MAST30027 Modern Applied Statistics Assignment 3

Tutorial:Wed 1-2PM, Yidi Deng

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1. $Z_i \sim \text{categorical}(\pi_1, \pi_2, 1-\pi_1-\pi_2)$

$$(X_i \mid Z_i = 1) \sim \text{Poisson}(\lambda_1)$$
$$(X_i \mid Z_i = 2) \sim \text{Poisson}(\lambda_2)$$
$$(X_i \mid Z_i = 3) \sim \text{Poisson}(\lambda_3)$$

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}$$

a) Let $X = (X_1, \cdots, X_{300})$ and $Z = (Z_1, \cdots, Z_{300})$

Derive $Q(\theta, \theta^0) = E_{Z|X,\theta^0}[\log(P(X, Z|\theta))]$

Let $n = 300$

$$P(X, Z \mid \theta) = \prod_{i=1}^{n} P(X_i \mid Z_i, \theta) P(Z_i \mid \theta)$$
$$= \prod_{i=1}^{n} \prod_{k=1}^{3} [P(X_i \mid Z_i = k, \theta) P(Z_i = k \mid \theta)]^{I(Z_i = k)}$$

$$\log P(X, Z \mid \theta) = \sum_{i=1}^{n} \sum_{k=1}^{3} I(Z_i = k)[\log P(X_i \mid Z_i = k, \theta) + \log P(Z_i = k \mid \theta)]$$

$$Q(\theta, \theta^0) = \sum_{i=1}^{n} \sum_{k=1}^{3} P(Z_i = k \mid X_i, \theta^0)[\log P(X_i \mid Z_i = k, \theta) + \log P(Z_i = k \mid \theta)]$$
$$= \sum_{i=1}^{n} \sum_{k=1}^{3} P(Z_i = k \mid X_i, \theta^0)[X_i \log(\lambda_k) - \lambda_k - \log(X_i!) + \log \pi_k]$$

where $\pi_3 = 1 - \pi_1 - \pi_2$

b) Let $\theta^0 = (\pi_1^0, \pi_2^0, \lambda_1^0, \lambda_2^0, \lambda_3^0)$

$$P(Z_i = k \mid X_i, \theta^0) = \frac{P(Z_i = k, X_i \mid \theta^0)}{P(X_i \mid \theta^0)}$$

$$= \frac{P(X_i \mid Z_i = k, \theta^0)\, P(Z_i = k \mid \theta^0)}{\sum_{k'=1}^{3} P(X_i \mid Z_i = k', \theta^0)\, P(Z_i = k' \mid \theta^0)}$$

$$P(Z_i = 1 \mid X_i, \theta^0) = \frac{P(X_i \mid Z_i = 1, \theta^0)\, P(Z_i = 1 \mid \theta^0)}{P(X_i \mid Z_i = 1, \theta^0)\, P(Z_i = 1 \mid \theta^0) + P(X_i \mid Z_i = 2, \theta^0)\, P(Z_i = 2 \mid \theta^0) + P(X_i \mid Z_i = 3, \theta^0)\, P(Z_i = 3 \mid \theta^0)}$$

$$P(Z_i = 2 \mid X_i, \theta^0) = \frac{P(X_i \mid Z_i = 2, \theta^0)\, P(Z_i = 2 \mid \theta^0)}{P(X_i \mid Z_i = 1, \theta^0)\, P(Z_i = 1 \mid \theta^0) + P(X_i \mid Z_i = 2, \theta^0)\, P(Z_i = 2 \mid \theta^0) + P(X_i \mid Z_i = 3, \theta^0)\, P(Z_i = 3 \mid \theta^0)}$$

$$P(Z_i = 3 \mid X_i, \theta^0) = 1 - P(Z_i = 1 \mid X_i, \theta^0) - P(Z_i = 2 \mid X_i, \theta^0)$$

where $P(X_i \mid Z_i = 1, \theta^0) = \dfrac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!}$, $P(X_i \mid Z_i = 2, \theta^0) = \dfrac{\lambda_2^{x_i} e^{-\lambda_2}}{x_i!}$, $P(Z_i = 1 \mid \theta^0) = \pi_1$, $P(Z_i = 2 \mid \theta^0) = \pi_2$

c) Let $\frac{\partial Q(\theta,\theta^\circ)}{\partial \pi_1}$

$$= \sum_{i=1}^{n} \left[\frac{P(z_i=1|x_i,\theta^\circ)}{\pi_1} - \frac{P(z_i=3|x_i,\theta^\circ)}{1-\pi_1-\pi_2} \right]$$

$$\Rightarrow \frac{(1-\pi_1-\pi_2)\left[\sum_{i=1}^{n} P(z_i=1|x_i,\theta^\circ)\right] - \pi_1\left[\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)\right]}{\pi_1\ (1-\pi_1-\pi_2)} = 0 \quad (1)$$

$$\frac{\partial Q(\theta,\theta^\circ)}{\partial \pi_2} = \sum_{i=1}^{n} \left[\frac{P(z_i=2|x_i,\theta^\circ)}{\pi_2} - \frac{P(z_i=3|x_i,\theta^\circ)}{1-\pi_1-\pi_2} \right]$$

$$\Rightarrow \frac{(1-\pi_1-\pi_2)\left[\sum_{i=1}^{n} P(z_i=2|x_i,\theta^\circ)\right] - \pi_2\left[\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)\right]}{\pi_2\ (1-\pi_1-\pi_2)} = 0 \quad (2)$$

from (1) $\Rightarrow (1-\pi_1-\pi_2)\left[\sum_{i=1}^{n} P(z_i=1|x_i,\theta^\circ)\right] = \pi_1\left[\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)\right]$ (1)'

from (2) $\Rightarrow (1-\pi_1-\pi_2)\left[\sum_{i=1}^{n} P(z_i=2|x_i,\theta^\circ)\right] = \pi_2\left[\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)\right]$ (2)'

