Limit Laws

Let f and g be real-valued functions and let $c \in \mathbb{R}$ be a constant. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then

1. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
2. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$.
3. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$.
4. $\lim_{x\to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided $\lim_{x\to a} g(x) \neq 0$.
5. $\lim_{x\to a} c = c$.
6. $\lim_{x\to a} x = a$.

combining functions

Continuity Theorem 1:

If the functions f and g are continuous at $x = a$, then the following functions are continuous at $x = a$:

1. $f + g$,
2. cf,
3. fg,
4. $\frac{f}{g}$ if $g(a) \neq 0$.

Continuity Theorem 2:

If f is continuous at $x = a$ and g is continuous at $x = f(a)$, then $g \circ f$ is continuous at $x = a$. *

$\sin(x^2)$ *

Continuity Theorem 3:

The following function types are continuous at every point in their domains:

- polynomials *
- trigonometric functions: $\sin x$, $\cos x$, $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$, $\arcsin x$, $\arccos x$, $\arctan x$ *
- exponential functions: a^x for $a > 0$
- logarithm functions: $\log_a x$ for $a > 0$
- nth root functions: $\sqrt[n]{x}$ for $n \in \{2, 3, 4, \ldots\}$
- hyperbolic functions: $\sinh x$, $\cosh x$, $\tanh x$, $\operatorname{sech} x$, $\operatorname{cosech} x$, $\coth x$, $\operatorname{arcsinh} x$, $\operatorname{arccosh} x$, $\operatorname{arctanh} x$

Divergence Test

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=1}^{\infty} a_n$ diverges.

Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive term series.

1. If $a_n \le b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $a_n \ge b_n$ for all n and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Compare with:

harmonic p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ ↗ converges $p > 1$ ↘ diverges $p \le 1$

geometric series $\sum_{n=1}^{\infty} r^n$ ↗ converges $|r| < 1$ ↘ diverges $|r| \ge 1$

Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series and

$$L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

1. If $L < 1$, $\sum_{n=1}^{\infty} a_n$ converges.
2. If $L > 1$, $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, the ratio test is inconclusive.

Useful for factorials / exponentials

Example 3.4: Express $\sin^5\theta$ in terms of the functions $\sin(n\theta)$ for integers n.

Solution:

$$\sin^5\theta = \left(\tfrac{1}{2i}(e^{i\theta} - e^{-i\theta})\right)^5$$
$$= \frac{1}{32i^5}\left[e^{5i\theta} - 5\boxed{e^{3i\theta}} + 10e^{i\theta} - 10e^{-i\theta} + 5e^{-3i\theta} - e^{-5i\theta}\right]$$

$(e^{i\theta})^4 \cdot (e^{-i\theta})^1 = e^{4i\theta}e^{-i\theta} = e^{3i\theta}$

$$= \frac{1}{32i}\left[(e^{5i\theta} - e^{-5i\theta}) - 5(e^{3i\theta} - e^{-3i\theta}) + 10(e^{i\theta} - e^{-i\theta})\right]$$
$$= \frac{1}{16}\left[\frac{1}{2i}(e^{5i\theta} - e^{-5i\theta}) - \frac{5}{2i}(e^{3i\theta} - e^{-3i\theta}) + \frac{10}{2i}(e^{i\theta} - e^{-i\theta})\right]$$
$$= \frac{1}{16}\left(\sin(5\theta) - 5\sin(3\theta) + 10\sin\theta\right)$$

Click for video. 135

De Moivre's Theorem:

If $z = re^{i\theta}$ and n is a positive integer then

$$\boxed{z^n = \left(re^{i\theta}\right)^n = r^n e^{in\theta}}$$

for $\theta \in \mathbb{R}$.

