THE UNIVERSITY OF MELBOURNE

Student number 1079860

Semester 2 Assessment, 2020

School of Mathematics and Statistics

MAST10006 Calculus 2

Reading time: 30 minutes — Writing time: 3 hours — Upload time: 30 minutes

This exam consists of 24 pages (including this page)

Permitted Materials

- This exam and/or an offline electronic PDF reader, one or more copies of the masked exam template made available earlier and blank loose-leaf paper.
- One double sided A4 page of notes (handwritten or printed).
- Calculators are not permitted.

Instructions to Students

- There are 12 questions with marks as shown. The total number of marks available is 127.
- During writing time you may only interact with the device running the Zoom session with supervisor permission. The screen of any other device must be visible in Zoom from the start of the session.
- If you have a printer, print the exam one-sided. If you cannot print, download the exam to a second device, which must then be disconnected from the internet.
- Write your answers in the boxes provided on the exam that you have printed or the masked exam template that has been previously made available. If you are unable to answer the whole question in the answer space provided then you can append additional handwritten solutions to the end after the 24 numbered pages. If you do this you MUST make a note in the correct answer space or page for the question, warning the marker that you have appended additional remarks at the end.
- If you have been unable to print the exam and do not have the masked template write your answers on A4 paper. The first page should contain only your student number, the subject code and the subject name. Write on one side of each sheet only. Start each question on a new page and include the question number at the top of each page.
- Assemble all exam pages (or masked template pages) in correct page number order and the correct way up. Add any extra pages with additional working at the end. Use a mobile phone scanning application to scan all pages to a single PDF file. Scan from directly above to reduce keystone effects. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Submit your PDF file to the Canvas Assignment corresponding to this exam using the Gradescope window. Before leaving Zoom supervision, confirm with your Zoom supervisor that you have Gradescope confirmation of submission.

Question 1 (11 marks)

Evaluate the following limits, if they exist.
In this question you must state if you use any standard limits, limit laws, continuity, l'Hôpital's rule or the sandwich theorem

(a) $\lim_{x\to\frac{3\pi}{2}} \left(\sin\left(x+\cos(x)\right)\right)$

$= \sin\left(\left(\lim_{x\to\frac{3\pi}{2}} x\right) + \cos\left(\lim_{x\to\frac{3\pi}{2}} x\right)\right)$, limit laws and continuity of sin and cos

$= \sin\left(\frac{3\pi}{2} + \cos\left(\frac{3\pi}{2}\right)\right)$, limit laws

$= \sin\left(\frac{3\pi}{2}\right)$

$= -1$

(b) $\lim_{n\to\infty} \left(e^{-3}\sin(\pi n)\right) \qquad n \in \mathbb{N}$

$-1 \le \sin(\pi n) \le 1$

$\Rightarrow -e^{-3} \le e^{-3}\sin(\pi n) \le e^{-3}$

The limit does not exist as $e^{-3}\sin(\pi n)$ oscillates between e^{-3} and $-e^{-3}$ and therefore diverges.

(c) $\lim_{x \to 0} \frac{3\sin(2x)}{6x - 4x^3}$

indeterminate form $\frac{0}{0}$

$= \lim_{x \to 0} \frac{6\cos(2x)}{6 - 12x^2}$, L'Hôpital's Rule

$= \frac{6\cos\left(\left(\lim_{x \to 0} x\right) \cdot 2\right)}{6 - 12\left(\lim_{x \to 0} x\right)^2}$, limit laws, continuity of polynomials and cos

$= \frac{6\cos(0)}{6}$, limit laws

$= 1$

(d) $\lim_{x \to \infty} \frac{e^{\cos\left(\frac{1}{x}\right)}}{x}$

$= \frac{\lim_{x \to \infty} \left(e^{\cos\left(\frac{1}{x}\right)}\right)}{\lim_{x \to \infty} (x)}$, limit laws

$= \frac{e^{\cos\left(\lim_{x \to \infty} \frac{1}{x}\right)}}{\lim_{x \to \infty} (x)}$, continuity of a^x for $a > 0$ and cos, limit laws

$= \frac{e^{\cos(0)}}{\lim_{x \to \infty} (x)}$, standard limit "$\lim_{x \to \infty} \frac{1}{x^p} = 0$, $p > 0$" limit laws

$= e \lim_{x \to \infty} \left(\frac{1}{x}\right)$, standard limit "$\lim_{x \to \infty} \frac{1}{x^p} = 0$, $p > 0$", limit laws

$= 0$

Question 2 (11 marks)

(a) Compute the first four partial sums of $\sum_{n=1}^{\infty}(n^2+1)$.