(1)' + (2)' $\Rightarrow (1-\pi_1-\pi_2)\left[\sum_{i=1}^{n}\left(P(z_i=1|x_i,\theta^\circ) + P(z_i=2|x_i,\theta^\circ)\right)\right]$

$$= (\pi_1+\pi_2)\left[\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)\right]$$

$$1-\pi_1-\pi_2 = \frac{\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)}{\sum_{i=1}^{n}\left(P(z_i=1|x_i,\theta^\circ)+P(z_i=2|x_i,\theta^\circ)+P(z_i=3|x_i,\theta^\circ)\right)}$$

$$= \frac{\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)}{n}$$

from (1)' $\Rightarrow \hat{\pi}_1 = (1-\pi_1-\pi_2)\left[\sum_{i=1}^{n} P(z_i=1|x_i,\theta^\circ)\right]$

$$= \frac{\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)}{n} \quad \frac{\left[\sum_{i=1}^{n} P(z_i=1|x_i,\theta^\circ)\right]}{\left[\sum_{i=1}^{n} P(z_i=3|x_i,\theta^\circ)\right]}$$

$$= \frac{\sum_{i=1}^{n} P(z_i=1|x_i,\theta^\circ)}{n}$$

from (2)' $\Rightarrow \hat{\pi}_2 = \frac{\sum_{i=1}^{n} P(z_i=2|x_i,\theta^\circ)}{n}$ $\qquad \hat{\pi}_3 = 1-\hat{\pi}_1-\hat{\pi}_2$

Let $\frac{\partial Q(\theta,\theta^\circ)}{\partial \lambda_k} = \sum_{i=1}^{n} P(z_i=k|x_i,\theta^\circ)\left[-1+\frac{x_i}{\lambda_k}\right]$

$$= \frac{\sum_{i=1}^{n} P(z_i=k|x_i,\theta^\circ)\, x_i - \lambda_k \sum_{i=1}^{n} P(z_i=k|x_i,\theta^\circ)}{\lambda_k} = 0$$

$$\Rightarrow \lambda_k \sum_{i=1}^{n} P(z_i=k|x_i,\theta^\circ) = \sum_{i=1}^{n} P(z_i=k|x_i,\theta^\circ)\, x_i$$

$$\Rightarrow \hat{\lambda}_k = \frac{\sum_{i=1}^{n} P(z_i=k|x_i,\theta^\circ)\, x_i}{\sum_{i=1}^{n} P(z_i=k|x_i,\theta^\circ)}$$

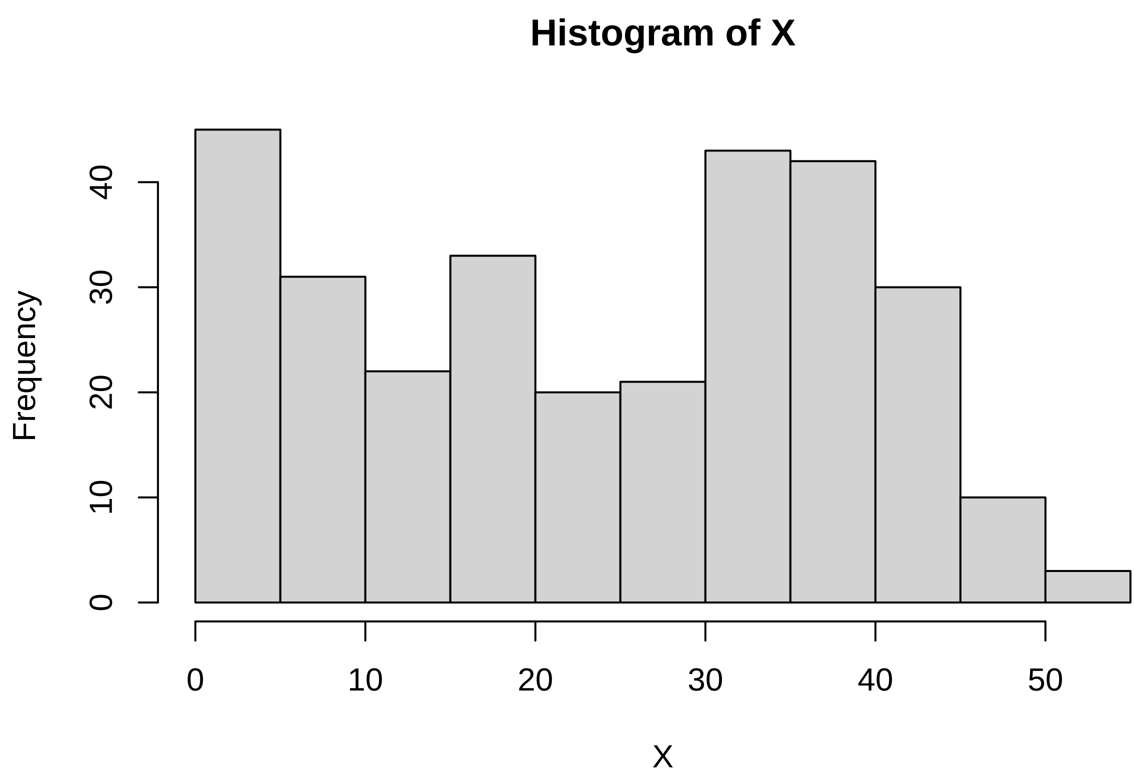
Question 1d

```r
X = scan(file="assignment3_prob1_2023.txt", what=double())

length(X)
```

```
## [1] 300
```

```r
hist(X)
```



