Markov Chain Monte Carlo

Model Fitting and Inference for
Infectious Disease Dynamics
short course, June 2018

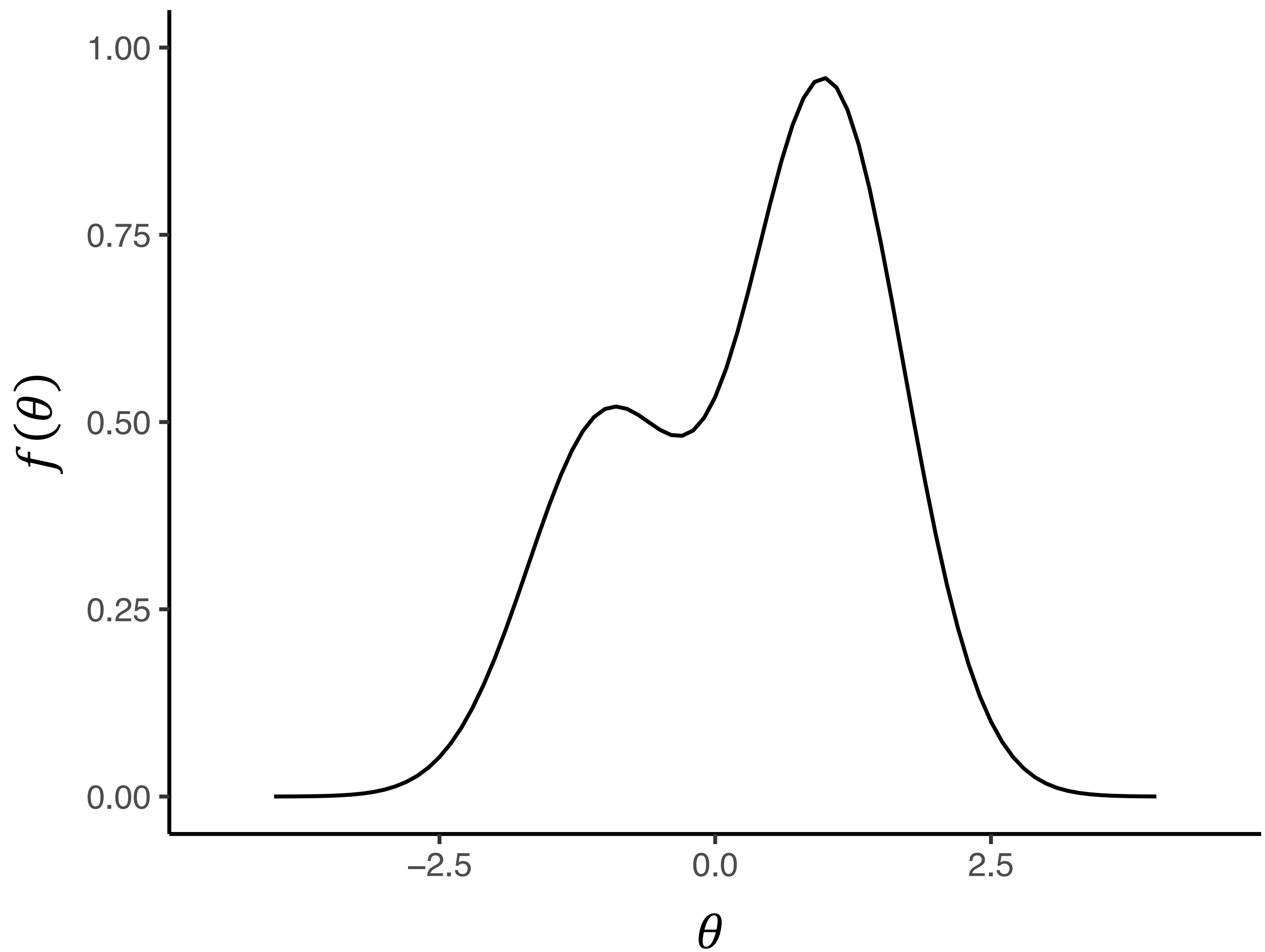
Recap

Last time we saw that the posterior distribution of θ, given observed data is

$$p(\theta|data) \propto p(data|\theta)p(\theta)$$

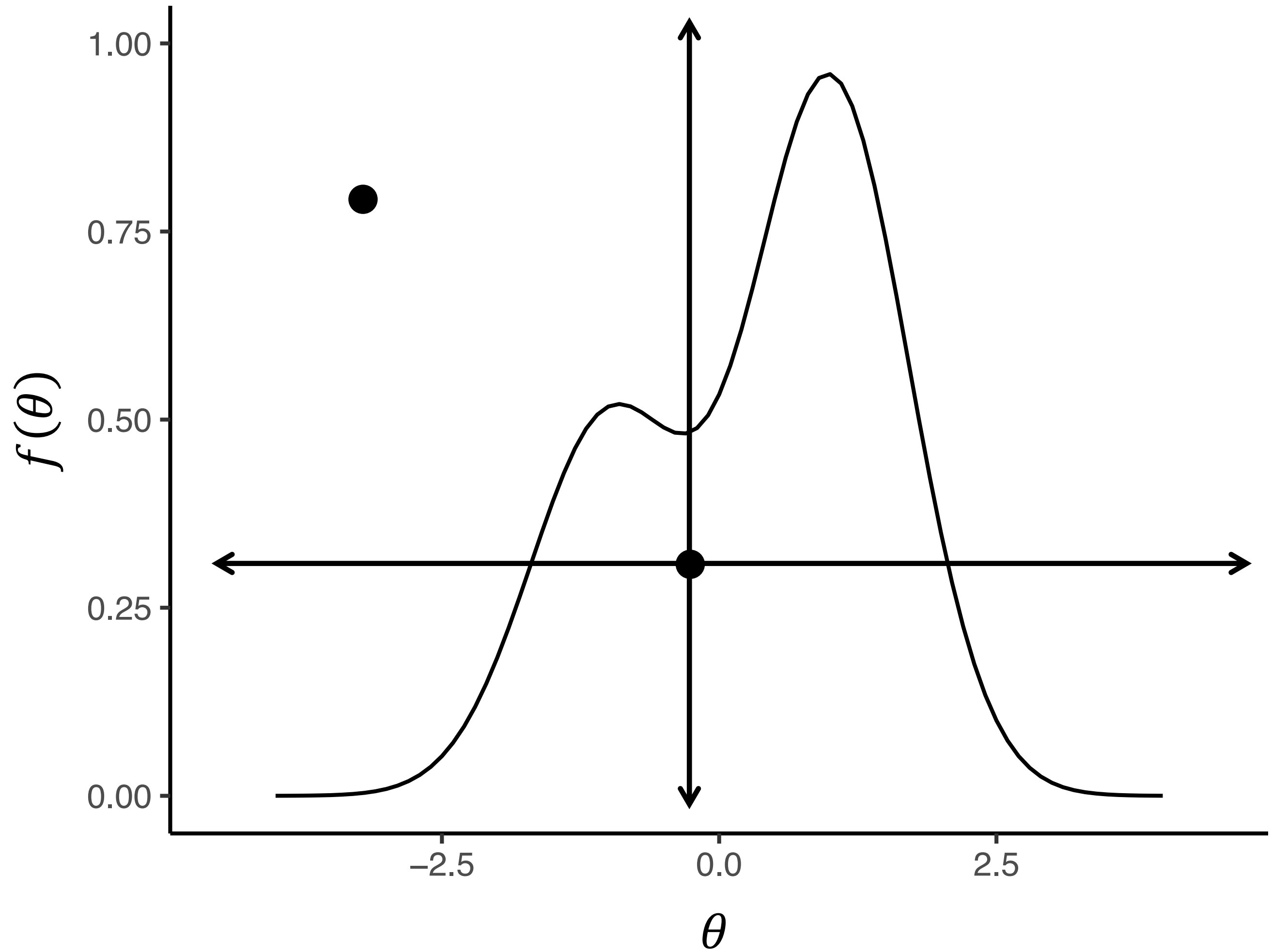
Our aim is to draw samples from this distribution, $p(data|\theta)p(\theta) = f(\theta)$.