$a_1 = (1^2+1) = 2$

$a_2 = (2^2+1)+a_1$

$= 5+2 = 7$

$a_3 = (3^2+1)+a_2$

$= 10+7 = 17$

$a_4 = (4^2+1)+a_3$

$= 17+17 = 34$

(b) For the following series, indicate whether they are convergent or divergent. Justify your answer with any relevant tests that you use.

(i) $\sum_{n=1}^{\infty} \frac{5^{2n+1}}{(2n+1)!}$

$n > 0$, so this is a positive term series

Ratio test:

$$\frac{5^{2(n+1)+1}}{(2(n+1)+1)!} \times \frac{(2n+1)!}{5^{2n+1}}$$

$$= \frac{5^{2n+3}}{(2n+3)!} \times \frac{(2n+1)(2n)!}{5^{2n+1}}$$

$$= \frac{5^2}{(2n+3)(2n+2)(2n+1)(2n)!} \times \frac{(2n+1)(2n)!}{1}$$

$$= \frac{25}{(2n+3)(2n+2)} = \frac{25}{4n^2+4n+6n+6}$$

$(\times \frac{1}{n^2})$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{\frac{25}{n^2}}{4+\frac{10}{n}+\frac{6}{n^2}} \right) = \frac{0}{4+0+0} = 0 < 1$$

Therefore the series is Convergent by the ratio test

standard limit "$\lim_{n\to\infty} \frac{1}{n^p} = 0$"

limit laws

(ii) $\sum_{n=1}^{\infty} \frac{n+1}{2n^2+1}$ $\gtrsim$ $\frac{n}{2n^2}$ $\simeq$ $\frac{1}{2n}$ expect divergence

$$\frac{n+1}{2n^2+1} \geq \frac{n}{2n^2+n^2}$$

, positive term series as $n > 0$

$$\geq \frac{n}{3n^2}$$

$\left(\frac{1}{3}\,\frac{1}{n}\right)$

$$\geq \frac{1}{3n}$$

$\sum_{n=1}^{\infty} \frac{1}{3n}$ is a harmonic p-series with $p \leq 1$ and therefore diverges

therefore

$\sum_{n=1}^{\infty} \frac{n+1}{2n^2+1}$ diverges by the comparison test

Question 3 (10 marks)

Let $z = a + ib$ be a complex number, $a, b \in \mathbb{R}$.

(a) For $t \in \mathbb{R}$, express the complex numbers e^{zt} and ze^{zt} in cartesian form.

$$e^{zt} = e^{(a+ib)t}$$
$$= e^{at}e^{ibt} = e^{at}\cos(bt) + e^{at}i\sin(bt)$$
$$\Rightarrow ze^{zt} = (a+ib)e^{at}(\cos(bt) + i\sin(bt))$$
$$= ae^{at}\cos(bt) + ibe^{at}\cos(bt)$$
$$+ ae^{at}i\sin(bt) - be^{at}\sin(bt)$$

(b) For differentiable functions $f, g : \mathbb{R} \to \mathbb{R}$, and $t \in \mathbb{R}$, we define

$$\frac{d}{dt}(f(t) + g(t)i) = f'(t) + g'(t)i.$$

Using this definition and the cartesian forms you determined in (a), prove

$$\frac{d}{dt}e^{zt} = ze^{zt}.$$

$$\frac{d}{dt}\left(e^{at}\cos(bt) + e^{at}i\sin(bt)\right)$$
$$= \frac{d}{dt}\left(e^{at}\cos(bt)\right) + \frac{d}{dt}\left(e^{at}i\sin(bt)\right)$$
$$= \left(ae^{at}\cos(bt) - be^{at}\sin(bt)\right) + \left(ae^{at}i\sin(bt) + ibe^{at}\cos(bt)\right), \quad \text{product rule}$$
$$= ae^{at}\cos(bt) + ibe^{at}\cos(bt) + ae^{at}i\sin(bt)$$
$$- be^{at}\sin(bt)$$
$$= ze^{zt} = \text{RHS}$$

(c) Name two real world applications where computations with the complex exponential become useful and explain why.