Implementation of the EM algorithm

We will assume that the observed data follows a mixture of three Poisson distributions. Specifically, for $i = 1 \ldots, n$,

$$Z_i \sim \text{categorical } (\pi_1, \pi_2, 1 - \pi_1 - \pi_2),$$

$$X_i | Z_i = 1 \sim \text{Poisson}(\lambda_1),$$
$$X_i | Z_i = 2 \sim \text{Poisson}(\lambda_2),$$
$$X_i | Z_i = 3 \sim \text{Poisson}(\lambda_3).$$

We aim to obtain MLE of parameters $\theta = (\pi_1, \pi_2, \lambda_1, \lambda_2, \lambda_3)$ using the EM algorithm.

We implement the E and M step in the `EM.iter` function below. The `compute.log.lik` function below computes the incomplete log-likelihood, assuming the parameters are known. We will check that the incomplete log-likelihoods increases at each step by plotting them. The `mixture.EM` function is the main function which runs multiple EM steps and checks for convergence by computing the incomplete log-likelihoods at each step.

```r
# w.init : initial value for pi
# lambda.init : initial value for lambda
# epsilon : If the incomplete log-likelihood has changed by less than epsilon,
# EM will stop.
# max.iter : maximum number of EM-iterations
mixture.EM <- function(X, w.init, lambda.init, epsilon=1e-5, max.iter=100) {

  w.curr = w.init
  lambda.curr = lambda.init

  # store incomplete log-likehoods for each iteration
  log_liks = c()

  # compute incomplete log-likehoods using initial values of parameters.
  log_liks = c(log_liks, compute.log.lik(X, w.curr, lambda.curr)$ill)

  # set the change in incomplete log-likelihood with 1
  delta.ll = 1

  # number of iteration
  n.iter = 1

  # If the log-likelihood has changed by less than epsilon, EM will stop.
  while((delta.ll > epsilon) & (n.iter <= max.iter)){

    # run EM step
    EM.out = EM.iter(X, w.curr, lambda.curr)

    # replace the current value with the new parameter estimate
    w.curr = EM.out$w.new
    lambda.curr = EM.out$lambda.new

    # incomplete log-likehoods with new parameter estimate
    log_liks = c(log_liks, compute.log.lik(X, w.curr, lambda.curr)$ill)

    # compute the change in incomplete log-likelihood
    delta.ll = log_liks[length(log_liks)]  - log_liks[length(log_liks)-1]

    # increase the number of iteration
    n.iter = n.iter + 1
  }
  return(list(w.curr=w.curr, lambda.curr=lambda.curr, log_liks=log_liks))
}
```

```r
EM.iter <- function(X, w.curr, lambda.curr) {

  # E-step: compute E_{Z|X,\theta_0}[I(Z_i = k)]
```

```r
  # for each sample $X_i$, compute $P(X_i, Z_i=k)$
  prob.x.z = compute.prob.x.z(X, w.curr, lambda.curr)$prob.x.z

  # compute P(Z_i=k | X_i)
  P_ik = prob.x.z / rowSums(prob.x.z)

  # M-step
  w.new = colSums(P_ik)/sum(P_ik)  # sum(P_ik) is equivalent to sample size
  lambda.new = colSums(P_ik*X)/colSums(P_ik)

  return(list(w.new=w.new, lambda.new=lambda.new))
}
```

Now we write a function to compute the incomplete log-likelihood, assuming the parameters are known.

$$\ell(\theta) = \sum_{i=1}^{n} \log \left(\sum_{k=1}^{3} \pi_k f(x_i; \lambda_k) \right)$$

```r
# Compute incomplete log-likehoods
compute.log.lik <- function(X, w.curr, lambda.curr) {

  # for each sample $X_i$, compute $P(X_i, Z_i=k)$
  prob.x.z = compute.prob.x.z(X, w.curr, lambda.curr)$prob.x.z

  # incomplete log-likehoods
  ill = sum(log(rowSums(prob.x.z)))

  return(list(ill=ill))
}

# for each sample $X_i$, compute $P(X_i, Z_i=k)$
compute.prob.x.z <- function(X, w.curr, lambda.curr) {

  # for each sample $X_i$, compute $P(X_i, Z_i=k)$. Store these values in the columns of L:
  L = matrix(NA, nrow=length(X), ncol= length(w.curr))
  for(k in seq_len(ncol(L))) {
    L[, k] = dpois(X, lambda=lambda.curr[k])*w.curr[k]
  }

  return(list(prob.x.z=L))
}
```

Apply the EM algorithm

Run EM algorithm with different initial values and check that the incomplete log-likelihoods increases at each step by plotting them.

