THE UNIVERSITY OF
MELBOURNE

Student Number 1079860

Semester 1 Assessment, 2020

School of Mathematics and Statistics

MAST10005 Calculus 1

This exam consists of 14 pages (including this page)

Authorised materials: printed one-sided copy of the Exam or the Masked Exam made available earlier (or an offline electronic PDF reader), two double-sided A4 handwritten sheet of notes, and blank A4 paper

Instructions to Students

- During exam writing time you may only interact with the device running the Zoom session with supervisor permission. The screen of any other device must be visible in Zoom from the start of the session.
- If you have a printer, print out the exam single-sided and hand write your solutions into the answer spaces.
- If you do not have a printer, or if your printer fails on the day of the exam,

  (a) download the exam paper to a second device (not running Zoom), disconnect it from the internet as soon as the paper is downloaded and read the paper on the second device;

  (b) write your answers on the Masked Exam PDF if you were able to print it single-sided before the exam day.

  If you do not have the Masked Exam PDF, write single-sided on blank sheets of paper.
- If you are unable to answer the whole question in the answer space provided then you can append additional handwritten solutions to the end of your exam submission. If you do this you MUST make a note in the correct answer space or page for the question, warning the marker that you have appended additional remarks at the end.
- Assemble all the exam pages (or template pages) in correct page number order and the correct way up, and add any extra pages with additional working at the end.
- Scan your exam submission to a single PDF file with a mobile phone or a scanner. Scan from directly above to avoid any excessive keystone effect. Check that all pages are clearly readable and cropped to the A4 borders of the original page. Poorly scanned submissions may be impossible to mark.
- Upload the PDF file via the Canvas Assignments menu and submit the PDF to the GradeScope tool by first selecting your PDF file and then clicking on Upload PDF.
- Confirm with your Zoom supervisor that you have GradeScope confirmation of submission before leaving Zoom supervision.
- You should attempt all questions.
- Calculators are not allowed.
- Check that all pages are included before submitting your paper in Gradescope.
- There are 8 questions with marks as shown. The total number of marks available is 68.

Useful Formulae

Pythagorean identity

$$\cos^2(x) + \sin^2(x) = 1$$

Compound angle formulae

$$\begin{aligned}\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y)\\ \cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y)\\ \tan(x+y) &= \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)}\end{aligned}$$

Derivatives of inverse trigonometric functions

$$\begin{aligned}\arcsin'(x) &= \frac{1}{\sqrt{1-x^2}}\\ \arccos'(x) &= -\frac{1}{\sqrt{1-x^2}}\\ \arctan'(x) &= \frac{1}{1+x^2}\end{aligned}$$

Antiderivatives from inverse trigonometric functions

$$\begin{aligned}\int \frac{1}{\sqrt{s^2-x^2}}\,dx &= \arcsin\left(\frac{x}{s}\right) + C\\ \int \frac{-1}{\sqrt{s^2-x^2}}\,dx &= \arccos\left(\frac{x}{s}\right) + C\\ \int \frac{1}{s^2+x^2}\,dx &= \frac{1}{s}\arctan\left(\frac{x}{s}\right) + C\end{aligned}$$

where s is a *positive* constant, and C is an arbitrary constant of integration.

Complex exponential formulae

$$\begin{aligned}e^{i\theta} &= \cos(\theta) + i\sin(\theta)\\ \cos(\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\\ \sin(\theta) &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)\end{aligned}$$

Vector projections

- The *vector projection* of $\mathbf{v}$ onto $\mathbf{u}$ is $\mathbf{v}_{\parallel} = (\hat{\mathbf{u}}\cdot\mathbf{v})\hat{\mathbf{u}} = k\mathbf{u}$, where $k \in \mathbb{R}$ is the unique solution of $\mathbf{u}\cdot(\mathbf{v} - k\mathbf{u}) = 0$.
- The *vector component of* $\mathbf{v}$ *perpendicular to* $\mathbf{u}$ is $\mathbf{v}_{\perp} = \mathbf{v} - \mathbf{v}_{\parallel}$.

Complex roots

The n-th roots of $w = se^{i\phi}$ are $s^{\frac{1}{n}}\,e^{i\left(\frac{1}{n}(\phi+2k\pi)\right)}$ for $k = 0, 1, \ldots, n-1$.

