MAST10006 Calculus 2, Semester 2, 2020

Assignment 3

School of Mathematics and Statistics, The University of Melbourne

- Submit your assignment solutions online in Canvas before **6pm, Monday 7 September 2020**
- This assignment is worth 2.22% of your final MAST10006 mark.
- Answer all questions below. Of these questions, one will be chosen for marking.
- Marks may be awarded for:
  - Correct use of appropriate mathematical techniques
  - Accuracy and validity of any calculations or algebraic manipulations
  - Clear justification or explanation of techniques and rules used
  - Clear communication of mathematical ideas through diagrams
  - Use of correct mathematical notation and terminology
- You must use methods taught in MAST10006 Calculus 2 to solve the assignment questions.

Question 1 marked. Total 12

1. Consider the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$$

(a) Calculate the first 11 partial sums, S_N for $N \in \{0, 1 \dots, 10\}$. You can write the values correct to 3 decimal places.
Sketch the partials sums on a graph with N on the horizontal axis and S_N on the vertical axis. Hand draw on grid paper or use an app of your choice such as Desmos or Mathematica. Make an educated guess about its convergence (or divergence) behaviour. You don't need to justify your guess.

(b) Which test from our class would you like to apply to test the convergence of this series, and what condition gets in the way?

(c) Find that test on Wikipedia. The test on Wikipedia will have exactly the same name, but there is a crucial difference compared to the way it is stated in class. State a version of the test from Wikipedia, and use it to test convergence of the series.

Solution.

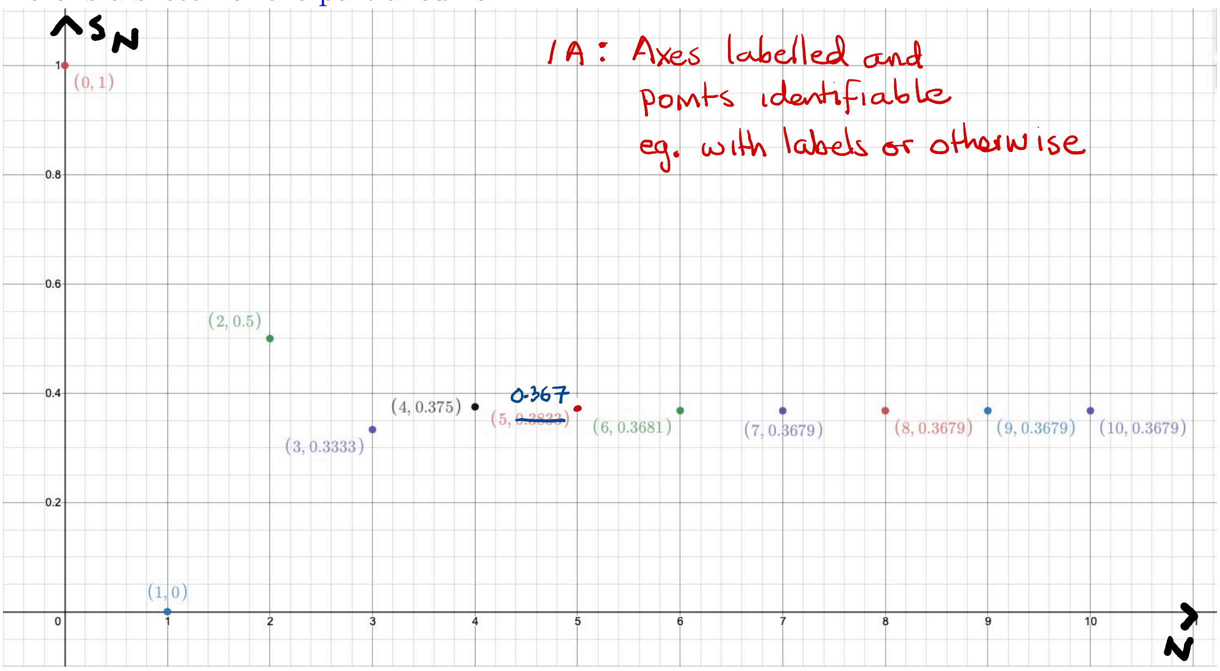
5 (a) Let $a_n = \frac{(-1)^n}{n!}$. Then $S_N = \sum_{n=0}^{N} a_n = a_0 + a_1 + \cdots + a_N$. We calculate these to 3 decimal places below.

