Introduction to Markov Chain Monte Carlo

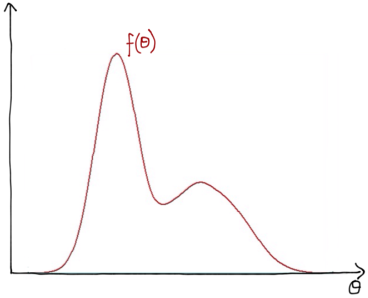
Recap. on Bayesian inference

Last time we saw that the posterior distribution of θ, given observed data is

$$p(\theta|\text{data}) \propto p(\text{data}|\theta)p(\theta)$$

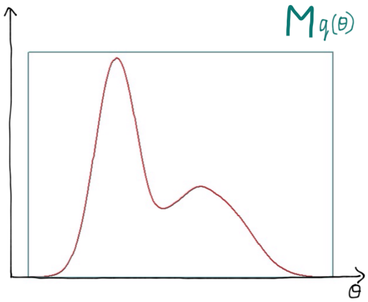
Our aim is to draw samples from this distribution.

Rejection sampling



- Consider a distribution $f(\theta)$,which we can evaluate for any θ
- How do we draw samples?

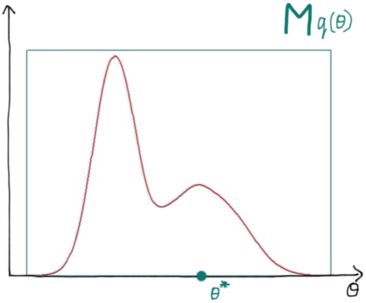
Rejection sampling



Rejection sampling uses a proposal distribution $q(\theta)$ which:

- is simple to evaluate
- is easy to sample from
- one can find $M > 1$ such that $f(\theta) < Mq(\theta)$ for all θ

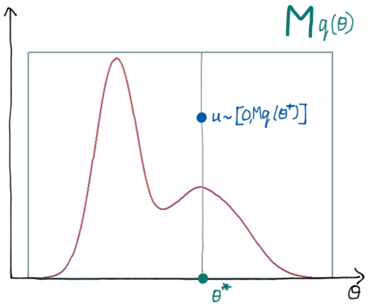
Rejection sampling



The algorithm proceeds as follows:

1. Sample θ^* from $q(\theta)$

Rejection sampling



The algorithm proceeds as follows:

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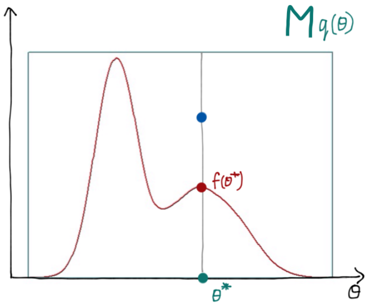

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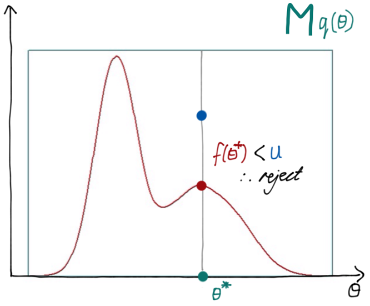

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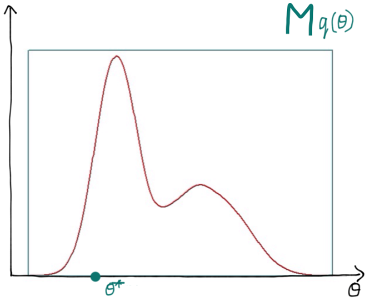

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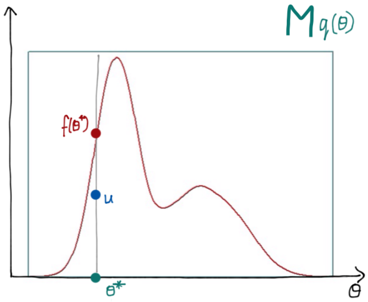

