Modal and Temporal Logic, 2012-2013
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1. (a)
   i. A is *valid in a model*, $\mathcal{M}$, if A is satisfied at every world of $\mathcal{M}$.
   ii. A is *valid in a frame*, $\mathcal{F}$, if for any model, $\mathcal{M}$, constructed from $\mathcal{F}$, A is valid in $\mathcal{M}$.
   iii. A is *valid* if A is valid in all frames.

   (b)
   i. Reflexive frames.
   ii. Let $A = \Box p \to p$. In the $\mathcal{M}$ given, A is valid because it is satisfied at all worlds. We can see that $1, \mathcal{M} \models A$ (1 sees only 2 and p is true at 2), and that $2, \mathcal{M} \models A$ (2 sees no worlds).
   However, A is not valid for the frame, $\mathcal{F}$. We can find a witness model $\mathcal{M}_0$ created from $\mathcal{F}$ and see that A is not valid in $\mathcal{M}_0$. Take $\mathcal{M}_0$ as below (where $h(p) = \emptyset$). $1, \mathcal{M} \not\models A$ so A is not valid in $\mathcal{M}_0$. Hence A cannot be valid in $\mathcal{F}$.

   (c)
   i. Assume without proof $\mathcal{M}, t \models A \to B$ iff $\mathcal{M}, t \models A \to \mathcal{M}, t \models B$.
   Take arbt. world t from arbt. model $\mathcal{M}$ constructed from arbt. frame $\mathcal{F}$.
   To show: $\mathcal{M}, t \models \Box(p \to q) \to (\Diamond p \to \Diamond q)$.
   Assume $\mathcal{M}, t \models \Box(p \to q)$. By Kripke semantics, if $R(t,u)$ then $\mathcal{M}, u \models p \to q$ for all u in $\mathcal{F}$. Now assume $\mathcal{M}, t \models \Diamond p$. Then $R(t, u_0)$ and $\mathcal{M}, u_0 \models p$ for some world u_0. But we also have $\mathcal{M}, u_0 \models p \to q$ by first assumption. So $\mathcal{M}, u_0 \models q$. Hence $\mathcal{M}, t \models \Diamond q$ So $\mathcal{M}, t \models \Diamond p \to \Diamond q$, and, $\mathcal{M}, t \models \Box(p \to q) \to (\Diamond p \to \Diamond q)$
   ii. *Not* valid.
   We can have $\mathcal{M}, t \models \Diamond(p \to q)$ but not $\mathcal{M}, t \models \Diamond p \to \Diamond q$. Assume $\mathcal{M}, t \models \Diamond(p \to q)$, then assume $\mathcal{M}, t \models \Diamond p$. There is some world that t can see that satisfies $p \to q$ and some world that t can see that satisfies p. However we cannot guarantee that these worlds as the *same*. We cannot apply modus ponens like before and say there is a a world that t can see that satisfies q.
   Counter-example:

   (d)
   i. $\neg\Diamond(\Diamond\Box p \wedge \neg p)$
   ii. Let $A = \Diamond(\Diamond\Box p \wedge \neg p)$
   We will apply Sahlqvist's algorithm to find the *Sahlqvist's correspondant* of A, $\alpha[t]$ and thus have $\mathcal{F}, u \models \neg A$ iff $\mathcal{F} \models \forall t \neg\alpha(t)$. *Boxed atoms*: $\Box p$. *Negative formulas*: $\neg p$. We want to find a lazy assignment that makes boxed atoms true without concern for negative formulas. Suppose arbt world t sees some world u and u sees some world v, then our assignment must make p true at worlds v can see in order to make boxed atom $\Box p$ true. So *lazy assignment* is $h^o(p) = \{x \mid R(v,x)\}$.

Now we take the *standard translation* of A:

$$\begin{aligned} A^t &= \exists u(R(t,u) \wedge (\Diamond\Box p \wedge \neg p)^u \\ &= \exists u(R(t,u) \wedge ((\Diamond\Box p)^u \wedge (\neg p)^u) \\ &= \exists u(R(t,u) \wedge ((\exists v(R(u,v) \wedge (\Box p)^v) \wedge (\neg p^u)) \\ &= \exists u(R(t,u) \wedge \exists v(R(u,v) \wedge (\Box p)^v \wedge \neg P(u)) \end{aligned}$$

