MAST30027: Modern Applied Statistics

Assignment 4 Solution 2023

1. (a) **Solution:** The posterior distribution of the precision τ is

$$\begin{aligned} f(\tau|x_1,\ldots,x_{100}) &\propto \prod_{i=1}^{100} P(x_i|\tau)f(\tau) \\ &\propto \tau^{100/2}\exp\left\{-\frac{\tau}{2}\sum_{i=1}^{100}(x_i-75)^2\right\}\tau^{2-1}\exp(-\tau) \\ &\propto \tau^{52-1}\exp\left\{-\left[\frac{1}{2}\sum_{i=1}^{100}(x_i-75)^2+1\right]\tau\right\} \end{aligned}$$

This is a kernel for Gamma(52, $\frac{1}{2}\sum_{i=1}^{100}(x_i-75)^2+1$).
Let's compute the rate parameter from the data.

```
> x = scan("assignment4_prob1_2023.txt", what=double(0))
> rate = sum((x-75)^2)/2 + 1
> rate
[1] 1805.65
```

Thus, $\tau|x_1,\ldots,x_{100} \sim$ Gamma$(52, 1805.65)$.

(b) **Solution:** Let $\alpha^* = 52$ and $\beta^* = 1805.65$.

$$\begin{aligned} p(\tilde{x}|x_1,...,x_{100}) &\propto \int p(\tilde{x}|\tau)p(\tau|x_1,...,x_{100})d\tau \\ &\propto \int \tau^{\frac{1}{2}}\exp\left\{-\frac{\tau}{2}(\tilde{x}-75)^2\right\}\tau^{\alpha^*-1}\exp(-\beta^*\tau)d\tau \\ &\propto \int \tau^{\alpha^*+\frac{1}{2}-1}\exp\left\{-\left[\beta^*+\frac{1}{2}(\tilde{x}-75)^2\right]\tau\right\}d\tau \end{aligned}$$

the integrand is proportional to the pdf of Gamma$(\alpha^*+\frac{1}{2}, \beta^*+\frac{1}{2}(\tilde{x}-75)^2)$

$$\begin{aligned} &\propto \frac{1}{\left(\beta^*+\frac{1}{2}(\tilde{x}-75)^2\right)^{\alpha^*+\frac{1}{2}}} \\ &\propto \left[1+\frac{(\tilde{x}-75)^2}{2\beta^*}\right]^{-(\alpha^*+\frac{1}{2})} \\ &\propto \left[1+\frac{(\tilde{x}-75)^2}{2\beta^*}\right]^{-\frac{2\alpha^*+1}{2}}. \end{aligned}$$

Therefore we conclude that

$$\tilde{x}|x_1,...,x_{100} \sim t(\nu=2\alpha^*, a=75, b=\frac{\beta^*}{\alpha^*}) = t(\nu=104, a=75, b=34.72404).$$

2. **Posterior inference using Gibbs sampling**

(a) **Solution** Let $n = 100$ and $m = 150$, we have

$$\begin{aligned}
&p(\mu_1|\mu_2, x_1, \cdots, x_n, y_1, \cdots, y_m) \\
\propto &p(x_1, \cdots, x_n|\mu_1)p(y_1, \cdots, y_m|\mu_2)p(\mu_1, \mu_2) \\
\propto &\exp\left[-\frac{1}{2}\left(\sum_{i=1}^{n}(x_i-\mu_1)^2 + 2\sum_{i=1}^{m}(y_i-\mu_2)^2 + (3\mu_1^2 + 4\mu_1\mu_2 + 3\mu_2^2)\right)\right] \\
\propto &\exp\left[-\frac{1}{2}\left((n+3)\mu_1^2 - 2\left(\sum_{i=1}^{n}x_i - 2\mu_2\right)\mu_1\right)\right] \\
\propto &\exp\left[-\frac{n+3}{2}\left(\mu_1^2 - 2\frac{\sum_{i=1}^{n}x_i - 2\mu_2}{n+3}\mu_1\right)\right],
\end{aligned}$$

Thus, we have $\mu_1|\mu_2, x_1, \cdots, x_n, y_1, \cdots, y_m \sim \text{Normal}(\frac{\sum_{i=1}^{n}x_i - 2\mu_2}{n+3}, \frac{1}{n+3})$.

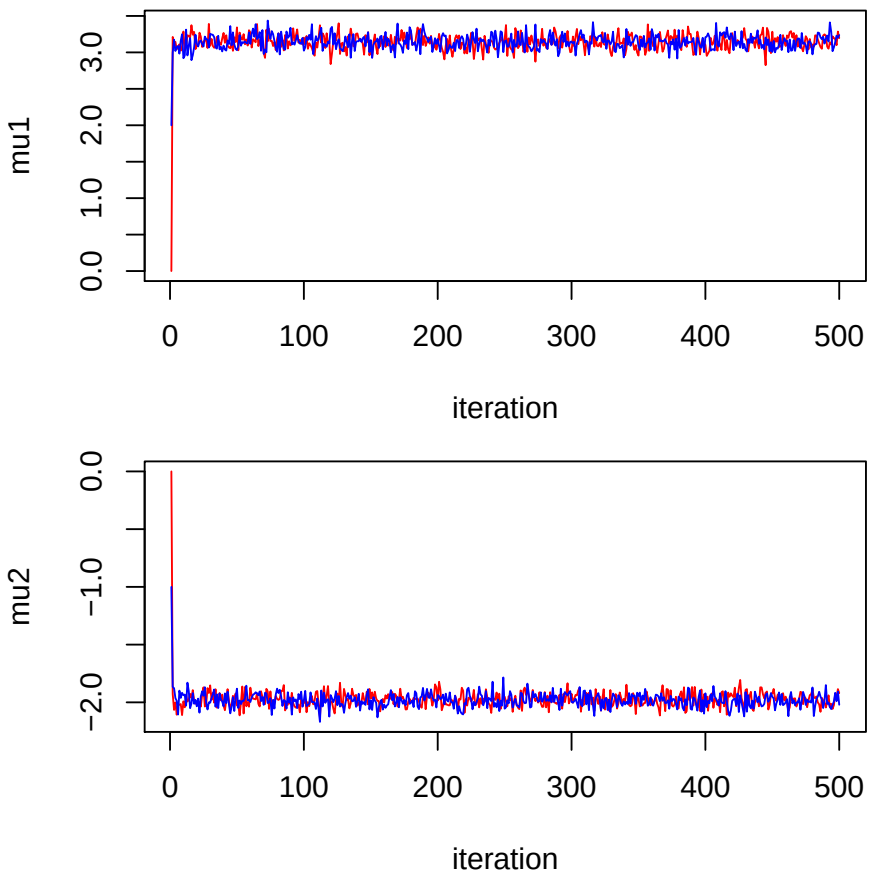
Similarly, we have $\mu_2|\mu_1, x_1, \cdots, x_n, y_1, \cdots, y_m \sim \text{Normal}(\frac{2\sum_{i=1}^{m}y_i - 2\mu_1}{2m+3}, \frac{1}{2m+3})$.

