Question 1 marked. Total marks: 11

MAST10006 Calculus 2, Semester 2, 2020

Assignment 5

School of Mathematics and Statistics, The University of Melbourne

- Submit your assignment solutions online in Canvas before **6pm, Monday 21 September 2020**
- Submit your solutions as a single PDF file with the pages in the right order and correct orientation. You may be penalised a mark if you do not.
- This assignment is worth 2.22% of your final MAST10006 mark.
- Answer all questions below. Of these questions, one will be chosen for marking.
- Marks may be awarded for:
  - Correct use of appropriate mathematical techniques
  - Accuracy and validity of any calculations or algebraic manipulations
  - Clear justification or explanation of techniques and rules used
  - Use of correct mathematical notation and terminology
- You must use methods taught in MAST10006 Calculus 2 to solve the assignment questions.

1. Consider the differential equation

$$x\frac{dy}{dx} = y + \sqrt{x^2+y^2}, \qquad x > 0 \qquad (1)$$

(a) Make the substitution $u = \frac{y}{x}$ and show that the differential equation reduces to

$$\frac{du}{dx} = \frac{\sqrt{1+u^2}}{x}. \qquad (2)$$

(b) Find the general solution to the ODE (2) for $u(x)$.

(c) Use your answer from part (b) to solve the ODE (1), subject to the condition $\lim_{x\to 0+} y(x) = -\frac{e}{2}$.

Solution.

(a) Let $u = \frac{y}{x}$. Therefore

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$$\left.\begin{aligned} y &= xu \\ \frac{dy}{dx} &= u + x\frac{du}{dx} \end{aligned}\right]$$

1M : Obtain an equation relating $\frac{du}{dx}$ and $\frac{dy}{dx}$

Substituting into (1) we have:

1A: Substitute into original ODE and demonstrate algebraic steps with no errors.

Does not need to be the same as written here

$$x\left(u + x\frac{du}{dx}\right) = xu + \sqrt{x^2 + (xu)^2}$$

$$\implies \quad x\left(u + x\frac{du}{dx}\right) = xu + |x|\sqrt{1+u^2}$$

Excellent if accounted for, but don't deduct if missing.

$$\implies \quad x\left(u + x\frac{du}{dx}\right) = xu + x\sqrt{1+u^2} \qquad \text{since } x > 0$$

$$\implies \quad u + x\frac{du}{dx} = u + \sqrt{1+u^2}$$

$$\implies \quad \frac{du}{dx} = \frac{\sqrt{1+u^2}}{x}$$

(b) First solve (2): The ODE is separable, so use separation of variables.

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$$\frac{du}{dx} = \frac{\sqrt{1+u^2}}{x}$$

$$\implies \frac{1}{\sqrt{1+u^2}}\frac{du}{dx} = \frac{1}{x}$$

$$\implies \int \frac{1}{\sqrt{1+u^2}}\,\mathrm{d}u = \int \frac{1}{x}\,\mathrm{d}x$$

1M: Use separation of variables

$$\implies \operatorname{arcsinh}(u) = \log|x| + c$$

1A: arcsinh and log

$$\implies u = \sinh(\log x + c) \qquad \text{since } x > 0$$

1M: Use exponential form for sinh or log form for arcsinh. Done now or later to assist with computing $\lim_{x\to 0^+} y(x)$.

$$\implies u = \frac{e^{(\log x + c)} - e^{-(\log x + c)}}{2}$$

$$\implies u = \frac{1}{2}\left(xe^c - \frac{1}{xe^c}\right)$$

$$\implies u = \frac{1}{2}\left(Ax - \frac{1}{Ax}\right) \qquad A = e^c$$

$$\implies u = \frac{A^2x^2 - 1}{2Ax}$$

1A: u written correctly as a function of x. Any of these are okay.

(c) Since $u = \frac{y}{x}$, we have

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$$\frac{y}{x} = \frac{A^2x^2 - 1}{2Ax}$$

$$\implies y = \frac{A^2x^2 - 1}{2A}$$

1M: Multiply previous answer by x to write y in terms of x.

Substitute the condition $\lim_{x\to 0+} y(x) = -\frac{e}{2}$ to find c:

$$-\frac{e}{2} = \lim_{x\to 0+} \frac{A^2x^2 - 1}{2A}$$

1M: Use condition to find c.

$$\implies -\frac{e}{2} = -\frac{1}{2A} \qquad \text{limit laws}$$

$$\implies A = \frac{1}{e}$$

1J: Limit laws or other justification for calculating the limit.