To preserve equivalence, we move $\exists v$ outside:
$\exists u \exists v(R(t,u) \wedge R(u,v) \wedge (\Box p)^v \wedge \neg P(u))$.
Our lazy assignment lets us replace $P(u)$ with $R(u,v)$ and $(\Box p)^v$ with $(\top)^v = \top$. $\exists u \exists v(R(t,u) \wedge R(u,v) \wedge \top \wedge \neg R(u,v)) = \alpha[t]$.
So $\forall t \neg \alpha(t) = \forall t \exists u \exists v(R(t,u) \wedge R(u,v) \wedge (\top)^v \wedge \neg R(u,v))$ which is logically equivalent to $\forall t \forall u \forall v(R(t,u) \wedge R(u,v) \rightarrow \neg R(u,v))$.
Since $(\mathcal{F},h),t \models A$ iff $\mathcal{F} \models (\mathcal{F},h^o),t \models A$ iff $\mathcal{F} \models \alpha[t]$.
we have: $(\mathcal{F},h),t \models \neg A$ iff $\mathcal{F} \models \forall t \neg \alpha[t]$ for any $\mathcal{F},h,t$. That is, B is valid in $\mathcal{F}$ iff $\mathcal{F}$ satisfies (using first-order semantics) $\forall t \neg \alpha[t]$

iii. B is valid in $\mathcal{F}$ iff $\forall t \forall u \forall v(R(t,u) \wedge R(u,v) \rightarrow \neg R(u,v))$. So we need to find a frame $\mathcal{F}_0$ that doesn't satisfy this first-order condition. The frame below doesn't satisfy the condition because there are some worlds x,y,z where x sees y, y sees z but we have z sees y.

2. (a) i. Since $\mathcal{F}'$ is a p-morphic image of $\mathcal{F}$, there is a p-morphism from $\mathcal{F}$ to $\mathcal{F}'$. Every p-morphism satisfies the forth property and back-property.
Forth property: $R(x,y) \rightarrow R'(f(x),f(y))$ for all $x,y \in W$
Back property: $R'(f(x),v) \rightarrow (R(x,y)$ and $f(y) = v$ for some $y \in W)$
for all $x \in W, v \in W'$.
Take arbt $x,y,z \in W'$. Assume $R'(x,y)$ and $R'(y,z)$ To show $R'(x,z)$.
x must the image of some world $t \in W$, $x = f(t)$. By the back-property, given $R'(f(t),y)$, we must have $R(t,t')$ where $f(t') = y$ for some world $t' \in W$. By the back-property, since $R'(f(t'),z)$, we have $R(t',t'')$ where $f(t'') = z$ for some world $t'' \in W$.
Since $\mathcal{F}$ is transitive, given that $R(t,t')$ and $R(t',t'')$, we must have $R(t,t'')$. And by the forth property, given that $R(t,t'')$ we must have $R'(f(t),f(t''))$, in other words, $R(x,z)$.

ii. No. Counter-example:

(b) i. Example:

ii. Let $\mathcal{F} \times (\mathbb{N},<) = (W^\times, R^\times)$ $(x,u)R^\times(y,v)$ iff $R(x,y)$ and $u < v$.
Assume $\mathcal{F}$ is triangle-free. To show: $R^\times(x,y)$ is triangle free.
Assume for contradiction, $R^\times(x,y)$, $R^\times(y,z)$, $R^\times(z,x)$ for some $x,y,z \in W^\times$.

Then by def. of $R^\times$, we have $x < y$, we have $y < z$ and $z < x$. By transitivity of $<$, we have $x < z$ but this contradicts $z < x$.
(*alternatively*: derive contradiction by seeing that there is a triangle in $\mathcal{F}$).
So there are no $x, y, z \in W^\times$ such that $R^\times(x,y)$, $R^\times(y,z)$, $R^\times(z,x)$.

iii. From lectures, there is a p-morphism from $\mathcal{F} \times (\mathbb{N}, <)$ to $\mathcal{F}$.
So $\mathcal{F}$ is a p-morphic image of $(\mathbb{N}, <)$. From lectures, p-morphic images preserve validity of modal formulas. That is, if A valid in $\mathcal{F} \times (\mathbb{N}, <)$ then A valid in $\mathcal{F}$.