We define the complex exponential using Euler's formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Example 3.5: Find $\frac{d^{56}}{dt^{56}}\left(e^{-t}\cos t\right)$. $t \in \mathbb{R}$

Solution:

$e^{-t}\cos t = e^{-t}\,\mathrm{Re}\,(e^{it})$
$= \mathrm{Re}\,(e^{-t}e^{it})$, since $e^{-t} \in \mathbb{R}$
$= \mathrm{Re}\,(e^{(-1+i)t})$, $-t + it = (-1+i)t$

$\frac{d^{56}}{dt^{56}}(e^{-t}\cos t) = \frac{d^{56}}{dt^{56}}\mathrm{Re}\,(e^{(-1+i)t})$
$= \mathrm{Re}\left[\frac{d^{56}}{dt^{56}}(e^{(-1+i)t})\right]$
$= \mathrm{Re}\left[(-1+i)^{56}e^{(-1+i)t}\right]$, $\frac{d}{dt}(e^{kt}) = ke^{kt}$

Click for video. 138 / 401

Since $(-1+i)^{56} = \left(\sqrt{2}\,e^{\frac{3\pi}{4}i}\right)^{56}$
$= (\sqrt{2})^{56} e^{\frac{168\pi i}{4}}$
$= 2^{\frac{56}{2}} e^{42\pi i}$
$= 2^{28}(\cos(42\pi) + i\sin(42\pi))$
$= 2^{28}(1 + 0i)$
$= 2^{28}$

Hence $\frac{d^{56}}{dt^{56}}(e^{-t}\cos t) = \mathrm{Re}\left(2^{28}(e^{(-1+i)t})\right)$
$= \mathrm{Re}\left(2^{28}e^{-t}(\cos t + i\sin t)\right)$
$= 2^{28}e^{-t}\cos t$

Derivative substitution

$\int f(g(x))\,g'(x)\,dx$

Let $u = g(x)$

Integration by parts

$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)\,g(x)\,dx$

Trig / hyp. substitution

$\int f(x)\,dx$ where f contains $(a^2 \pm x^2)^{1/2}$, $(x^2 \pm a^2)^{1/2}$

$1 + x^2$: $\cosh^2 x = 1 + \sinh^2 x$

let $x = \sinh x$

Integrand	Substitution
$\sqrt{a^2-x^2}$, $\frac{1}{\sqrt{a^2-x^2}}$, $(a^2-x^2)^{\frac{3}{2}}$ etc.	$x = a\sin\theta$ or $x = a\cos\theta$
$\sqrt{a^2+x^2}$, $\frac{1}{\sqrt{a^2+x^2}}$, $(a^2+x^2)^{-\frac{3}{2}}$ etc.	$x = a\sinh\theta$
$\sqrt{x^2-a^2}$, $\frac{1}{\sqrt{x^2-a^2}}$, $(x^2-a^2)^{\frac{5}{2}}$ etc.	$x = a\cosh\theta$
$\frac{1}{a^2+x^2}$, $\frac{1}{(a^2+x^2)^2}$ etc.	$x = a\tan\theta$

Example 4.4: Evaluate $\int \sqrt{9-4x^2}\,dx$ if $|x| \le \frac{3}{2}$.

Solution:

$\int\sqrt{9-4x^2}\,dx = \int\sqrt{4(\frac{9}{4}-x^2)}\,dx = \int 2\sqrt{\frac{9}{4}-x^2}\,dx$

Let $x = \frac{3}{2}\sin\theta$, $\sin\theta = \frac{2x}{3}$

Then $\theta = \arcsin\left(\frac{2x}{3}\right)$ is valid when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $-1 \le \frac{2x}{3} \le 1$
$= -\frac{3}{2} \le x \le \frac{3}{2}$

- $\frac{dx}{d\theta} = \frac{3}{2}\cos\theta$
- $\sqrt{\frac{9}{4}-x^2} = \sqrt{\frac{9}{4}-\frac{9}{4}\sin^2\theta} = \sqrt{\frac{9}{4}(1-\sin^2\theta)} = \sqrt{\frac{9}{4}\cos^2\theta}$
$= \frac{3}{2}|\cos\theta|$
$= \frac{3}{2}\cos\theta$, since $\cos\theta \ge 0$ when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$\int\sqrt{9-4x^2}\,dx = 2\int\sqrt{\frac{9}{4}-x^2}\,dx$
$= 2\int\frac{3}{2}\cos\theta\cdot\frac{3}{2}\cos\theta\,d\theta$
$= \frac{9}{2}\int\cos^2\theta\,d\theta$
$= \frac{9}{2}\cdot\frac{1}{2}\int\cos(2\theta) + 1\,d\theta$
$= \frac{9}{4}\left[\frac{1}{2}\sin(2\theta) + \theta\right] + C$
$= \frac{9}{4}\left[\sin\theta\cos\theta + \theta\right] + C$
$= \frac{9}{4}\left[\sin\theta\sqrt{1-\sin^2\theta} + \theta\right] + C$
$= \frac{9}{4}\left(\frac{2x}{3}\sqrt{1-\frac{4x^2}{9}} + \arcsin\left(\frac{2x}{3}\right)\right) + C$
$= x\sqrt{\frac{9}{4}-x^2} + \frac{9}{4}\arcsin\left(\frac{2x}{3}\right) + C$