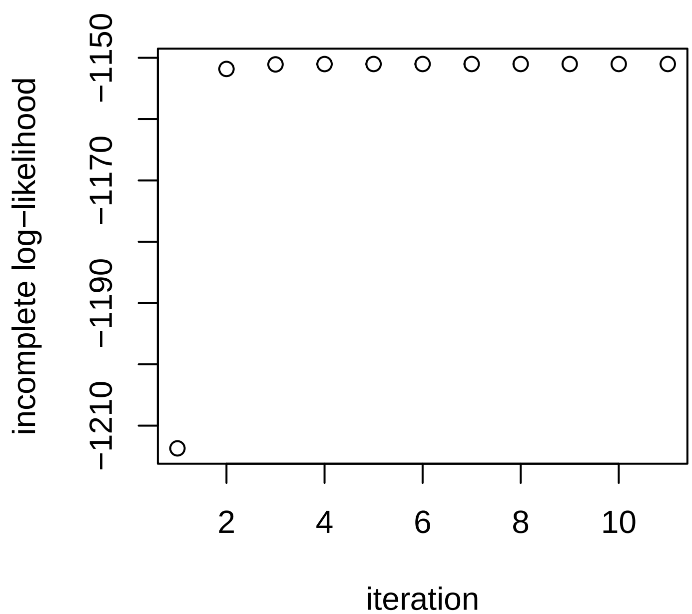
```r
EM1 <- mixture.EM(X, w.init=c(0.3,0.3, 0.4), lambda.init=c(3, 20, 35), epsilon=1e-5, max.iter=100)
ee = EM1
print(paste("Estimate pi = (", round(ee$w.curr[1],2), ",",
            round(ee$w.curr[2],2), ",",
            round(ee$w.curr[3],2), ")", sep=""))
```

```
## [1] "Estimate pi = (0.25,0.25,0.5)"
```

```r
print(paste("Estimate lambda = (", round(ee$lambda.curr[1],2), ",",
            round(ee$lambda.curr[2],2), ",",
            round(ee$lambda.curr[3],2), ")", sep=""))
```

```
## [1] "Estimate lambda = (5.17,18.09,36.94)"
```

```r
plot(ee$log_liks, ylab='incomplete log-likelihood', xlab='iteration')
```



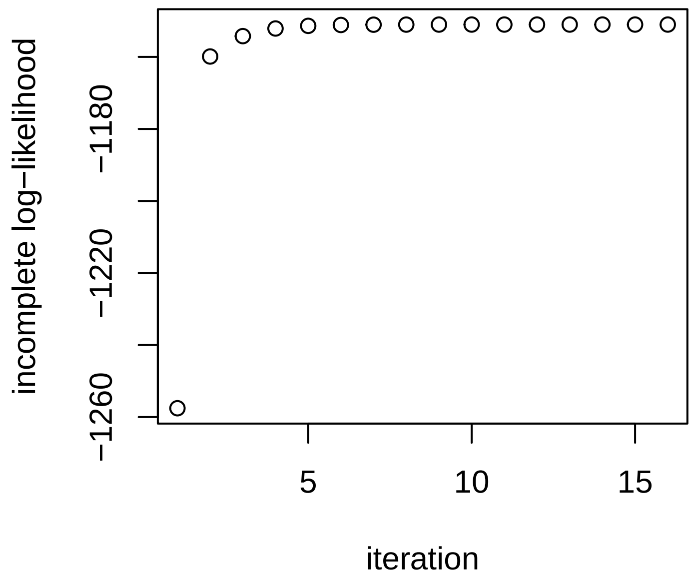

```r
EM2 <- mixture.EM(X, w.init=c(0.1,0.2, 0.7), lambda.init=c(5, 25, 40), epsilon=1e-5, max.iter=100)
ee = EM2
print(paste("Estimate pi = (", round(ee$w.curr[1],2), ",",
            round(ee$w.curr[2],2), ",",
            round(ee$w.curr[3],2), ")", sep=""))
```

```
## [1] "Estimate pi = (0.25,0.25,0.5)"
```

```r
print(paste("Estimate lambda = (", round(ee$lambda.curr[1],2), ",",
            round(ee$lambda.curr[2],2), ",",
            round(ee$lambda.curr[3],2), ")", sep=""))
```

```
## [1] "Estimate lambda = (5.17,18.09,36.94)"
```

```r
plot(ee$log_liks, ylab='incomplete log-likelihood', xlab='iteration')
```



Check which estimators have the highest incomplete log-likelihood.

```r
EM1$log_liks[length(EM1$log_liks)]
```

```
## [1] -1151.015
```

```r
EM2$log_liks[length(EM2$log_liks)]
```

```
## [1] -1151.015
```

Estimators from the two EM runs have (equally) highest incomplete log-likelihoods. You can see that the estimators from the EM runs are the same, so it doesn't matter which estimators we choose. Lets choose the estimators from the first EM run - $\hat{\pi}_1 = 0.25,\ \hat{\pi}_2 = 0.25,\ \hat{\lambda}_1 = 5.17,\ \hat{\lambda}_2 = 18.09,\ \hat{\lambda}_3 = 36.94$.