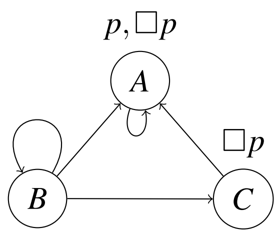
(c) i. Take some arbitrary time t, in $\mathcal{M} = ((\mathbb{N}, <), h)$ for some arbitrary h.
Assume $\mathcal{M}, t \models FGq$. Then there is some future time f, where q is true for the all future times of f.
Now take any time t' in the future of t. If $t' < f$ then there is indeed a future time where q is true (f). So t' must satisfy Fq. If $t' \geqslant f'$, q is always true in the future of f', and so must always be true in the future of t'. So there must be a time when q is true in the future t'. Hence t' must satisfy Fq.
Any future time of t, t', satisfies Fq, so we have GFq.
Hence $FGq \to GFq$ is satisfied at all worlds t of $\mathcal{M}$.

ii. Not valid. We can find some model $\mathcal{M}_0$ constructed from $(\mathbb{N}, <)$ where at some world, t, GFq holds but FGq doesn't. Let $\mathcal{M}_0$ have assignment $h(q) = \{\, t \in \mathbb{N} \mid t \text{ is even} \,\}$.
GFq is satisfied at t, as for all future times of t, t', there will be some future time of t' where q is true (say, at the next even time larger than t').
However FGq isn't satisfied at t. We cannot find a future time of t, f, where for all times in future of f, q is true. There will be some odd time in the future of f where q is false.

3.(a) Lemmon filtration for A:

- W^f = set of $\sim$-equivalence classes, where $\sim$ is a relation on W where $t \sim u \iff [\mathcal{M}, t \models B \Leftrightarrow \mathcal{M}, u \models B$, for all subformulas, B, of $A]$
- $R^\ell(X,Y) \iff [\mathcal{M}, x \models \Box B \Rightarrow \mathcal{M}, y \models \Box B \wedge B$ for all $x \in X, y \in Y$, subformulas, $\Box B$, of $A]$
- $h^f(p) = \{\, X \in W^f \mid \mathcal{M}, x \models p$ for some $x \in X \,\}$

(b) Lemmon filtration $\mathcal{N}^\ell_{\Box p}$:



(c) Take any arbitrary $X \in \mathcal{F}_A^\ell$.
Take any arbitrary world $x \in X$. Let B be any subformula of A.
Assume $\mathcal{M}, x \models \Box B$.

$\mathcal{F}$ is serial, so there is some world $y \in W$ with $R(x,y)$. So by our assumption, $\mathcal{M}, y \models B$. Take some arbt $z \in W$. Assume $R(y,z)$. By transitivity of $\mathcal{F}$, we have $R(x,z)$. And so by our first assumption, $\mathcal{M}, z \models B$. Hence $\forall z(R(y,z) \to \mathcal{M}, z \models B)$, so $\mathcal{M}, y \models \Box B$. Hence $\mathcal{M}, y \models \Box B \wedge B$

Let $Y \in \mathcal{F}_A^\ell$ be the $\sim$-equivalence class that y is in. Take any y' in Y. Since y and y' agree on the satisfiability of all subformulas of A, they agree on the satisfisfiability of $B \wedge \Box B$. So must have for all $y' \in Y$, $\mathcal{M}, y' \models B \wedge \Box B$. So $\mathcal{M}, x \models \Box B$ implies [$\mathcal{M}, y' \models B \wedge \Box B$ for all $x \in X$, $y' \in Y$, all subformulas B.]
Hence $R^\ell(X,Y)$.

So for any $X \in \mathcal{F}_A^\ell$, there is some $Y \in \mathcal{F}_A^\ell$ such that $R^\ell(X,Y)$.

(d) $KTS = K + (\Diamond\top, \Box p \to \Box\Box p)$ is sound and complete for $\mathcal{TS}$. We claim Thm(KTS) has the strong finite model property.

Take some $A \notin$ Thm(KTS), Since KTS is sound and complete over $\mathcal{TS}$, $A \notin$ Log$(\mathcal{TS})$, in other words, there is some model $\mathcal{M}$ constructed from a frame in $\mathcal{TS}$ with $\mathcal{M}, t \models \neg A$. We can take this $\mathcal{M}$ and find the Lemmon filtration, $\mathcal{M}_A^\ell$. By the Filtration lemma, $\neg A$ is satisfied in $\mathcal{M}_A^\ell$

From part (c), $\mathcal{F}_A^\ell$ is serial. From lectures, The Lemmon filtration relation R^ℓ is transitive (Lemma 9.19), so $\mathcal{F}_A^\ell$ is transitive. Hence $\mathcal{M}_A^\ell$ is serial and transitive and so validates Thm(KTS).

