Transient Analysis for RC Network

EE 225

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Contents

1 Transient Analysis Problem

The transient analysis summarized here for the RC circuit shown in Figure 1a corresponds to the case of a step function input, with the input voltage $V_{in}(t)$ changing abruptly at $t = 1$ from $V_{in}(t < 0) = 0$ volts at times less that $t = 0$ to $V_{in}(t \geq 0) = V_0$ at times $t \geq 0$. This input voltage is shown in Figure 1b.

2 Steady-State Conditions

First, consider the *steady state* values of the voltage V_r across the resistor with resistance R and of the voltage V_c across the capacitor with capacitance C. Before the voltage switches, the input voltage V_{in} has been 0 volts "forever" and all currents and voltages are constant (i.e., in a "steady state" rather than in a changing state). Long after the transition to $V_{in} = V_0$ at $t = 0$, the currents and voltages will again be in a steady state. Under such steady state conditions, no current is flowing through the capacitor (look at the symbol of the capacitor, which represents two metal
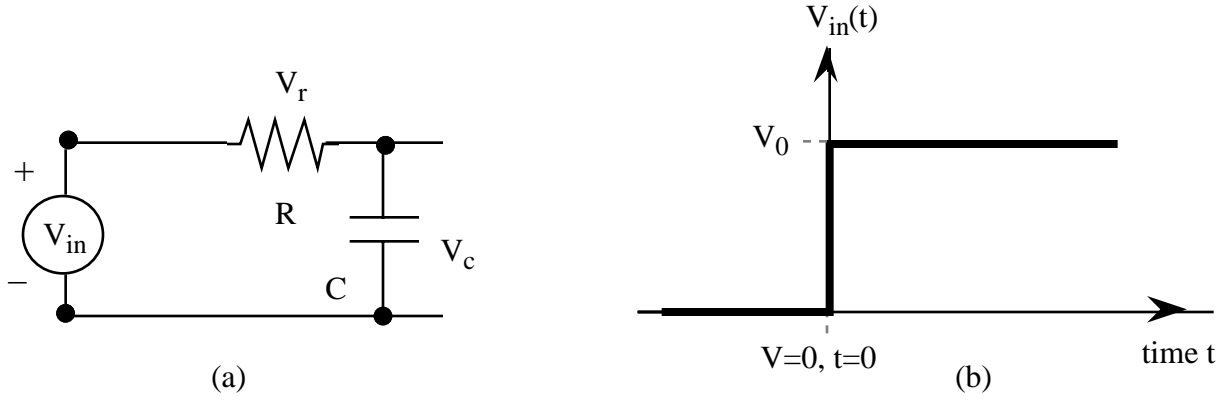
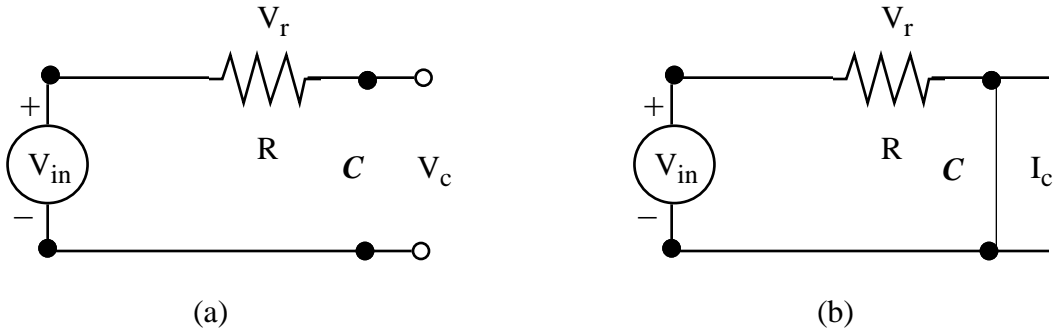Figure 1: (a) RC network. (b) Step function input voltage.

Figure 2: (a) At DC, no current can flow through insulator between plates of capacitor and the capacitor in Figure 1a is replaced by an open circuit. (b) At extremely high frequencies no voltage develops across the capacitor and the capacitor in Figure 1a is replaced by an short circuit.

plates and a perfect insulator, allowing no DC current flow, between the plates). We can therefore replace the capacitance by an "open circuit" as shown in Figure 2a.[1]

Must Know.

* *Under steady state conditions, no current can flow through a capacitor.* A perfect insulator (infinite resistance) separates the two plates of the capacitor.

