theta)$	$1/3$	$1/3$	$1/3$

(b) Using this prior, calculate the posterior probability that the balls in the packet were yellow.

$$f(\theta|x=0) = \frac{f(x=0|\theta)\frac{1}{3}}{f(x=0|\theta=\text{yellow})\cdot\frac{1}{3} + f(x=0|\theta=\text{white})\cdot\frac{1}{3} + f(x=0|\theta=\text{blue})\cdot\frac{1}{3}}$$

$X \sim Bi(2, 4/5)$

y number of yellow balls picked by student

=

(c) Before Martina's class, Howard peeked inside Martina's locker and noticed there were 8 packets of new balls, with exactly one of these packets having yellow balls. Howard believes that Martina selected one of these packets at random for setting up her box. Using an appropriate prior, what is Howard's posterior probability that the balls in the opened packet were yellow?

$$f(\theta)\begin{cases} 1/8 & \theta = \text{yellow} \\ 7/16 & \theta = \text{white} \\ 7/16 & \theta = \text{blue} \end{cases}$$

$$f(\theta = \text{yellow} \mid x = 0) = \frac{f(x=0 \mid \theta=\text{yellow}) \times \frac{1}{8}}{f(x=0 \mid \theta=\text{yellow}) \times \frac{1}{8} + f(x=0 \mid \theta=\text{blue}) \times 7/16 + f(x=0 \mid \theta=\text{white}) \times 7/16}$$

$$= \frac{\left(\frac{2}{10}\right)^2 \times \frac{1}{8}}{\left(\frac{2}{10}\right)^2 \times \frac{1}{8} + \left(\frac{9}{10}\right)^2 \times \frac{7}{16} + \left(\frac{9}{10}\right)^2 \times \frac{7}{16}}$$

$$=$$

Question 6 (18 marks)

P. Cortez and A. Silva examined students' achievement in secondary education in two Portuguese schools during 2006. Using the following R code (with some of the details removed), the authors tried to examine the relationship between the students' final grade (G3) and their first grade (G1). Each grade is an integer between 0 and 20, inclusive.

```
> reg <- lm(formula = G3 ~ G1 , data = grades)
> summary(reg)

Call:
lm(formula = G3 ~ G1, data = grades)

Coefficients:
             Estimate   Std. Error t value Pr(>|t|)
(Intercept) -1.85100
G1           1.12680
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.784 on 380 degrees of freedom
Multiple R-squared:  0.6482,    Adjusted R-squared:  0.6473
F-statistic: 700.3 on 1 and 380 DF,  p-value: < 2.2e-16
```

In addition, we have the following statistics of the students' first grades (G1): the sample mean was 10.86, and the sample standard deviation was 3.349.

(a) Write the ANOVA table for this regression model.
(Hint: the F statistic is given in the R output.)

		df	SS	MS	F
1	Model	1	5427.8152	5427.8152	
n-2	Error	380	2945.266	7.7507	
n-1	Total	381	8373.1172	5435.5659	700.3

(b) Dana's first grade was 15. Calculate a 95% prediction interval for her final grade.

$\bar{x} = 10.86$

$\hat{\beta} = 1.1268$ $\hat{\alpha} = -1.851$ $\hat{\sigma} = 2.784$ $s_x = 3.349$

95% PI: $\hat{\mu}(x^*) \pm \underbrace{c}_{\text{0.975 quantile from } t_{n-2}} \hat{\sigma} \sqrt{1 + \frac{1}{n} + \frac{(x^* - \bar{x})^2}{\underbrace{k}_{(n-1)s_x^2}}}$

$\underbrace{\hat{\mu}(15)}_{-1.851 + 1.1268 \times 15} \pm 1.966 \times 2.784 \sqrt{1 + \frac{1}{382} + \frac{(15 - 10.86)^2}{381 \times 3.349^2}}$

$= (9.5595, 20.5425)$

(c) Cortez & Silva used a simple linear regression model in their analysis. The slope parameter, β, can be estimated by least squares estimation or maximum likelihood estimation. Show that the estimators obtained from these two methods are identical. (Hint: you do not need to provide full mathematical expressions for these estimators, you simply need to prove they are equal.)

$$\hat{\beta} \sim N\left(\beta, \frac{\sigma^2}{K}\right)$$

$$\hat{\beta} = \frac{\sum_{i=1}^{\hat{}} (x_i - \bar{x}) y_i}{\sum_{i=1}^{\hat{}} (x_i - \bar{x})^2}$$

$$= \frac{\sum_{i=1}^{\hat{}} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{\hat{}} (x_i - \bar{x})^2}$$

$$= \frac{\sum_{i=1}^{\hat{}} (x_i - \bar{x})(y_i - \bar{y})}{K}$$

(d) Derive the variance of $\hat{\beta} = \frac{1}{K}\sum_{i=1}^{n}(x_i - \bar{x})Y_i$, where $K = \sum_{i=1}^{n}(x_i - \bar{x})^2$.