$\cos^2\theta = \frac{1}{2}(\cos(2\theta)+1)$

$\sin 2\theta = 2\sin\theta\cos\theta$

Powers of Hyperbolic Functions

Consider the integral:

$$\int \sinh^m x\cosh^n x\,dx$$

where m, n are integers (≥ 0).

eg $\int\sinh^3 x\,dx = \int\sinh x\cdot\sinh^2 x\,dx = \int\sinh x\,(\cosh^2 x - 1)\,dx$

- If m or n is odd, use a "derivative substitution" after rewriting one of the odd power terms using identities if necessary.
- If m and n are even, use double angle formulae.

Partial Fractions

Quotient of polynomials

Denominator Factor	Partial Fraction Expansion
$(x-a)$	$\frac{A}{x-a}$
$(x-a)^r$	$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_r}{(x-a)^r}$
(x^2+bx+c)	$\frac{Ax+B}{x^2+bx+c}$
$(x^2+bx+c)^r$	$\frac{A_1x+B_1}{x^2+bx+c} + \frac{A_2x+B_2}{(x^2+bx+c)^2} + \cdots + \frac{A_rx+B_r}{(x^2+bx+c)^r}$

$$\frac{6x^2+7x-20}{2x+5}$$

$$\begin{array}{r} 3x - 4 \\ 2x+5\overline{)\,6x^2+7x-20} \\ -(6x^2+15x) \\ \hline 0 - 8x - 20 \\ -(-8x - 20) \\ \hline 0 \end{array}$$

$6x^2 \div 2x = 3x$

$3x\,(2x+5)$ subtract

$-8x \div 2x = -4$

$-4\,(2x+5)$ subtract

$\Rightarrow 3x - 4$ is quotient

$\Rightarrow 0$ is remainder

Separable ODEs

A separable first order ODE has the form:

$$\frac{dy}{dx} = M(x)N(y), \quad (M(x) \neq 0,\ N(y) \neq 0)$$

Use of sep. of variable

$\Rightarrow \frac{1}{N(y)}\frac{dy}{dx} = M(x)$

$\Rightarrow \int\frac{1}{N(y)}\frac{dy}{dx}\,dx = \int M(x)\,dx$

$\Rightarrow \int\frac{1}{N(y)}\,dy = \int M(x)\,dx$

Find integrating factor:

$I = e^{\int P(x)\,dx}$

$I\frac{dy}{dx} + IP(x)\,y = IQ(x)$

$\Rightarrow \frac{d}{dx}(Iy) = IQ(x)$

$\Rightarrow Iy = \int I\,Q(x)\,dx$

A linear first order ODE has the form:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Doomsday model with harvesting.

$$\frac{dp}{dt} = kp - h, \quad h > 0.$$

Logistic model.

$$\frac{dp}{dt} = \underbrace{kp}_{\text{net birth rate}} - \underbrace{\frac{k}{a}p^2}_{\text{competition term}} = kp\left(1 - \frac{p}{a}\right)$$

Logistic model with harvesting.

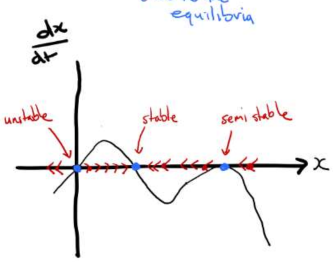
$$\frac{dp}{dt} = kp\left(1 - \frac{p}{a}\right) - h, \quad h > 0, \ a > 0$$

Phase plots

A phase plot is a plot of $\frac{dx}{dt}$ as a function of x.

