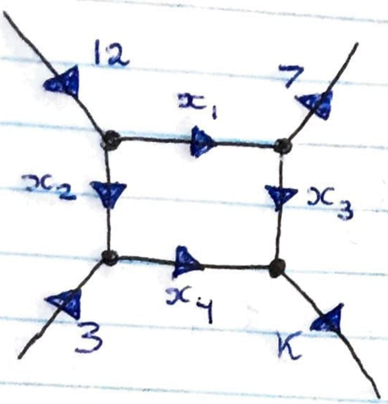
1.



(a)

out	in
x_1+x_2	12
$7+x_3$	x_1
x_4	$3+x_2$
k	x_3+x_4

$$\boxed{\begin{aligned} x_1 + x_2 &= 12 \\ x_1 - x_3 &= 7 \\ x_4 - x_2 &= 3 \\ x_3 + x_4 &= k \end{aligned}}$$

(b)

$$\begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ \boxed{1} & 0 & -1 & 0 & 7 \\ 0 & -1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & k \end{array}\right] \underset{\sim}{R_2 \to R_2 - R_1} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & -1 & -1 & 0 & -5 \\ 0 & -1 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & k \end{array}\right]$$

$$\underset{\sim}{R_2 \to -R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & \boxed{-1} & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & k \end{array}\right] \underset{\sim}{R_3 \to R_3 + R_2} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & \boxed{1} & 1 & k \end{array}\right]$$

$$\underset{\sim}{R_4 \to R_4 - R_3} \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 12 \\ 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 & 8 \\ 0 & 0 & 0 & 0 & k-8 \end{array}\right]$$

t

(i) $\boxed{K=8}$ (consistent)

$x_4 = t$

$x_3 + x_4 = 8 \Rightarrow x_3 + t = 8 \Rightarrow \boxed{x_3 = 8-t}$

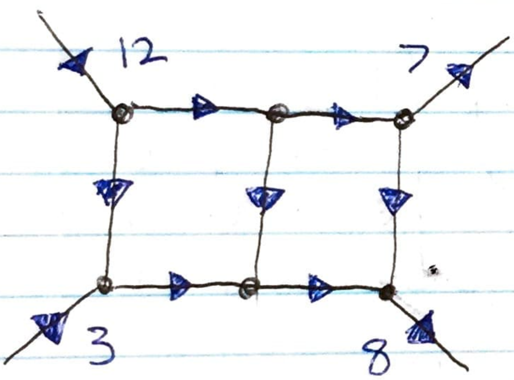
$x_2 + x_3 = 5 \Rightarrow x_2 + 8 - t = 5 \Rightarrow \boxed{x_2 = t-3}$

$x_1 + x_2 = 12 \Rightarrow x_1 + t - 3 = 12 \Rightarrow \boxed{x_1 = 15-t}$

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 15-t \\ t-3 \\ 8-t \\ t \end{pmatrix} = \begin{pmatrix} 15 \\ -3 \\ 8 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}}$$

(ii) $\boxed{K \neq 8}$ (inconsistent)

(c)



sum of flows in = sum of flows out

$12 + 3 = 7 + 8$

$15 = 15$

Therefore the system has at least 1 solution and thus is consistent.

(2.)

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

If A is invertible then $[A|I_3] \sim [I_3|B]$ where B is A^{-1}

(a) $$[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & -2 & 2 & -1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right]$$

$R_2 \to -\frac{1}{2}R_2$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$R_3 \to R_3 - 2R_2 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 2 & 0 & 1 & 1 \end{array}\right]$$

$$R_3 \to \frac{1}{2} R_3 \quad \sim \left[\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 + R_3 \\ R_2 \to R_2 + R_3 \end{array} \quad \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$$R_1 \to R_1 - R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right]$$

$[A \mid I_3] \sim [I_3 \mid B]$ where $B = A^{-1}$

$$\Rightarrow \boxed{A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}}$$

(b) $A_5 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad B_5 = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

$$B_5 A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\cdot 1 + 0\cdot 1 + 0\cdot(-1) & 1\cdot 1 + 0\cdot(-1) + 0\cdot 1 & 1\cdot(-1) + 0\cdot 1 + 0\cdot 1 \\ \frac{1}{2}\cdot 1 + (-\frac{1}{2})\cdot 1 + 0\cdot(-1) & \frac{1}{2}\cdot 1 + (-\frac{1}{2})\cdot(-1) + 0\cdot 1 & \frac{1}{2}\cdot(-1) + (-\frac{1}{2})\cdot 1 + 0\cdot 1 \\ 0\cdot 1 + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot(-1) & 0\cdot 1 + \frac{1}{2}\cdot(-1) + \frac{1}{2}\cdot 1 & 0\cdot(-1) + \frac{1}{2}\cdot 1 + \frac{1}{2}\cdot 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{B_5 A - A_5 = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = 0}$$