Electrical engineering
and
physics as it makes
calculations in these fields
easier to manipulate and
work with

Question 4 (11 marks)

Evaluate the following integrals:

(a) $\int \cosh(4x)\exp(2\sinh(4x))\,dx$

$= \int \cosh(4x)\, e^{2\sinh(4x)}\, dx$ let $u = 2\sinh(4x)$

$\frac{du}{dx} = 8\cosh(4x)$

$\frac{1}{8}\frac{du}{dx} = \cosh(4x)$

$= \int \frac{1}{8}\frac{du}{dx} e^{u}\, dx$

$= \frac{1}{8}\int e^{u}\, du$, derivative substitution

$= \frac{1}{8} e^{u}$

$= \frac{1}{8} e^{2\sinh(4x)}$

(b) $\int e^{x}\cos(7x)\,dx$

$e^{x}\cos(7x) = e^{x}\mathrm{Re}(e^{i7x}) = \mathrm{Re}(e^{x}e^{i7x})$, $e^{x} \in \mathbb{R}$

$\Rightarrow \int e^{x}\cos(7x) = \mathrm{Re}\int e^{(1+7i)x}\, dx$

$= \mathrm{Re}\left[\frac{1}{1+7i} e^{x}(\cos(7x) + i\sin(7x))\right] + C$

$= \mathrm{Re}\left[\frac{e^{x}}{50}(1-7i)(\cos(7x) + i\sin(7x))\right] + C$

$= \mathrm{Re}\left[\frac{e^{x}}{50}\left(\cos(7x) - 7i\cos(7x) + i\sin(7x) + 7\sin(7x)\right)\right] + C$

$= \frac{e^{x}}{50}(\cos(7x) + 7\sin(7x))$

$= \frac{e^{x}}{50}\cos(7x) + \frac{7e^{x}}{50}\sin(7x)$

complex exponential

(c) $\int \operatorname{arcsinh}(x)\, dx$

$= \int \operatorname{arcsinh}(x)$

Let $u = \operatorname{arcsinh}(x)$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{x^2+1}}$ $\quad v = x$, Integration by parts

$= x \operatorname{arcsinh}(x) - \int \frac{x}{\sqrt{x^2+1}}\, dx$

$- \int \frac{1}{\sqrt{u}} \frac{1}{2} \frac{du}{dx}$, derivative substitution

$- \frac{1}{2} \int u^{-1/2}\, du$

$- \frac{1}{2} \times 2u^{1/2}$

$\Rightarrow \int \operatorname{arcsinh}(x)$

$= x \operatorname{arcsinh}(x) - \sqrt{x^2+1} + C$

Question 5 (14 marks)

Consider the curve C in the x-y plane described by the equations $x = 5\cos(\theta)$, $y = 4\sin(\theta)$, where $0 \leq \theta \leq 2\pi$.

(a) Find a relation relating y and x.

$\frac{x^2}{25} + \frac{y^2}{16} = 1$, trig identity ($\cos^2\theta + \sin^2\theta = 1$)

(b) Sketch the graph of the curve C in the x-y plane.

x-int

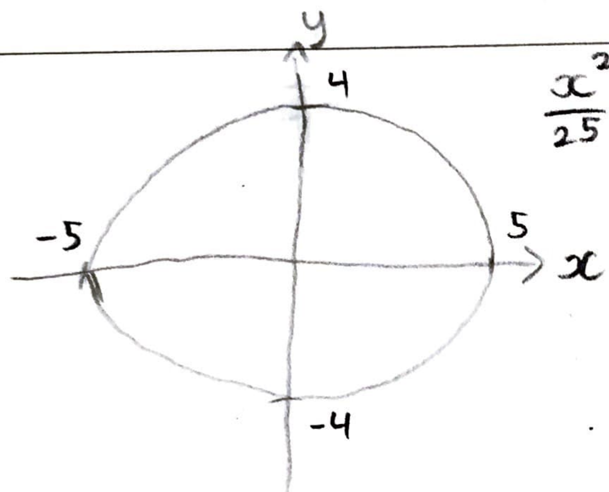
$\frac{x^2}{25} = 1$

$x = \pm 5$

y-int

$\frac{y^2}{16} = 1$

$y = \pm 4$



$\frac{x^2}{25} + \frac{y^2}{16} = 1$

(c) Find functions f and g such that

$$C = \{(x, y) \mid y = f(x)\} \cup \{(x, y) \mid y = g(x)\}.$$

In other words, express C as the union of graphs of functions of x.

(d) Evaluate $\int \sqrt{25 - s^2}\, ds$, carefully documenting each step of your calculation.
Hence, calculate the area enclosed by C.