2. $Z_i \sim \text{categorical}(\pi_1, \pi_2, 1-\pi_1-\pi_2)$

$(X_i \mid Z_i = 1) \sim \text{Poisson}(\lambda_1)$

$(X_i \mid Z_i = 2) \sim \text{Poisson}(\lambda_2)$ for $i = 1, \dots, 300$

$(X_i \mid Z_i = 3) \sim \text{Poisson}(\lambda_3)$

$f(x; \lambda) = \dfrac{\lambda^x e^{-\lambda}}{x!}$ $X_i \sim \text{Poisson}(\lambda_2)$ for $i = 301, \dots, 400$

a) Let $X = (X_1, \dots, X_{400})$ and $Z = (Z_1, \dots, Z_{300})$

Derive $Q(\theta, \theta^0) = E_{Z|X,\theta^0}[\log(P(X, Z \mid \theta))]$

Let $n = 300$

$$P(X, Z \mid \theta) = \prod_{i=1}^{n} \left[P(X_i \mid Z_i, \theta) P(Z_i \mid \theta)\right] \prod_{i=301}^{400} P(X_i)$$

$$= \prod_{i=1}^{n} \prod_{k=1}^{3} \left[P(X_i \mid Z_i = k, \theta) P(Z_i = k \mid \theta)\right]^{I(Z_i = k)} \prod_{i=301}^{400} P(X_i)$$

$$\log P(X, Z \mid \theta) = \sum_{i=1}^{n} \sum_{k=1}^{3} \left(I(Z_i = k)\left[\log P(X_i \mid Z_i = k, \theta) + \log P(Z_i = k \mid \theta)\right]\right) + \sum_{i=301}^{400} \log P(X_i)$$

$$Q(\theta, \theta^0) = \sum_{i=1}^{n} \sum_{k=1}^{3} P(Z_i = k \mid X_i, \theta^0)\left[\log P(X_i \mid Z_i = k, \theta) + \log P(Z_i = k \mid \theta)\right] + \sum_{301}^{400} \log P(X_i)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{3} P(Z_i = k \mid X_i, \theta^0)\left[X_i \log(\lambda_k) - \lambda_k - \log(X_i!) + \log \pi_k\right] + \sum_{301}^{400} \left[X_i \log(\lambda_2) - \lambda_2 - \log(X_i!)\right]$$

where $\pi_3 = 1 - \pi_1 - \pi_2$

b) E step:

Let $\theta^0 = (\pi_1^0, \pi_2^0, \lambda_1^0, \lambda_2^0, \lambda_3^0)$

$$P(Z_i = k \mid X_i, \theta^0) = \frac{P(Z_i = k, X_i \mid \theta^0)}{P(X_i \mid \theta^0)}$$

$$= \frac{P(X_i \mid Z_i = k, \theta^0)\, P(Z_i = k \mid \theta^0)}{\sum_{k'=1}^{3} P(X_i \mid Z_i = k', \theta^0)\, P(Z_i = k' \mid \theta^0)}$$

$$P(Z_i = 1 \mid X_i, \theta^0) = \frac{P(X_i \mid Z_i = 1, \theta^0)\, P(Z_i = 1 \mid \theta^0)}{P(X_i \mid Z_i = 1, \theta^0)\, P(Z_i = 1 \mid \theta^0) + P(X_i \mid Z_i = 2, \theta^0)\, P(Z_i = 2 \mid \theta^0) + P(X_i \mid Z_i = 3, \theta^0)\, P(Z_i = 3 \mid \theta^0)}$$

$$P(Z_i = 2 \mid X_i, \theta^0) = \frac{P(X_i \mid Z_i = 2, \theta^0)\, P(Z_i = 2 \mid \theta^0)}{P(X_i \mid Z_i = 1, \theta^0)\, P(Z_i = 1 \mid \theta^0) + P(X_i \mid Z_i = 2, \theta^0)\, P(Z_i = 2 \mid \theta^0) + P(X_i \mid Z_i = 3, \theta^0)\, P(Z_i = 3 \mid \theta^0)}$$

$$P(Z_i = 3 \mid X_i, \theta^0) = 1 - P(Z_i = 1 \mid X_i, \theta^0) - P(Z_i = 2 \mid X_i, \theta^0)$$

where $P(X_i \mid Z_i = 1, \theta^0) = \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!}$, $P(X_i \mid Z_i = 2, \theta^0) = \frac{\lambda_2^{x_i} e^{-\lambda_2}}{x_i!}$, $P(Z_i = 1 \mid \theta^0) = \pi_1$, $P(Z_i = 2 \mid \theta^0) = \pi_2$

M step:

The new summation is removed from all partial derivatives except for $\frac{\partial Q(\theta, \theta^0)}{\partial \lambda_2}$ so we copy our answers from 1c

$$\Rightarrow \hat{\pi}_1 = \frac{\sum_{i=1}^{n} P(Z_i = 1 \mid X_i, \theta^0)}{n}$$

$$\Rightarrow \hat{\pi}_2 = \frac{\sum_{i=1}^{n} P(Z_i = 2 \mid X_i, \theta^0)}{n} \qquad \hat{\pi}_3 = 1 - \hat{\pi}_1 - \hat{\pi}_2$$

