Assignment 1: Solutions and marking scheme

1. [2 marks for submitting, 3 marks for peer reviews]

   (a) Since a product of two real numbers is positive precisely when either both are positive or both are negative:

   $$\begin{aligned} x^2 - x - 6 = (x-3)(x+2) > 0 \\ &\Rightarrow (x-3>0 \text{ and } x+2>0) \text{ or } (x-3<0 \text{ and } x+2<0) \\ &\Rightarrow (x>3 \text{ and } x>-2) \text{ or } (x<3 \text{ and } x<-2) \\ &\Rightarrow x>3 \text{ or } x<-2 \end{aligned}$$

   Hence $A = (-\infty, -2) \cup (3, \infty)$.

   (b) Since there are only three possibilities for the sign of $x^2 - x - 6$:

   $$\begin{aligned} x^2 - x - 6 < 0 &\Rightarrow x \notin A \text{ and } x^2 - x - 6 \neq 0 \\ &\Rightarrow x \in [-2, 3] \text{ and } x \notin \{-2, 3\} \\ &\Rightarrow x \in (-2, 3) \end{aligned}$$

2. (a) $\boxed{1A}$ *Correct example for one of $S \setminus C$ and $C \setminus S$.*

   $\frac{3}{5} \in P,\ \frac{\pi}{6} \in S \setminus C,\ \frac{\pi}{3} \in C \setminus S,\ 0 \in C \cap S.$

   (b) $\boxed{1A}$ *Any valid counterexample.* Observe that $0 \in S$ since $\sin(0) = 0 \in \mathbb{Q}$. Suppose $0 \in T$. This would give $\sin(0) = \frac{a}{c} \in P$, where $a, c \in \mathbb{N}$. But this implies $a = 0$ contradicting the fact that $a \in \mathbb{N}$. We conclude that $S \not\subseteq T$.

   Many counterexamples are possible. Proof is slightly harder for non-zero examples.

   (c) Let $x \in T$. This means $\sin(x) \in P$, so $\sin(x) = \frac{a}{c}$ where $a, c \in \mathbb{N}$ and $a^2 + b^2 = c^2$ for some $b \in \mathbb{N}$. $\boxed{1M}$ *Set up proof hypothesis.*

   Now

   $$\cos^2(x) = 1 - \sin^2(x) = 1 - \frac{a^2}{c^2} = \frac{c^2 - a^2}{c^2} = \frac{b^2}{c^2} \Rightarrow \cos(x) = \pm\frac{b}{c}$$

   This shows $x \in C$, since $-\frac{b}{c}, \frac{b}{c} \in \mathbb{Q}$. $\boxed{2M}$

3. Let $x \in D$. There are two cases to consider.

   $\boxed{1M}$ *Two cases for denominator (or two cases for numerator and denominator if 2 is moved to LHS).*

   $\boxed{2M}$ *Overall proof argument*

Case 1: If $\mathbf{x^2 - x - 6 > 0}$, Q1(a) gives $x \in (-\infty, -2) \cup (3, \infty)$ and:

$$\begin{aligned} \frac{2x^2}{x^2 - x - 6} > 2 &\Rightarrow 2x^2 > 2x^2 - 2x - 12 \\ &\Rightarrow 2x > -12 \Rightarrow x > -6 \\ &\Rightarrow x \in (-6, \infty) \end{aligned}$$

so in this case we have $x \in [(-\infty, -2) \cup (3, \infty)] \cap (-6, \infty) = (-6, -2) \cup (3, \infty)$.

Case 2: If $\mathbf{x^2 - x - 6 < 0}$, Q1(b) gives $x \in (-2, 3)$ and:

$$\begin{aligned}\frac{2x^2}{x^2 - x - 6} < 2 &\Rightarrow 2x^2 < 2x^2 - 2x - 12 \\ &\Rightarrow 2x < -12 \Rightarrow x < -6 \\ &\Rightarrow x \in (-\infty, -6)\end{aligned}$$

so in this case we have $x \in (-2, 3) \cap (-\infty, -6) = \varnothing$.

Combining these answers gives $D = (-6, -2) \cup (3, \infty)$. $\boxed{\textbf{1A}}$

$\boxed{\textbf{1L}}$ *For the whole assignment: ALL mathematical notation is correct.*