Let $s = 5\sin\theta$, $\sin\theta = \frac{s}{5}$

Then $\theta = \arcsin\left(\frac{s}{5}\right)$ is valid when $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$-1 \le \frac{s}{5} \le 1$

$-5 \le s \le 5$

• $\frac{dx}{d\theta} = 5\cos\theta$

• $\sqrt{25-s^2} = \sqrt{25 - 25\sin^2\theta} = \sqrt{25(1-\sin^2\theta)}$
$= \sqrt{25\cos^2\theta} = 5|\cos\theta|$
$= 5\cos\theta$ as $\cos\theta \ge 0$ for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Trigonometric substitution

$\Rightarrow \int \sqrt{25-s^2}\, ds = \int 5\cos\theta \cdot 5\cos\theta \, d\theta$

$= \int 25\cos^2\theta \, d\theta$

$= 25\int \cos^2\theta \, d\theta$

$= \frac{25}{2}\int \cos(2\theta)+1 \, d\theta$, Double Angle Formula "$\cos(2\theta) = 2\cos^2\theta - 1$"

$= \frac{25}{2}\left[\frac{1}{2}\sin(2\theta) + \theta\right] + C$

$= \frac{25}{2}\left[\sin\theta\cos\theta + \theta\right] + C$, Double Angle formula "$\sin(2\theta) = 2\sin\theta\cos\theta$"

$= \frac{25}{2}\left[\sin\theta\sqrt{1-\sin^2\theta} + \theta\right] + C$

$= \frac{25}{2}\left[\frac{s}{5}\sqrt{1 - \frac{s^2}{25}} + \arcsin\left(\frac{s}{5}\right)\right] + C$

$= \frac{25}{2}\left[\frac{s}{25}\sqrt{25-s^2} + \arcsin\left(\frac{s}{5}\right)\right] + C$

$= \frac{s}{2}\sqrt{25-s^2} + \frac{25}{2}\arcsin\left(\frac{s}{5}\right) + C$

$A = 2\left[\frac{s}{2}\sqrt{25-s^2} + \frac{25}{2}\arcsin\left(\frac{s}{5}\right)\right]_{-5}^{5}$

$A = 2\left[\left(\frac{5}{2}\sqrt{0} + \frac{25}{2}\arcsin(1)\right) - \left(\frac{-5}{2}\sqrt{0} + \frac{25}{2}\arcsin(-1)\right)\right]$

$= 2(25\pi - 75\pi) = -100\pi = \boxed{100\pi}$

Question 6 (6 marks)

Consider the ODE

$$\frac{dy}{dx} = y(y-5)(y-2)^2.$$

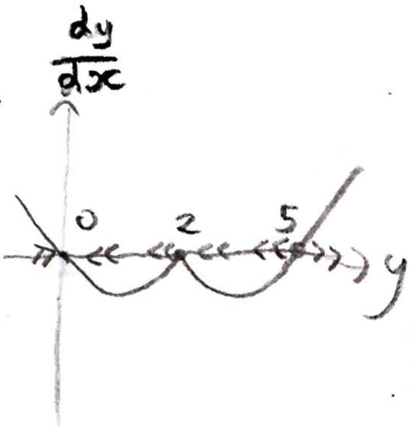
(a) Find and classify the equilibrium solutions.

equilibrium solutions occur when $\frac{dy}{dx} = 0$

$\Rightarrow y(y-5)(y-2)^2 = 0$

$y=0$, $y=5$, $y=2$

- $y<0$, $\frac{dy}{dx} > 0$
- $y \in (0,2)$, $\frac{dy}{dx} < 0$
- $y \in (2,5)$, $\frac{dy}{dx} < 0$
- $y > 5$, $\frac{dy}{dx} > 0$



Therefore $y=0$ is a stable equilibria, $y=2$ is a semistable equilibria, $y=5$ is an unstable equilibria

(b) Describe the long term behaviour of the solution to this ODE with the initial condition $y(0) = \pi$.

The solution to this ODE will approach $y=2$ as $x \to \infty$ with the initial condition $y(0) = \pi$

Question 7 (8 marks)

Consider the Initial Value Problem

$$\frac{dy}{dx} + xy = xy^3, \qquad y(0) = \frac{1}{2}.$$

(a) By making the substitution $z = y^{-2}$, show that the given ODE reduces to

$$\frac{dz}{dx} - 2xz = -2x.$$

$y = z^{-1/2}$

$\frac{dy}{dx} = -\frac{1}{2} z^{-3/2} \cdot \frac{dz}{dx}$

sub into ODE:

$= -\frac{1}{2} z^{-3/2} \frac{dz}{dx} + x z^{-1/2} = x z^{-3/2}$

$(\times -2) = z^{-3/2} \frac{dz}{dx} - 2x z^{-1/2} = -2x z^{-3/2}$

$(\div z^{-3/2}) = \frac{dz}{dx} - 2xz = -2x$ ✓

(b) Hence solve the original initial value problem.

