(Hidden) assumptions of simple compartmental ODE models

Rebecca Borchering, PhD
U.S. Centers for Disease Control and Prevention (CDC)
University of Florida and MMED Alumnus
ICI3D Faculty

MMED 2023

Goals

- Review the main uses of applied epidemiological modelling
- Introduce our conceptual framework for applied modelling
- Review commonly overlooked assumptions that are inherent in the structure of simple compartmental ODE models
- Discuss when these assumptions might be problematic, and when they may be desirable
- Begin to explore some alternative model structures that relax the assumptions

Applied Epi. Modelling

For application to the real world

Related to the distribution and determinants of health-related states and events

The use of simplification to represent the key components of something you're trying to understand more clearly

Applied Epi. Modelling

- INSIGHT
- ESTIMATION
- PREDICTION
- PLANNING
- ASSESSMENT

Applied Epi. Modelling

Insight

- Improving understanding of the dynamics of health and disease
- Translation of results into decision-making and communication tools

Applied Epi. Modelling

- Insight
- Estimation
  - Improving measurement and interpretation of key health indicators at the population and individual levels

Applied Epi. Modelling

Insight

Estimation

Prediction

- Projection and forecasting of expected future trends

Applied Epi. Modelling

- Insight
- Estimation
- Prediction
- Planning
  - Guiding study design and intervention roll-out
  - Informing decisions through analysis and comparison of policy scenarios

Applied Epi. Modelling

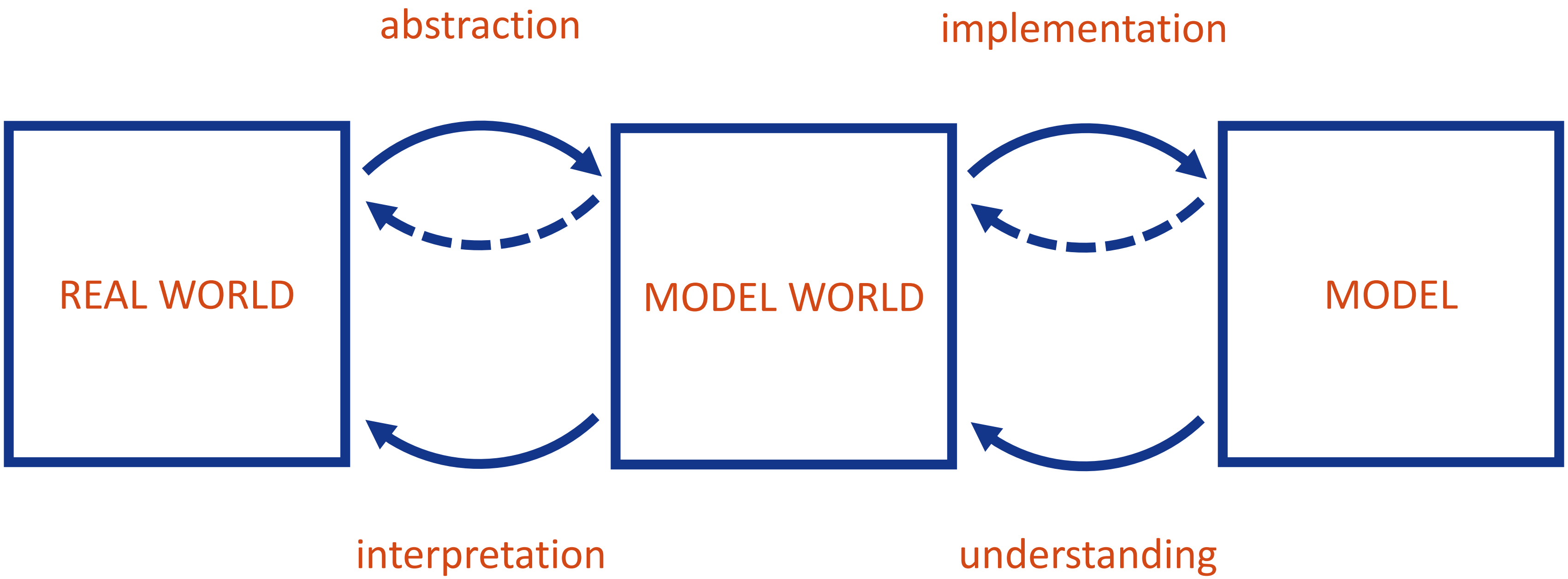
- Insight
- Estimation
- Prediction
- Planning
- Assessment
  - Evaluating the impact of public health interventions
  - Assessing risk of future public health events

Applied Epi. Modelling

Insight	• Improving understanding of the dynamics of health and disease • Translation of results into decision-making and communication tools
Estimation	• Improving measurement and interpretation of key health indicators at the population and individual levels
Prediction	• Projection and forecasting of expected future trends
Planning	• Guiding study design and intervention roll-out • Informing decisions through analysis and comparison of policy scenarios
Assessment	• Evaluating the impact of public health interventions • Assessing risk of future public health events

Goals

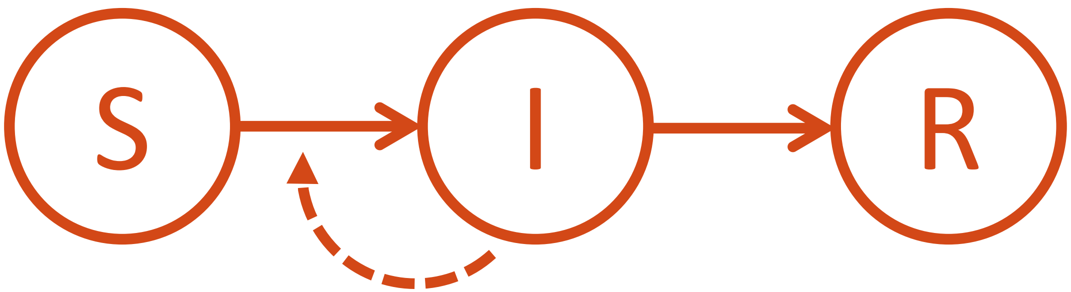
- Review the main uses of applied epidemiological modelling
- Introduce our conceptual framework for applied modelling
- Review commonly overlooked assumptions that are inherent in the structure of simple compartmental ODE models
- Discuss when these assumptions might be problematic, and when they may be desirable
- Begin to explore some alternative model structures that relax the assumptions


abstraction
implementation
REAL WORLD
MODEL WORLD
MODEL
interpretation
understanding