Let $$\frac{\partial Q(\theta, \theta^0)}{\partial \lambda_k} = \sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0)\left[-1 + \frac{X_i}{\lambda_k}\right] \quad \text{for } k \neq 2$$

$$= \frac{\sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0)\, X_i - \lambda_k \sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0)}{\lambda_k}$$

$$\lambda_k \sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0) = \sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0)\, X_i$$

$$\Rightarrow \hat{\lambda}_k = \frac{\sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0)\, X_i}{\sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0)} \quad \text{for } k \neq 2$$

$$\frac{\partial Q(\theta, \theta^0)}{\partial \lambda_2} = \sum_{i=1}^{n} P(Z_i = 2 \mid X_i, \theta^0)\left[-1 + \frac{X_i}{\lambda_2}\right] + \sum_{i=301}^{400}\left[-1 + \frac{X_i}{\lambda_2}\right]$$

$$= \frac{\sum_{i=1}^{n}\left[P(Z_i = 2 \mid X_i, \theta^0)\, X_i - \lambda_k \sum_{i=1}^{n} P(Z_i = 2 \mid X_i, \theta^0)\right] + \sum_{i=301}^{400}\left[X_i - \lambda_2\right]}{\lambda_2} = 0$$

$$= \frac{\sum_{i=1}^{n}\left[P(Z_i = 2 \mid X_i, \theta^0)\, X_i - \lambda_2 \sum_{i=1}^{n} P(Z_i = 2 \mid X_i, \theta^0)\right] + \sum_{i=301}^{400} X_i - 100\lambda_2}{\lambda_2} = 0$$

$$\Rightarrow \lambda_2 \sum_{i=1}^{n} P(Z_i = 2 \mid X_i, \theta^0) + 100\lambda_2 = \sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0)\, X_i + \sum_{i=301}^{400} X_i$$

$$\Rightarrow \lambda_2 \left(\sum_{i=1}^{n}\left[P(Z_i = 2 \mid X_i, \theta^0)\right] + 100\right) = \sum_{i=1}^{n} P(Z_i = k \mid X_i, \theta^0)\, X_i + \sum_{i=301}^{400} X_i$$

$$\Rightarrow \hat{\lambda}_2 = \frac{\sum_{i=1}^{n} P(Z_i = 2 \mid X_i, \theta^0)\, X_i + \sum_{i=301}^{400} X_i}{\sum_{i=1}^{n} P(Z_i = 2 \mid X_i, \theta^0) + 100}$$

Question 2c

```
X = scan(file="assignment3_prob1_2023.txt", what=double())

X0 = scan(file="assignment3_prob2_2023.txt", what=double())

length(X)
```

```
## [1] 300
```

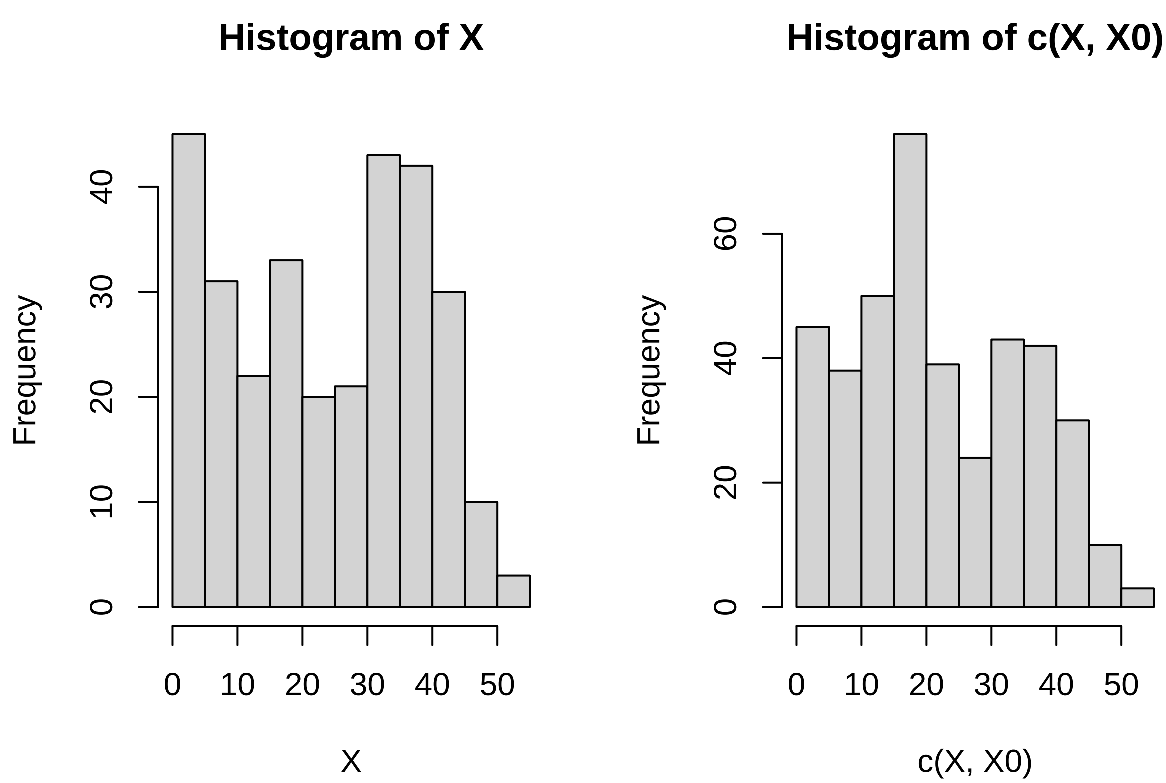
```
length(X0)
```

```
## [1] 100
```

```
par(mfrow=c(1,2))

hist(X)

hist(c(X,X0))
```