$\frac{dz}{dx} - 2xz = -2x$, first order linear ODE - use integrating factor

$I = e^{\int -2x \, dx}$

$I = e^{-x^2}$

So multiply ODE by I:

$e^{-x^2} \frac{dz}{dx} + 2x e^{-x^2} z = -2x e^{-x^2}$,

$\frac{d}{dx}\left(e^{-x^2} z\right) = -2x e^{-x^2}$

$e^{-x^2} z = \int -2x e^{-x^2} \, dx$ $\quad u = -x^2, \ \frac{du}{dx} = -2x$

$= e^{-x^2} + C$, derivative substitution

$z = \frac{e^{-x^2} + C}{e^{-x^2}} = \frac{1}{e^{-x^2}}\left(e^{-x^2} + C\right)$

$z(0) = 4$ as $y(0) = \frac{1}{2}$

$4 = \frac{1}{e^0}\left(e^0 + C\right) \Rightarrow 4 = 1 + C \Rightarrow C = 3$

$\boxed{y = \sqrt{\frac{e^{-x^2}}{e^{-x^2} + 3}}}$

Question 8 (11 marks)

A tank contains a soluble fertilizer solution consisting initially of 20 kg of fertilizer dissolved in 10 gallons of water. Pure fresh water is being poured into the tank at a rate of 3 gallons/min and the solution (kept uniform by stirring) is flowing out at 2 gallons/min.

(a) Show that the amount of fertilizer in the tank $Q(t)$ is given by

$$\frac{dQ}{dt} = -\frac{2Q}{10+t}.$$

$C = \frac{Q}{V(t)}$, $V(t) = V_0 + V_{in} - V_{out} = 10 + 3t - 2t$
$= 10 + t$

$C = \frac{Q}{10+t}$

$\frac{dQ}{dt}$ = rate of inflow − rate of outflow
$= 3\text{gal/min} \times 0\% - 2\text{gal/min} \times \frac{Q}{10+t}$

$\frac{dQ}{dt} = -\frac{2Q}{10+t}$ ✓

(b) Find the amount of fertilizer in the tank after 5 minutes.

separable - use sep. of variables

$-\frac{1}{2Q}\frac{dQ}{dt} = \frac{1}{10+t}$, $2Q \neq 0$

$-\int \frac{1}{2Q}\, dQ = \frac{1}{10+t}\, dt$

$-\log(2Q) = \log(10+t) + C$, $2Q > 0$, $10+t > 0$

$e^{\log\left(\frac{1}{2Q}\right)} = e^{\log(10+t) + C}$

$\frac{1}{2Q} = A(10+t)$, $A = \pm e^C$

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More space for part (b)

$$2Q = \frac{1}{A(10+t)}$$

$$Q = \frac{1}{2A(10+t)}$$

$$Q(0) = 20 \Rightarrow 20 = \frac{1}{2A(10+0)}$$

$$20 = \frac{1}{20A}$$

$$400A = 1$$

$$A = \frac{1}{400}$$

$$Q = \frac{1}{\frac{1}{200}(10+t)} = \frac{1}{\frac{10+t}{200}} = \frac{200}{10+t}$$

So $Q(5) = \frac{200}{10+5} = \frac{200}{15} = \boxed{\frac{40}{3} \text{ kg of fertilizer}}$

(c) How long will it take to reach 25% of the initial amount of fertilizer in the tank?

$$Q(t) = \frac{1}{4} \times 20$$

$$Q(t) = 5$$

$$\Rightarrow 5 = \frac{200}{10+t}$$

$$50 + 5t = 200$$

$$5t = 150$$

$$t = 30$$

30 minutes to reach 25% of the initial amount of fertilizer

Question 9 (13 marks)

Consider a series RLC circuit with a resistor, an inductor and a capacitor with a driving electromotive force E. Set $L = 1, R = 4$ and $C = 1/4$.
The current equation is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = E. \qquad (*)$$

(a) Find the general solution to the corresponding homogeneous ODE.

$I'' + 4I' + 4I = 0$ (homogeneous ODE)

Try $I = e^{\lambda t}$, $I' = \lambda e^{\lambda t}$, $I'' = \lambda^2 e^{\lambda t}$

sub into ODE:

$e^{\lambda t}(\lambda^2 + 4\lambda + 4) = 0$

$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$, characteristic equation

$\Rightarrow \lambda = \frac{-4 \pm \sqrt{16-16}}{2} = \frac{-4}{2} = -2$, GS(H) $\Rightarrow I(t) = Ae^{-2t} + Bte^{-2t}$

(b) Suppose you wish to drive the system to produce in the long term an oscillatory current of the form $I_p = \sin t$.

(i) Find the electromotive force E that produces the particular solution $I_p = \sin t$.

(ii) Write down the steady state term for $I(t)$ in this case.

(i) $I_p = \sin t$

$I_p' = \cos t$

$I_p'' = -\sin t$

sub into ODE:

$\Rightarrow -\sin t + 4\cos t + 4\sin t = E$

$\Rightarrow E = 4\cos t + 3\sin t$

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More space for part (b).