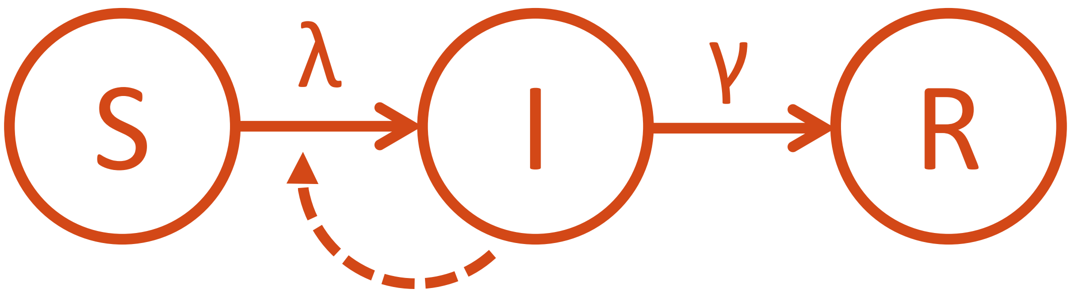
Model Worlds

- **A model world is an abstraction of the world that is simple and fully specified, which we construct to help us understand particular aspects of the real world**
- **A mathematical model is a formal description of the assumptions that define a model world**
  - **We know exactly what assumptions we've made, and we can follow those assumptions to their logical conclusions to address research questions**