Implementation of the EM algorithm

We will assume that the observed data follows a mixture of three Poisson distributions. Additionally, for samples 301 to 400, we assume that they follow a known Poisson distribution. Specifically, for $i = 1 \ldots, n$,

$$Z_i \sim \text{categorical } (\pi_1, \pi_2, 1 - \pi_1 - \pi_2),$$

$$X_i | Z_i = 1 \sim \text{Poisson}(\lambda_1),$$
$$X_i | Z_i = 2 \sim \text{Poisson}(\lambda_2),$$
$$X_i | Z_i = 3 \sim \text{Poisson}(\lambda_3),$$

and for i=301, ..., 400,

$$X_i \sim \text{Poisson}(\lambda_2).$$

We aim to obtain MLE of parameters $\theta = (\pi_1, \pi_2, \lambda_1, \lambda_2, \lambda_3)$ using the EM algorithm.

We implement the E and M step in the `EM.iter` function below. The `compute.log.lik` function below computes the incomplete log-likelihood, assuming the parameters are known. We will check that the incomplete log-likelihoods increases at each step by plotting them. The `mixture.EM` function is the main function which runs multiple EM steps and checks for convergence by computing the incomplete log-likelihoods at each step.

```
# w.init : initial value for pi
# lambda.init : initial value for lambda
# epsilon : If the incomplete log-likelihood has changed by less than epsilon,
# EM will stop.
# max.iter : maximum number of EM-iterations
mixture.EM <- function(X, w.init, lambda.init, epsilon=1e-5, max.iter=100) {

  w.curr = w.init
  lambda.curr = lambda.init

  # store incomplete log-likehoods for each iteration
  log_liks = c()

  # compute incomplete log-likehoods using initial values of parameters.
  log_liks = c(log_liks, compute.log.lik(X, w.curr, lambda.curr)$ill)

  # set the change in incomplete log-likelihood with 1
  delta.ll = 1

  # number of iteration
  n.iter = 1

  # If the log-likelihood has changed by less than epsilon, EM will stop.
  while((delta.ll > epsilon) & (n.iter <= max.iter)){

    # run EM step
    EM.out = EM.iter(X, w.curr, lambda.curr)

    # replace the current value with the new parameter estimate
    w.curr = EM.out$w.new
    lambda.curr = EM.out$lambda.new

    # incomplete log-likehoods with new parameter estimate
```

```r
    log_liks = c(log_liks, compute.log.lik(X, w.curr, lambda.curr)$ill)

    # compute the change in incomplete log-likelihood
    delta.ll = log_liks[length(log_liks)]  - log_liks[length(log_liks)-1]

    # increase the number of iteration
    n.iter = n.iter + 1
  }
  return(list(w.curr=w.curr, lambda.curr=lambda.curr, log_liks=log_liks))
}
```

```r
EM.iter <- function(X, w.curr, lambda.curr) {

  # E-step: compute E_{Z|X,\theta_0}[I(Z_i = k)]

  # for each sample $X_i$, compute $P(X_i, Z_i=k)$
  prob.x.z = compute.prob.x.z(X, w.curr, lambda.curr)$prob.x.z

  # compute P(Z_i=k | X_i)
  P_ik = prob.x.z / rowSums(prob.x.z)

  # M-step
  w.new = colSums(P_ik)/sum(P_ik)  # sum(P_ik) is equivalent to sample size
  lambda.new = colSums(P_ik*X)/colSums(P_ik)
  lambda.new[2] = (colSums(P_ik*X)[2] + sum(X0))/(colSums(P_ik)[2] + 100)

  #print(paste("lambda.new = (", lambda.new, ")", sep=""))


  return(list(w.new=w.new, lambda.new=lambda.new))
}
```

Now we write a function to compute the incomplete log-likelihood, assuming the parameters are known.

$$\ell(\theta) = \sum_{i=1}^{n} \log \left(\sum_{k=1}^{3} \pi_k f(x_i; \lambda_k) \right)$$

```r
# Compute incomplete log-likehoods
compute.log.lik <- function(X, w.curr, lambda.curr) {

  # for each sample $X_i$, compute $P(X_i, Z_i=k)$
  prob.x.z = compute.prob.x.z(X, w.curr, lambda.curr)$prob.x.z

  # incomplete log-likehoods
  ill = sum(log(rowSums(prob.x.z)))

  return(list(ill=ill))
}

# for each sample $X_i$, compute $P(X_i, Z_i=k)$
compute.prob.x.z <- function(X, w.curr, lambda.curr) {

  # for each sample $X_i$, compute $P(X_i, Z_i=k)$. Store these values in the columns of L:
```

```
  L = matrix(NA, nrow=length(X), ncol= length(w.curr))
  for(k in seq_len(ncol(L))) {
    L[, k] = dpois(X, lambda=lambda.curr[k])*w.curr[k]
  }

  return(list(prob.x.z=L))
}
```

Apply the EM algorithm

Run EM algorithm with different initial values and check that the incomplete log-likelihoods increases at each step by plotting them.