(b) **Solution**

```
> x = scan(file="assignment4_prob2_x_2023.txt", what=double())
> y = scan(file="assignment4_prob2_y_2023.txt", what=double())
> set.seed(30027)
> # Implement Gibbs Sampler
> GibbsS <- function(mu1.0, mu2.0, nreps, x, y){
+
+   Gsamples <- matrix(nrow=nreps, ncol=2)
+   Gsamples[1,] <- c(mu1.0, mu2.0)
+
+   # main loop
+   n = length(x)
+   m = length(y)
+   for (i in 2:nreps) {
+     mu1 = Gsamples[i-1,1]
+     mu2 = Gsamples[i-1,2]
+     mu1 = rnorm(1, (sum(x)-2*mu2)/(n+3), sqrt(1/(n+3)))
+     mu2 = rnorm(1, (2*sum(y)-2*mu1)/(2*m+3), sqrt(1/(2*m+3)))
+     Gsamples[i,] <- c(mu1, mu2)
+   }
+   return(Gsamples=Gsamples)
+ }
> # number of iterations
> nreps <- 500
> # run two Gibbs sampling chains
> GibbsS1 = GibbsS(mu1.0=0, mu2.0=0, nreps, x, y)
> GibbsS2 = GibbsS(mu1.0=2, mu2.0=-1, nreps, x, y)
```

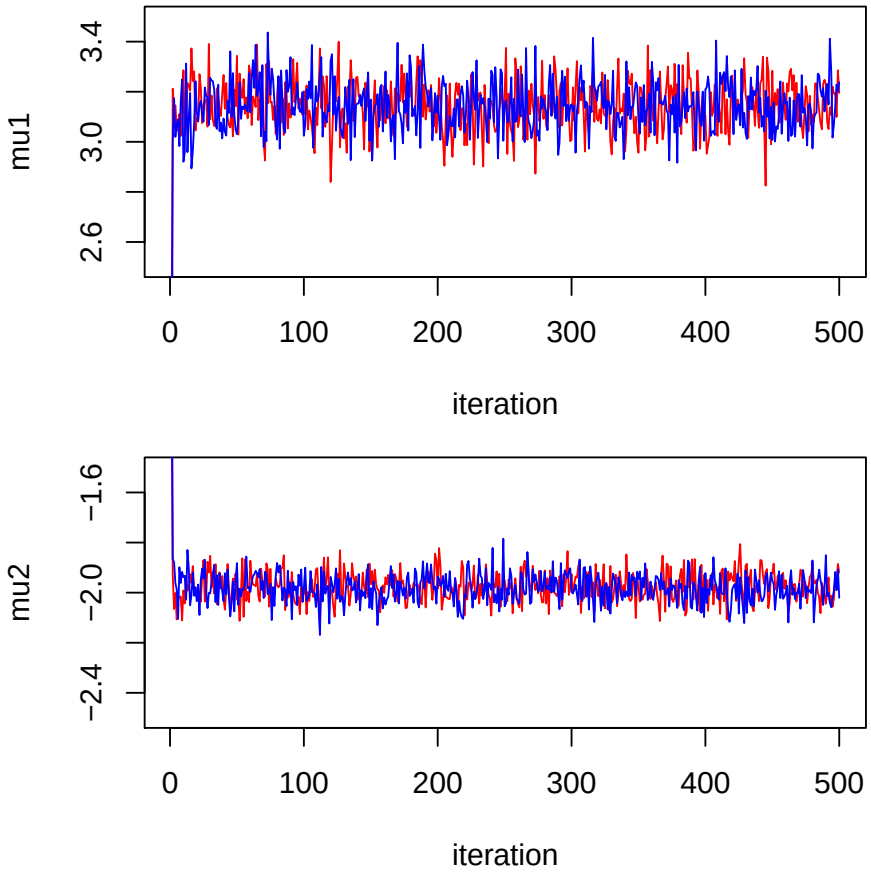
Make a trace plot for each of parameters.

```
> par(mfrow=c(2,1), mar=c(4,4,1,1))
> plot(1:nreps, GibbsS1[,1], type="l", col="red",
+      ylim = c(min(GibbsS1[,1],GibbsS2[,1]), max(GibbsS1[,1],GibbsS2[,1])),
+      xlab = "iteration", ylab ="mu1")
> points(1:nreps, GibbsS2[,1], type="l", col="blue")
> plot(1:nreps, GibbsS1[,2], type="l", col="red",
+      ylim = c(min(GibbsS1[,2],GibbsS2[,2]), max(GibbsS1[,2],GibbsS2[,2])),
+      xlab = "iteration", ylab ="mu2")
> points(1:nreps, GibbsS2[,2], type="l", col="blue")
```