The SIR Model World



SIR: ODE Model



$$/ = \frac{bI}{N}$$

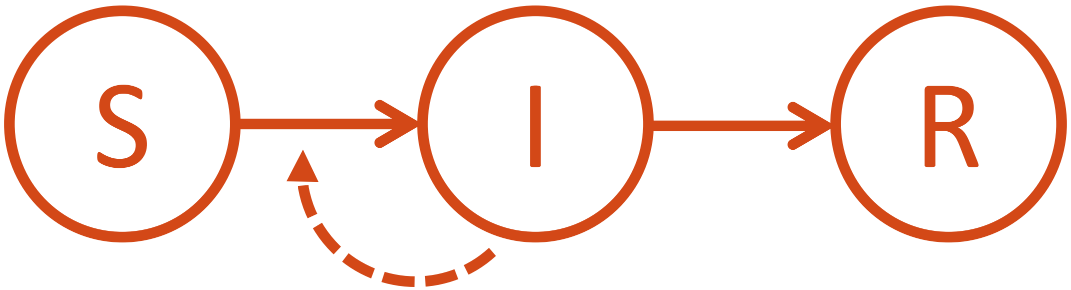
N=S+I+R

$$\frac{dS}{dt} = - \frac{bSI}{N}$$

$$\frac{dI}{dt} = \frac{bSI}{N} - gI$$

$$\frac{dR}{dt} = gI$$

SIR: Reed-Frost Model



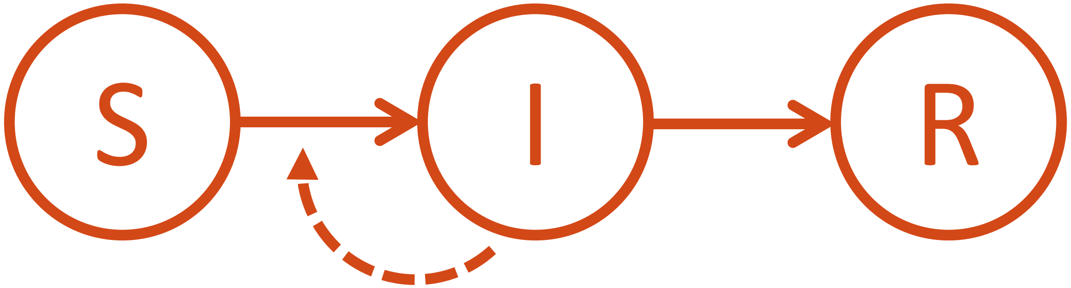
$$S_{t+1} = S_t - I_{t+1}$$

$$I_{t+1} = S_t(1 - q^{I_t})$$

$$R_{t+1} = R_t + I_t$$

N=S+I+R

SIR: Stochastic Reed-Frost

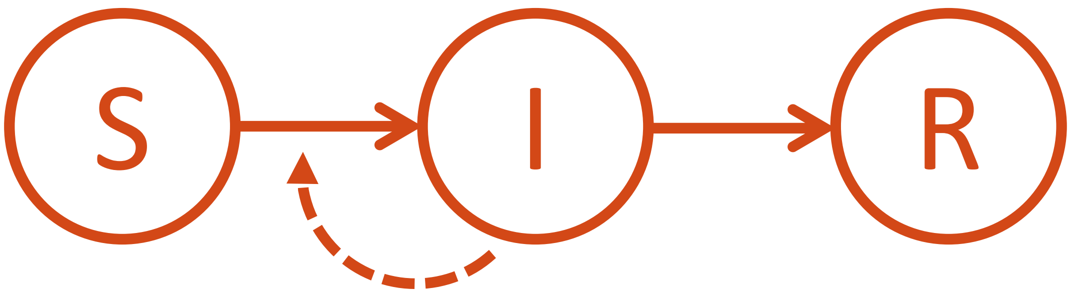


$$S_{t+1} = S_t - I_{t+1}$$

$$\mathbb{P}(I_{t+1} = x) = \binom{S_t}{x}(1 - q^{I_t})^x (q^{I_t})^{S_t - x}$$

$$R_{t+1} = R_t + I_t$$

SIR: Chain Binomial Model

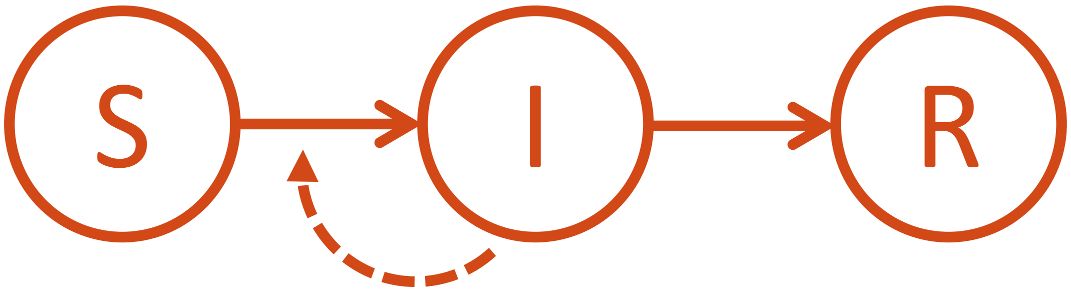


$$S_{t+\Delta t} = S_t - X$$
$$I_{t+\Delta t} = I_t + X - Y$$
$$R_{t+\Delta t} = R_t + Y$$

$$\mathbb{P}(X = x) = \binom{S_t}{x} p^x (1-p)^{S_t - x}$$

$$\mathbb{P}(Y = y) = \binom{I_t}{y} r^y (1-r)^{I_t - y}$$

The SIR Model Family



A mathematical model is formal description of the assumptions that define a model world

Taxonomy of **compartmental** models

	Continuous treatment of individuals (averages, proportions, or population densities)	Discrete treatment of individuals
Deterministic	**Continuous time** • Ordinary differential equations • Partial differential equations **Discrete time** • Difference equations (eg, Reed-Frost type models)	
Stochastic	**Continuous time** • Stochastic differential equations **Discrete time** • Stochastic difference equations	**Continuous time** • Gillespie algorithm **Discrete time** • Chain binomial type models (eg, Stochastic Reed-Frost models)

Taxonomy of **compartmental** models

	Continuous treatment of individuals (averages, proportions, or population densities)	**Discrete treatment of individuals**
Deterministic	**CONTINUOUS TIME** • Ordinary differential equations • Partial differential equations **DISCRETE TIME** • Difference equations (eg, Reed-Frost type models)	
Stochastic	**CONTINUOUS TIME** • Stochastic differential equations **DISCRETE TIME** • Stochastic difference equations	**CONTINUOUS TIME** • Gillespie algorithm **DISCRETE TIME** • Chain binomial type models (eg, Stochastic Reed-Frost models)

Taxonomy of **compartmental** models

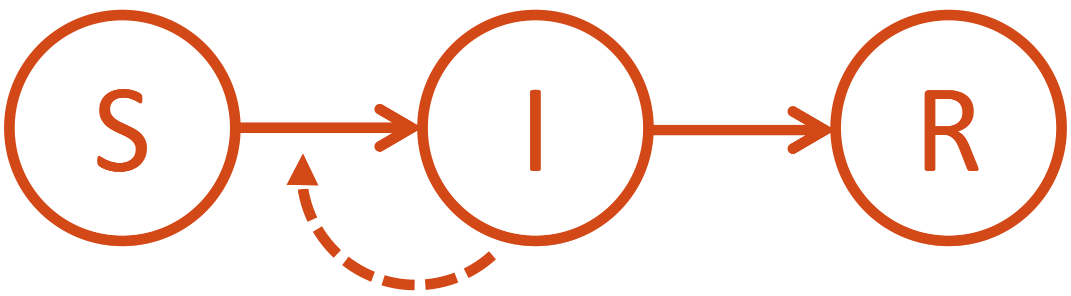
	CONTINUOUS TREATMENT OF INDIVIDUALS (averages, proportions, or population densities)	DISCRETE TREATMENT OF INDIVIDUALS
DETERMINISTIC	CONTINUOUS TIME • Ordinary differential equations • Partial differential equations DISCRETE TIME • Difference equations (eg, Reed-Frost type models)	
STOCHASTIC	CONTINUOUS TIME • Stochastic differential equations DISCRETE TIME • Stochastic difference equations	CONTINUOUS TIME • Gillespie algorithm DISCRETE TIME • Chain binomial type models (eg, Stochastic Reed-Frost models)

Taxonomy of **compartmental** models

	Continuous treatment of individuals (averages, proportions, or population densities)	**Discrete treatment of individuals**
Deterministic	**Continuous time** • Ordinary differential equations • Partial differential equations **Discrete time** • Difference equations (eg, Reed-Frost type models)	
Stochastic	**Continuous time** • Stochastic differential equations **Discrete time** • Stochastic difference equations	**Continuous time** • Gillespie algorithm **Discrete time** • Chain binomial type models (eg, Stochastic Reed-Frost models)

Taxonomy of **compartmental** models

What is a **compartmental** model?