(e) Calculate a 90% confidence interval for β.

$$\frac{\hat{\beta} - \beta}{\hat{\sigma}/\sqrt{K}} \sim t_{n-2}$$

90% CI: $\hat{\beta} \pm c \frac{\hat{\sigma}}{\sqrt{K}}$

$c \to$ 0.95 quantile from t_{380}

$$1.1268 \pm 1.649 \frac{2.784}{\sqrt{381 \times 3.349^2}}$$

$(1.0566, 1.1970)$

Question 7 (12 marks)

In the first week of semester we did a 'fun quiz' to predict how long the border between NSW and Victoria will be closed. The responses to this were saved in a data frame that contains two columns: `subject` (either 'MAST20005' or 'MAST90058' for each student, depending on which subject they were enrolled in) and `answer` (the number of days predicted by each student). The data was analysed by Julia using R. Some output from this (with some of the details removed) is shown below.

```
> table(quiz$subject)
    x          y
MAST20005 MAST90058
       99        17
> mast20005 <- quiz$answer[quiz$subject == "MAST20005"]
> mast90058 <- quiz$answer[quiz$subject == "MAST90058"]
> summary(mast20005)                  x̄
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  10.00   39.00   60.00   67.79   85.00  243.00
> summary(mast90058)                  ȳ
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   (??)   43.00   63.00   63.29   75.00  145.00
> sd(mast20005)
[1] 42.01998
> sd(mast90058)
[1] 31.30049
```

(a) Assume the data are a random sample from each group of students. Calculate a 90% confidence interval for the difference in the two population means, $\mu_{\text{MAST20005}} - \mu_{\text{MAST90058}}$, you may assume the responses from each group follow a normal distribution, with equal variances. Is it plausible that the two groups have similar means?

$$\frac{\bar{X} - \bar{Y} - (\mu_x - \mu_y)}{s_p\sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t_{n+m-2}$$

$\bar{x} = 67.79$

$\bar{y} = 63.29$

$s_x = 42.01998$

$s_y = 31.30049$

$n = 99 \quad m = 17$

90% CI:

$$\bar{x} - \bar{y} \pm \overset{\text{0.95 quantile from } t_{114}}{C} \sqrt{\frac{98 s_x^2 + 16 s_y^2}{114}\left(\frac{1}{99} + \frac{1}{17}\right)}$$

$(-13.2100, 22.2100)$

$\Rightarrow$ It is plausible that the two groups have similar means

(b) Julia believes that the gamma distribution Gamma($\alpha = 3, \beta = 0.04$) is a good approximation to the data in the **answer** column for the MAST90058 group. Find the probability that the sample minimum is larger than 10, i.e. $\Pr(X_{(1)} > 10)$.
Note: α and β are the shape and the rate parameters, respectively.

$$Pr(X_{(1)} > x) = (1 - F(x))^n$$

$$Pr(X_{(1)} > 10) = (1 - Pr(X \leq 10))^{17}$$
$$= (1 - 0.007926)^{17}$$
$$= 0.8735$$

(c) Damjan believes that a log-normal distribution is a good approximation to the data in the **answer** column for the MAST90058 group. Derive the method of moments (MM) estimators for μ and σ^2, and then give the MM estimates using the R output given at the beginning of the question.

$E(Y) = e^{\mu + \frac{\sigma^2}{2}}$ $\quad var(Y) = (e^{\sigma^2} - 1)\, e^{2\mu + \sigma^2}$