(ii) $I(t) = Ae^{-2t} + Bte^{-2t} + \sin t$

the steady state term is $\sin(t)$ as $\sin(t)$ does not decay to 0 as $t \to \infty$ it simply oscillates between -1 and 1

(c) Write down the solution to the ODE $(*)$ with the electromotive force found in part (b), given the initial conditions $I(0) = 2$ and $\frac{dI}{dt}\big|_{t=0} = 4$.

$I(t) = Ae^{-2t} + Bte^{-2t} + \sin(t)$ ①

$\frac{dI}{dt} = -2Ae^{-2t} + Be^{-2t} - 2Bte^{-2t} + \cos(t)$ ②

① $2 = Ae^{0} + B \cdot 0 \cdot e^{0} + \sin(0)$

$\boxed{2 = A}$

② $4 = -4e^{0} + Be^{0} - 2Be^{0} + \cos(0)$

$4 = -4 - B + 1$

$\boxed{B = -7}$

Therefore $I(t) = 2e^{-2t} - 7te^{-2t} + \sin(t)$

Question 10 (13 marks)

Let

$$f(x,y) = xy^2 + x^3y.$$

(a) Calculate

$$\lim_{(x,y)\to(1,2)} f(x,y).$$

$$\lim_{(x,y)\to(1,2)} xy^2 + x^3y$$

$= 1\times(2)^2 + (1)^3\times 2$, polynomials are continuous

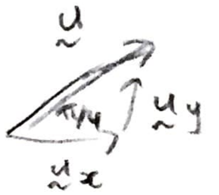
-limit laws

$= 1\times 4 + 1\times 2$

$= 6$

(b) Find the directional derivative of f at $(1,2)$ in the direction $\frac{\pi}{4}$ anticlockwise from the positive x-axis.

$$D_{\hat{u}} f\big|_{(1,2)} = \nabla f\big|_{(1,2)} \cdot \hat{u}$$



$\tan\frac{\pi}{4} = \frac{u_y}{u_x} \Rightarrow u_y = 1,\ u_x = 1$

$\Rightarrow u = (1,1)$, $\hat{u} = \frac{1}{\sqrt{2}}(1,1) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$f_x = y^2 + 3x^2y$

$f_y = 2xy + x^3$

$f_x(1,2) = 4 + 3\times 2 = 10$

$f_y(1,2) = 2\times1\times2 + 1 = 5$

$\Rightarrow \nabla f\big|_{(1,2)} = (10,5)$

$\Rightarrow D_{\hat{u}} f\big|_{(1,2)} = (10,5)\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$= \frac{10}{\sqrt{2}} + \frac{5}{\sqrt{2}}$

$= \frac{15}{\sqrt{2}}$

$= \boxed{\frac{15\sqrt{2}}{2}}$

(c) Starting at $(1,2)$, in which direction does f decrease the fastest? Give your answer as a unit vector.

f decreases fastest in the $-\nabla f$ direction

$-\nabla f = (-10, -5)$

$\hat{-\nabla f}$ $\frac{1}{\sqrt{100+25}}(-10,-5) = \frac{1}{\sqrt{125}}(-10,-5)$

$= \left(\frac{-10}{5\sqrt{5}}, -\frac{5}{5\sqrt{5}}\right)$

$= \left(-\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$

(d) Find $\frac{df}{dt}$ at $t = 0$ given

$$x(t) = t\cosh(2t) \quad \text{and} \quad y(t) = t^3e^t + 2e.$$

$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$= (y^2+3x^2y) \times (2t\sinh(2t)+\cosh(2t))$

$+ (2xy+x^3) \times (t^3e^t + 3t^2e^t)$

$(t=0)$

$x(0) = 0, \quad y(0) = 2e$

$\Rightarrow 4e^2 \times (0+1) + (0) \times (0^3e^0 + 3(0)^2e^0)$

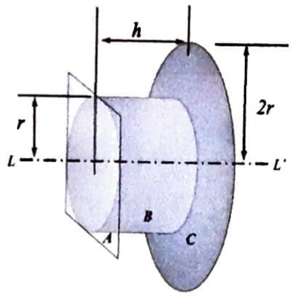
$\boxed{= 4e^2}$

Question 11 (12 marks)

We wish to design a peg to fit a square or a round hole as shown. The peg has three parts:

- a square end A on the left (side length $2r$ cm)
- a solid cylinder B in the middle (radius r cm and length h cm)
- a round circular end C on the right (radius $2r$ cm)

The centres of both ends and the axis of the middle cylinder must lie along the line $\overline{LL'}$.