Goals

- Review the main uses of applied epidemiological modelling
- Introduce our conceptual framework for applied modelling
- Review commonly overlooked assumptions that are inherent in the structure of simple compartmental ODE models
- Discuss when these assumptions might be problematic, and when they may be desirable
- Begin to explore some alternative model structures that relax the assumptions

Taxonomy of **compartmental** models

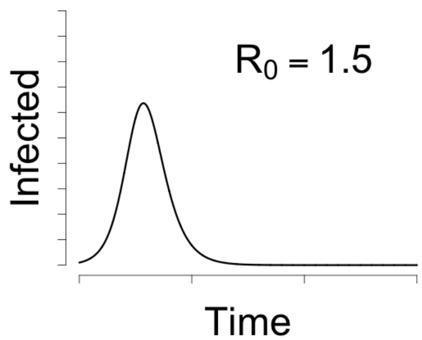
CONTINUOUS TREATMENT OF INDIVIDUALS

(<u>averages</u>, proportions, or population densities)

DISCRETE TREATMENT OF INDIVIDUALS

DETERMINISTIC

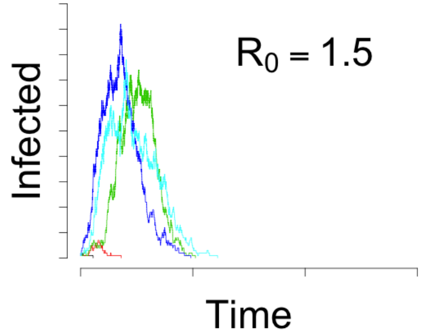
CONTINUOUS TIME



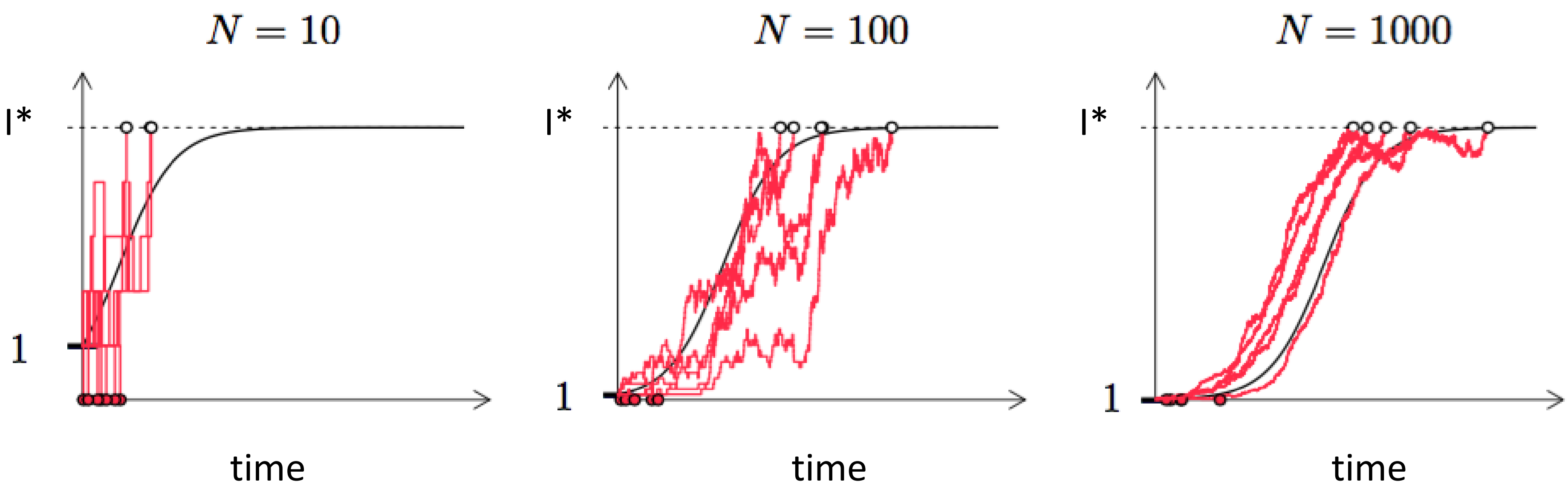
large population size

STOCHASTIC

CONTINUOUS TIME



Demographic stochasticity



Borchering & McKinley (2018) *Multiscale Modeling and Simulation*
DOI: 10.1137/17M1155259

Taxonomy of **compartmental** models

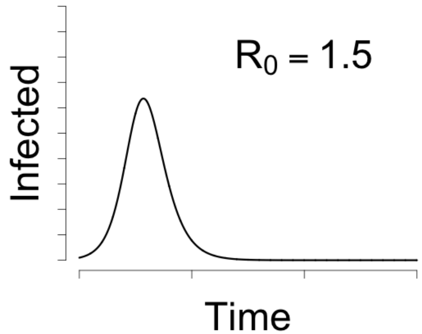
CONTINUOUS TREATMENT OF INDIVIDUALS

(averages, proportions, or population densities)