1M: Demonstrate understanding of the difference between a partial sum and a term.

$S_0 = 1$
$S_1 = 0$
$S_2 = 0.5$
$S_3 = 0.333$
$S_4 = 0.375$
$S_5 = 0.367$
$S_6 = 0.368$
$S_7 = 0.368$
$S_8 = 0.368$
$S_9 = 0.368$
$S_{10} = 0.368$

1A: small rounding errors okay here

Here is a sketch of the partial sums: 1M : sketch of points (Not a curve)



1A: A reasonable guess of convergence or divergence based on graph.

From the graph, it looks like the partial sums (and so the series) converges.

① (b) The presence of the factorials suggests that we could apply the ratio test. The condition that gets in the way is the requirement of a positive series ~~[illegible]~~, since not all terms in ~~this~~ the given series ~~is~~ are positive.

1A: A correct reason why the test written (any of div. test, ratio test, comparison test) is not applicable to $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

⑤ (c) The only test on Wikipedia with the same name is the ratio test. The version on Wikipedia says something along these lines:

1A: The test is the ratio test

Let

$$\sum_{n=0}^{\infty} a_n$$

be a series.

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=0}^{\infty} a_n$ converges.

1A: A correct version of this part of the ratio test is stated. Must be applicable to $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$

- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ or $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ diverges to infinity, then $\sum_{n=0}^{\infty} a_n$ diverges.
- The test is inconclusive if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

Apply the ratio test (Wikipedia version) to $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$:

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{(-1)^{n+1}/((n+1)!)}{1/(n!)}\right| &= \lim_{n\to\infty} \frac{n!}{(n+1)!} \\
&= \lim_{n\to\infty} \frac{1}{n+1} \\
&= \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n}} \\
&= 0 \qquad \text{limit laws and standard limit } \frac{1}{n} \to 0 \\
&< 1
\end{aligned}$$

1M: Simplifying factorials. (or other valid method if ratio test is not used)

1J: Has written $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ to justify convergence. (or other statement written that checks test condition if ratio test not used)

By the ratio test, $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ converges.

1A: limit is zero, and series converges and justification included. Must specify standard limit used (or other correct to calculation if ratio test not used)

1N Notation correct throughout question 1

Question 2 and 3 for self reflection. Key parts of the solution are highlighted for you to focus on.

2. Consider the function

$$f(x) = \tanh^2(x) - 5\,\text{sech}(x)$$

(a) Find the axis intercepts of the graph $y = f(x)$.

(b) Find the stationary points of $y = f(x)$.

(c) Determine if f is odd, even or neither.

(d) For which value(s) of $x \in \mathbb{R}$ is the function f continuous? Justify your answer with reference to continuity theorems from lectures.

(e) Hence sketch the graph of $y = f(x)$.

Give numerical answers as exact values, in terms of inverse hyperbolic functions if necessary. In your graphs, label all curves, axis intercepts and asymptotes (if any).

Solution.