A phase plot will give
- the equilibria $\frac{dx}{dt} = 0$
- the behaviour of solutions close to the equilibria

see $\frac{dx}{dt}$ close to the equilibria

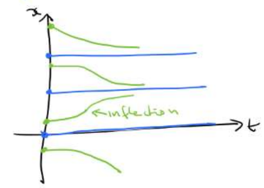


Note:
Phase plots are only useful for ODEs of the form

$$\frac{dx}{dt} = f(x)$$

i.e., when the right-hand side has no explicit dependence on t.
ODEs of this form are called autonomous.

• equilibrium



Mixing Problems

x = amount of stuff often linear

$$\frac{dx}{dt} = \text{rate of inflow} - \text{rate of outflow}$$

$$c = \frac{x(t)}{V(t)} \qquad V(t) = V_0 + V_{in} - V_{out}$$

Definitions

1. Transient terms: terms decaying to 0 as $t \to \infty$.
2. Steady state terms: terms NOT decaying to 0 as $t \to \infty$.

$$y = \underbrace{e^{-t}\sin t}_{\text{transient terms}} + \underbrace{\cos t}_{\text{steady state terms}}$$

Homogeneous 2^{nd} Order Linear ODEs with Constant Coefficients

General form: $ay'' + by' + cy = 0$

where a, b, c are constants.

Try $y(x) = e^{\lambda x}$

$\Rightarrow y'(x) = \lambda e^{\lambda x}, \quad y''(x) = \lambda^2 e^{\lambda x}$

so $(a\lambda^2 + b\lambda + c)\underbrace{e^{\lambda x}}_{\neq 0} = 0$

$\Rightarrow \boxed{a\lambda^2 + b\lambda + c = 0}$ Characteristic Equation

$\Rightarrow \lambda = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

• 2 distinct real values λ_1, λ_2

$$\boxed{y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}}$$

• 1 real value $\lambda = \frac{-b}{2a}$

$$\boxed{y(x) = Ae^{\lambda x} + Bxe^{\lambda x}}$$

• 2 complex conjugate values

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$

$$\boxed{y(x) = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)}$$

Inhomogeneous 2^{nd} Order Linear ODEs

Theorem:
The general solution of

$$y'' + \mathcal{P}(x)y' + \mathcal{Q}(x)y = \mathcal{R}(x)$$

is the function y given by

$$y(x) = y_{\mathcal{H}}(x) + y_{\mathcal{P}}(x)$$

try $y_p = \ldots$
$y_p' = \ldots$
$y_p'' = \ldots$
sub into ODE

$R(x)$	try y_p
1	a
x	$ax + b$
x^2	$ax^2 + bx + c$
$\vdots$	
etc	
e^{kx}	ae^{kx}
$\sin(kx)$	$a\sin(kx) + b\cos(kx)$
$\cos(kx)$	$a\sin(kx) + b\cos(kx)$
$R(x)$	general sum of $R(x)$ and its derivatives

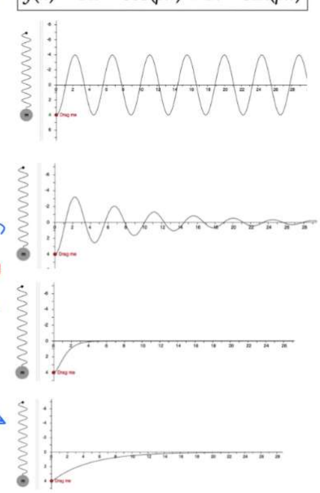
if $R(x)$ part of $GS(H)$ try multiplying by x . . .

Springs - Free Vibrations

$W = mg$
$T = -k \cdot$ extension
$R = -\beta\dot{y}$

Eqtn of motion: $m\ddot{y} + \beta\dot{y} + ky = 0$

- If $\beta = 0$: $\lambda = \pm ib$ simple harmonic motion
- If $0 < \beta < 2\sqrt{mk}$: $\lambda = a \pm ib$ underdamped, weak damping
- If $\beta = 2\sqrt{mk}$: $\lambda = a, a$ critical damping
- If $\beta > 2\sqrt{mk}$: $\lambda = a, b$ overdamped, strong damping



Springs - Forced Vibrations

$m\ddot{y} + \beta\dot{y} + ky = F(t)$

Definition

Resonance: Resonance occurs when the external force f has the same form as one of the terms in the $GS(H)$.