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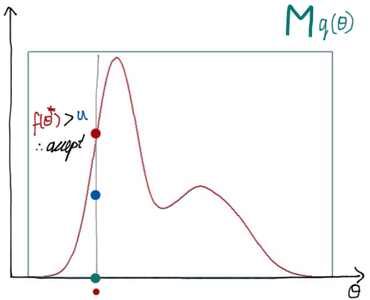

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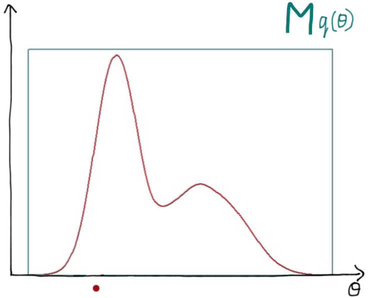

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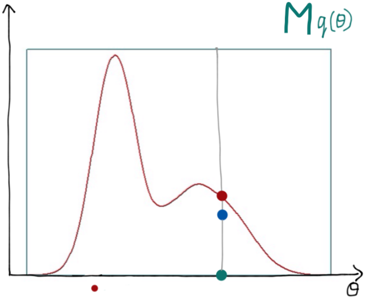

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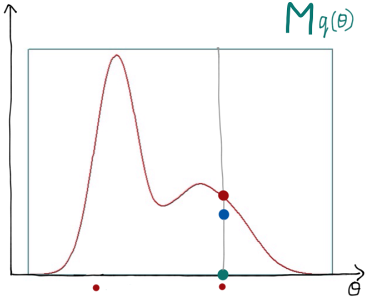

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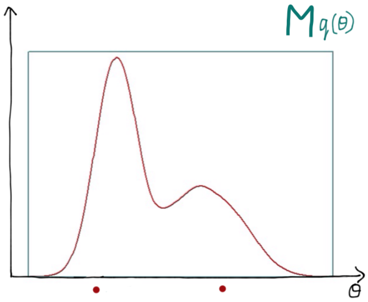

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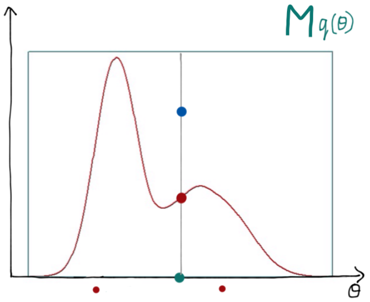

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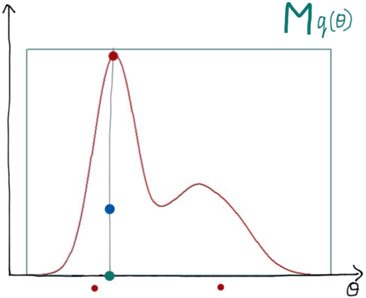

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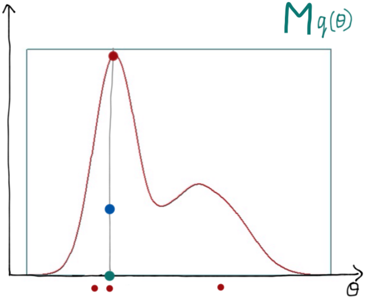

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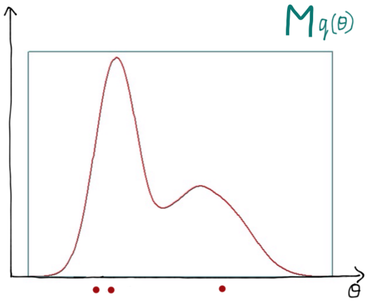

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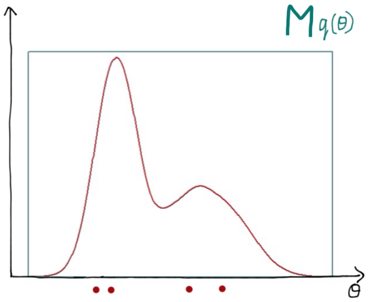