Characterising an unknown distribution

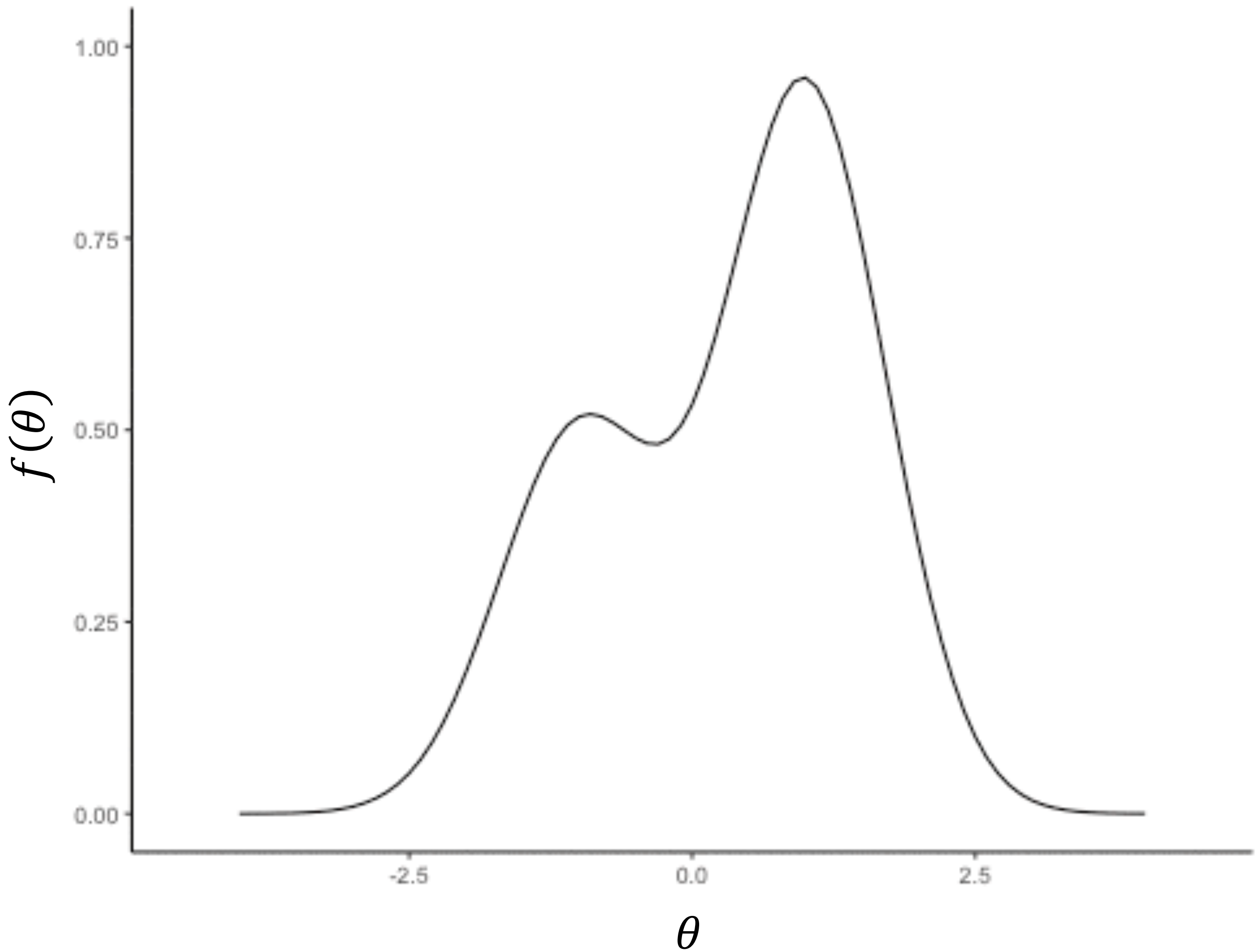


How do we draw samples from this distribution?

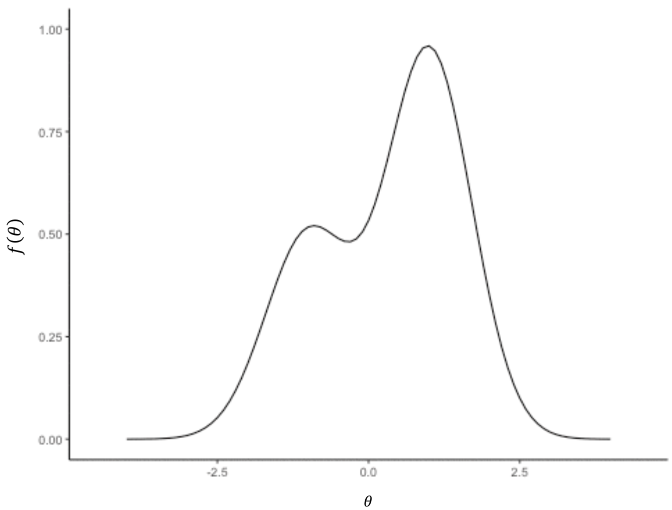
Rejection sampling



Rejection sampling



Problems with rejection sampling



Need to characterise the bounds of the distribution

Curse of dimensionality

Markov Chain Monte Carlo (MCMC)

Markov chain: stochastic sequence of states in which the next state depends only upon the current state

$$\theta_{t+1} \sim \mathbb{D}(\theta_t)$$

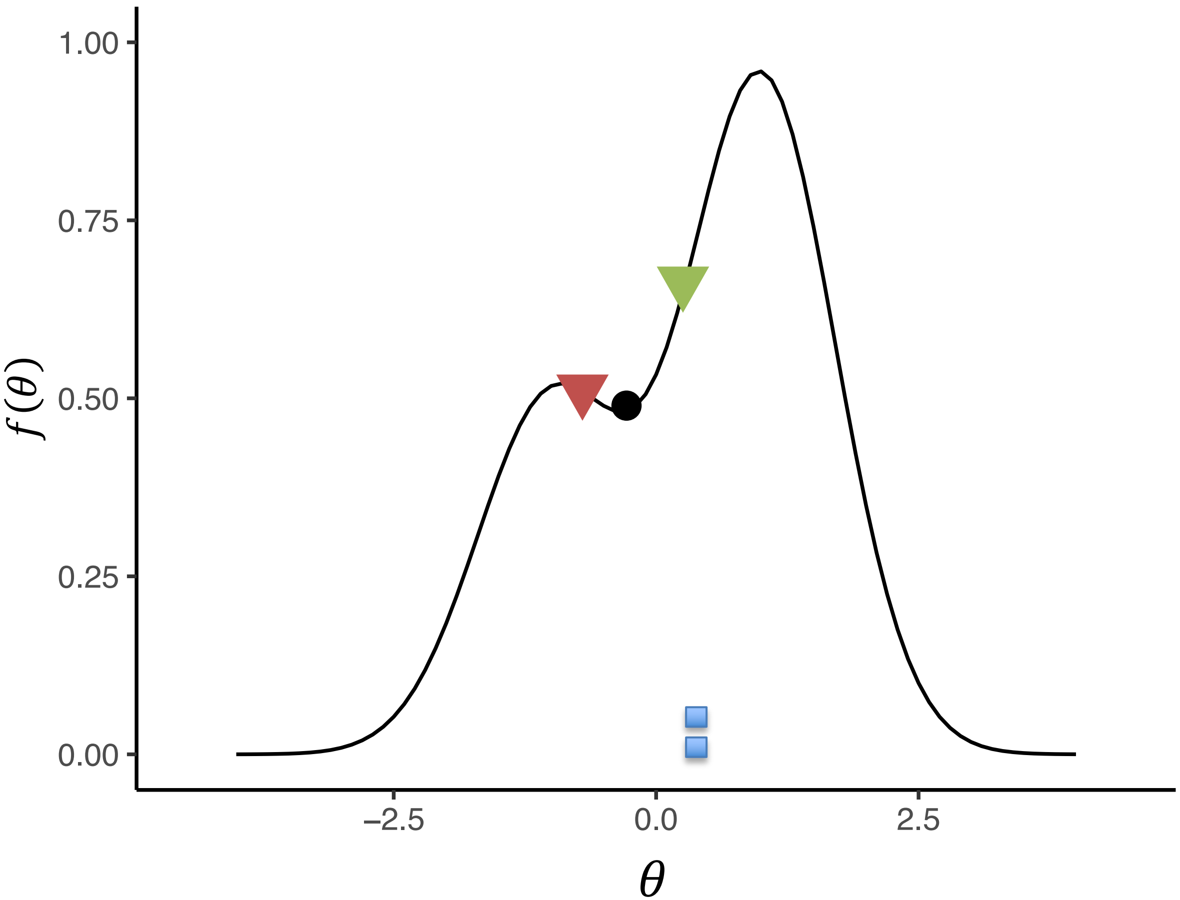
Monte Carlo: a famous casino. Also a class of algorithms in which random sampling is used to solve problems.