Let's zoom in on the trace plot.

```
> par(mfrow=c(2,1), mar=c(4,4,1,1))
> plot(1:nreps, GibbsS1[,1], type="l", col="red", ylim = c(2.5,3.5),
+      xlab = "iteration", ylab ="mu1")
> points(1:nreps, GibbsS2[,1], type="l", col="blue")
> plot(1:nreps, GibbsS1[,2], type="l", col="red", ylim = c(-2.5, -1.5),
+      xlab = "iteration",ylab ="mu2")
> points(1:nreps, GibbsS2[,2], type="l", col="blue")
```

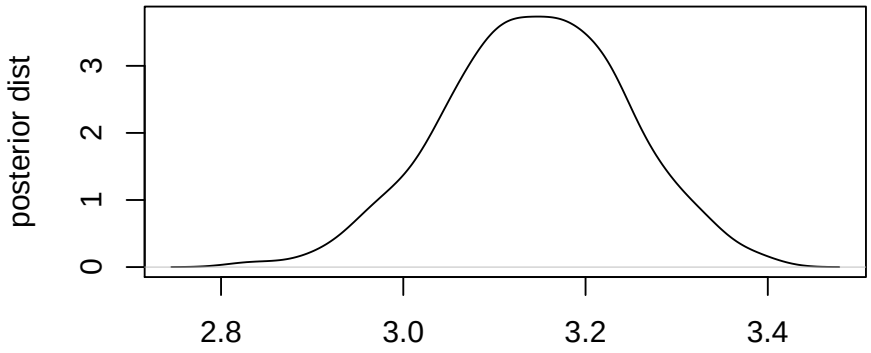


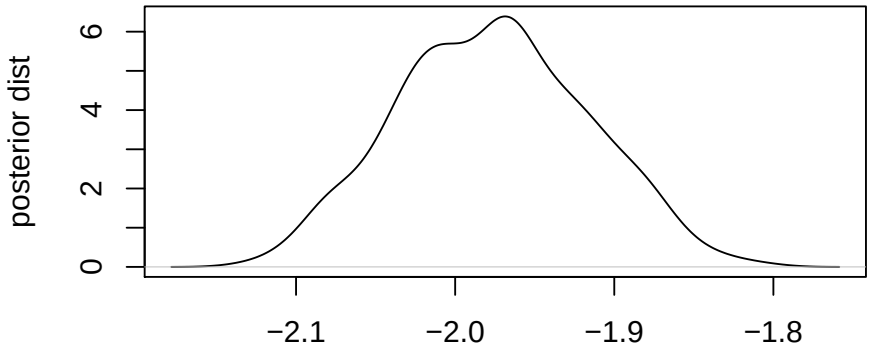
The trace plots show that samples from different chains are mixed well and behave similarly.

(c) **Solution** We will remove the first 50 samples as burn-in period.

1) make a plot that shows empirical (estimated) marginal posterior distribution.

```
> par(mfrow=c(2,1), mar=c(4,4,1,1))
> plot(density(GibbsS1[-(1:50),1]), ylab="posterior dist", xlab="mu1", main="")
> plot(density(GibbsS1[-(1:50),2]), ylab="posterior dist", xlab="mu2", main="")
```





2) estimate marginal posterior mean.

```
> # for mu1
> mean(GibbsS1[-(1:50),1])
[1] 3.144036
> # for mu2
> mean(GibbsS1[-(1:50),2])
[1] -1.976291
```

3) report a 90% credible interval for the marginal posterior distribution.

```
> quantile(GibbsS1[-(1:50),1], probs=c(0.05, 0.95))
      5%      95%
2.974518 3.306848
> # for mu2
> quantile(GibbsS1[-(1:50),2], probs=c(0.05, 0.95))
       5%       95%
-2.076106 -1.876740
```

Posterior inference using the Metropolis-Hastings (MH) algorithm

(d) **Solution** The posterior distribution is proportional to the product of the likelihood and prior distribution. Thus, we will use the product of the likelihood and prior distribution as a target distribution $\pi(\mu_1, \mu_2)$ in the Metropolis-Hastings algorithm.

