Q1.

(a)
$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} = \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}$$ expand along 1st column

$$= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= \begin{vmatrix} b-a & (b+a)(b-a) \\ c-a & (c+a)(c-a) \end{vmatrix}$$

$$= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (b-a)(c-a)\left((c+a)-(b+a)\right)$$
$$= (b-a)(c-a)(c-b)$$

(b) For any three points (x_1, y_1), (x_2, y_2), (x_3, y_3) on xy-plane with $x_1 \neq x_2$, $x_1 \neq x_3$ and $x_2 \neq x_3$, a curve $y = \alpha + \beta x + \gamma x^2$ passes through them if and only iff

$$\alpha + \beta x_1 + \gamma x_1^2 = y_1$$
$$\alpha + \beta x_2 + \gamma x_2^2 = y_2$$
$$\alpha + \beta x_3 + \gamma x_3^2 = y_3$$

that is,
$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad (*)$$

By part (a) we have
$$\begin{vmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{vmatrix} = (x_2 - x_1)(x_3 - x_1)(x_3 - x_2) \neq 0$$

as $x_1 \neq x_2$, $x_1 \neq x_3$ and $x_2 \neq x_3$. This implies that the coefficient matrix in (*) is invertible.

Hence (*) has a unique solution for $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$,

and so there exists a unique curve of the form $y = \alpha + \beta x + \gamma x^2$ passing through (x_1, y_1), (x_2, y_2) and (x_3, y_3)

Q2.

(a) The unit vectors parallel to $\overrightarrow{PQ} = (1, 2, 3)$ are

$$\pm \frac{1}{\|\overrightarrow{PQ}\|} \overrightarrow{PQ} = \pm \frac{1}{\sqrt{1^2+2^2+3^2}} (1, 2, 3) = \pm \frac{1}{\sqrt{14}} (1, 2, 3) = \pm \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$$

(b) $\overrightarrow{QR} = (-1, 1, 0)$ Let $\hat{\underset{\sim}{u}} = \frac{1}{\|\overrightarrow{PQ}\|} \overrightarrow{PQ} = \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right)$

Then $\hat{\underset{\sim}{u}} \cdot \overrightarrow{QR} = \frac{1}{\sqrt{14}} \times (-1) + \frac{2}{\sqrt{14}} \times 1 + \frac{3}{\sqrt{14}} \times 0 = \frac{1}{\sqrt{14}}$

and $\text{proj}_{\overrightarrow{PQ}} \overrightarrow{QR} = (\hat{\underset{\sim}{u}} \cdot \overrightarrow{QR}) \hat{\underset{\sim}{u}} = \frac{1}{\sqrt{14}} \left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}} \right) = \left(\frac{1}{14}, \frac{1}{7}, \frac{3}{14} \right)$

(c)
$$\overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR}) = \begin{vmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \begin{vmatrix} 1 & -1 & -2 \\ 0 & 3 & 5 \\ 0 & 3 & 3 \end{vmatrix} \; R_3 \to R_3 - R_2$$

$$= \begin{vmatrix} 1 & -1 & -2 \\ 0 & 3 & 5 \\ 0 & 0 & -2 \end{vmatrix}$$

$$= 1 \times 3 \times (-2)$$

$$= -6$$

Hence the volume is $\left| \overrightarrow{OP} \cdot (\overrightarrow{OQ} \times \overrightarrow{OR}) \right| = |-6| = 6$