(a) A, C and the curved side of the cylinder B are to be made from sheet plastic. Suppose that the cost in dollars of sheet plastic is given by

$$c(r,h) = \frac{a(r,h)}{r^2h} + rh, \qquad \text{for } r > 0, h > 0$$

where $a(r,h)$ is the area in cm^2 of sheet plastic used to make the peg.
Show that

$$c(r,h) = \frac{4+4\pi}{h} + \frac{2\pi}{r} + rh.$$

(b) Find the critical point (r,h) of the cost function and show that it is a local minimum.

More space for part (b) on the next page

More space for part (b)

(c) For any value of r and h, find the mass M (in grams) of the cylinder B, given

$$M = \int_0^h \int_0^r (1 + x^2 + y^2)\, dx\, dy.$$

$$M = \int_0^h \left[x + \frac{x^3}{3} + xy^2 \right]_{x=0}^{x=r} dy$$

$$= \int_0^h \left[r + \frac{r^3}{3} + ry^2 \right] dy$$

$$= \left[ry + \frac{r^3 y}{3} + \frac{ry^3}{3} \right]_{y=0}^{y=h}$$

$$= rh + \frac{r^3 h}{3} + \frac{rh^3}{3}$$

Question 12 (7 marks)

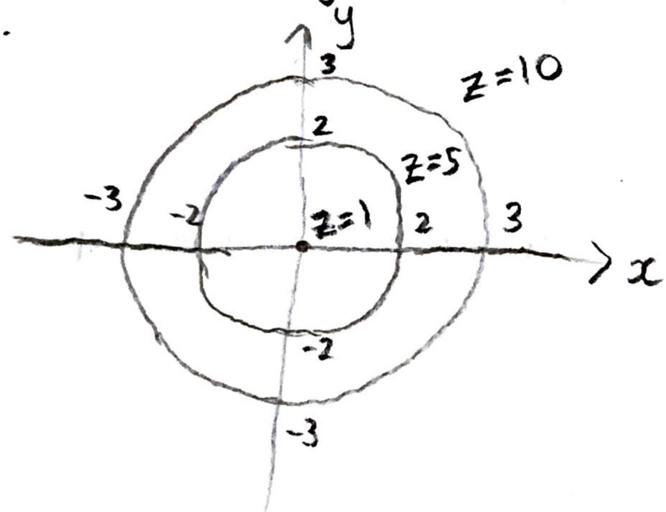
Consider the surface $z = f(x, y)$ where $f(x, y) = 1 + x^2 + y^2$.

(i) Sketch the level curves of f corresponding to $z = 1, 5, 10$ on the same axes.

$z=1$: $1 = 1 + x^2 + y^2 \Rightarrow x^2 + y^2 = 0$

$z=5$: $5 = 1 + x^2 + y^2 \Rightarrow x^2 + y^2 = 4$

$z=10$: $10 = 1 + x^2 + y^2 \Rightarrow x^2 + y^2 = 9$

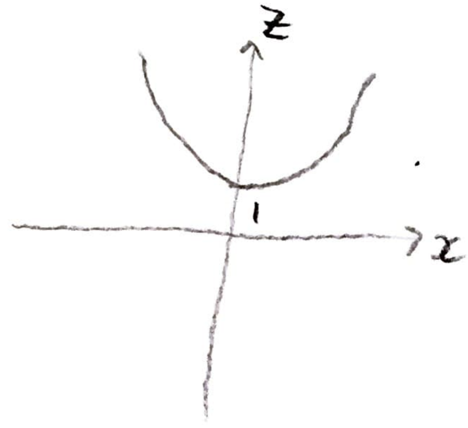


(ii) Sketch the x-z and y-z cross sections.