$\bar{Y} = e^{\mu + \sigma^2/2}$

$\ln(\bar{Y}) = \mu + \frac{\sigma^2}{2}$

$\hat{\mu} = \ln(\bar{Y}) - \frac{\sigma^2}{2}$

$\hat{\mu} = \ln(63.29) - \frac{31.30049^2}{2}$

$= -485.7126$

$s^2 = (e^{\sigma^2} - 1)\, e^{2\mu + \sigma^2}$

$s^2 = (e^{\sigma^2} - 1)\, e^{2\mu} e^{\sigma^2}$

$\frac{s^2}{e^{2\mu}} = e^{2\sigma^2} - e^{\sigma^2}$

End of Exam — Total Available Marks = 80

Turn the page for appended material

Appendix

Distributions

- The pdf of $X \sim \mathrm{N}(\mu, \sigma^2)$ is
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}.$$
- The pdf of $X \sim \text{log-normal}(\mu, \sigma^2)$ is
$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left\{-\frac{(\ln x-\mu)^2}{2\sigma^2}\right\}, \quad (x \geqslant 0),$$
and it has $\mathbb{E}(X) = \exp\left(\mu + \frac{\sigma^2}{2}\right)$ and $\mathrm{var}(X) = [\exp(\sigma^2) - 1]\exp(2\mu + \sigma^2)$.
- The pdf of $X \sim \mathrm{Exp}(\lambda)$ is
$$f(x) = \lambda e^{-\lambda x}, \quad (x \geqslant 0),$$
and it has $\mathbb{E}(X) = 1/\lambda$ and $\mathrm{var}(X) = 1/\lambda^2$.
- The pdf of $X \sim \mathrm{Gamma}(\alpha, \beta)$ is
$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-x\beta}, \quad (x \geqslant 0),$$
and it has $\mathbb{E}(X) = \alpha/\beta$ and $\mathrm{var}(X) = \alpha/\beta^2$.
- The pdf of $X \sim \mathrm{Beta}(\alpha, \beta)$ is
$$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \quad (0 \leqslant x \leqslant 1),$$
and it has $\mathbb{E}(X) = \alpha/(\alpha+\beta)$ and $\mathrm{var}(X) = \alpha\beta/\{(\alpha+\beta)^2(\alpha+\beta+1)\}$.

R output

```
> p1 <- c(0.01, 0.025, 0.05, 0.1, 0.9, 0.95, 0.975, 0.99)

> qnorm(p1)
[1] -2.326 -1.960 -1.645 -1.282  1.282  1.645  1.960  2.326

> qt(p1, df = 8)
[1] -2.896 -2.306 -1.860 -1.397  1.397  1.860  2.306  2.896
> qt(p1, df = 9)
[1] -2.821 -2.262 -1.833 -1.383  1.383  1.833  2.262  2.821
> qt(p1, df = 10)
[1] -2.764 -2.228 -1.812 -1.372  1.372  1.812  2.228  2.764
> qt(p1, df = 114)
[1] -2.360 -1.981 -1.658 -1.289  1.289  1.658  1.981  2.360
> qt(p1, df = 115)
[1] -2.359 -1.981 -1.658 -1.289  1.289  1.658  1.981  2.359
> qt(p1, df = 116)
[1] -2.359 -1.981 -1.658 -1.289  1.289  1.658  1.981  2.359
> qt(p1, df = 380)
[1] -2.336 -1.966 -1.649 -1.284  1.284  1.649  1.966  2.336
> qt(p1, df = 381)
[1] -2.336 -1.966 -1.649 -1.284  1.284  1.649  1.966  2.336
> qt(p1, df = 382)
[1] -2.336 -1.966 -1.649 -1.284  1.284  1.649  1.966  2.336
```

```
> qchisq(p1, df = 1)
[1] 0.0001571 0.0009821 0.0039321 0.0157908 2.7055435 3.8414588 5.0238862 6.6348966
> qchisq(p1, df = 2)
[1] 0.02010 0.05064 0.10259 0.21072 4.60517 5.99146 7.37776 9.21034
> qchisq(p1, df = 3)
[1]  0.1148  0.2158  0.3518  0.5844  6.2514  7.8147  9.3484 11.3449
> qchisq(p1, df = 4)
[1]  0.2971  0.4844  0.7107  1.0636  7.7794  9.4877 11.1433 13.2767
> qchisq(p1, df = 5)
[1]  0.5543  0.8312  1.1455  1.6103  9.2364 11.0705 12.8325 15.0863
> qchisq(p1, df = 6)
[1]  0.8721  1.2373  1.6354  2.2041 10.6446 12.5916 14.4494 16.8119

> qf(p1, 1, 380)
[1] 0.0001573 0.0009834 0.0039373 0.0158119 2.7187835 3.8660462 5.0639382 6.7020355
> qf(p1, 98, 16)
[1] 0.4568 0.5149 0.5722 0.6480 1.7578 2.0696 2.3977 2.8649
> qf(p1, 98, 17)
[1] 0.4640 0.5219 0.5787 0.6537 1.7263 2.0216 2.3301 2.7661
> qf(p1, 99, 16)
[1] 0.4572 0.5153 0.5725 0.6483 1.7574 2.0690 2.3969 2.8638
> qf(p1, 99, 17)
[1] 0.4645 0.5223 0.5791 0.6540 1.7259 2.0210 2.3293 2.7650

> qgamma(p1, shape = 3, rate = 0.04)
[1]  10.90  15.47  20.44  27.55 133.06 157.39 180.62 210.15

> pgamma(c(9, 10, 11), shape = 3, rate = 0.04)
[1] 0.005951 0.007926 0.010245
```