Since $\mathcal{F}_A^\ell$ is finite with at most 2^n worlds (where n is the number of subformulas of $\neg A$), KTS has the strong finite model property. There is an algorithm to decide whether $\mathcal{F}_A^\ell$ validates KTS (KTS has finite many axioms - by Theorem 8.3, it is enough to just check each axiom true for all finite worlds taken from finitely possible models build from $\mathcal{F}_A^\ell$) - so by Theorem 9.8, KTS is decidable.

Algorithm:
Compute $s(A) = 2^n$, where n is the number of subformulas of A.
Enumerate all models whose frame is $\mathcal{F}_A^\ell$.
If we find a model satisfies $\neg A$, halt and print $A \notin L$.
If no models are found that satisfy $\neg A$, halt and print $A \in L$

4. (a) i. For any $\Gamma, \Delta \in W_C$,
$R_C(\Gamma, \Delta)$ iff [if $\Box A \in \Gamma$ then $A \in \Delta$, for all formulas, A].

ii. ($\Rightarrow$) Assume $R_C(\Gamma, \Delta)$.
Take any A. Assume $A \in \Delta$. By the truth lemma, $\mathcal{M}_C, \Delta \models A$. By Kripke semantics, $\mathcal{M}_C, \Gamma \models \Diamond A$. By the truth lemma, $\Diamond A \in \Gamma$.
($\Leftarrow$) Assume $A \in \Delta$ implies $\Diamond A \in \Gamma$ for any A.
Assume $\Box A \in \Gamma$.

Suppose for contradiction, $A \notin \Delta$.
Δ is a MCS, so by MCS properties (lemma 7.20), $\neg A \in \Delta$. So by our first assumption, $\Diamond\neg A \in \Gamma$. But $\Diamond B$ is an abbreviation for $\neg\Box\neg B$, so we have $\neg\Box\neg\neg A \in \Gamma$. By the truth lemma, $\mathcal{M}_C, \Gamma \models \neg\Box\neg\neg A$. By equivalence in Kriple semantics $\mathcal{M}_C, \Gamma \models \neg\Box A$. And by the truth lemma we have $\neg\Box A \in \Gamma$. Γ is a MCS, so by properties of MCSs, $\Box A \notin \Gamma$. Contradiction. So $A \in \Delta$. Hence $R(\Gamma, \Delta)$.

iii. Take any worlds $\Gamma, \Delta_1, \Delta_2 \in W_C$.
Assume $R_C(\Gamma, \Delta_1)$ and $R_C(\Gamma, \Delta_2)$.
To show $R_C(\Delta_1, \Delta_2)$.
Assume $\Box A \in \Delta_1$
From part ii, since $R_C(\Gamma, \Delta_1)$, we have $\Diamond\Box A \in \Gamma$. Since Γ is an MCS, by Lemma 7.19, $\Gamma \vdash_C \Diamond\Box A$. Now, $\Diamond\Box p \to \Box p$ is an axiom of C, so by sub, $\vdash_C \Diamond\Box A \to \Box A$. Since $\Gamma \vdash_C \Diamond\Box A$ and $\vdash_C \Diamond\Box A \to \Box A$, we have $\Gamma \vdash_C \Box A$, which by MCS properties, gives $\Box A \in \Gamma$. But since $R(\Gamma, \Delta_2)$, $A \in \Delta_2$.
Hence $R_C(\Delta_1, \Delta_2)$.

iv. The *Completeness Theorem* states that the if a class of frames $\mathcal{C}$ contains the canonical frame $\mathcal{F}_H$, then H is complete over $\mathcal{C}$.
In part iii, we showed that the class of balloon-like frames contains $\mathcal{F}_C$, and so our Hilbert system is complete over the class.

