

## Written Assignment 4 (Gradescope)

● Graded

Student

James La Fontaine

Total Points

11 / 19 pts

Question 1

Q1 matrix subspace

5 / 10 pts

1.1 1(a) give element of  $S$

1 / 1 pt

✓ + 1 pt Correct answer

+ 0 pts scored zero

1.2 1(b) prove  $S$  subspace

4 / 5 pts

✓ + 1 pt showing  $S$  non-empty by giving specific example

✓ + 1 pt method for proving closure under addition

✓ + 1 pt correct proof of closure under addition

✓ + 1 pt correct proof of closure under scalar multiplication

+ 1 pt conclusion, quoting subspace theorem

+ 0 pts scored zero

3

Must quote subspace thm!

1.3 1(c) find basis for  $S$ ,  $\dim S$

0 / 4 pts

+ 1 pt correct general solution to system (needn't be matrices)

+ 1 pt correct basis (must be matrices)

+ 1 pt correct dimension

+ 1 pt correct notation & mathematical accuracy throughout Q1

✓ + 0 pts scored zero

## Question 2

Q2 basis for polynomials

6 / 9 pts

### 2.1 2(a) polynomial in terms of basis

3 / 4 pts

4

+ 4 pts perfect answer

✓ + 3 pts mostly correct with some error

+ 2 pts part of proof understood

+ 1 pt some correct maths but far from proof

+ 0 pts nothing correct

1

is this equals to what?

### 2.2 2(b) polynomials give basis

3 / 5 pts

✓ + 1 pt showing vectors span  $P_2$

✓ + 1 pt method for testing linear independence

+ 1 pt proof of linear independence

+ 1 pt conclusion: spans + linear independent hence basis

✓ + 1 pt correct notation & mathematical accuracy throughout Q2

+ 0 pts scored zero

2

Must proof linear independence

School of Mathematics and Statistics  
MAST10007 Linear Algebra, Semester 1 2020  
Written assignment 4

Submit your assignment online in Canvas before 12 noon on Monday 11th May.

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Student ID: 1079860

- This assignment is worth  $1\frac{1}{9}\%$  of your final MAST10007 mark.
- Your solutions should be neatly handwritten in blue or black pen, then uploaded as a single PDF file in **GradeScope**.
- Full explanations and working must be shown in your solutions.
- Marks may be deducted in every question for incomplete working, insufficient justification of steps and incorrect mathematical notation.
- You must use methods taught in MAST10007 Linear Algebra to solve the assignment questions.

**New submission guidelines:**

- This assignment is being handled using a similar process to that planned for the final exam so you can start to become familiar with it.
- If you have access to a printer, then you should print out this assignment sheet and handwrite your solutions into the answer boxes.
- If you do not have access to a printer, but you can annotate a PDF file using an iPad/Android tablet/Graphics tablet or using Adobe Acrobat, then annotate your answers directly in the boxes on the assignment PDF and save a copy for submission.
- Otherwise, you may handwrite your answers as normal on blank paper and then scan for submission.
- The answer boxes should typically provide sufficient space for your solution, but if you find you need extra space please take a blank sheet of paper and continue your solution there, clearly indicating which question this refers to. Also indicate in the corresponding box that the solution continues at the end.
- Scan your assignment to a PDF file using your mobile phone or scanner, then upload by going to the Assignments menu on Canvas and submit the PDF to the **GradeScope** tool by first selecting your PDF file and then clicking on 'Upload pdf'.

1. Let  $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$  and  $M_{2,2}$  be the vector space of  $2 \times 2$  matrices with real entries. Define

$$S = \{A \in M_{2,2} \mid \text{there exists } r \in \mathbb{R} \text{ such that } A\mathbf{v} = r\mathbf{v}\}.$$

(a) Write down an element of  $S$  that is not a scalar multiple of the identity matrix  $I$ .

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \in S, \text{ since } \begin{bmatrix} 1+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\left( \begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} r \\ 2r \end{bmatrix} \right) \quad \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \neq \alpha \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) Prove that  $S$  is a subspace of  $M_{2,2}$ .

$$\text{Let } \mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in S, \mathbf{v} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \in S, \alpha \in \mathbb{R}$$

Then there exists a  $r \in \mathbb{R}$  such that

$$\mathbf{u} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = r \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a+2b \\ c+2d \end{bmatrix} = \begin{bmatrix} r \\ 2r \end{bmatrix} \Rightarrow \begin{matrix} a+2b=r \\ c+2d=2r \end{matrix}$$

$$\mathbf{v} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = r \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \begin{bmatrix} a'+2b' \\ c'+2d' \end{bmatrix} = \begin{bmatrix} r \\ 2r \end{bmatrix} \Rightarrow \begin{matrix} a'+2b'=r \\ c'+2d'=2r \end{matrix}$$