Metropolis-Hastings algorithm: a particular way of using MCMC to sample from a distribution

MCMC: Outline

- What the algorithm is
- Why it works.

MCMC algorithm



1. PROPOSE
2. MOVE OR STAY ("acceptance")
3. SAVE LOCATION

Choose a starting point, $\theta = \theta_0$

PROPOSE

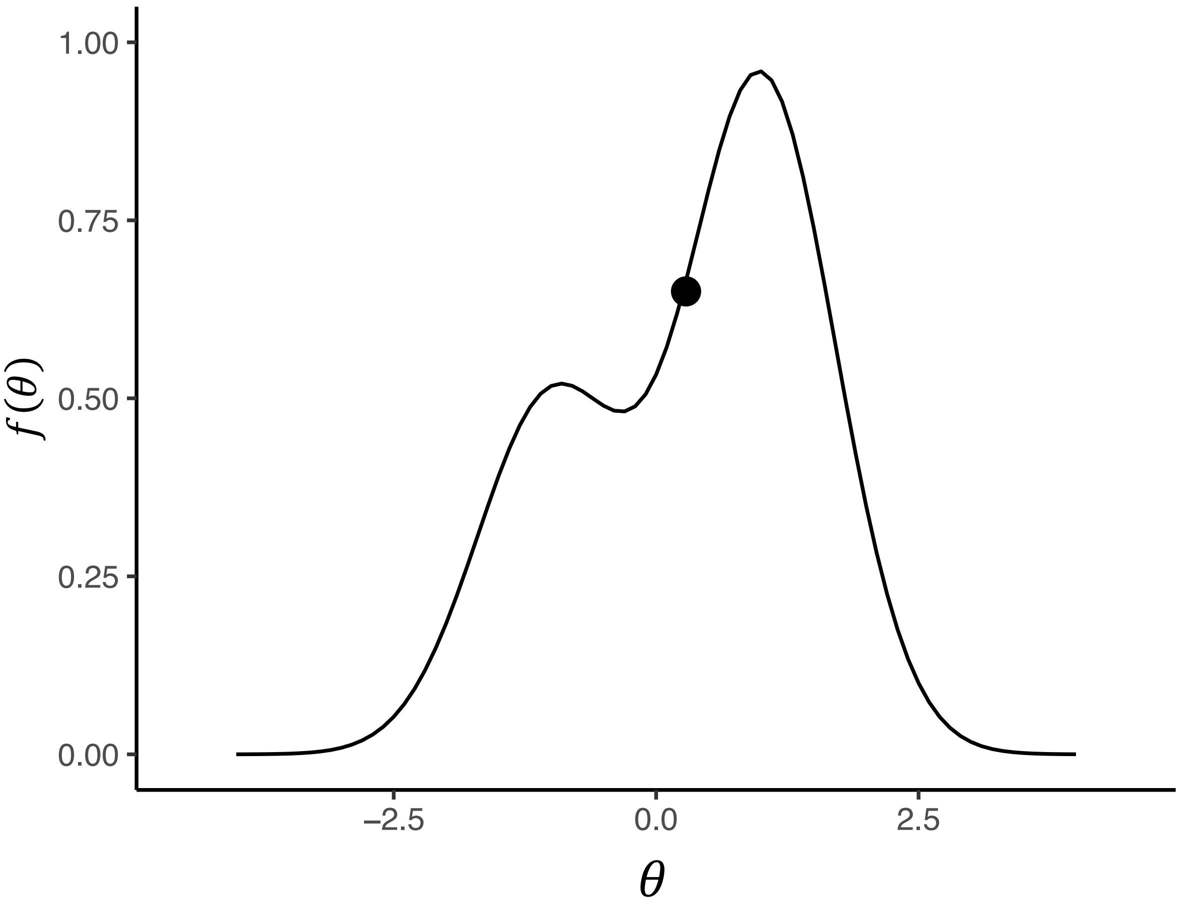
$$\theta' = \theta + \varepsilon \mid \varepsilon \sim Q$$

MOVE OR STAY

If $f(\theta') > f(\theta)$, definitely move.

If $f(\theta') < f(\theta)$, move with probability $f(\theta')/f(\theta)$, otherwise stay.

MCMC algorithm



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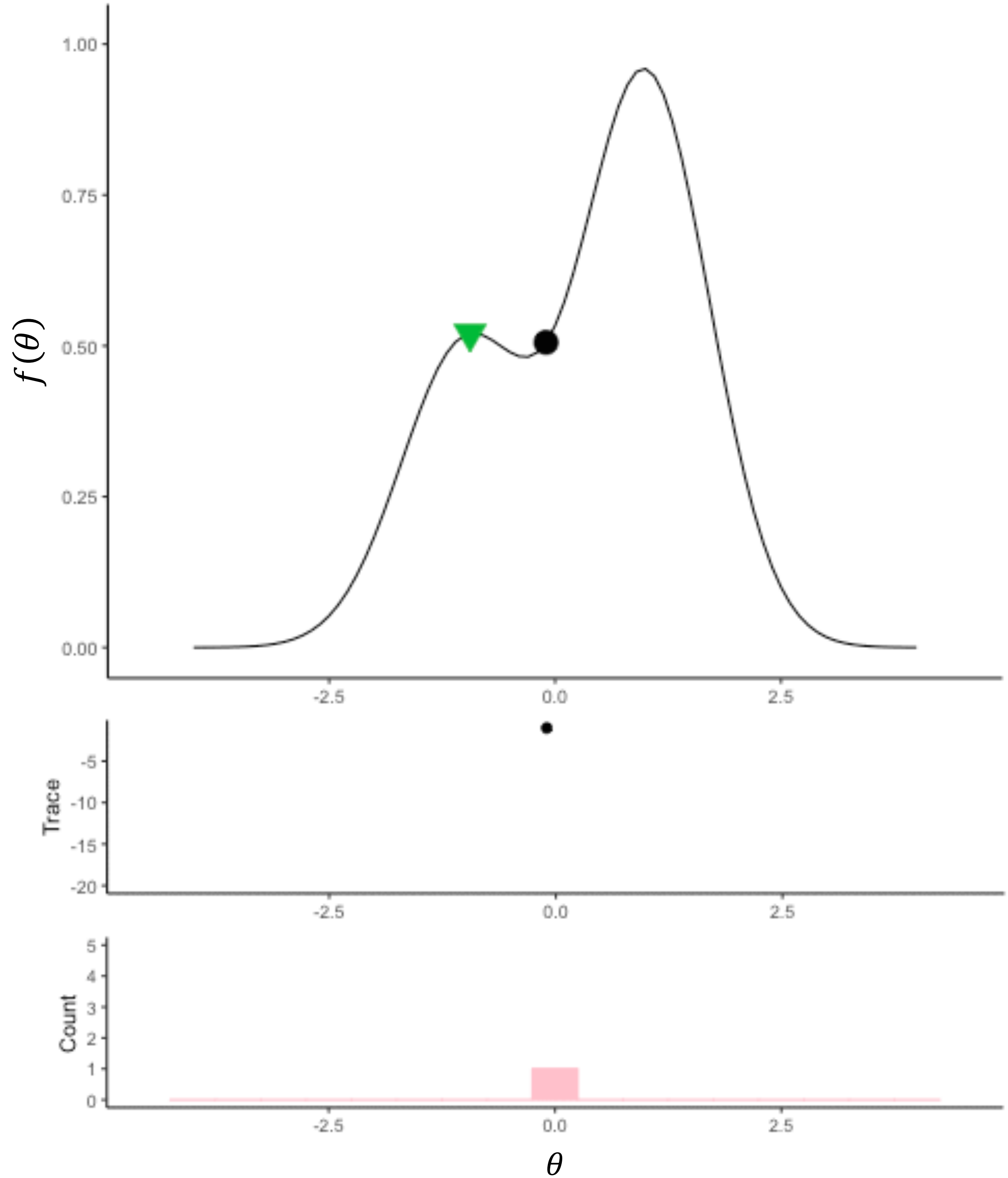
MOVE OR STAY