DISCRETE TREATMENT OF INDIVIDUALS

DETERMINISTIC

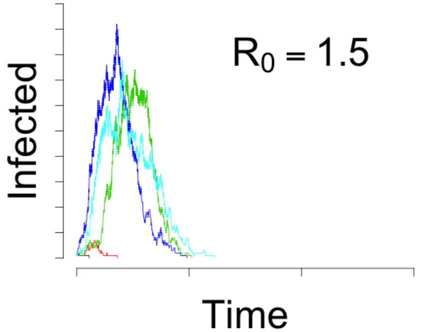
CONTINUOUS TIME



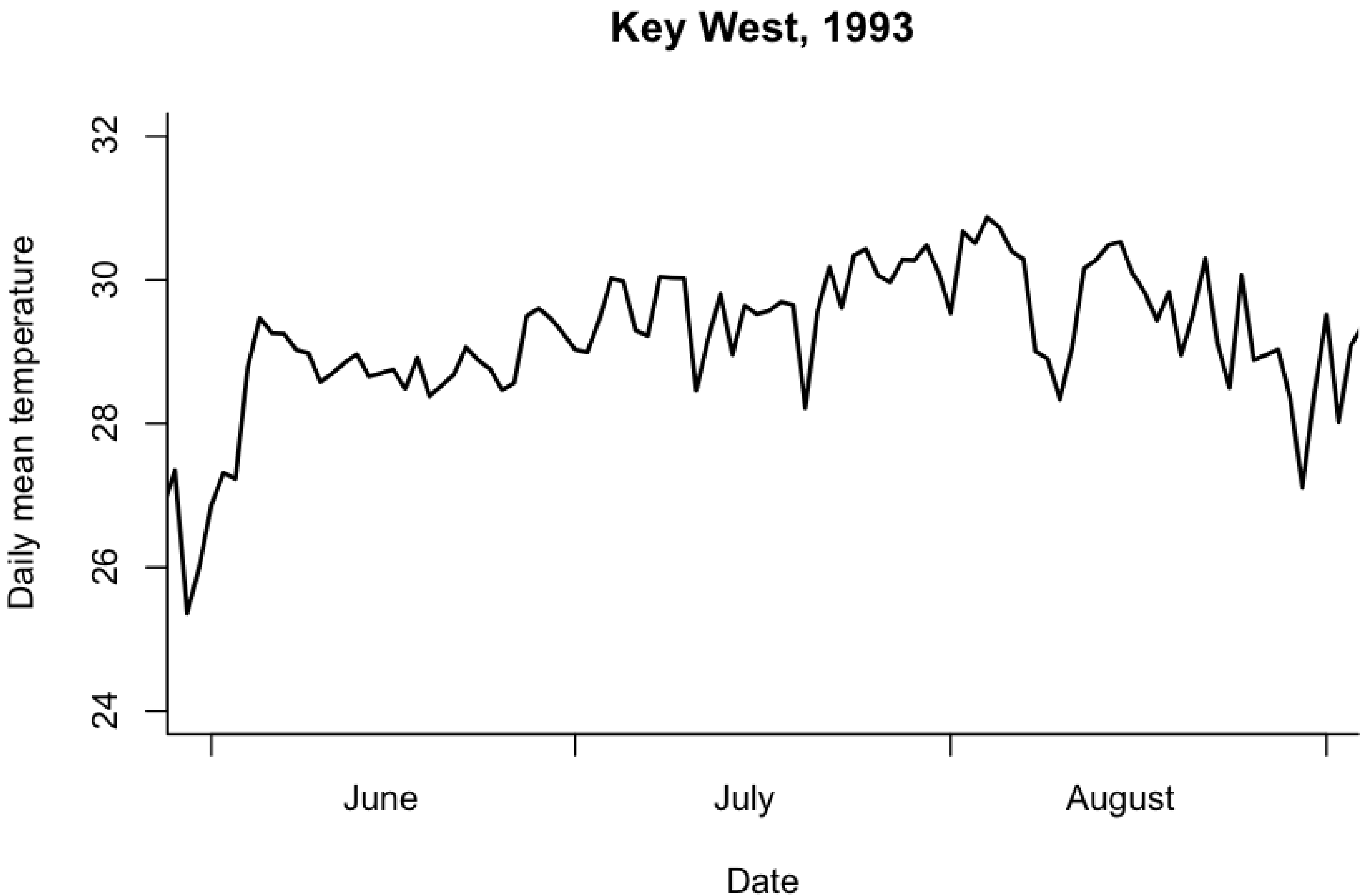
large population size

STOCHASTIC

CONTINUOUS TIME



Environmental stochasticity



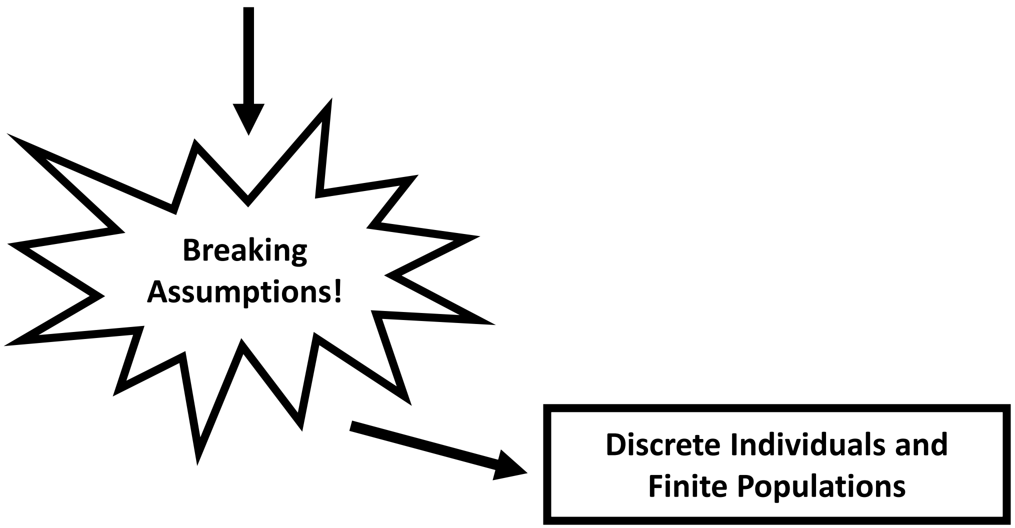
Compartmental ODE models assume

- Large population size
- Deterministic progression
  - for a given set of initial conditions and parameter values, a deterministic model always gives the same outcome

Benefit: Simplicity and consequences of assumptions facilitate quick assessments of what model outcomes are possible, and when