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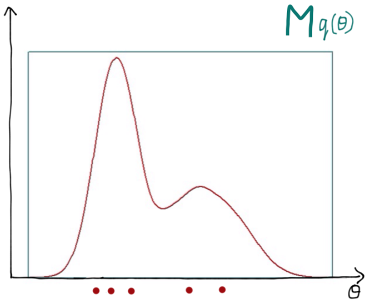

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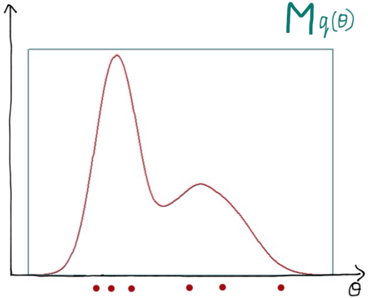

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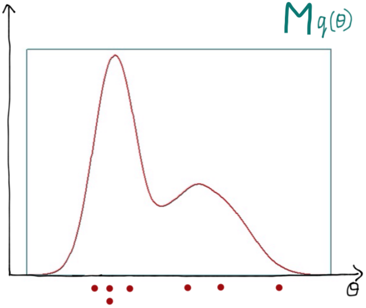

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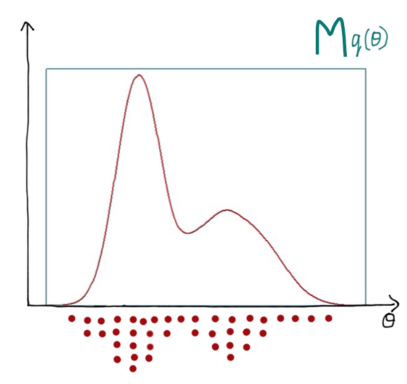

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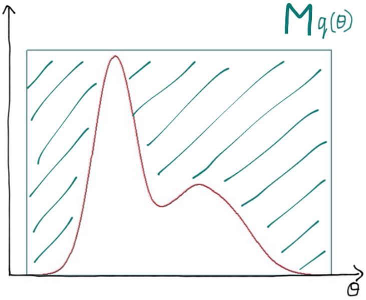
Rejection sampling

- Rejection sampling works best if $q(\theta) \approx f(\theta)$ ($M \gtrapprox 1$)
- Acceptance rate of rejection sampler is $\frac{1}{M}$
- Requiring $f(\theta) < Mq(\theta)$ for all θ can make rejection rate v. high
- Even more limited in high dimensions



Markov Chain Monte Carlo

- In Markov Chain Monte Carlo (MCMC) we do not define one proposal density $q(\theta)$ such that $f(\theta) < Mq(\theta)$.

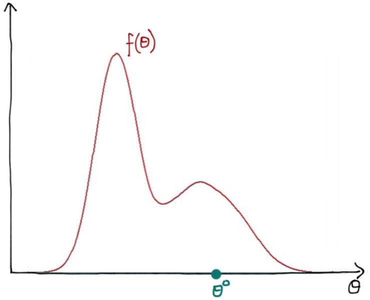
- Rather we build up a chain of samples where each proposed θ^* depends on the previous one

  i.e the proposal density takes the form $q(\theta^*|\theta)$

- A commonly used MCMC algorithm is Metropolis-Hastings (M-H).

- The acceptance rate of M-H is carefully derived to ensure unbiased samples.