(a) Find y intercept: $f(0) = -5$. So the y-intercept is $(0, -5)$
Find x intercepts by solving $f(x) = 0$ for x.

$$\begin{aligned}
&\tanh^2 x - 5\,\text{sech}\, x = 0 \\
\implies &\sinh^2 x - 5\cosh x = 0 \\
\implies &\cosh^2 x - 5\cosh x - 1 = 0 \\
\implies &\cosh x = \frac{5 \pm \sqrt{29}}{2} \\
\implies &\cosh x = \frac{5 + \sqrt{2}}{2} && \text{since } \cosh x \geq 1 \\
\implies &x = \pm \text{arccosh}\left(\frac{5+\sqrt{29}}{2}\right)
\end{aligned}$$

So the x-intercepts are

$$\left(\text{arccosh}\left(\frac{5+\sqrt{29}}{2}\right), 0\right) \text{ and } \left(-\,\text{arccosh}\left(\frac{5+\sqrt{29}}{2}\right), 0\right)$$

(b) Find the stationary points by solving $f'(x) = 0$ for x:

$$\begin{aligned}
f'(x) &= 2\tanh x \frac{d}{dx}(\tanh(x)) - 5\tanh x\,\text{sech}\, x \\
&= 2\tanh x\,\text{sech}^2 x - 5\tanh x\,\text{sech}\, x \\
&= \tanh x\,\text{sech}\, x(2\,\text{sech}\, x - 5)
\end{aligned}$$

Since $0 < \text{sech}(x) \leq 1$, we have $\text{sech}\, x \neq 0$ and $2\,\text{sech}\, x - 5 \neq 0.$
Therefore, $f'(x) = 0 \implies \tanh x = 0 \implies x = 0.$
There is one stationary point:

$$(0, -5)$$

(c)

$$\begin{aligned}
f(-x) &= \tanh^2(-x) - 5\,\text{sech}(-x) \\
&= (-1)^2\tanh^2(x) - 5\,\text{sech}(x) \\
&= f(x)
\end{aligned}$$

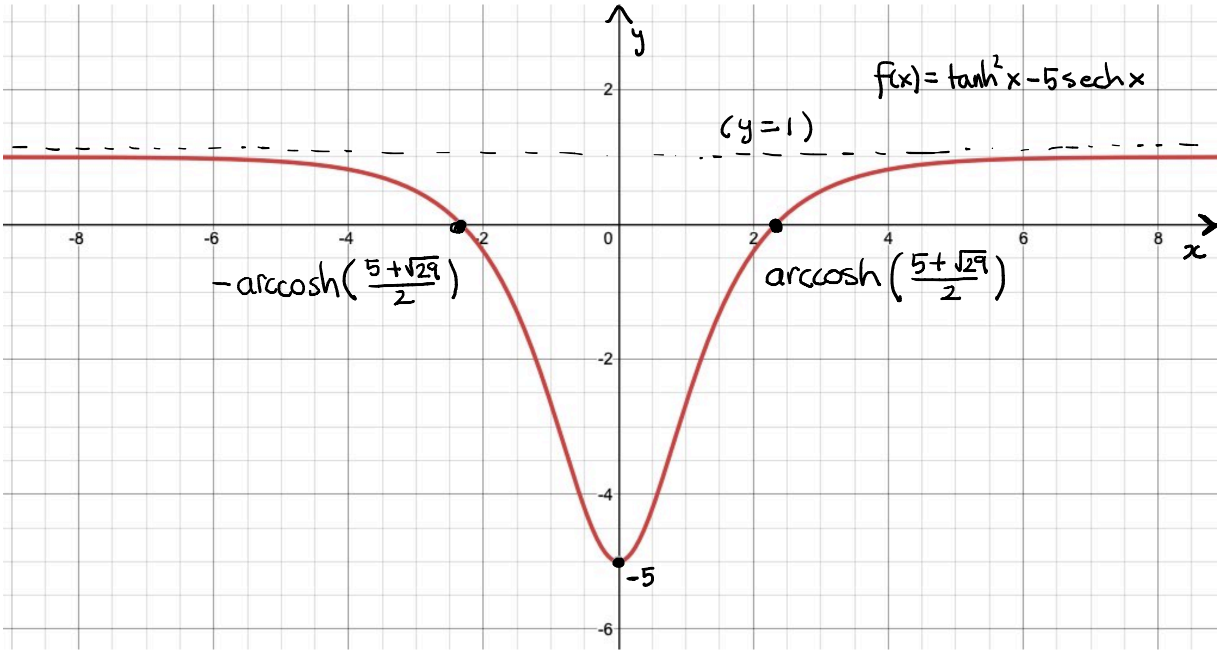
So f is even.