Since no current is flowing in the steady state, there is no voltage drop across the resistor and the voltage at the right hand end of the resistor is exactly the same as the input, i.e., the steady state voltages $V_r(ss)$ and $V_c(ss)$ are as follows.

Before the input voltage switches (i.e., for $t < 0$),

$$V_r(ss) = 0 \quad \text{for} t < 0$$
$$V_c(ss) = 0 \quad \text{for} t < 0$$

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[1]As discussed in class, at very high frequencies there is insufficient time for a voltage to appear across the capacitance even though a current is flowing. For the high frequency case, the capacitor is replaced by a short circuit, as shown in Figure 2b.

Long after the input voltage has switched (i.e., for $t \gg 0$),

$$V_r(ss) = 0 \quad \text{for} t \gg 0$$
$$V_c(ss) = V_0 \quad \text{for} t \gg 0$$

These steady state voltages are shown in Figure 3a and b. The steady state conditions establish the boundary conditions used to complete some solutions of differential equations.

Must Know.

* *The voltage across a capacitor can not change instantaneously. To determine steady state conditions, replace capacitor with an open circuit.* An open circuit corresponds to no connection between the nodes.

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3 Conditions Immediately After Input Voltage Transition

We also need to easily see the initial value of the voltages V_r and V_c immediately after the transition (e.g., at time $t = \epsilon$ where $\epsilon > 0$ is arbitrarily close to 0). The formal comment above means IMMEDIATELY after, rather than after a few nanoseconds, the INSTANTANEOUS change in the input voltage. The current-voltage behavior of a capacitor is

$$i(t) = C \cdot dV_c/dt$$

and, by integrating this equation, the voltage is

$$V_c(t) = V_c(t = 0) + \frac{1}{C} \int_{T=0}^{T=t} i(T)dT. \tag{1}$$

Equation 1 shows that the voltage can not instantaneously change, for a finite current, and therefore, immediately after the input voltage V_{in} changes, the voltage V_c across the capacitance has not changed. In particular,

$$V_c(t + \epsilon) = V_c(t - \epsilon).$$

Since the voltage can not change instantaneously across the capacitor, all the voltage change initially appears across the resistor, establishing the conditions at $t + \epsilon$ shown in Figure 3.

Must Know.

* *The voltage across a capacitor can not change instantaneously.*

* *The current into a capacitor can change instantaneously.* Note that if charge enters one side of the capacitor, equal charge leaves through the other side, having the effect of a current flowing "through" the capacitor.

In Figure 3, the trend in V_r and V_c between the initial state immediately following the change in V_{in} is shown as dashed, shaded lines. The change is **not** linear but instead, as we shown next by solving the differential equation for the circuit in Figure 1a, exponential.

4 Solving the Differential Equation for the RC Circuit

The Kirchoff voltage equation around the RC circuit in Figure 1a is given by

$$V_{in}(t) = i(t) \cdot R + Q(t)/C \tag{2}$$

where Q, the charge on the capacitor, is related to the current $i(t)$ by $i(t) = dQ(t)/dt$ or

$$Q = \int_0^t i(x)dx. \tag{3}$$

Substituting (3) in (2) eliminates the variable Q, giving the following equation in terms of only $V_{in}(t)$ and $i(t)$:

$$V_{in}(t) = i(t) \cdot R + \frac{1}{C} \int_0^t i(x)dx. \tag{4}$$

We can eliminate the integral in (4) by differentiation, leading to

$$\frac{dV_{in}(t)}{dt} = R \cdot \frac{di(t)}{dt} + \frac{1}{C}i(t). \tag{5}$$

Since $V_{in}(t > 0) = V_0$, V_0 a constant, $dV_{in}/dt \equiv 0$ and (following division by R), (5) becomes
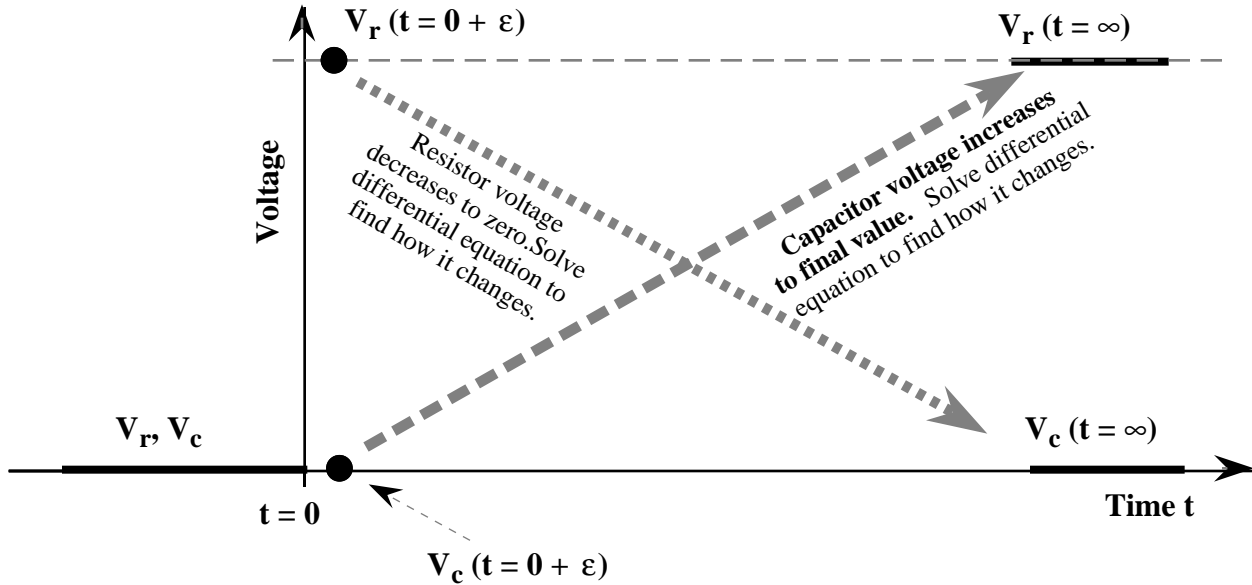
$$\frac{di(t)}{dt} + \frac{1}{RC}i(t) = 0. \tag{6}$$

Figure 3: Illustration of voltages in RC circuit including (i) voltages appearing before the input voltage (Figure 1b) changes, (ii) voltages immediately after $(t = 0+\epsilon)$, and (iii) steady state voltages long after $(t = \infty)$ the input voltage changes. The dashed, shaded lines represent the unknown behavior of the voltages between the initial and final states, requiring solution to the differential equation.

We will use the parameter $\alpha = 1/(RC)$ in (6), giving the final equation that we will solve, namely,

$$\frac{di(t)}{dt} + \alpha \cdot i(t) = 0. \tag{7}$$

Rearranging terms in (7) gives

$$\frac{di(t)}{i} = -\alpha \cdot dt. \tag{8}$$

Integrating (8) gives, using the initial current $i(t = 0 + \epsilon) = i_0$,

$$\int_0^t \frac{di}{i} = -\alpha \int_0^t dt$$
$$ln(i(t)) - ln(i_0) = -\alpha \cdot t$$
$$\text{or}$$
$$i(t) = i_0 e^{-\alpha t} \tag{9}$$

Equation 9 shows that the current at time $t = 0$ starts with the value $i(t) = i_0$ and "decays" to zero as $t \to \infty$. However, we have not determined the value of i_0, the current flowing immediately after the input voltage changes. This is readily obtained by looking at Figure 3 where, since the voltage across the capacitor can not change instantaneously, the input voltage initially appears fully across the resistor. Therefore, the initial current is given by the voltage across the resistor ($V_r(t = 0 + \epsilon) = V_0$ where V_0 was the value of the input voltage after the switching) divided by the resistance R or

$$i_0 = V_0/R$$

. Using this value of i_0 in 9 gives the value of current in terms of the **known** input voltage and the resistance, i.e.,

$$i(t) = \frac{V_0}{R} e^{-\alpha t}$$
$$ = \frac{V_0}{R} e^{-t/RC} \tag{10}$$

where we have replaced α by its definition $\alpha - 1/RC$. The quantity RC has units of time (time in seconds if R is in ohms and C is in farads) and is usually called the *RC time constant* of the circuit.