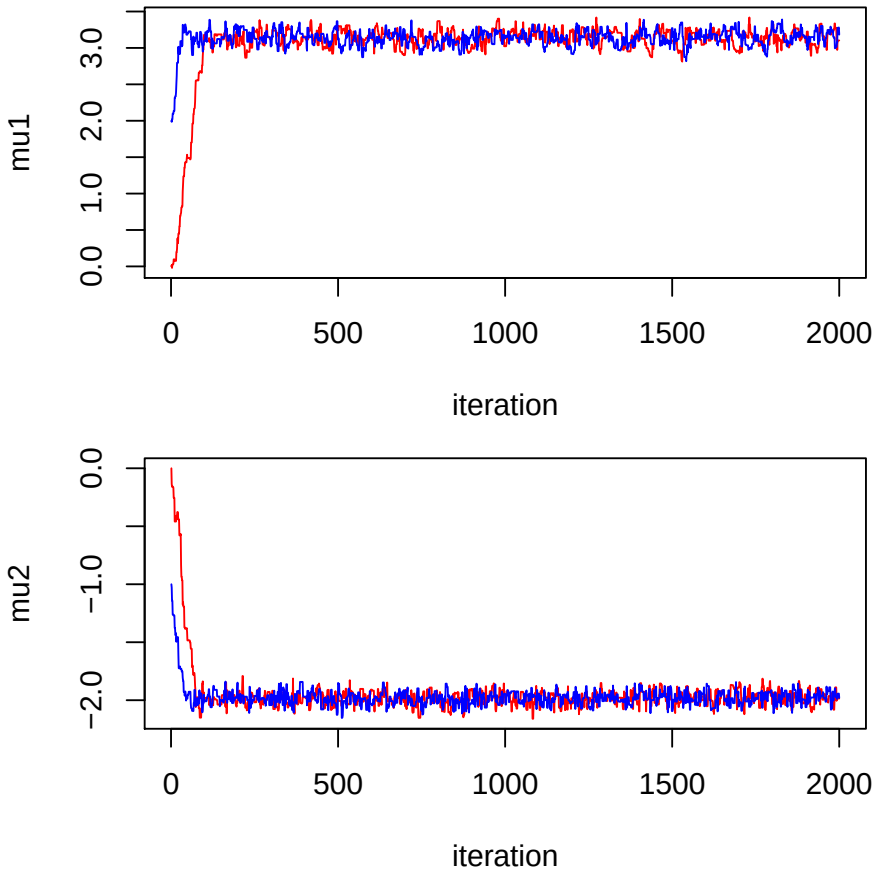
```
> # log likelihood
> likelihood <- function(param, x, y){
```

```
+     mu1 = param[1]
+     mu2 = param[2]
+
+     singlelikelihoodsX = dnorm(x, mean = mu1, sd = 1, log = T)
+     singlelikelihoodsY = dnorm(y, mean = mu2, sd = sqrt(1/2), log = T)
+     sumll = sum(singlelikelihoodsX) + sum(singlelikelihoodsY)
+     return(sumll)
+ }
> # log prior
> prior <- function(param){
+     mu1 = param[1]
+     mu2 = param[2]
+     return(-0.5*(3*mu1^2 + 4*mu1*mu2 + 3*mu2^2))
+ }
> # log posterior distribution up to a constant
> posterior <- function(param, x, y){
+    return (likelihood(param, x, y) + prior(param))
+ }
> # proposal function
> proposalfunction <- function(param, delta){
+   mu1 = param[1]
+   mu2 = param[2]
+   new.mu1 = rnorm(1, mu1, delta)
+   new.mu2 = rnorm(1, mu2, delta)
+
+   return(as.vector(c(new.mu1, new.mu2)))
+ }
> # log prob of proposal function
> proposal.prob <- function(old.param, new.param, delta){
+   mu1 = old.param[1]
+   mu2 = old.param[2]
+   new.mu1 = new.param[1]
+   new.mu2 = new.param[2]
+
+   return(dnorm(new.mu1, mean = mu1, sd = delta, log = T) +
+            dnorm(new.mu2, mean = mu2, sd= delta, log=T))
+ }
> # Metropolis algorithm
> run_metropolis_MCMC <- function(x, y, startvalue, iterations, delta){
+     chain = array(dim = c(iterations,2))
+     chain[1,] = startvalue
+     for (i in 1:(iterations-1)){
+         proposal = proposalfunction(chain[i,], delta)
+         probab = exp(posterior(proposal,x, y) +
+                         proposal.prob(proposal, chain[i,], delta) -
+                         posterior(chain[i,], x, y) -
+                         proposal.prob(chain[i,], proposal, delta))
+         if (runif(1) < probab){
+             chain[i+1,] = proposal
+         }else{
+             chain[i+1,] = chain[i,]
+         }
+     }
+     return(chain)
+ }
```

```
> # number of iterations
> nreps <- 2000
> # run two MH chains
> MHS1 = run_metropolis_MCMC(x, y, startvalue = c(0,0), nreps, delta = 1/10)
> MHS2 = run_metropolis_MCMC(x, y, startvalue = c(2,-1), nreps, delta = 1/10)
```

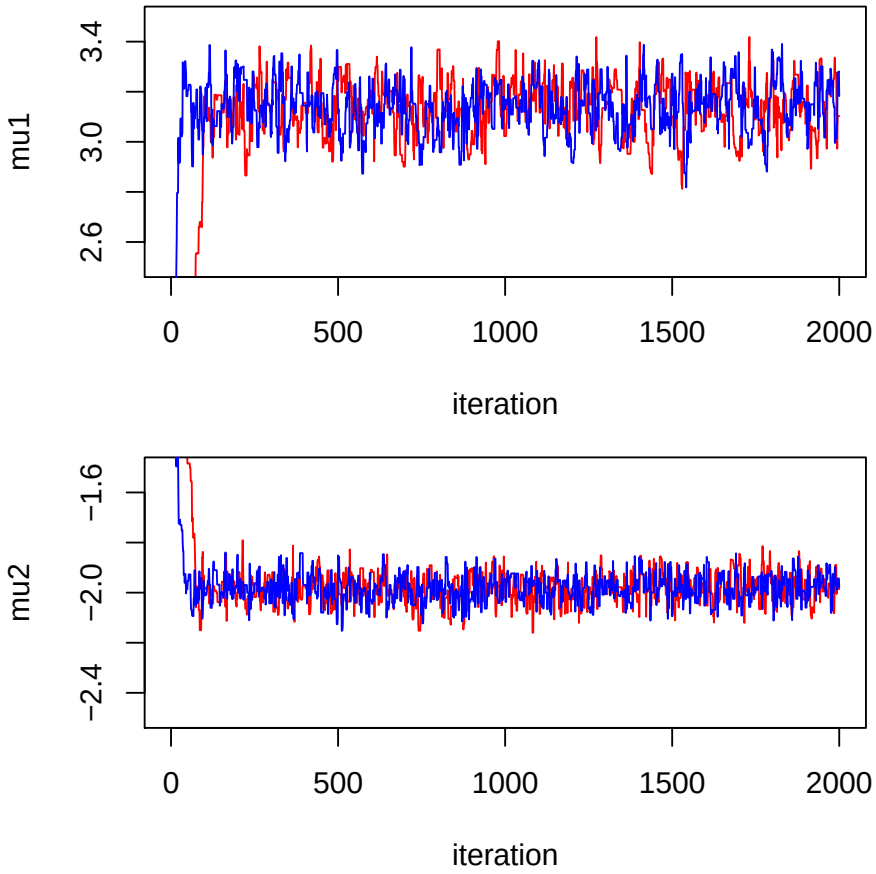
Make a trace plot for each of parameters.

```
> par(mfrow=c(2,1), mar=c(4,4,1,1))
> plot(1:nreps, MHS1[,1], type="l", col="red",
+      ylim = c(min(MHS1[,1],MHS2[,1]), max(MHS1[,1],MHS2[,1])),
+      xlab = "iteration", ylab ="mu1")
> points(1:nreps, MHS2[,1], type="l", col="blue")
> plot(1:nreps, MHS1[,2], type="l", col="red",
+      ylim = c(min(MHS1[,2],MHS2[,2]), max(MHS1[,2],MHS2[,2])),
+      xlab = "iteration", ylab ="mu2")
> points(1:nreps, MHS2[,2], type="l", col="blue")
```