Changes in speed

Provided $\mathbf{r}'(t) \neq \mathbf{0}$, the speed function $\|\mathbf{r}'(t)\|$ is decreasing when $\mathbf{r}'(t)\cdot\mathbf{r}''(t) < 0$ and increasing when $\mathbf{r}'(t)\cdot\mathbf{r}''(t) > 0$.

Question 1 (7 marks)

(a) Define $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $\mathbf{r}(t) = (1 - t^2)\mathbf{i} + (t - t^3)\mathbf{j}$ and let

$$C = \{(x, y) \in \mathbb{R}^2 \mid x^3 + y^2 = x^2\}.$$

Prove that $\text{range}(\mathbf{r}) \subseteq C$.

range$(\vec{r}) \subseteq C$

$\rightarrow x(t) = 1 - t^2 \Rightarrow x = 1 - t^2 \Rightarrow -t^2 = x - 1$

$y(t) = t - t^3$ $\qquad t^2 = 1 - x$

$t = \sqrt{1-x}$

$\Rightarrow y = \sqrt{1-x} - \sqrt{1-x}\sqrt{1-x}\sqrt{1-x}$

$y = \sqrt{1-x} - (1-x)\sqrt{1-x}$

$y = \sqrt{1-x} - \sqrt{1-x} + x\sqrt{1-x}$

$y = x\sqrt{1-x}$

$\Rightarrow x^3 + y^2 = x^2$

$x^3 + (x\sqrt{1-x})^2 = x^2$

$x^3 + (x^2(1-x)) = x^2$

$x^3 + x^2 - x^3 = x^2 \Rightarrow x^2 = x^2$ ✓

Therefore range$(\vec{r}) \subseteq C$ $\square$

Question 1 continued next page . . .

... Question 1 continued.

(b) For $f : \mathbb{R} \longrightarrow \mathbb{R}$ with formula $f(x) = e^{-x^2}$, define

$$I = \{x \in \mathbb{R} : f'(x) > 0\} \text{ and } U = \{x \in \mathbb{R} : f''(x) > 0\}.$$

Is $U \subseteq I$? (In other words, is f increasing at all points where it is is concave up?) If your answer is yes, prove it. If no, give a counterexample.

$f'(x) = e^{-u} \cdot 2x \quad u = x^2 \quad \frac{du}{dx} = 2x$

$= -2xe^{-u}$

$= -2xe^{-x^2}$

$f''(x) = 4x^2 e^{-x^2} \quad u = x^2 \quad \frac{du}{dx} = 2x$

No, when $x = 1$, $f'(x) = -2e^1 < 0$

$= -2e$ f is decreasing

$f''(x) = 4 \cdot 1e^1$

$= 4e > 0$ f is concave up

$\therefore U \not\subseteq I$

$\square$

Question 2 (5 marks)

Sketch the following subset of $\mathbb{C}$

$$\overset{A}{\{z \in \mathbb{C} : \mathrm{Re}((1+2i)z) \geq -1\}} \cap \overset{B}{\{z \in \mathbb{C} : |z-(1+i)| < 1\}}$$

in the complex plane, clearly indicating which boundaries are included. Show your reasoning.