(b) i. f is monotonic iff for all sets $U, V \subseteq W$:
if $U \subseteq V$ then $f(U) \subseteq f(V)$.

ii. $U \subseteq W$ is a fixed point of f iff $f(U) = U$.

iii. $f^0(\emptyset) = \emptyset$ by def of f. But $\emptyset \subseteq f^1(\emptyset)$, as the emptyset is a subset of all sets. Since f is monotonic, we have $f^1(\emptyset) \subseteq f^2(\emptyset)$. By monotonicity of f we have $f^2(\emptyset) \subseteq f^3(\emptyset)$ and $f^3(\emptyset) \subseteq f^4(\emptyset)$, ...
As W is finite, there must be some m such that $f^m(\emptyset) = f^{m+1}(\emptyset)$. Hence $Z = f^m(\emptyset)$ is a fixed point of f. Since $f^0(\emptyset) \cup f^1(\emptyset) \cup \ldots \cup f^m(\emptyset) = Z$ and $f^m(\emptyset) \cup f^{m+1}(\emptyset) \cup \ldots = Z$, we have $Z = Z \cup Z = \cup_{n \in \mathbb{N}} f^n(\emptyset)$.

(c) i. Let $A = p \wedge \Box q$. Then $[\![\mu q A]\!]_h = \text{LFP}(A_q^h)$ where $A_q^h(U) = [\![A]\!]_{h[p \mapsto U]}$.

- Let $h_0 = h[q \mapsto \emptyset]$. Then $A_q^h(\emptyset)$
$= [\![p \wedge \Box q]\!]_{h_0} = [\![p]\!]_{h_0} \cap [\![\Box q]\!]_{h_0} = \{2,3,4,5\} \cap \Box [\![q]\!]_{h_0}$
$= \{2,3,4,5\} \cap \Box\emptyset = \{2,3,4,5\} \cap \{5\} = \{5\}$
- Let $h_1 = h[q \mapsto \{5\}]$. Then $A_q^h(A_q^h(\emptyset))$
$= [\![p \wedge \Box q]\!]_{h_1} = [\![p]\!]_{h_1} \cap [\![\Box q]\!]_{h_1} = \{2,3,4,5\} \cap \Box [\![q]\!]_{h_1}$
$= \{2,3,4,5\} \cap \Box\{5\} = \{2,3,4,5\} \cap \{4,5\} = \{4,5\}$.
- Let $h_2 = h[q \mapsto \{4,5\}]$. Then $A_q^h(A_q^h(A_q^h(\emptyset)))$
$= [\![p \wedge \Box q]\!]_{h_2} = [\![p]\!]_{h_2} \cap [\![\Box q]\!]_{h_2} = \{2,3,4,5\} \cap \Box [\![q]\!]_{h_2}$
$= \{2,3,4,5\} \cap \Box\{4,5\} = \{2,3,4,5\} \cap \{4,5\} = \{4,5\}$.

So $\mathrm{LFP}(A_q^h) = \cup_{n\in\mathbb{N}}(A_q^h)^n(\emptyset) = \{4,5\}$. So $[\![\mu qA]\!]_h = \{4,5\}$.

ii. $[\![\nu qA]\!]_h = \mathrm{GFP}(A_q^h)$. Let $W = \{1,2,3,4,5\}$.

- Let $h_0 = h[q \mapsto W]$.
  Then: $A_q^h(W) = [\![p \wedge \Box q]\!]_{h_0} = [\![p]\!]_{h_0} \cap [\![\Box q]\!]_{h_0} = \{2,3,4,5\} \cap \Box[\![q]\!]_{h_0}$
  $= \{2,3,4,5\} \cap \Box W = \{2,3,4,5\} \cap W = \{2,3,4,5\}$.
- Let $h_1 = h[q \mapsto \{2,3,4,5\}]$.
  Then: $A_q^h(W) = [\![p \wedge \Box q]\!]_{h_1} = [\![p]\!]_{h_0} \cap [\![\Box q]\!]_{h_1} = \{2,3,4,5\} \cap \Box[\![q]\!]_{h_1}$
  $= \{2,3,4,5\} \cap \Box\{2,3,4,5\} = \{2,3,4,5\} \cap \{3,4,5\} = \{3,4,5\}$.
- Let $h_2 = h[q \mapsto \{3,4,5\}]$.
  Then: $A_q^h(W) = [\![p \wedge \Box q]\!]_{h_2} = [\![p]\!]_{h_0} \cap [\![\Box q]\!]_{h_2} = \{2,3,4,5\} \cap \Box[\![q]\!]_{h_2}$
  $= \{2,3,4,5\} \cap \Box\{3,4,5\} = \{2,3,4,5\} \cap \{3,4,5\} = \{3,4,5\}$.

So $\mathrm{LFP}(A_q^h) = \cap_{n\in\mathbb{N}}(A_q^h)^n(W) = \{3,4,5\}$. So $[\![\nu qA]\!]_h = \{3,4,5\}$.