Let's zoom in on the trace plot.

```
> par(mfrow=c(2,1), mar=c(4,4,1,1))
> plot(1:nreps, MHS1[,1], type="l", col="red", ylim = c(2.5, 3.5),
+      xlab = "iteration", ylab ="mu1")
> points(1:nreps, MHS2[,1], type="l", col="blue")
> plot(1:nreps, MHS1[,2], type="l", col="red", ylim = c(-2.5, -1.5),
+      xlab = "iteration", ylab ="mu2")
> points(1:nreps, MHS2[,2], type="l", col="blue")
```

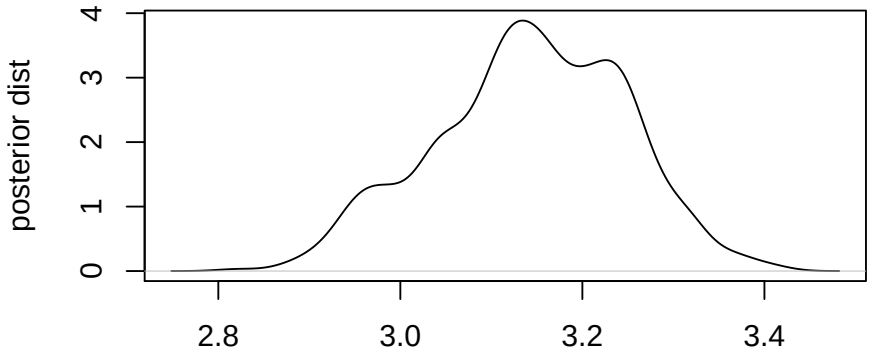


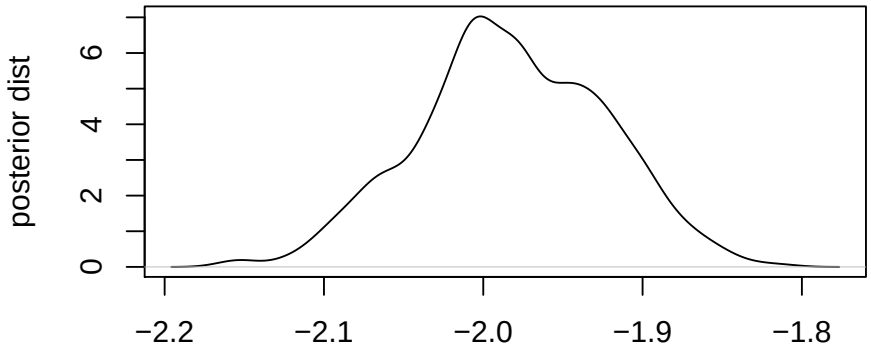
The trace plots show that samples from different chains are mixed well and behave similarly after 2000 iterations.

(e) **Solution** We will remove the first 300 samples as burn-in period.

1) make a plot that shows empirical (estimated) marginal posterior distribution.

```
> par(mfrow=c(2,1), mar=c(4,4,1,1))
> plot(density(MHS1[-(1:300),1]), ylab="posterior dist", xlab="mu1", main="")
> plot(density(MHS1[-(1:300),2]), ylab="posterior dist", xlab="mu2", main="")
```





2) estimate marginal posterior mean.

```
> # for mu1
> mean(MHS1[-(1:300),1])
[1] 3.142711
> # for mu2
> mean(MHS1[-(1:300),2])
[1] -1.982071
```

3) report a 90% credible interval for the marginal posterior distribution.

```
> # for mu1
> quantile(MHS1[-(1:300),1], probs=c(0.05, 0.95))
      5%      95%
2.955025 3.305431
> # for mu2
> quantile(MHS1[-(1:300),2], probs=c(0.05, 0.95))
       5%       95%
-2.081424 -1.889318
```

Posterior inference using Variational Inference (VI)

(f) **Solution** From problem 1, we have

$$\begin{aligned}&\log p(\mu_1, \mu_2, x_1, \cdots, x_n, y_1, \cdots, y_m)\\ =&-\frac{1}{2}\left(\sum_{i=1}^{n}(x_i-\mu_1)^2+2\sum_{i=1}^{m}(y_i-\mu_2)^2+(3\mu_1^2+4\mu_1\mu_2+3\mu_2^2)\right)+\text{const}\end{aligned}$$