$x - z$ cross section

$y = 0$

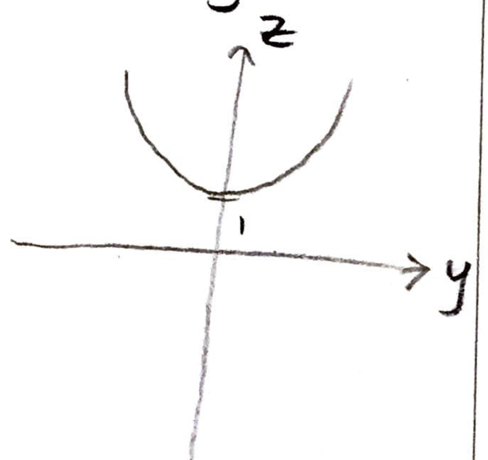
$z = 1 + x^2$



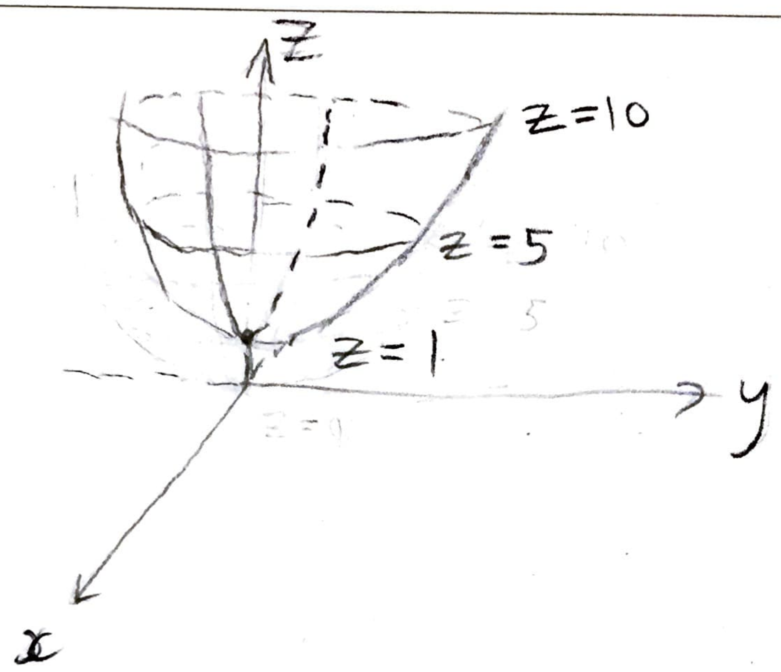
$y - z$ cross section

$x = 0$

$z = 1 + y^2$



(iii) Sketch the surface, and describe/name the surface.



This is a paraboloid shape, essentially an upside down bowl in this case

End of Exam — Total Available Marks = 127

MAST10006 Calculus 2 Formulae Sheet

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \log|\sec x + \tan x| + C \qquad \int \operatorname{cosec} x\,dx = \log|\operatorname{cosec} x - \cot x| + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \operatorname{cosec}^2 x\,dx = -\cot x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

$$\int \operatorname{sech}^2 x\,dx = \tanh x + C \qquad \int \operatorname{cosech}^2 x\,dx = -\coth x + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2+a^2}}\,dx = \operatorname{arcsinh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \arccos\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{x^2-a^2}}\,dx = \operatorname{arccosh}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{a^2-x^2}\,dx = \frac{1}{a}\operatorname{arctanh}\left(\frac{x}{a}\right) + C$$

where $a > 0$ is constant and C is an arbitrary constant of integration.

$$\cos^2 x + \sin^2 x = 1 \qquad \cosh^2 x - \sinh^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$
$$\cot^2 x + 1 = \operatorname{cosec}^2 x \qquad \coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \cosh(2x) = \cosh^2 x + \sinh^2 x$$
$$\cos(2x) = 2\cos^2 x - 1 \qquad \cosh(2x) = 2\cosh^2 x - 1$$
$$\cos(2x) = 1 - 2\sin^2 x \qquad \cosh(2x) = 1 + 2\sinh^2 x$$
$$\sin(2x) = 2\sin x\cos x \qquad \sinh(2x) = 2\sinh x\cosh x$$

$$\cos(x+y) = \cos x\cos y - \sin x\sin y \qquad \cosh(x+y) = \cosh x\cosh y + \sinh x\sinh y$$
$$\sin(x+y) = \sin x\cos y + \cos x\sin y \qquad \sinh(x+y) = \sinh x\cosh y + \cosh x\sinh y$$

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$$

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \qquad \sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$$

$$e^{ix} = \cos x + i\sin x \qquad \operatorname{arctanh} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$\operatorname{arcsinh} x = \log(x + \sqrt{x^2+1}) \qquad \operatorname{arccosh} x = \log(x + \sqrt{x^2-1})$$

$$\lim_{n\to\infty} \frac{1}{n^p} = 0 \quad (p > 0) \qquad \lim_{n\to\infty} r^n = 0 \quad (|r| < 1)$$

$$\lim_{n\to\infty} a^{\frac{1}{n}} = 1 \quad (a > 0) \qquad \lim_{n\to\infty} n^{\frac{1}{n}} = 1$$

$$\lim_{n\to\infty} \frac{a^n}{n!} = 0 \quad (a \in \mathbb{R}) \qquad \lim_{n\to\infty} \frac{\log n}{n^p} = 0 \quad (p > 0)$$

$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a \quad (a \in \mathbb{R}) \qquad \lim_{n\to\infty} \frac{n^p}{a^n} = 0 \quad (p \in \mathbb{R},\ a > 1)$$