$$\theta \to \begin{cases} \theta' & \Pr(a) & Accept \\ \theta & \Pr(1-a) & Reject \end{cases}$$

$$a = \min\left(1, \frac{f(\theta')}{f(\theta)}\right)$$

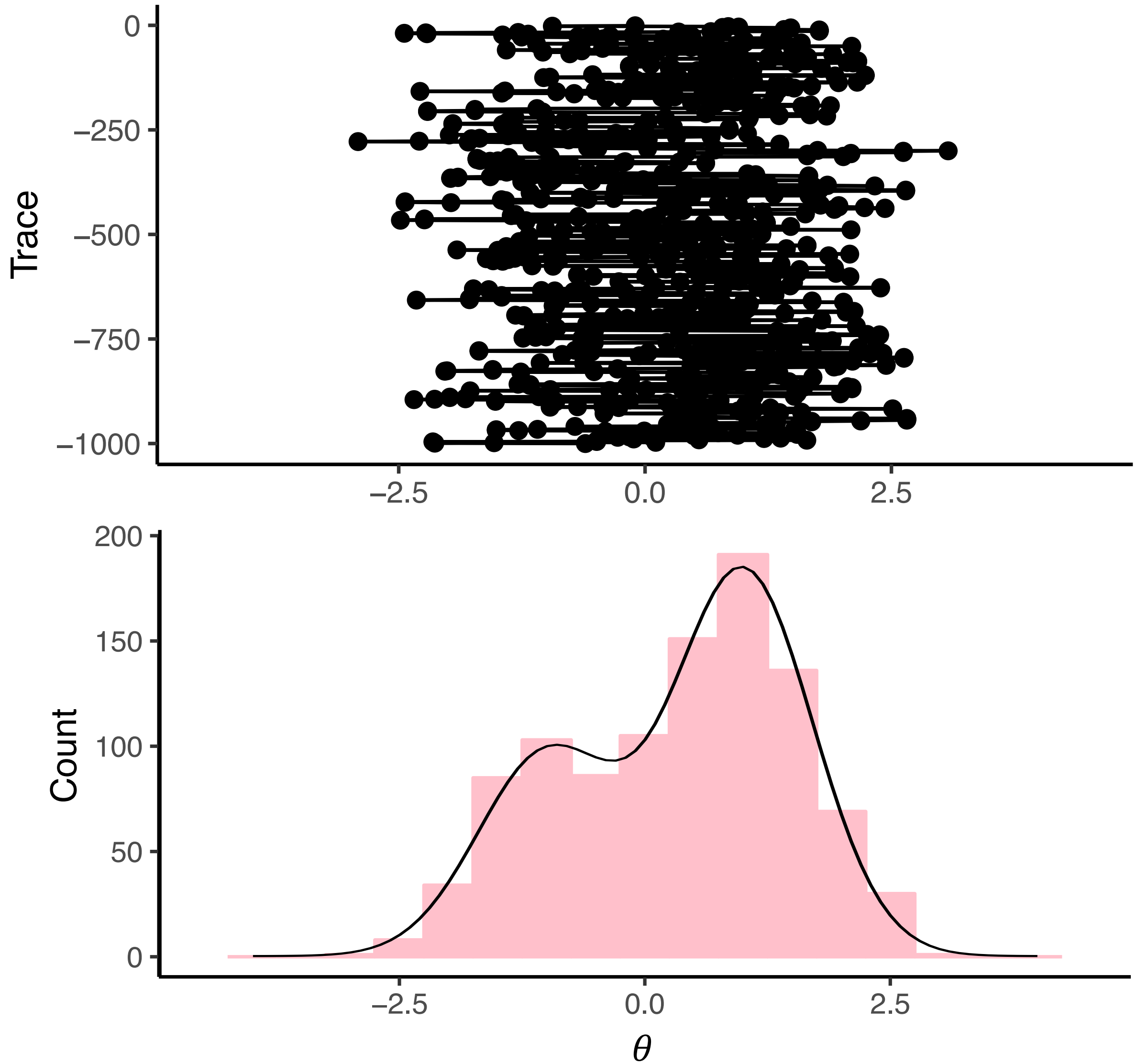
SAVE LOCATION

$$\theta_t \leftarrow \theta$$

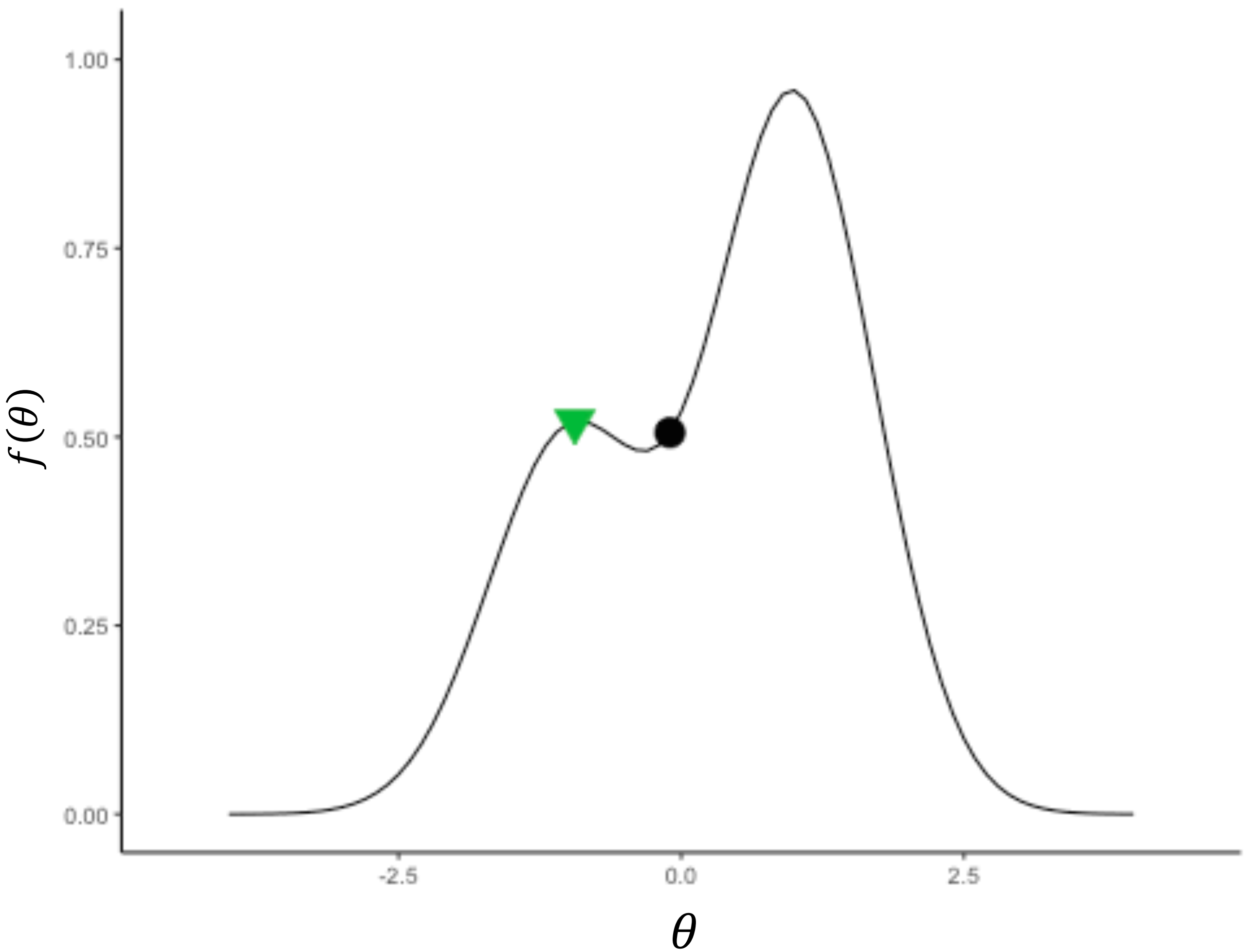


$f(\theta)$
1.00
0.75
0.50
0.25
0.00
-2.5
0.0
2.5
Trace
-5
-10
-15
-20
Count
5
4
3
2
1
0
θ