(d)
- Firstly $\tanh x$ and $\text{sech}\, x$ are hyperbolic functions, with domain $\mathbb{R}$ so are continuous for all $x \in \mathbb{R}$.

and $-5\,\text{sech}\,x$ are

- Therefore $\tanh^2 x$ ~~is a~~ products of continuous functions, so ~~is~~ are continuous for all $x \in \mathbb{R}$. (or refer to continuity theorem 1)
- Therefore $f(x) = \tanh^2 x$ ~~+~~ -5 $\text{sech}\, x$ is a sum of continuous functions so is continuous for all $x \in \mathbb{R}$.

(e) Here is a sketch:



Note that the asymptote is calculate by computing the limit

$$
\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} (\tanh^2 x - 5\,\text{sech}\,x) \\
&= \lim_{x\to\infty} \left(\left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 - \frac{10}{e^x + e^{-x}} \right) \\
&= \lim_{x\to\infty} \left(\left(\frac{1 - e^{-2x}}{1 + e^{-2x}}\right)^2 - \frac{10e^{-x}}{1 + e^{-2x}} \right) \\
&= 1
\end{aligned}
$$

Last line uses limit laws, and the standard limit $\frac{1}{a^x} \to 0$ as $x \to \infty$.
Similarly $\lim_{x\to-\infty} f(x) = 1$, which we can calculate by calculating the limit, or noticing that the function f is even.

3. Use the complex exponential to evaluate the derivative

$$\frac{d^{61}}{dt^{61}}\left(e^{-t+1}\cos(t)\right)$$

Solution.

$$
\begin{aligned}
\frac{d^{61}}{dt^{61}}\left(e^{-t+1}\cos(t)\right) &= Re\left[\frac{d^{61}}{dt^{61}}\left(e^{-t+1+it}\right)\right] \\
&= Re\left[\frac{d^{61}}{dt^{61}}\left(e^{(-1+i)t+1}\right)\right] \\
&= Re\left[(-1+i)^{61} e^{(-1+i)t+1}\right]
\end{aligned}
$$

Calculate $(-1+i)^{61}$ by converting to polar form:

$$\begin{aligned}(-1+i)^{61} &= \left(\sqrt{2}e^{\frac{3\pi i}{4}}\right)^{61} \\ &= 2^{\frac{61}{2}} e^{\frac{-\pi i}{4}} \\ &= 2^{\frac{61}{2}} \frac{1}{\sqrt{2}}(1-i) \\ &= 2^{30}(1-i)\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d^{61}}{dt^{61}}\left(e^{-t+1}\cos(t)\right) &= Re\left[2^{30}(1-i)e^{(-1+i)t+1}\right] \\ &= Re\left[2^{30}(1-i)e^{-t+1}(\cos t + i\sin t)\right] \\ &= Re\left[2^{30}e^{1-t}(\cos t + \sin t + i(\sin t - \cos t))\right] \\ &= 2^{30}e^{1-t}(\cos t + \sin t)\end{aligned}$$

Points for self reflection:

2a) Did you get both x-intercepts? Note it is possible to have the x-intercepts written in terms of arcsech. Check that your solution matches with the one given.

2b) After factorising the derivative, did you give a reason as to why the factors involving sech were never zero?

2c) Make sure at some point you have written the definition of even: f(-x) = f(x)

2d) Make sure you are mentioning all the highlighted text.

2e) Did your sketch include all labels, including the asymptotes?

3. This should be a straightforward, albeit long, calculation that you can do now. The parts were errors often occur are highlighted, which involve not include errors in calculating the modulus or argument, and not finding the cartesian form of a complex number before finding its real or imaginary part.

End of assignment