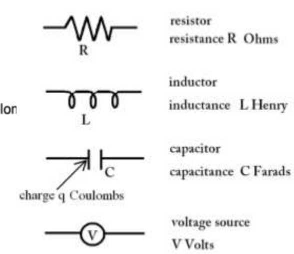
If $\beta = 0$, then the $PS(IH)$ will grow without bound as $t \to \infty$.

RLC series electric circuit

Let $q(t)$ be the charge on the capacitor (measured in Coulon at time t seconds.

The charge satisfies the second-order ODE

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V$$

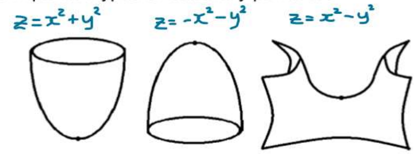


$$\boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}}$$

- $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$
- $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$

Three important types of stationary points are

$z = x^2 + y^2$ $\quad z = -x^2 - y^2$ $\quad z = x^2 - y^2$



Local Minimum | Local Maximum | Saddle Point

the tangent plane has equation

$$\boxed{z - z_0 = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0).}$$

Linear Approximations

If f is differentiable at (x_0, y_0), we can approximate $z = f(x, y)$ by its tangent plane at (x_0, y_0, z_0).

This linear approximation of f near (x_0, y_0) is:

$$\boxed{f(x,y) \approx \underbrace{f(x_0,y_0) + \left.\tfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}(x - x_0) + \left.\tfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}(y - y_0)}_{\text{eqtn of tangent plane}}}$$

Let $\Delta x = x - x_0$, $\quad \Delta y = y - y_0$, $\quad \Delta f = z - z_0 = f(x, y) - f(x_0, y_0)$.

Then the approximate change in f near (x_0, y_0), for given small changes in x and y, is:

$$\boxed{\Delta f \approx \left.\tfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}\Delta x + \left.\tfrac{\partial f}{\partial y}\right|_{(x_0,y_0)}\Delta y}$$

$\Delta f = f(x, y) - f(x_0, y_0)$

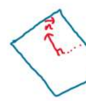
Click for video.

Equations of a Plane

The Cartesian equation of a plane has the form

$$ax + by + cz = d$$

where a, b, c, d are real constants.



$\mathbf{n} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is a normal vector to the plane.

$\vec{n} = (a, b, c)$ perpendicular

In fact, the plane passing through a point (x_0, y_0, z_0) with a normal vector (a, b, c) consists of the points (x, y, z) such that (a, b, c) is perpendicular to $(x - x_0, y - y_0, z - z_0)$ and thus has equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

that is,

$$ax + by + cz = \overbrace{ax_0 + by_0 + cz_0}^{d}.$$

Sketching Functions of Two Variables

The key steps in drawing a graph of a function of two variables $z = f(x, y)$ are:

1. Draw the x, y, z axes.
   For right handed axes: the positive x axis is towards you, the positive y axis points to the right, and the positive z axis points upward.
2. Draw the $y - z$ cross section.
3. Draw some level curves and their contours.
4. Draw the $x - z$ cross section.
5. Label any x, y, z intercepts and key points.

Gradient Vectors

If $f : \mathbb{R}^2 \to \mathbb{R}$ is a differentiable function, we can define the gradient of f to be the vector

nabla

$$\boxed{\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)}$$

Then the directional derivative of f at the point P_0 in the direction $\hat{\mathbf{u}}$ is the dot product

$$\boxed{D_{\hat{\mathbf{u}}}f\big|_{P_0} = \nabla f\big|_{P_0} \cdot \hat{\mathbf{u}}}$$

Second Derivative Test

If $\nabla f(x_0, y_0) = \mathbf{0}$ and the second partial derivatives of f are continuous on an open disk centred at (x_0, y_0), consider the Hessian function

$$H(x, y) = f_{xx}f_{yy} - (f_{xy})^2$$

evaluated at (x_0, y_0).

Then (x_0, y_0) is a
1. local minimum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$.
2. local maximum if $H(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$.
3. saddle point if $H(x_0, y_0) < 0$.

Note: Test is inconclusive if $H(x_0, y_0) = 0$.

Fubini's Theorem:

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function over the domain $R = [a, b] \times [c, d]$. Then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

So order of integration is not important.

+ gradf is fastest increase of z at a point

- gradf is fastest decrease of z at a point

If degree of denominator is <= degree of numerator, long division first