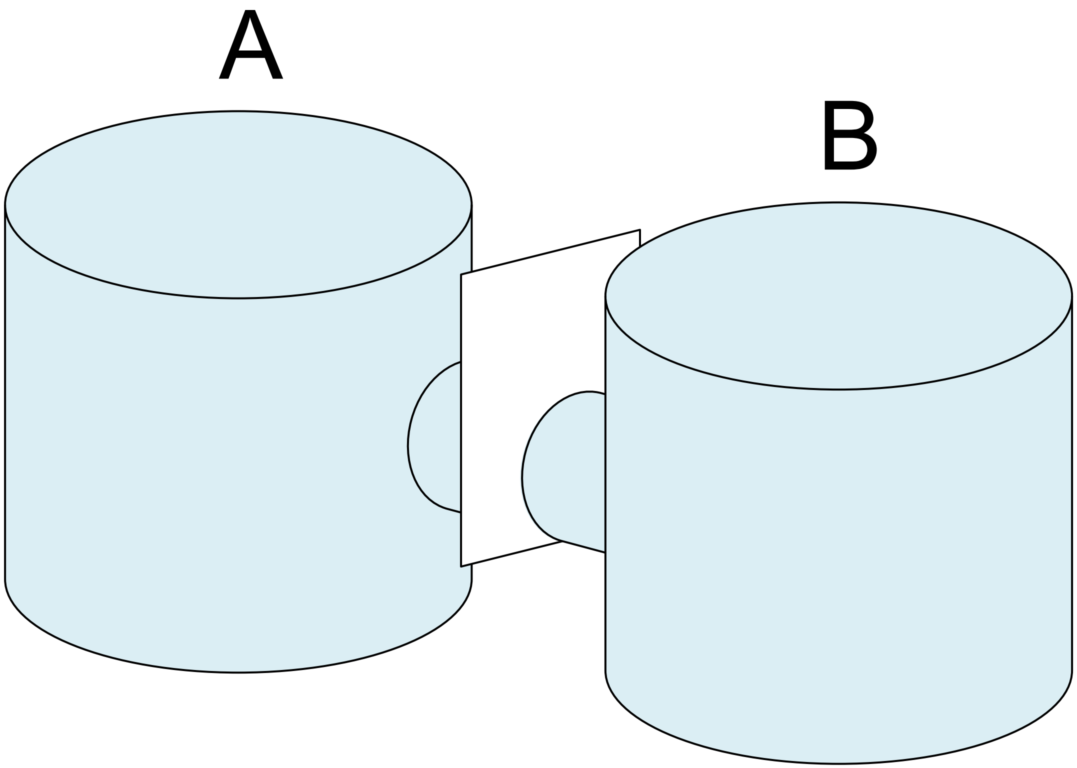
After enough iterations…



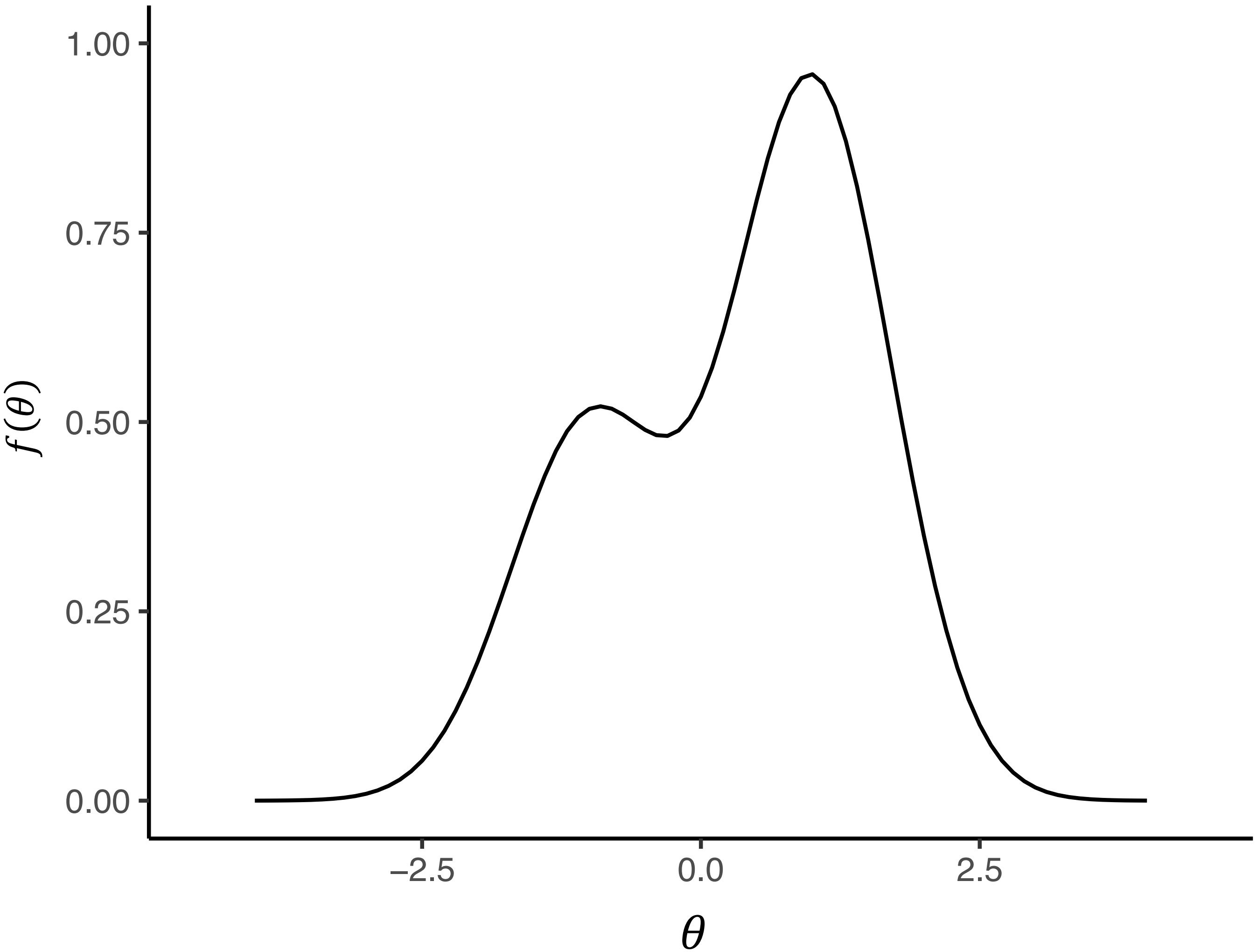
A sloppy hill-climbing algorithm



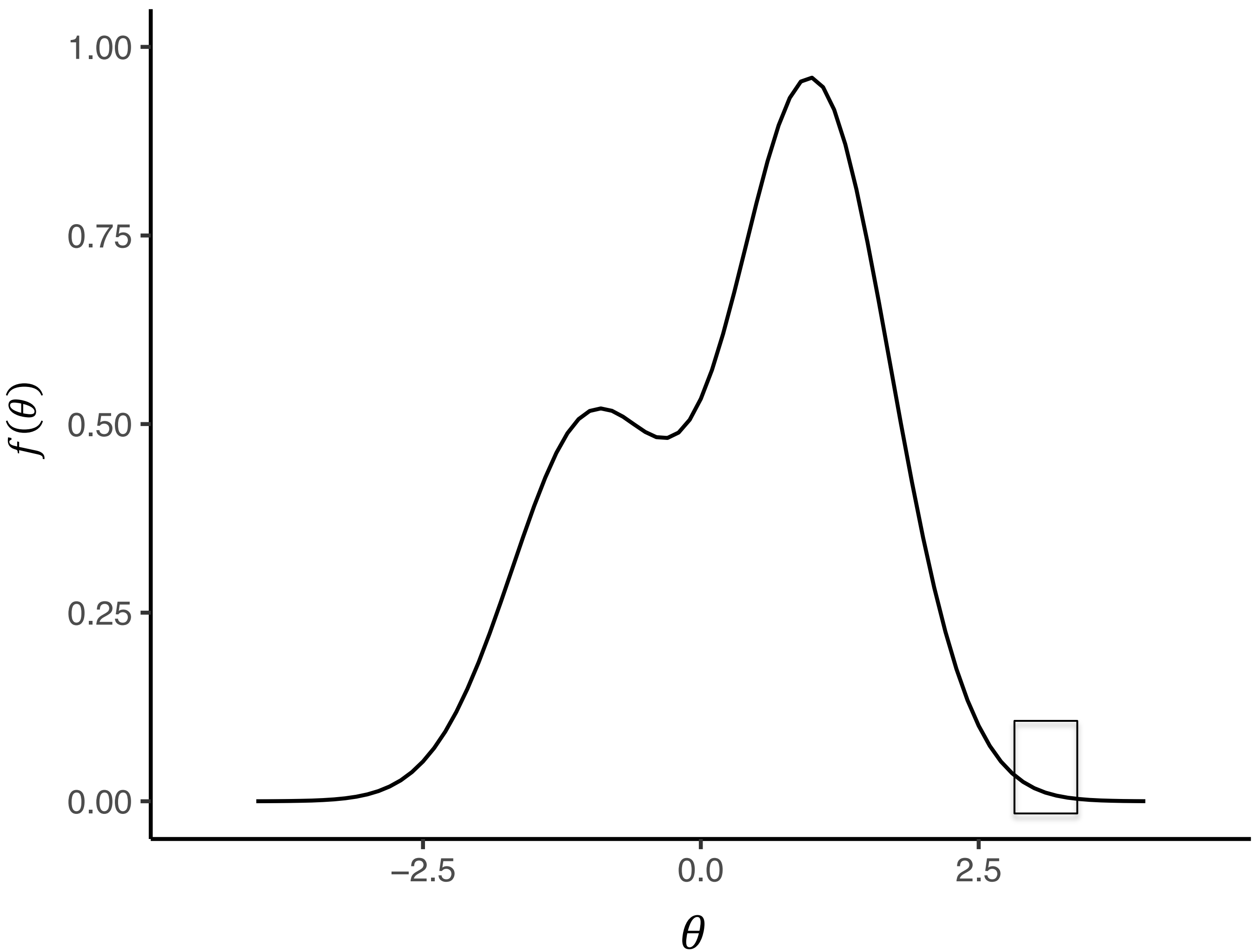
A thought experiment…



How does the dye know to stop flowing?