①  $S$  is not empty:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \in S \text{ since } \begin{bmatrix} 1+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, 3 \in \mathbb{R}$$

$\Rightarrow S$  is not empty

① Closure under vector addition:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a+a' & b+b' \\ c+c' & d+d' \end{bmatrix}$$

$$(a+a') + 2(b+b') = r \in \mathbb{R} \text{ as } a, a', b, b' \in \mathbb{R}$$

$$(c+c') + 2(d+d') = 2r \in \mathbb{R} \text{ as } c, c', d, d' \in \mathbb{R}$$

$$\Rightarrow \mathbf{u} + \mathbf{v} \in S$$

② Closure under scalar multiplication:

$$\alpha \mathbf{u} = \alpha \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{bmatrix}$$

$$\Rightarrow \alpha a + \alpha 2b = r$$

$$\alpha(a+2b) = r \in \mathbb{R}$$

$$\Rightarrow \alpha c + \alpha 2d = 2r$$

$$\alpha(c+2d) = 2r \in \mathbb{R}$$

$$\Rightarrow \alpha \mathbf{u} \in S$$

$\Rightarrow S$  is a subspace of  $M_{2,2}$

(c) Find a basis for  $S$  and calculate the dimension of  $S$ .

*Hint.* there exists  $r \in \mathbb{R}$  such that  $A\mathbf{v} = r\mathbf{v} \Leftrightarrow \begin{bmatrix} -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$ .

2. Let  $\mathcal{P}_2$  be the real vector space of polynomials of degree at most 2. Define

$$p_0(x) = \frac{1}{2}(x-1)(x-2), \quad p_1(x) = -x(x-2), \quad p_2(x) = \frac{1}{2}x(x-1).$$

(a) Prove that every polynomial  $f(x) \in \mathcal{P}_2$  satisfies

$$f(x) = f(0)p_0(x) + f(1)p_1(x) + f(2)p_2(x).$$

$$\begin{aligned} p_0(x) &= 1 - \frac{3}{2}x + \frac{1}{2}x^2, \quad p_1(x) = 0 + 2x - x^2, \quad p_2(x) = 0 - \frac{1}{2}x + \frac{1}{2}x^2 \\ f(0) &= \alpha_1, \quad f(1) = \alpha_2, \quad f(2) = \alpha_3 \\ a_0 + a_1x + a_2x^2 &= \alpha_1 p_0(x) + \alpha_2 p_1(x) + \alpha_3 p_2(x) \\ \Rightarrow (\alpha_1 + 0\alpha_2 + 0\alpha_3, -\frac{3}{2}\alpha_1 + 2\alpha_2 - \frac{1}{2}\alpha_3, \frac{1}{2}\alpha_1 - \alpha_2 + \frac{1}{2}\alpha_3) & \\ A = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix} \begin{matrix} R_2 \rightarrow R_2 + \frac{3}{2}R_1 \\ R_3 \rightarrow R_3 - \frac{1}{2}R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} \end{bmatrix} \\ R_3 \rightarrow R_3 + \frac{1}{2}R_2 &\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{4} \end{bmatrix} \end{aligned}$$

rank(A) = no. of vectors  
So there is a unique solution for all  
 $f(x) = a_0 + a_1x + a_2x^2$

$\Rightarrow$  Equation is satisfied for all  $f(x) \in \mathcal{P}_2$  //

(b) Use (a), or otherwise, to prove that  $\{p_0(x), p_1(x), p_2(x)\}$  is a basis of  $\mathcal{P}_2$ .

from (a),  $p_0(x), p_1(x), p_2(x)$  span  $\mathcal{P}_2$ .

$$\begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{bmatrix}$$

$$\begin{aligned} (1, -\frac{3}{2}, \frac{1}{2}) &\neq \alpha(0, 2, -1) \text{ for } \alpha \in \mathbb{R} \\ (1, -\frac{3}{2}, \frac{1}{2}) &\neq \alpha(0, -\frac{1}{2}, \frac{1}{2}) \text{ for } \alpha \in \mathbb{R} \\ (0, 2, -1) &\neq \alpha(0, -\frac{1}{2}, \frac{1}{2}) \text{ for } \alpha \in \mathbb{R} \end{aligned}$$

$\Rightarrow$  They are linearly independent.

$\Rightarrow \{p_0(x), p_1(x), p_2(x)\}$  is a basis of  $\mathcal{P}_2$ . //