$A: \mathrm{Re}\left(\left(1+2i\right)\left(x+yi\right)\right) \geq -1$

$= \mathrm{Re}\left(x+yi+2xi+2yi^2\right) \geq -1$

$\mathrm{Re}\left(x-2y+yi+2xi\right) \geq -1$

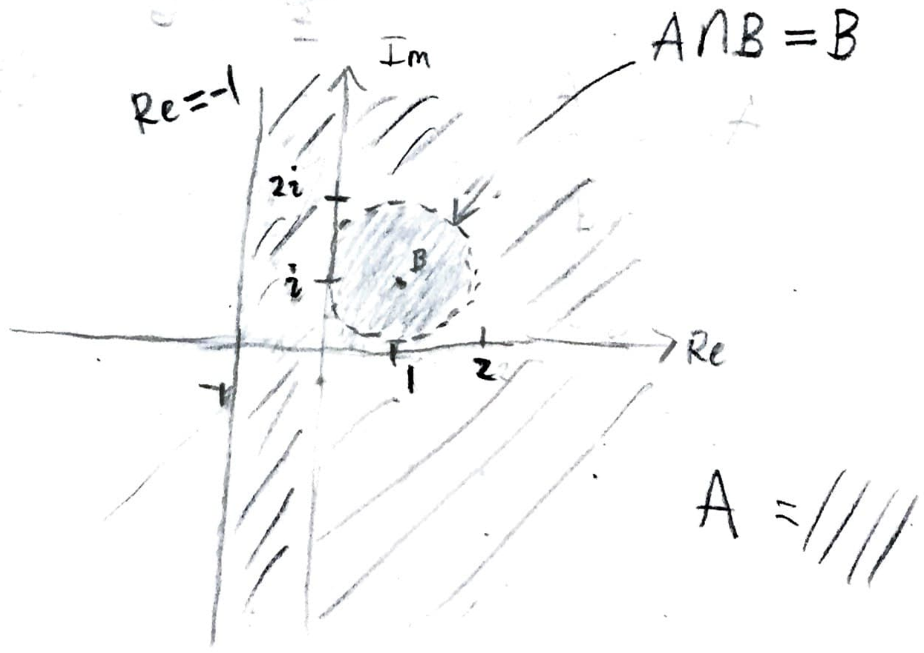
$x-2y \geq -1$

$\mathrm{Re} \geq -1$

$B = |z-(1+i)| < 1$

circle of radius 1 non-inclusiv

with centre $(1,i)$



Question 3 (7 marks)

Consider the polynomial $P : \mathbb{C} \longrightarrow \mathbb{C}$ defined by $P(z) = (z-2)^3 - 8$.

(a) Find all $z \in \mathbb{C}$ such that $P(z) = 0$. Express your answers in both Cartesian form and exponential polar form.

$(z-2)^3 - 8 = 0$

$(z-2)^3 = 8$

$(z-2) = \sqrt[3]{8}$

$z - 2 = 2$

$z = 4$

(b) Express $P(z)$ as a product of linear factors.

$P(z) = (z-4)(\quad)(\quad)$

(c) Express $P(z)$ as the product of a linear factor and a quadratic factor with all coefficients in $\mathbb{R}$.

$P(z) = (z-4)(z^2 \quad)$

Question 4 (12 marks)

Define a curve $C = \{(x, y) \in \mathbb{R}^2 \mid x^3 + y^2 = x^2\}$.

(a) Use implicit differentiation to find a formula for $\frac{dy}{dx}$.

$$x^3+y^2=x^2$$

$$3x^2+2y\frac{dy}{dx}=2x$$

$$\frac{dy}{dx}(2y)=2x-3x^2$$

$$\frac{dy}{dx}=\frac{2x-3x^2}{2y}$$

(b) Find all places where we can be sure that the tangent line to C is *vertical*.

The tangent line is vertical where $\frac{dy}{dx}$ is undefined:

$$\left.\begin{array}{l} ⓐ\ 2x-3x^2\neq 0 \\ ⓑ\ 2y=0 \\ ⓒ\ x^3+y^2=x^2 \end{array}\right\}\ \text{simultaneous solutions}$$

ⓐ $x(2-3x)\neq 0$

$x\neq 0,\ x\neq \frac{2}{3}$

ⓑ $y=0$

ⓒ if $y=0$,

$x^3+0=x^2$

$x^3-x^2=0$

$x^2(x-1)=0$

$\swarrow \quad \downarrow$

$x=0 \quad x=1$

∴ the tangent line to C is vertical at the point (1, 0)

However, $x\neq 0$

Question 4 continued next page ...

...Question 4 continued.

(c) Find all places where we can be sure that the tangent line to C is *horizontal.*

The tangent line to C is horizontal $\Rightarrow \frac{dy}{dx} = 0$

$$\left.\begin{array}{l} ⓐ\ 2x - 3x^2 = 0 \\ ⓑ\ 2y \neq 0 \\ ⓒ\ x^3 + y^2 = x^2 \end{array}\right\} \text{Simultaneous solutions}$$

ⓐ $x(2-3x) = 0$

$x = 0,\ x = \frac{2}{3}$

ⓑ $y \neq 0$

ⓒ (if $x=0$)

$0 + y^2 = 0$

$y^2 = 0$

$y = 0$

(However, $y \neq 0$)

(if $x = \frac{2}{3}$)

$\frac{8}{27} + y^2 = \frac{4}{9}$

$y^2 = \frac{4}{9} - \frac{8}{27}$

$y^2 = \frac{12}{27} - \frac{8}{27}$

$y^2 = \frac{4}{27}$

$y = \pm \frac{2}{\sqrt{27}} = \pm \frac{2}{3\sqrt{3}}$

$\therefore$ the tangent line to C is horizontal at the points $\left(\frac{2}{3}, \frac{2}{3\sqrt{3}}\right)$, $\left(\frac{2}{3}, -\frac{2}{3\sqrt{3}}\right)$

Question 5 (10 marks)

Define $\mathbf{r} : \mathbb{R} \longrightarrow \mathbb{R}^2$ by $\mathbf{r}(t) = (1 - t^2)\mathbf{i} + (t - t^3)\mathbf{j}$.