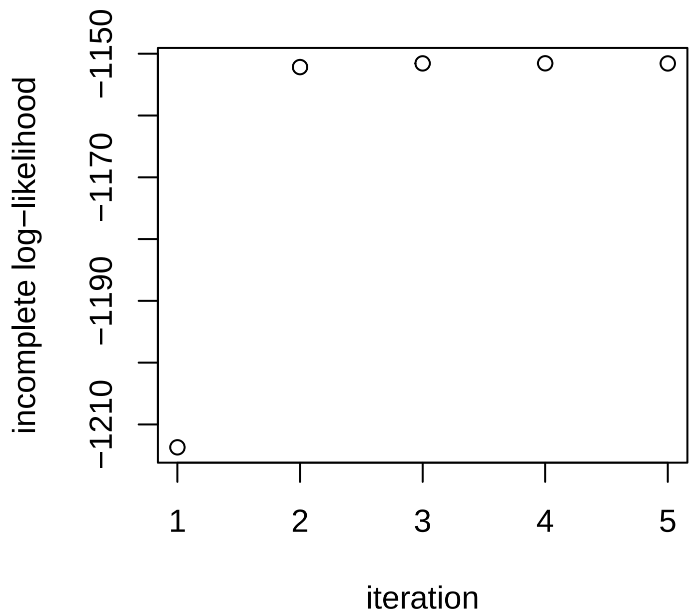
```
EM1 <- mixture.EM(X, w.init=c(0.3,0.3, 0.4), lambda.init=c(3, 20, 35), epsilon=1e-5, max.iter=100)
ee = EM1
print(paste("Estimate pi = (", round(ee$w.curr[1],2), ",",
            round(ee$w.curr[2],2), ",",
            round(ee$w.curr[3],2), ")", sep=""))
```

```
## [1] "Estimate pi = (0.25,0.25,0.51)"
```

```
print(paste("Estimate lambda = (", round(ee$lambda.curr[1],2), ",",
            round(ee$lambda.curr[2],2), ",",
            round(ee$lambda.curr[3],2), ")", sep=""))
```

```
## [1] "Estimate lambda = (5.12,17.38,36.79)"
```

```
plot(ee$log_liks, ylab='incomplete log-likelihood', xlab='iteration')
```



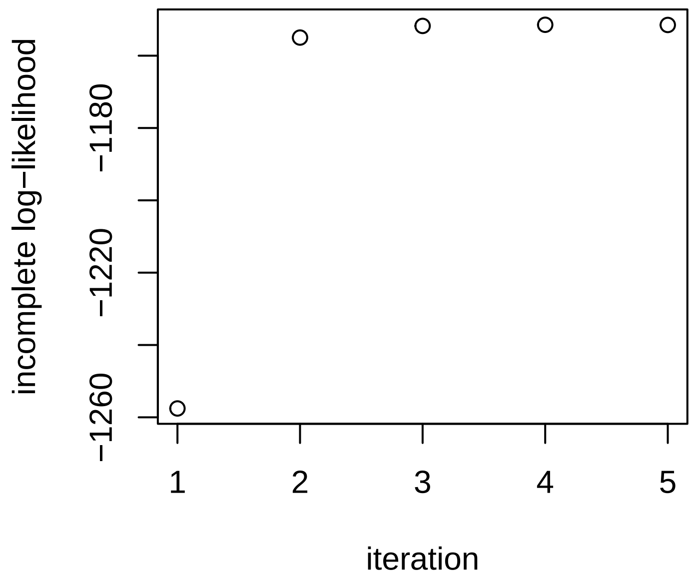

```
EM2 <- mixture.EM(X, w.init=c(0.1,0.2, 0.7), lambda.init=c(5, 25, 40), epsilon=1e-5, max.iter=100)
ee = EM2
print(paste("Estimate pi = (", round(ee$w.curr[1],2), ",",
            round(ee$w.curr[2],2), ",",
            round(ee$w.curr[3],2), ")", sep=""))
```

```
## [1] "Estimate pi = (0.25,0.25,0.5)"
```

```
print(paste("Estimate lambda = (", round(ee$lambda.curr[1],2), ",",
            round(ee$lambda.curr[2],2), ",",
            round(ee$lambda.curr[3],2), ")", sep=""))
```

```
## [1] "Estimate lambda = (5.13,17.43,36.86)"
```

```
plot(ee$log_liks, ylab='incomplete log-likelihood', xlab='iteration')
```



Check which estimators have the highest incomplete log-likelihood.

```
EM1$log_liks[length(EM1$log_liks)]
```

```
## [1] -1151.57
```

```
EM2$log_liks[length(EM2$log_liks)]
```

```
## [1] -1151.507
```

Estimators from the two EM runs nearly have equally highest incomplete log-likelihoods. You can see that the incomplete log-likelihood is slightly higher for the estimators from the second EM run and so we will choose the estimators from the second EM run - $\hat{\pi}_1 = 0.25,\ \hat{\pi}_2 = 0.25,\ \hat{\lambda}_1 = 5.13,\ \hat{\lambda}_2 = 17.43,\ \hat{\lambda}_3 = 36.86$.