Compartmental ODE models assume

- Large population size
- Deterministic progression
  - for a given set of initial conditions and parameter values, a deterministic model always gives the same outcome



Taxonomy of **compartmental** models

CONTINUOUS TREATMENT OF INDIVIDUALS

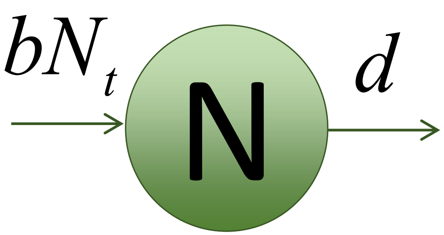
(averages, proportions, or population densities)

DISCRETE TREATMENT OF INDIVIDUALS

DETERMINISTIC

CONTINUOUS TIME

- Ordinary differential equations



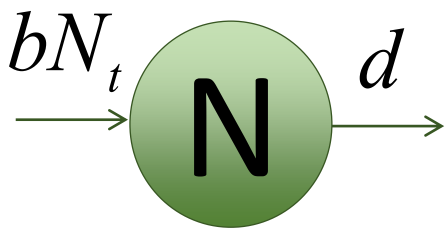
STOCHASTIC

Simple ODE models assume

- Time proceeds in a continuous manner
- Parameter values remain constant

CONTINUOUS TIME

- Ordinary differential equations



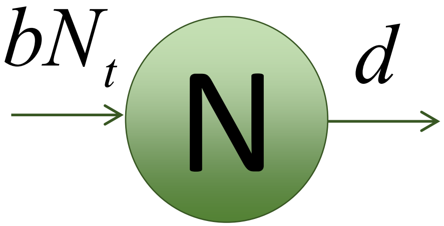
$$\frac{dN_t}{dt} = bN_t - dN_t$$

Simple ODE models assume

- Time proceeds in a continuous manner
- Parameter values remain constant

CONTINUOUS TIME

- Ordinary differential equations



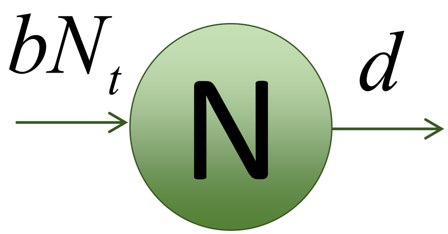
$$r = b - d$$

$$\frac{dN_t}{dt} = rN_t$$

$$N_{t+\Delta t} = N_t e^{r\Delta t}$$

CONTINUOUS TIME

- Ordinary differential equations



$$r = b - d$$

$$\frac{dN}{dt} = rN$$

$$N_{t+\Delta t} = N_t e^{r\Delta t}$$

DISCRETE TIME

- Difference equations

$$\lambda_{\Delta t} = e^{r\Delta t}$$

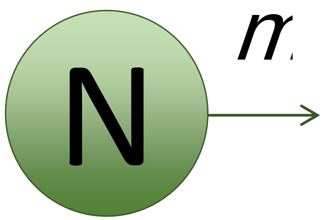
$$N_{t+\Delta t} = \lambda_{\Delta t} N_t$$

Simple compartmental ODE models assume

- Homogeneity within compartments
- Large population size
- Deterministic progression
- Time proceeds in a continuous manner
- Parameter values remain constant
- Memory-less processes