(a) Find $\mathbf{r}'(t)$ and $\mathbf{r}''(t)$.

$$r'(t) = x'(t)i + y'(t)j$$
$$= -2t\,i + (1-3t^2)j$$
$$r''(t) = x''(t)i + y''(t)j$$
$$= -2i - 6tj$$

(b) Express the set of t values for which the *speed* of $\mathbf{r}$ is strictly decreasing as a union of intervals.

$\|r'(t)\|$ is decreasing when $r'(t) \cdot r''(t) < 0$

$$\Rightarrow (-2t \cdot -2) + (-6t(1-3t^2)) < 0$$
$$4t + (-6t + 18t^3) < 0$$
$$-2t + 18t^3 < 0$$
$$-2t(1-9t^2) < 0$$

$-2t < 0$

$t > 0$

$1-9t^2 < 0$

$-9t^2 < -1$

$t^2 > \frac{1}{9}$

case when $-2t<0$:

$t>0,\ t<\frac{1}{3},\ t>-\frac{1}{3}$

case when $1-9t^2<0$:

$t>\frac{1}{3},\ t<-\frac{1}{3},\ t<0$

$$t \in \left(-\infty, -\tfrac{1}{3}\right) \cup \left(0, \tfrac{1}{3}\right)$$

(c) Does $\mathbf{r}$ have any *cusps*? Explain your reasoning.

The curve $r(t)$ has a cusp precisely if $x'(t) = y'(t) = 0$ and either $x'(t)$ or $y'(t)$ change sign at t

$-2t = 0 \Rightarrow t = 0$

$(1-3t^2) = 0 \Rightarrow -3t^2 = -1$

$t^2 = \frac{1}{3}$

$t = \frac{1}{\sqrt{3}}$

$\Rightarrow x'(0) = 0 \neq y'(0)$

$y'\left(\frac{1}{\sqrt{3}}\right) = 0 \neq x'\left(\frac{1}{\sqrt{3}}\right)$

$\therefore \vec{r}$ does not have any cusps.

Question 6 (7 marks)

Find the implied domain and range of the function

$$f(x) = \arcsin(4\cos^2 x - 3),$$

where

$$\arcsin : [-1, 1] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}].$$

$$\mathrm{dom}(f(x)) = \{x \in \mathrm{dom}(4\cos^2 x - 3) \mid 4\cos^2(x) - 3 \in [-1, 1]\}$$

$$= \{x \in \mathbb{R} \mid -1 \leq 4\cos^2(x) - 3 \leq 1\}$$

$$= -1 \leq 4\cos^2(x) - 3 \leq 1$$

$$2 \leq 4\cos^2(x) \leq 4$$

$$\frac{1}{2} \leq \cos^2(x) \leq 1$$

$$\frac{1}{\sqrt{2}} \leq \cos(x) \leq 1$$

$$\frac{\pi}{4} \leq x \leq 2\pi \quad = [\frac{\pi}{4}, 2\pi]$$

$$\mathrm{ran}(f(x)) = f(\mathrm{range}(4\cos^2 x - 3) \cap \mathrm{dom}(\arcsin))$$

$$= f([-3, 1] \cap [-1, 1])$$

$$= f([-1, 1])$$

$$= [-\frac{\pi}{2}, \frac{\pi}{2}]$$

Question 7 (8 marks)

Calculate the antiderivative $\displaystyle\int \frac{2x^3 - 14x^2 + 24x + 3}{x^2 - 7x + 12}\,dx$.