Then

$$\begin{aligned}\log q^*_{\mu_1}(\mu_1) &= -\frac{1}{2}E_{\mu_2}\left[\sum_{i=1}^{n}(x_i-\mu_1)^2+2\sum_{i=1}^{m}(y_i-\mu_2)^2+(3\mu_1^2+4\mu_1\mu_2+3\mu_2^2)\right]+\text{const}\\ &= -\frac{1}{2}\left(\sum_{i=1}^{n}(x_i-\mu_1)^2+E_{\mu_2}\left[2\sum_{i=1}^{m}(y_i-\mu_2)^2\right]+3\mu_1^2+4\mu_1E_{\mu_2}(\mu_2)\right)+\text{const}\\ &= -\frac{1}{2}\left((n+3)\mu_1^2-2\left(\sum_{i=1}^{n}x_i-2E_{\mu_2}(\mu_2)\right)\mu_1\right)+\text{const}\end{aligned}$$

Hence, $q^*_{\mu_1}(\mu_1)$ is the pdf of $N(\mu_1^*,\sigma_1^{2*})$, where $\mu_1^*=\frac{\sum_{i=1}^{n}x_i-2E_{\mu_2}(\mu_2)}{n+3}$ and $\sigma_1^{2*}=\frac{1}{n+3}$. Similarly,

$$\begin{aligned}\log q^*_{\mu_2}(\mu_2) &= -\frac{1}{2}E_{\mu_1}\left[\sum_{i=1}^{n}(x_i-\mu_1)^2+2\sum_{i=1}^{m}(y_i-\mu_2)^2+(3\mu_1^2+4\mu_1\mu_2+3\mu_2^2)\right]+\text{const}\\ &= -\frac{1}{2}\left(E_{\mu_1}\left[\sum_{i=1}^{n}(x_i-\mu_1)^2\right]+2\sum_{i=1}^{m}(y_i-\mu_2)^2+4E_{\mu_1}(\mu_1)\mu_2+3\mu_2^2\right)+\text{const}\\ &= -\frac{1}{2}\left((2m+3)\mu_2^2-2\left(2\sum_{i=1}^{m}y_i-2E_{\mu_1}(\mu_1)\right)\mu_2\right)+\text{const}\end{aligned}$$

Hence, $q^*_{\mu_2}(\mu_2)$ is the pdf of $N(\mu_2^*,\sigma_2^{2*})$, where $\mu_2^*=\frac{2\sum_{i=1}^{m}y_i-2E_{\mu_1}(\mu_1)}{2m+3}$ and $\sigma_2^{2*}=\frac{1}{2m+3}$.

(g) **Solution**

$$\begin{aligned}&ELBO(q^*_{\mu_1}(\mu_1),q^*_{\mu_2}(\mu_2))=ELBO(\mu_1^*,\mu_2^*)\\ =&E_{\mu_1,\mu_2}\left[\log p(\mu_1,\mu_2,x_1,\cdots,x_n,y_1,\cdots,y_m)\right]-E_{\mu_1,\mu_2}\left[\log(q^*_{\mu_1}(\mu_1))\right]-E_{\mu_1,\mu_2}\left[\log(q^*_{\mu_2}(\mu_2))\right]\end{aligned}$$

Each term is computed as follow,

$$\begin{aligned}&E_{\mu_1,\mu_2}\left[\log p(\mu_1,\mu_2,x_1,\cdots,x_n,y_1,\cdots,y_m)\right]\\ =&-\frac{1}{2}E_{\mu_1,\mu_2}\left[\sum_{i=1}^{n}(x_i-\mu_1)^2+2\sum_{i=1}^{m}(y_i-\mu_2)^2+(3\mu_1^2+4\mu_1\mu_2+3\mu_2^2)\right]+\text{const}\\ =&-\frac{1}{2}\left[\sum_{i=1}^{n}E_{\mu_1}(x_i-\mu_1)^2+2\sum_{i=1}^{m}E_{\mu_2}(y_i-\mu_2)^2+3(\sigma_1^{2*}+\mu_1^{*2})+4\mu_1^*\mu_2^*+3(\sigma_2^{2*}+\mu_2^{*2})\right]+\text{const}\\ =&-\frac{1}{2}\left[\sum_{i=1}^{n}\left(\sigma_1^{2*}+\mu_1^{*2}-2x_i\mu_1^*\right)+2\sum_{i=1}^{m}\left(\sigma_2^{2*}+\mu_2^{*2}-2y_i\mu_2^*\right)+3(\sigma_1^{2*}+\mu_1^{*2})+4\mu_1^*\mu_2^*+3(\sigma_2^{2*}+\mu_2^{*2})\right]+\text{con}\end{aligned}$$

$$\begin{aligned}&E_{\mu_1,\mu_2}\left[\log(q^*_{\mu_1}(\mu_1))\right]\\ =&-\frac{1}{2}E_{\mu_1,\mu_2}\left[\frac{(\mu_1-\mu_1^*)^2}{\sigma_1^{2*}}\right]+\text{const}\\ =&-\frac{1}{2}+\text{const}\end{aligned}$$

$$\begin{aligned}&E_{\mu_1,\mu_2}\left[\log(q^*_{\mu_2}(\mu_2))\right]\\ =&-\frac{1}{2}E_{\mu_1,\mu_2}\left[\frac{(\mu_2-\mu_2^*)^2}{\sigma_2^{2*}}\right]+\text{const}\\ =&-\frac{1}{2}+\text{const}\end{aligned}$$