Metropolis-Hastings



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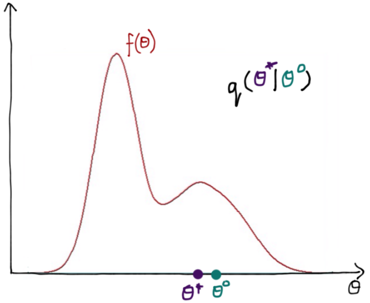

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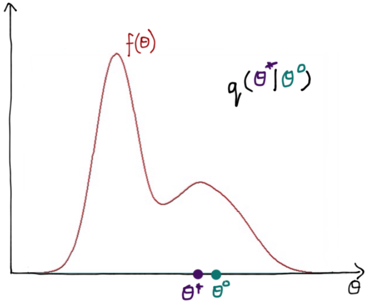

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Acceptance

- If $q(\theta^*|\theta)$ symmetric, then

$$r = min\left(1, \frac{f(\theta^*)}{f(\theta)}\right)$$

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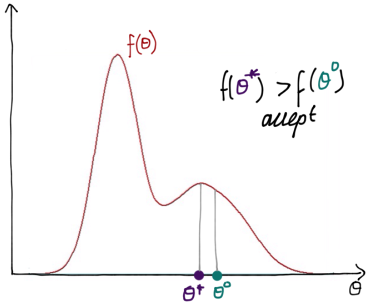

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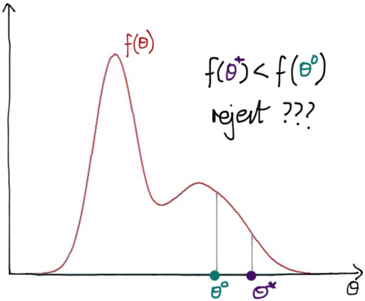

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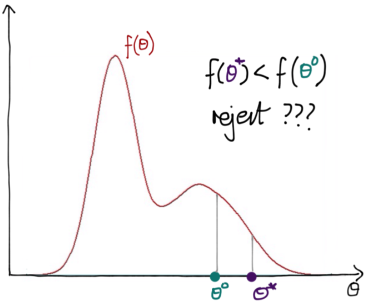

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$$r = min\left(1, \frac{f(\theta^*)q(\theta|\theta^*)}{f(\theta)q(\theta^*|\theta)}\right)$$

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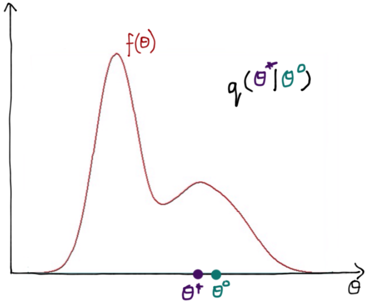

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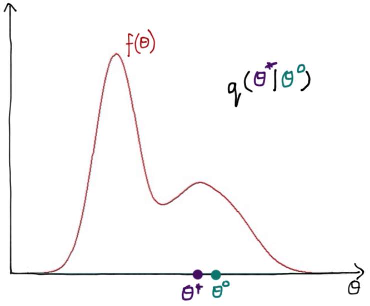

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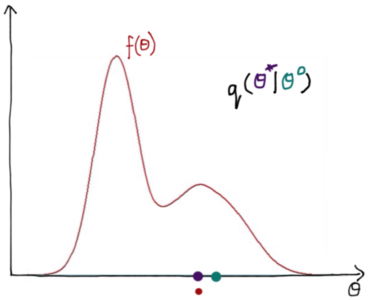

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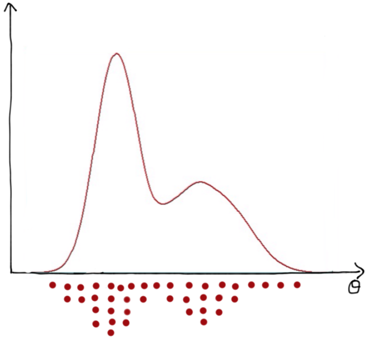
$$\theta^{(s+1)} = \begin{cases} \theta^*, & \text{if } u < r \\ \theta^{(s)}, & \text{if } u \geqslant r \end{cases}$$

Metropolis-Hastings



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