Therefore

$$y(x) = \frac{x^2 - e^2}{2e}.$$

1A: Accept also $y(x) = x\sinh(\log(x) - 1)$

1N: Notation is correct.

2. Find the general solution to the ODE

$$t\log(t)\frac{dr}{dt} + r = \frac{t}{(2t^2-9)^{\frac{3}{2}}}.$$

Solution. The ODE is linear so we rewrite it in standard form and find an integrating factor.

$$\frac{dr}{dt} + \frac{1}{t\log(t)}r = \frac{1}{\log(t)}\frac{1}{(2t^2-9)^{\frac{3}{2}}} \qquad (*)$$

An integrating factor is

$$I(t) = e^{\int \frac{1}{t\log t}dt}$$
$$= \log(t)$$

Multiply the equation $(*)$ by I:

$$\begin{aligned}
\log t \frac{dr}{dt} + \frac{1}{t} r &= \frac{1}{(2t^2-9)^{\frac{3}{2}}} \\
\implies \frac{d}{dt}\left[r\log(t)\right] &= \frac{1}{(2t^2-9)^{\frac{3}{2}}} \\
\implies r\log(t) &= \int \frac{1}{(2t^2-9)^{\frac{3}{2}}}\,\mathrm{d}t
\end{aligned}$$

To calculate the integral $\displaystyle\int \frac{1}{(2t^2-9)^{\frac{3}{2}}}\,\mathrm{d}t$, we use a hyperbolic substitution.

Let

- $t = \frac{3}{\sqrt{2}}\cosh\theta$
- $\theta = \operatorname{arccosh}(\frac{t\sqrt{2}}{3})$
- $\frac{dt}{d\theta} = \frac{3}{\sqrt{2}}\sinh\theta$

We need $2t^2 - 9 \neq 0$, $\theta \in \text{range}(\operatorname{arccosh})$, and $\frac{t\sqrt{2}}{3} \in \text{domain}(\operatorname{arccosh})$.
Therefore, this substitution is valid for $\theta > 0$ and $t > \frac{3}{\sqrt{2}}$.

$$\begin{aligned}
\int \frac{1}{(2t^2-9)^{\frac{3}{2}}}\,\mathrm{d}t &= \int \frac{1}{(9\cosh^2(\theta)-9)^{\frac{3}{2}}}\frac{3}{\sqrt{2}}\sinh\theta\,\mathrm{d}\theta \\
&= \frac{1}{9\sqrt{2}}\int \frac{\sinh\theta}{|\sinh^3\theta|}\,\mathrm{d}\theta \\
&= \frac{1}{9\sqrt{2}}\int \frac{\sinh\theta}{\sinh^3\theta}\,\mathrm{d}\theta && \sinh\theta > 0 \text{ when } \theta > 0 \\
&= \frac{1}{9\sqrt{2}}\int \frac{1}{\sinh^2\theta}\,\mathrm{d}\theta \\
&= \frac{1}{9\sqrt{2}}\int \operatorname{cosech}^2\theta\,\mathrm{d}\theta \\
&= -\frac{1}{9\sqrt{2}}\coth\theta + c \\
&= -\frac{1}{9\sqrt{2}}\frac{\cosh\theta}{\sinh\theta} + c \\
&= -\frac{1}{9\sqrt{2}}\frac{\cosh\theta}{\sqrt{\cosh^2\theta - 1}} + c \\
&= -\frac{1}{9\sqrt{2}}\frac{t\frac{\sqrt{2}}{3}}{\sqrt{t^2\frac{2}{9} - 1}} + c \\
&= -\frac{t}{9\sqrt{2t^2-9}} + c
\end{aligned}$$

Therefore, continuing our solution of the ODE $(*)$ we have

$$\begin{aligned}
r\log(t) &= \int \frac{1}{(2t^2-9)^{\frac{3}{2}}}\,\mathrm{d}t \\
&= -\frac{t}{9\sqrt{2t^2-9}} + c \\
\implies r(t) &= -\frac{t}{9\log(t)\sqrt{2t^2-9}} + \frac{c}{\log(t)}
\end{aligned}$$

End of assignment

Notes on question 2 for reflection

- Did you rewrite the linear ODE in standard form before using the integrating factor formula?

- Did you multiply both sides of the ODE by the integrating factor (in particular, the right hand side!)?

- Did you consider the domain for which the integral was valid?

- Did you write your final answer in terms of t?

- Check that your constant is being divided by log(t)