$$\begin{array}{r|l} & 2x \\ \hline x^2-7x+12 & 2x^3-14x^2+24x+3 \\ & -(2x^3-14x^2+24x) \\ \hline & 0+0+0+3 \end{array}$$

$$\therefore \int \frac{2x^3-14x^2+24x+3}{x^2-7x+12}\,dx = \int 2x\,dx + \int \frac{3}{x^2-7x+12}\,dx$$

$$\Rightarrow \frac{2x^2}{2} = x^2 \qquad \Rightarrow \int \frac{3}{x^2-7x+12}\,dx$$

$$= \int \frac{3}{(x-4)(x-3)}$$

Partial Fractions

$$\frac{3}{(x-4)(x-3)} = \frac{A}{x-4} + \frac{B}{x-3}$$

$$3 = \frac{A}{x-4}(x-4)(x-3) + \frac{B}{x-3}(x-4)(x-3)$$

$$3 = A(x-3) + B(x-4)$$

let $x=3$:

$3 = B(-1)$

$-B = 3$

$B = -3$

let $x=4$:

$3 = A$

$A = 3$

$$\Rightarrow \int \frac{3}{(x-4)(x-3)}\,dx = \int \frac{3}{x-4}\,dx - \int \frac{3}{x-3}\,dx$$

$$= 3\int \frac{1}{x-4}\,dx - 3\int \frac{1}{x-3}\,dx$$

$$\therefore \text{Answer} = x^2 + 3\log|x-4| - 3\log|x-3| + C = \boxed{x^2 + 3\log\left|\frac{x-4}{x-3}\right| + C}$$

Question 8 (12 marks)

Consider the following separable differential equation:

$$\frac{dy}{dx} = y(y-1)\cos(x).$$

(a) Find all constant solutions of this differential equation.

Constant Solutions satisfy $G(y)=0$:

$\Rightarrow y(y-1)=0$

$\boxed{y\equiv 0,\ y\equiv 1}$

(b) Suppose $y = f(x)$ is a solution of this differential equation for which $y = \frac{1}{2}$ when $x = 0$. Explain why the range of f is contained in $(0,1)$.

The range of f is contained in $(0, 1)$ in accordance with the theorem that the curves of distinct solutions of DEs never cross. Therefore, as $y\equiv 0, y\equiv 1$ are constant solutions, we know that there are solutions contained within this interval. $(0, 1)$.

Question 8 continued next page . . .

. . . Question 8 continued.

(c) Find the *general solution* of this differential equation, considering only solutions $y = f(x)$ where $0 < y < 1$ for all $x \in \mathbb{R}$. You should explain where you use the fact that $0 < y < 1$.

$$\frac{dy}{dx} = \overbrace{y(y-1)}^{G(y)} \; \overbrace{\cos(x)}^{f(x)}$$

$$\frac{1}{G(y)} \frac{dy}{dx} = f(x)$$

$$\int \frac{1}{G(y)} \frac{dy}{dx} = \int \cos(x) \, dx$$

$$\int \frac{1}{y(y-1)} \frac{dy}{dx} = \sin(x) + C$$

(assume $y \neq 0, 1$ which is true as $y \equiv 0$, $y \equiv 1$ are constant solutions of the DE.)

$$\int \frac{1}{y^2 - y} \frac{dy}{dx} = \sin(x) + C$$

$$\int (y^2 - y)^{-1} \, dy = \sin(x) + C$$

$$\int y^{-2} - y^{-1} \, dy = \sin(x) + C$$

$$-y^{-1} - \log(y) = \sin(x) + C$$

(assume $y \neq 0$ and $y > 0$ which is true as $0 < y < 1$)

$$-\frac{1}{y} - \log(y) = \sin(x) + C$$

$$\frac{1}{y} + \log(y) = -\sin(x) + C$$

Question 8 continued next page . . .

...Question 8 continued.

Extra answer space for 8 (c) if needed.

$$e^{y^{-1}} + y = e^{-\sin(x)} + e^{c}$$

$$e^{y^{-1}} + y = A e^{-\sin(x)} \quad \text{where } A > 0$$

(d) What is the solution to the initial value problem with $y = \frac{1}{2}$ when $x = 0$?

$$e^{2} + \frac{1}{2} = A e^{-\sin(0)}$$

$$e^{2} + \frac{1}{2} = A e^{0}$$

$$A = e^{2} + \frac{1}{2}$$

$$e^{y^{-1}} + y = \left(e^{2} + \frac{1}{2}\right) e^{-\sin(x)}$$

$$= e^{-2\sin(x)} + \frac{1}{2} e^{-\sin(x)}$$

End of Exam—Total Available Marks = 68