(h) **Solution**

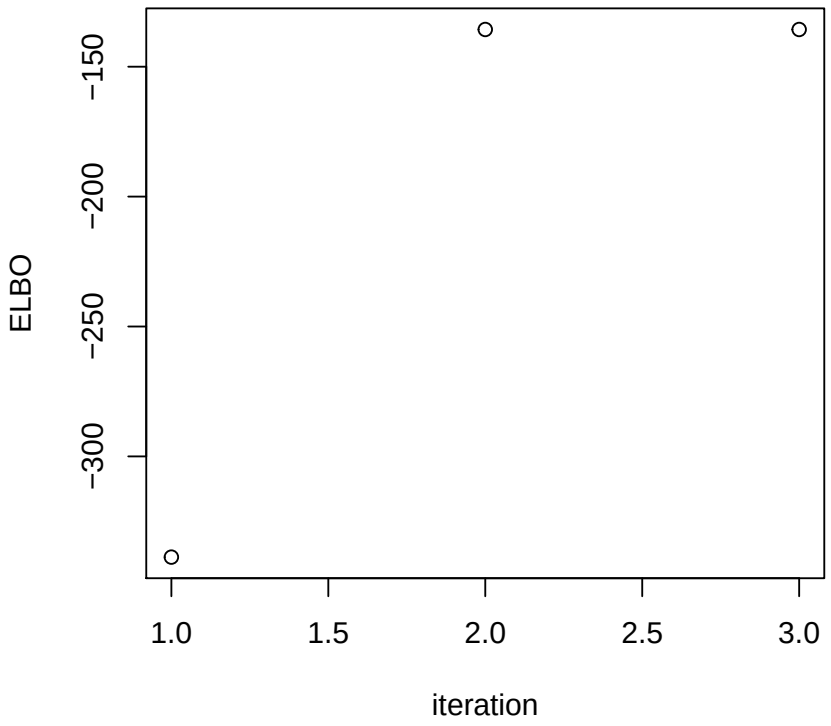
Implementation CAVI algorithm

```
> # x y : data
> # initial values for mu1*, sigma1.2*, mu2*, sigma2.2*:
> #           mu1.vi.init, sigma1.2.vi.init, mu2.vi.init, sigma2.2.vi.init
> # epsilon : If the ELBO has changed by less than epsilon,
> #           the CAVI algorithm will stop
> # max.iter: maximum number of iteration
> cavi.normal <- function(x, y, mu1.vi.init, sigma1.2.vi.init,
+                         mu2.vi.init, sigma2.2.vi.init, epsilon=1e-5, max.iter=100){
+
+   n = length(x)
+   m = length(y)
+
+   mu1.vi = mu1.vi.init
+   sigma1.2.vi = sigma1.2.vi.init
+   mu2.vi = mu2.vi.init
+   sigma2.2.vi = sigma2.2.vi.init
+
+   # store the ELBO for each iteration
+   elbo = c()
+
+   # I will store mu1*, sigma1.2*, mu2*, sigma2.2* for each iteration
+   mu1.vi.list = sigma1.2.vi.list = mu2.vi.list = sigma2.2.vi.list = c()
+
+   # compute the ELBO using initial values of mu1*, sigma1.2*, mu2*, sigma2.2*
+   Elogq.mu1 = Elogq.mu2 = -1/2
+   # Elogp.x.y.mu1.mu2
+   A = sigma1.2.vi + mu1.vi^2 - 2*x*mu1.vi + x*x
+   B = sigma2.2.vi + mu2.vi^2 - 2*y*mu2.vi + y*y
+   Elogp.x.y.mu1.mu2 = -0.5*sum(A) - sum(B) -
+     0.5*(3*(sigma1.2.vi + mu1.vi^2) + 4*mu1.vi*mu2.vi + 3*(sigma2.2.vi + mu2.vi^2))
+
+   elbo = c(elbo, Elogp.x.y.mu1.mu2 -Elogq.mu1 - Elogq.mu2)
+
+   mu1.vi.list = c(mu1.vi.list, mu1.vi)
+   sigma1.2.vi.list = c(sigma1.2.vi.list, sigma1.2.vi)
+   mu2.vi.list = c(mu2.vi.list, mu2.vi)
+   sigma2.2.vi.list = c(sigma2.2.vi.list, sigma2.2.vi)
+
+   # set the change in the ELBO with 1
+   delta.elbo = 1
+
+   # number of iteration
+   n.iter = 1
+
+   # If the elbo has changed by less than epsilon, the CAVI will stop.
+   while((delta.elbo > epsilon) & (n.iter <= max.iter)){
+
+     # Update mu1.vi and sigma1.2.vi
+     mu1.vi = (sum(x) - 2*mu2.vi)/(n + 3)
+     sigma1.2.vi = 1/(n+3)
+
+     # Update mu2.vi and sigma2.2.vi
+     mu2.vi = (2*sum(y) - 2*mu1.vi)/(2*m + 3)
```