1.00
0.75
0.50
0.25
0.00
$f(\theta)$
−2.5
0.0
2.5
θ



1.00
0.75
0.50
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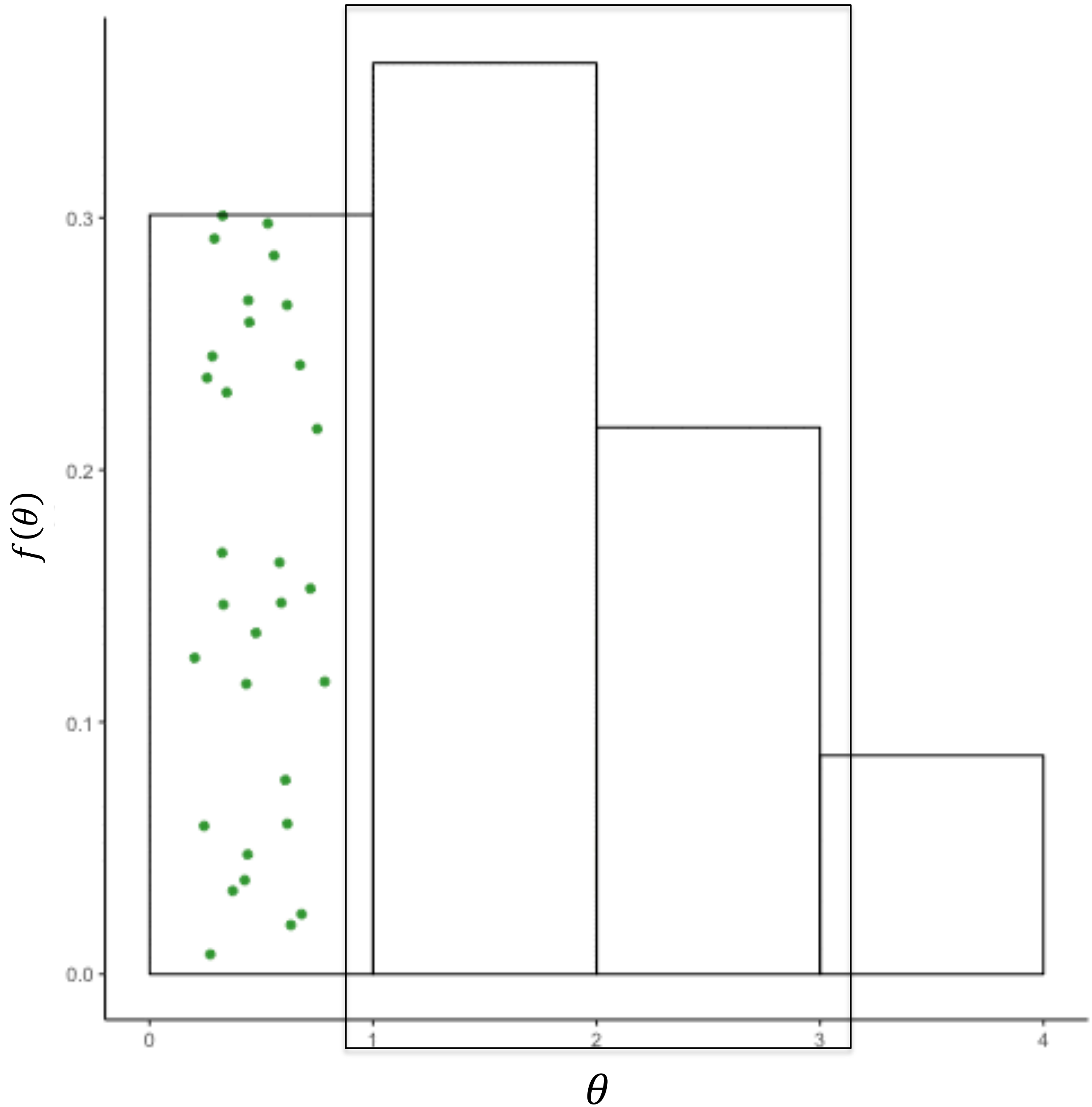
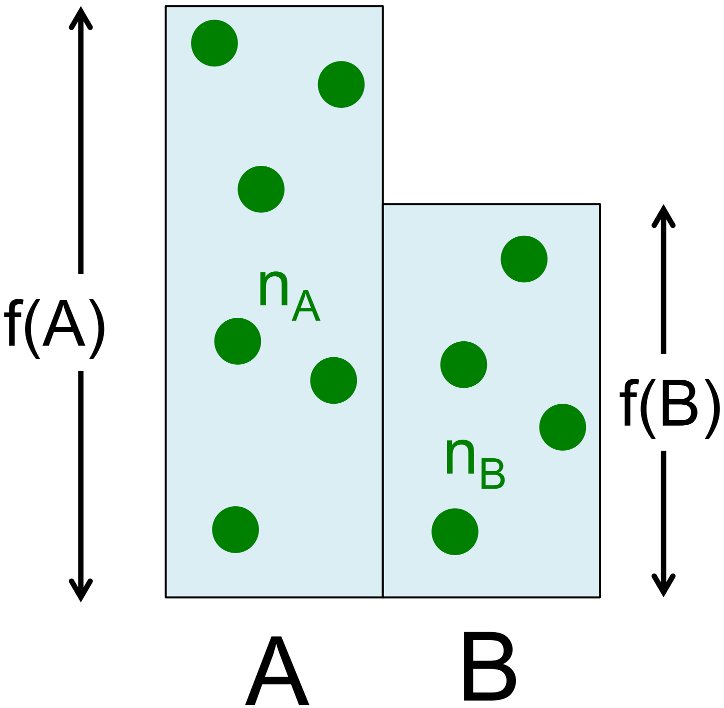
0.00
$f(\theta)$
−2.5
0.0
2.5
θ