FINALLY! We are now able to give the solutions for the voltages across the resistor and capacitor indicated by the dashed, shaded lines in Figure 3. In particular,

$$V_r(t) = i(t) \cdot R = V_0 e^{-t/RC} \tag{11}$$
$$V_c(t) = V_0 - V_r = V_0 \left(1 - e^{-t/RC}\right) \tag{12}$$

The number "e" is equal to 2.72 (approximately). The voltage across the resistor decreases to $1/e$ of its initial value at $t = RC$ (i.e., to 37% of its initial value in one time constant). In two time constants ($t = 2 \cdot RC$, the voltage has decayed to 13% of its initial value. In four time constants, it has dropped to a little under 2% of its initial value. Figure 4 shows the voltages $V_r(t)$ and $V_c(t)$,
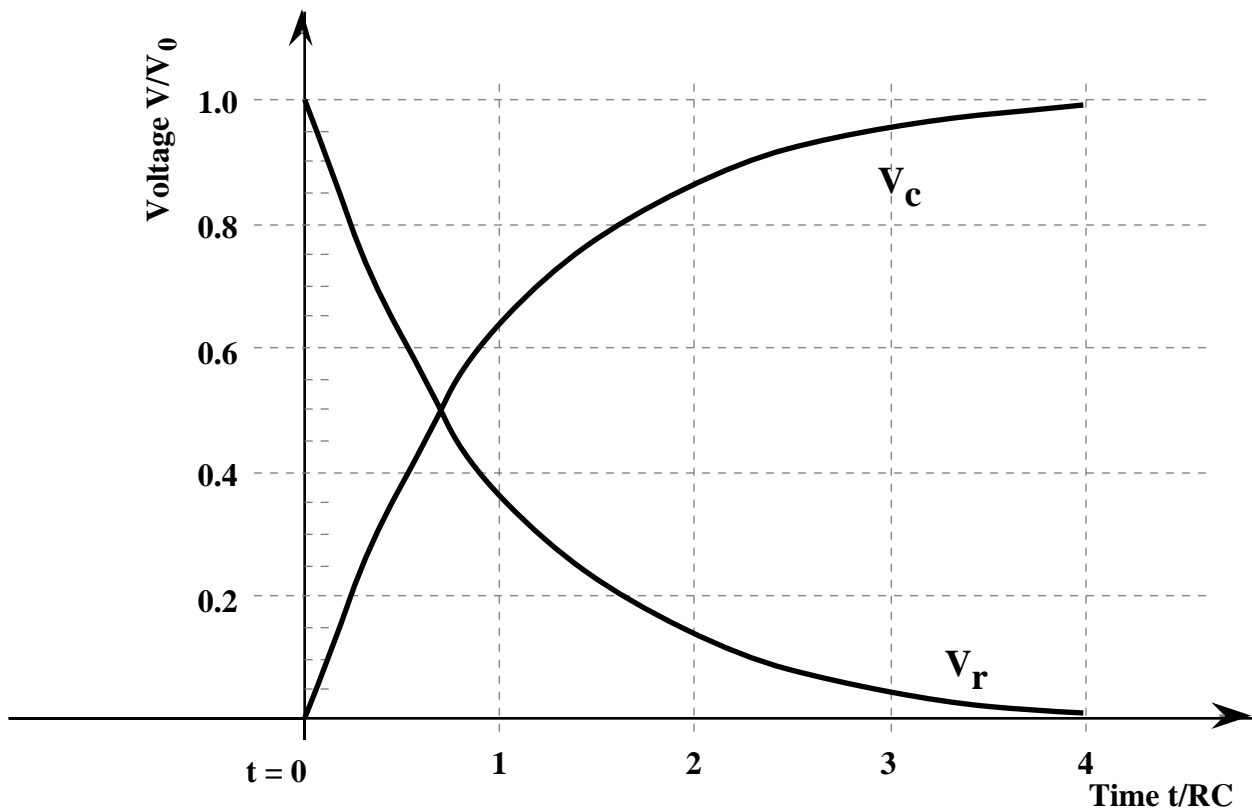
Figure 4: Exponential decay of voltage across resistor and exponential growth of voltage across capacitor of simple RC circuit in Figure 1a.

showing the exponential characteristic shown earlier by the unknown dashed, shaded lines in Figure 3.

Must Know.

* *The time dependence of the changes of voltages and currents in the simple circuit of 1a is exponential, of the form $exp(-t/RC)$, where RC is the circuit's "time constant."*

* *Initially, all the voltage is dropped across the resistor*

* *After a long time (relative to $\tau = RC$), all the voltage is dropped across the capacitor.*

* *Initially, a current (initially limited by the resistance to V_0/R) flows.*

* *After a long time, the current is zero.*