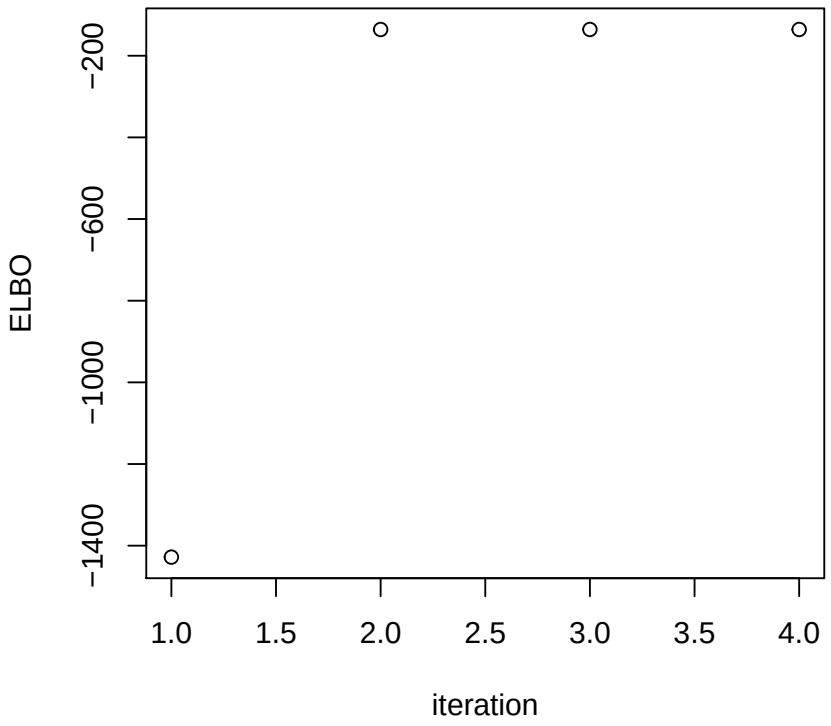
```
+     sigma2.2.vi = 1/(2*m + 3)
+
+     # compute the ELBO using the current values of mu1*, sigma1.2*, mu2*, sigma2.2*
+     Elogq.mu1 = Elogq.mu2 = -1/2
+     # Elogp.x.y.mu1.mu2
+     A = sigma1.2.vi + mu1.vi^2 - 2*x*mu1.vi + x*x
+     B = sigma2.2.vi + mu2.vi^2 - 2*y*mu2.vi + y*y
+     Elogp.x.y.mu1.mu2 = -0.5*sum(A) - sum(B) -
+       0.5*(3*(sigma1.2.vi + mu1.vi^2) + 4*mu1.vi*mu2.vi + 3*(sigma2.2.vi + mu2.vi^2))
+
+     elbo = c(elbo, Elogp.x.y.mu1.mu2 -Elogq.mu1 - Elogq.mu2)
+
+     mu1.vi.list = c(mu1.vi.list, mu1.vi)
+     sigma1.2.vi.list = c(sigma1.2.vi.list, sigma1.2.vi)
+     mu2.vi.list = c(mu2.vi.list, mu2.vi)
+     sigma2.2.vi.list = c(sigma2.2.vi.list, sigma2.2.vi)
+
+     # compute the change in the elbo
+     delta.elbo = elbo[length(elbo)]  - elbo[length(elbo)-1]
+
+     # increase the number of iteration
+     n.iter = n.iter + 1
+   }
+
+   return(list(elbo = elbo,
+               mu1.vi.list = mu1.vi.list, sigma1.2.vi.list=sigma1.2.vi.list,
+               mu2.vi.list = mu2.vi.list, sigma2.2.vi.list=sigma2.2.vi.list))
+ }
```

Applying the implemented algorithm. Run the CAVI algorithm with different initial values and check that the ELBO increases at each step by plotting them.

```
> cavi1 = cavi.normal(x, y,
+                     mu1.vi.init = 3, sigma1.2.vi.init = 1,
+                     mu2.vi.init = -2, sigma2.2.vi.init = 1,
+                     epsilon=1e-5, max.iter=100)
> cavi.res = cavi1
> cavi.res$elbo
[1] -338.7661 -135.6699 -135.6699
> plot(cavi.res$elbo, ylab='ELBO', xlab='iteration')
> print(paste("mu1* and sigma1.2* = (",
+             round(cavi.res$mu1.vi.list[length(cavi.res$mu1.vi.list)],2), ",",
+             round(cavi.res$sigma1.2.vi.list[length(cavi.res$sigma1.2.vi.list)],4),
+             ")", sep=""))
[1] "mu1* and sigma1.2* = (3.14,0.0097)"
> print(paste("mu2* and sigma2.2* = (",
+             round(cavi.res$mu2.vi.list[length(cavi.res$mu2.vi.list)],2), ",",
+             round(cavi.res$sigma2.2.vi.list[length(cavi.res$sigma2.2.vi.list)],4),
+             ")", sep=""))
[1] "mu2* and sigma2.2* = (-1.98,0.0033)"
```




```
> cavi2 = cavi.normal(x, y,
+                     mu1.vi.init = 0, sigma1.2.vi.init = 1,
+                     mu2.vi.init = 0, sigma2.2.vi.init = 1,
+                     epsilon=1e-5, max.iter=100)
> cavi.res = cavi2
> cavi.res$elbo
[1] -1428.0667  -135.7461  -135.6699  -135.6699
> plot(cavi.res$elbo, ylab='ELBO', xlab='iteration')
> print(paste("mu1* and sigma1.2* = (",
+             round(cavi.res$mu1.vi.list[length(cavi.res$mu1.vi.list)],2), ",",
+             round(cavi.res$sigma1.2.vi.list[length(cavi.res$sigma1.2.vi.list)],4),
+             ")", sep=""))
[1] "mu1* and sigma1.2* = (3.14,0.0097)"
> print(paste("mu2* and sigma2.2* = (",
+             round(cavi.res$mu2.vi.list[length(cavi.res$mu2.vi.list)],2), ",",
+             round(cavi.res$sigma2.2.vi.list[length(cavi.res$sigma2.2.vi.list)],4),
+             ")", sep=""))
[1] "mu2* and sigma2.2* = (-1.98,0.0033)"
```



The two CAVI runs have equally highest ELBO. You can see that approximated posterior distributions from the runs are the same. I will use the output from the first run: $q^*_{\mu_1}(\mu_1)$ is a pdf of $\mathrm{N}(3.14, 0.0097)$ and $q^*_{\mu_2}(\mu_2)$ is a pdf of $\mathrm{N}(-1.98, 0.0033)$.