0.3
0.2
0.1
0.0
$f(\theta)$
0
1
2
3
4
θ



assume $f(A) > f(B)$

$$\Pr(\mathrm{A} \to \mathrm{B}) = q_{A\to B} \cdot \frac{f(B)}{f(A)}$$

$$\Pr(\mathrm{B} \to \mathrm{A}) = q_{B\to A} \cdot 1$$

$$q_{A\to B} = q_{B\to A}$$

$$\Pr(A \to B) \propto \frac{f(B)}{f(A)}$$

$$\Pr(B \to A) \propto 1$$

Net movement from A → B

$$n_A \Pr(\mathrm{A} \to \mathrm{B}) - n_B \Pr(\mathrm{B} \to \mathrm{A})$$

there will be flow A→B:

$$n_A \frac{f(B)}{f(A)} - n_B > 0$$

$$\frac{n_A}{n_B} > \frac{f(A)}{f(B)}$$

there will be flow B→A:

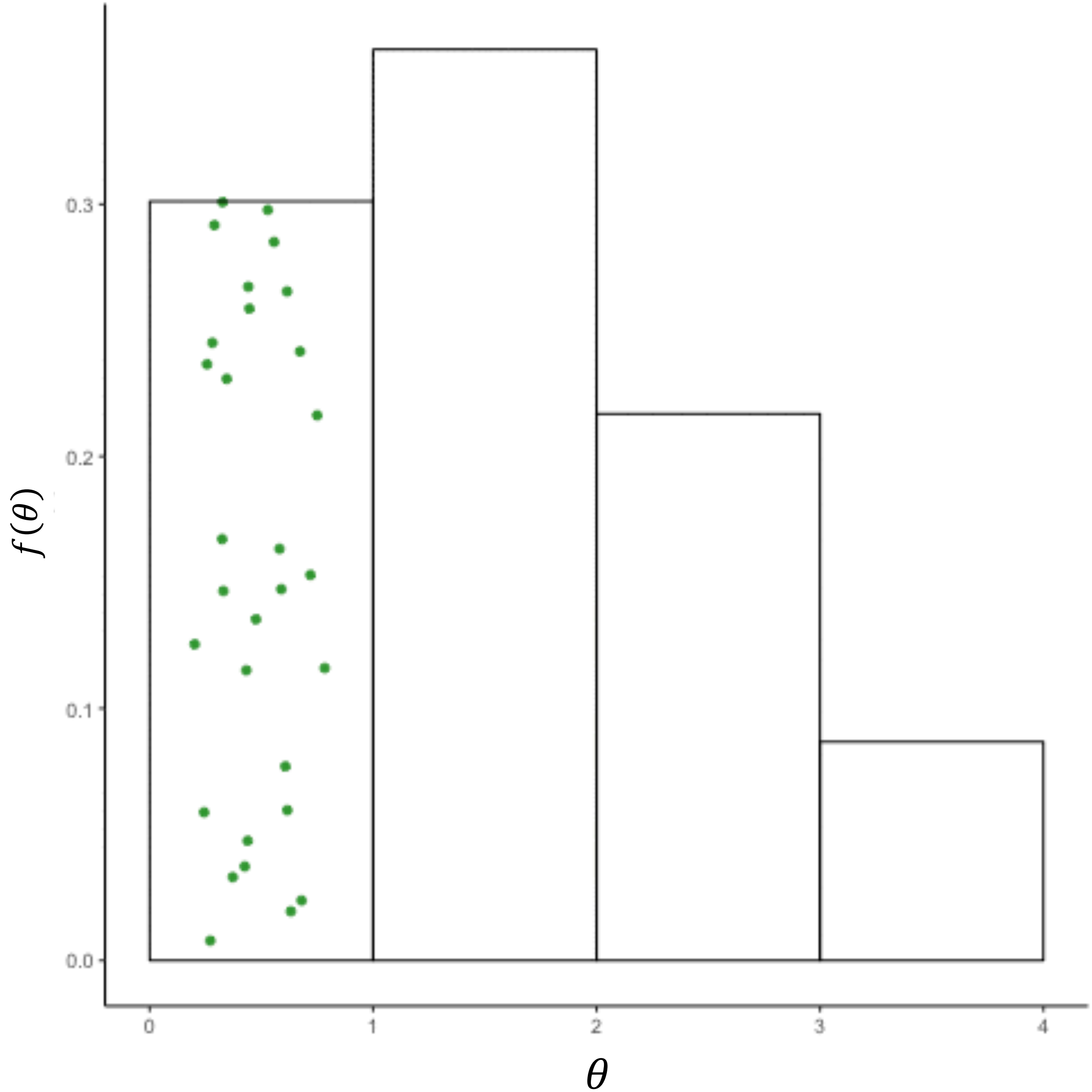
$$n_A \frac{f(B)}{f(A)} - n_B < 0$$

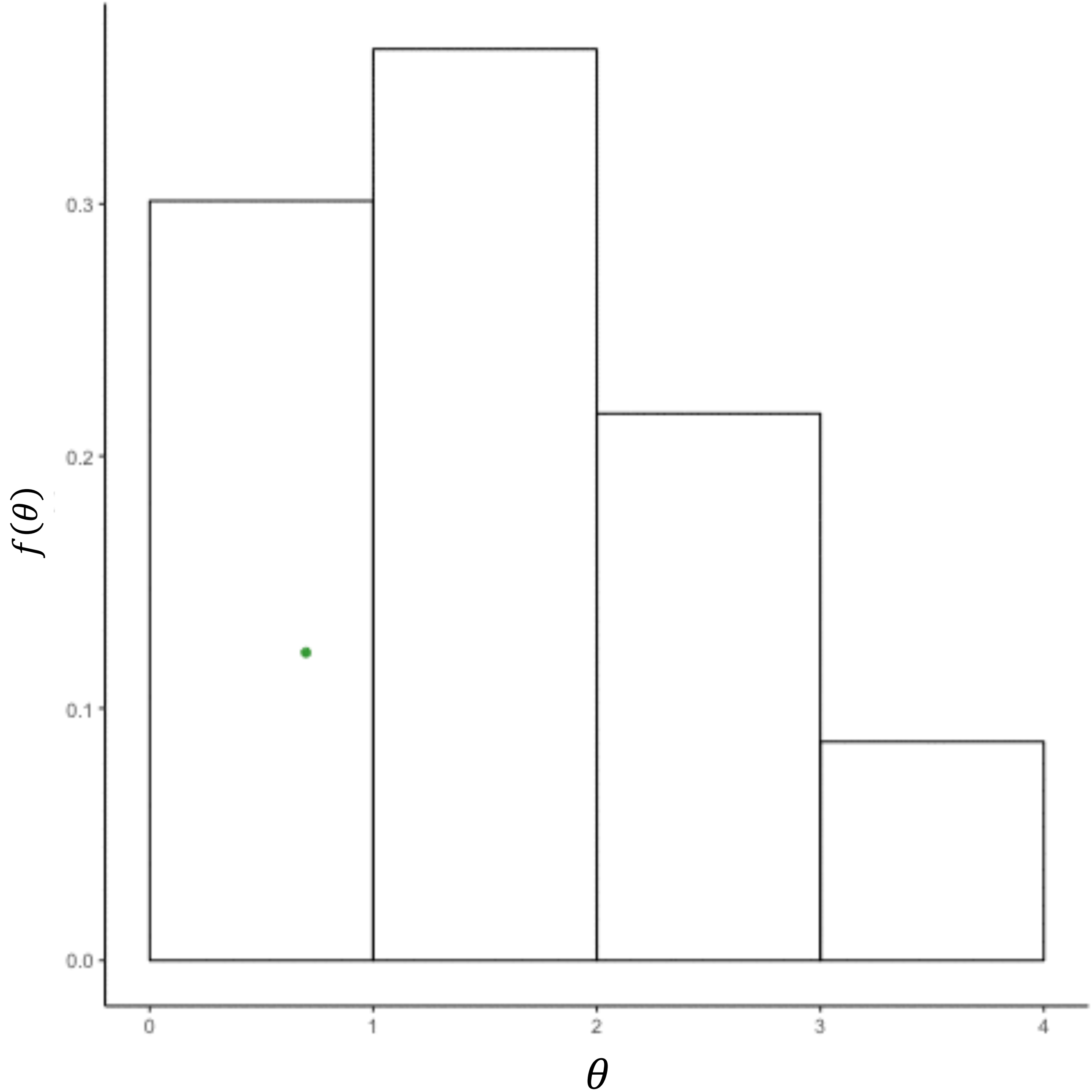
$$\frac{n_A}{n_B} < \frac{f(A)}{f(B)}$$

no net flow A</>B:

$$n_A \frac{f(B)}{f(A)} - n_B = 0$$

$$\frac{n_A}{n_B} = \frac{f(A)}{f(B)}$$


0.0
0.1
0.2
0.3
$f(\theta)$
0
1
2
3
4
θ



0.3
0.2
0.1
0.0
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0
1
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θ