Simple ODE models assume

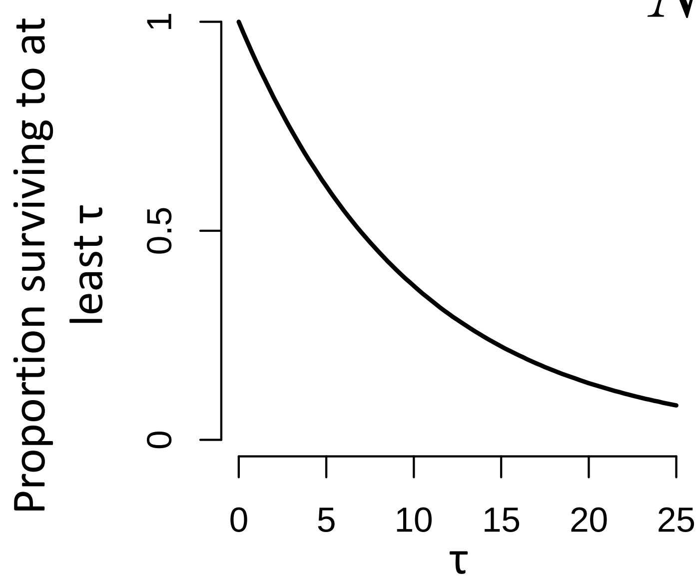
- Memory-less processes



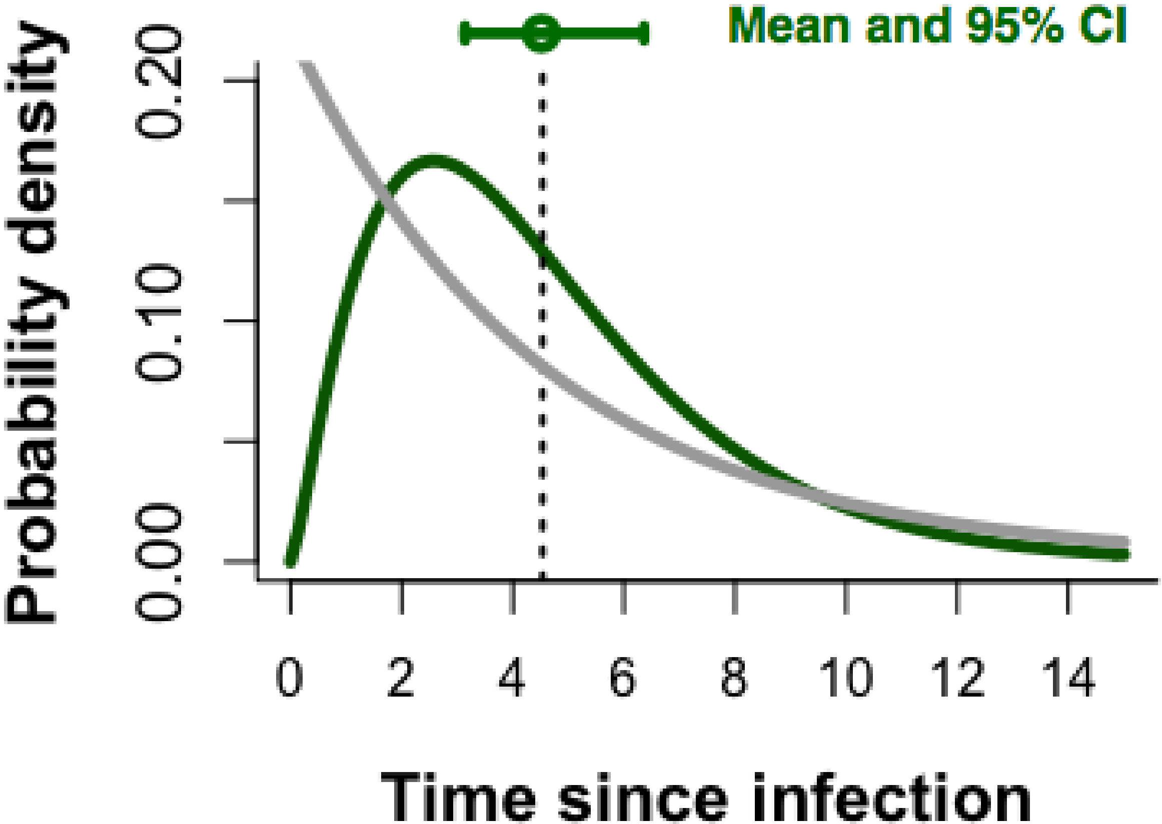
$$\frac{dN_t}{dt} = -mN_t$$

$$N_t = N_0 e^{-mt}$$

$$\frac{N_t}{N_0} = e^{-mt}$$



Realistic waiting times



Estimate based on data from Joshi *et al.* (2009) *Transactions of the Royal Society of Tropical Medicine and Hygiene*

Simple compartmental ODE models assume

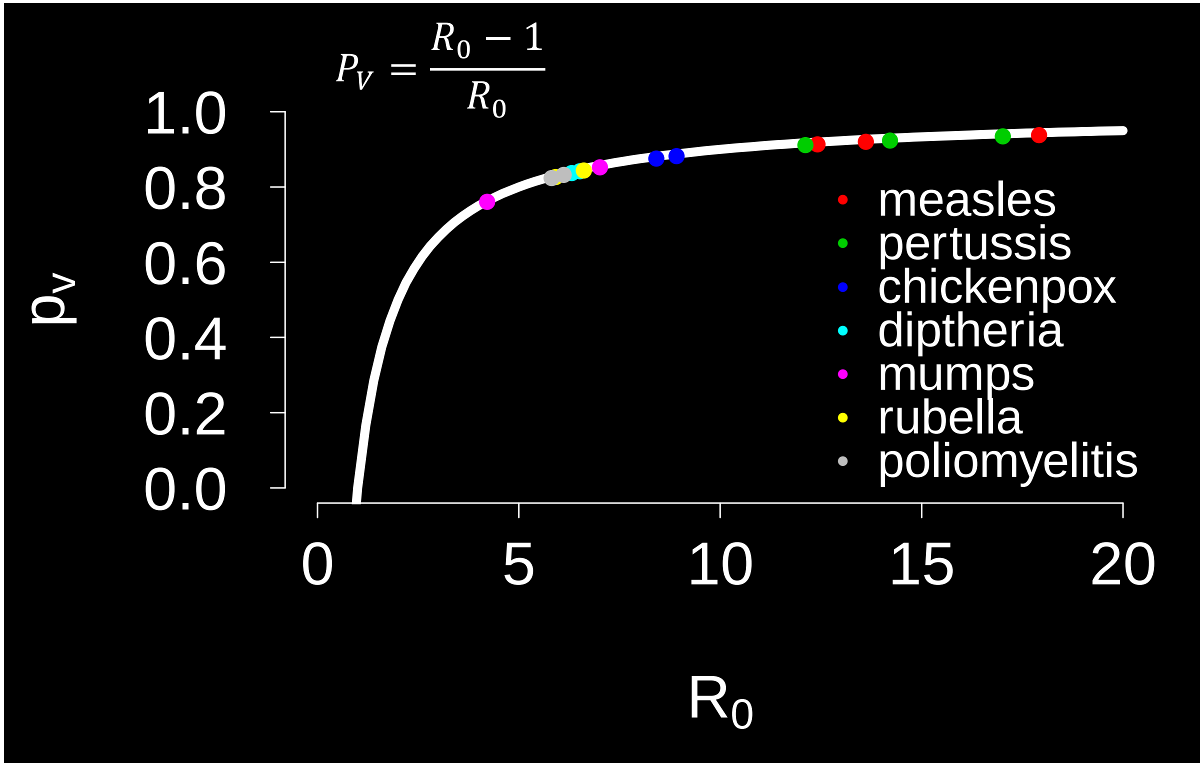
- Homogeneity within compartments
- Large population size
- Deterministic progression
- Time proceeds in a continuous manner
- Parameter values remain constant
- Memory-less processes

Breaking Assumptions!