Requirements

Symmetry of proposal distribution

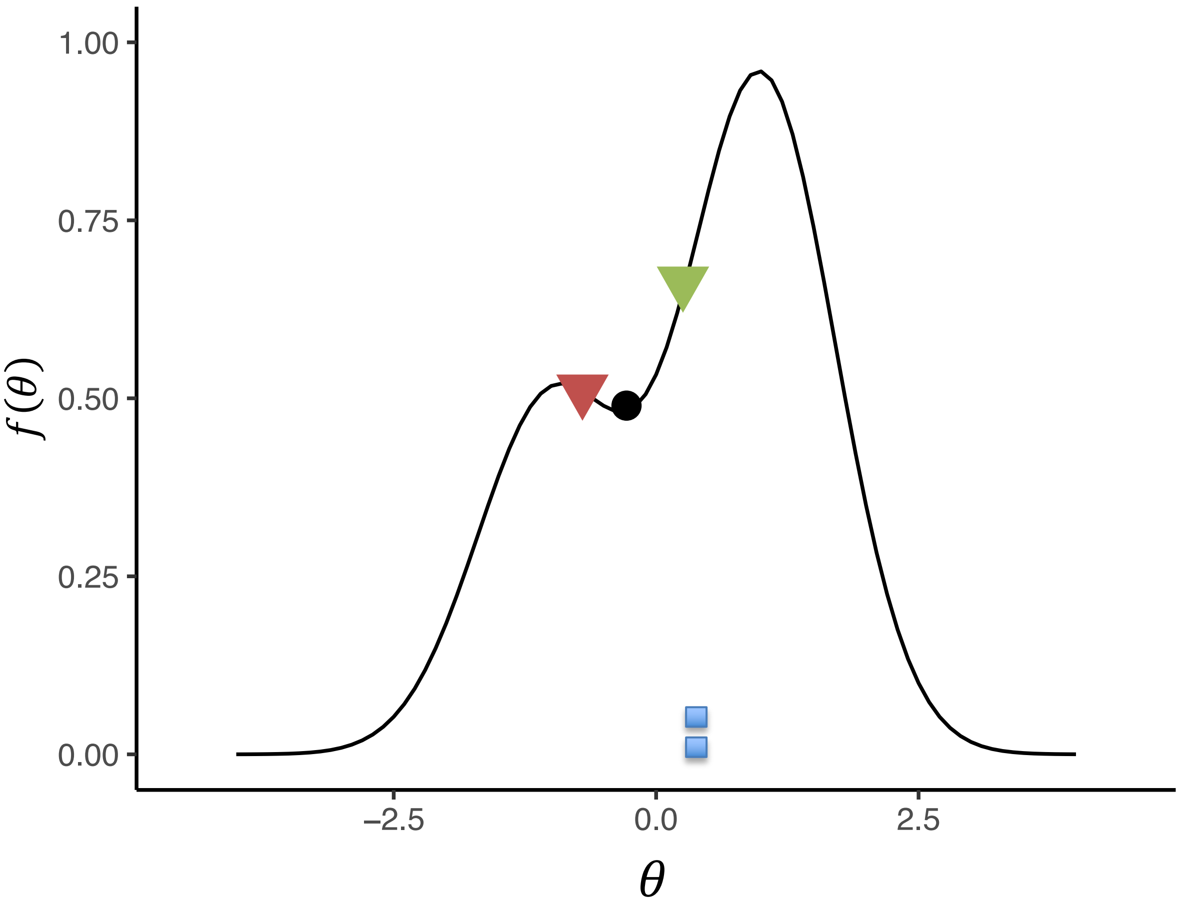
- "detailed balance" $q_{A \to B} = q_{B \to A}$
- if $q_{A \to B} \neq q_{B \to A}$, use acceptance ratio

$$A = \min\left(1, \frac{f(\theta')}{f(\theta)} \frac{q_{\theta' \to \theta}}{q_{\theta \to \theta'}}\right)$$

Connectedness of distribution

- ensures "ergodicity"

MCMC algorithm



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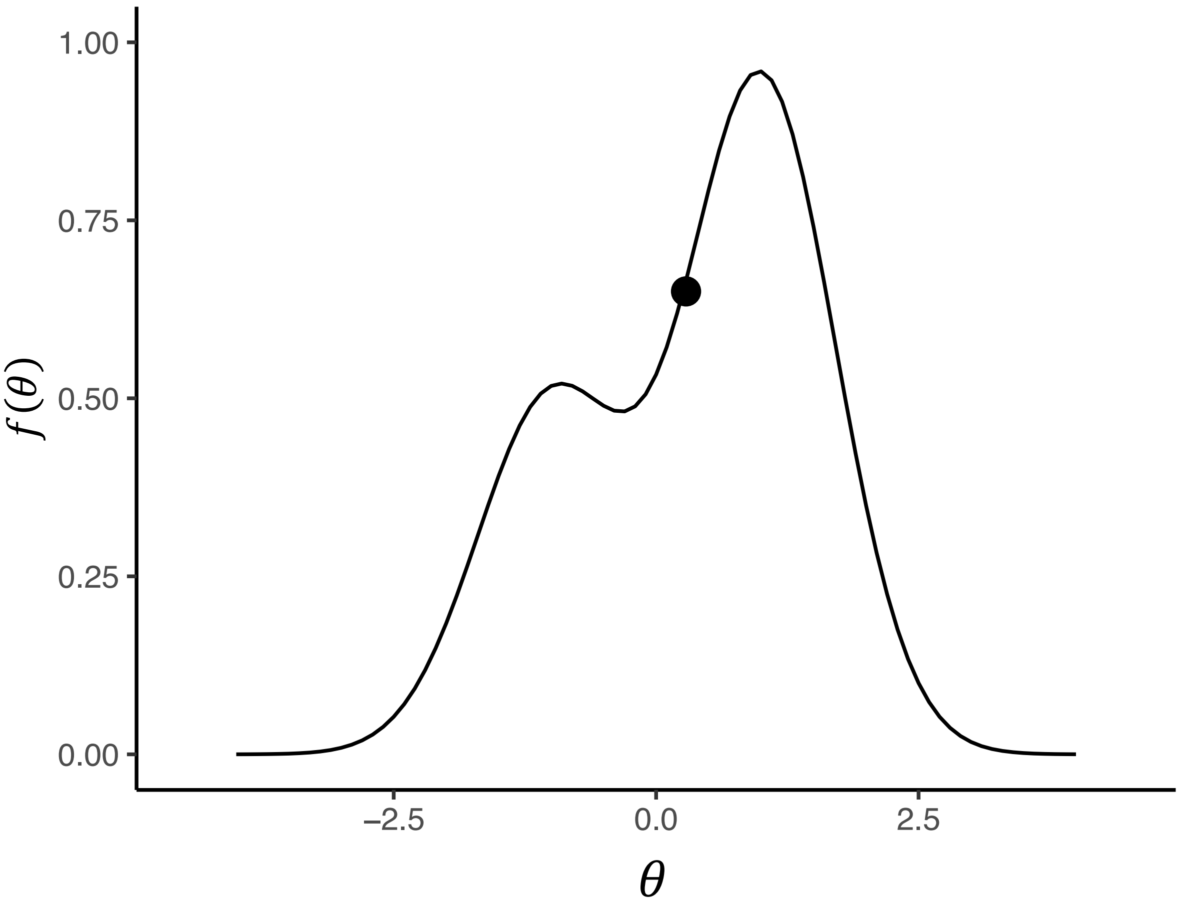

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