Non-exponential waiting times

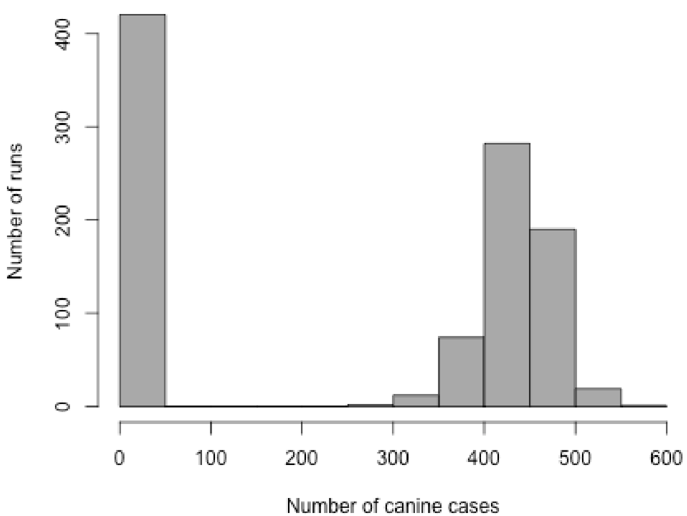
Summary

- Simple ODE models are important tools for building understanding



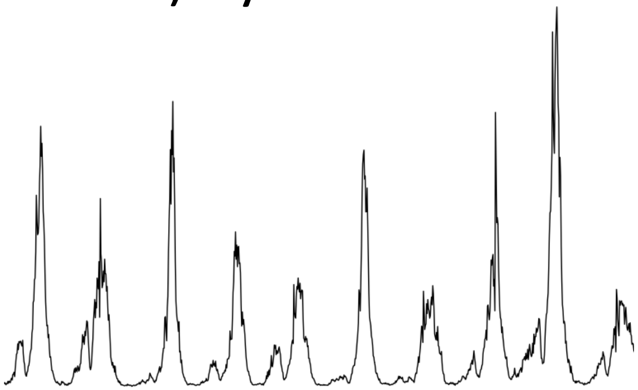
Summary

- Simple ODE models are important tools for building understanding
- It's important to recognize the assumptions built into these models
  - When populations are small, average behaviors can be misleading



Summary

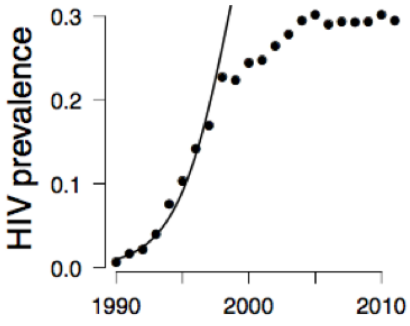
- Simple ODE models are important tools for building understanding
- It's important to recognize the assumptions built into these models
  - When populations are small, average behaviors can be misleading
  - When rates vary, simple ODEs can fail to reproduce important (observed) dynamics



Data from Earn et al. 2000 *Science*

Summary

- Simple ODE models are important tools for building understanding
- It's important to recognize the assumptions built into these models
  - When populations are small, average behaviors can be misleading
  - When rates vary, simple ODEs can fail to reproduce important (observed) dynamics



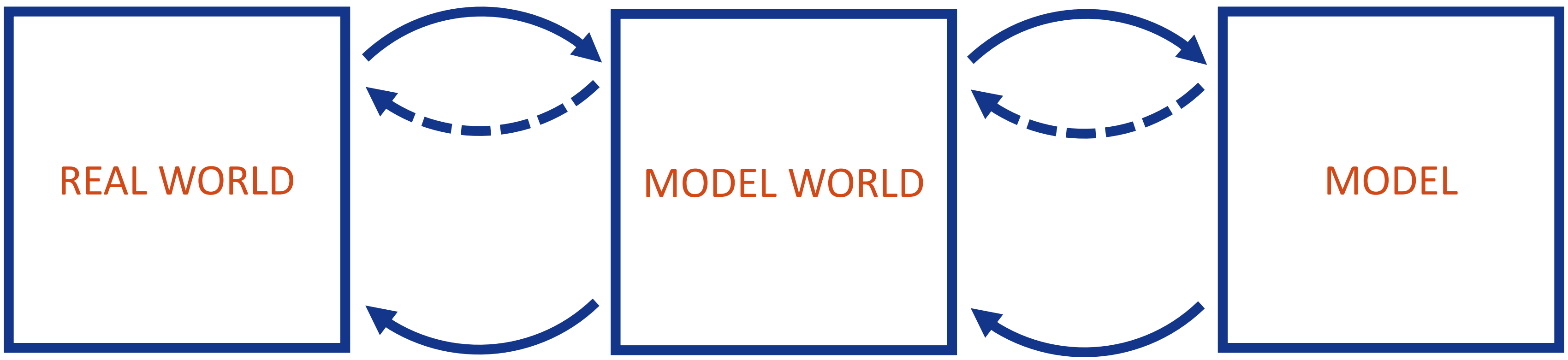
Dushoff lecture on heterogeneity (Wed)

Summary

- Simple ODE models are important tools for building understanding
- It's important to recognize the assumptions built into these models
  - When populations are small, average behaviors can be misleading
  - When rates vary, simple ODEs can fail to reproduce important (observed) dynamics

Summary

- The applied epidemiological modelling process requires
  - abstraction
  - specification and implementation
  - gaining an understanding of the dynamics
  - interpretation



(Hidden) assumptions of simple compartmental ODE models. DOI: 10.6084/m9.figshare.5044606

Attribution:

Juliet Pulliam & Rebecca Borchering
Clinic on the Meaningful Modeling of Epidemiological Data

Source URL:
https://figshare.com/articles/Mathematical_Assumptions_of_Simple_ODE_models/5044606

For further information please contact admin@ici3d.org.