Relativity

Special Relativity

Lorentz transform:

$$X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu}$$

Where Lorentz Transformation matrix, Λ is:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Where γ is:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Recovering Lorentz equations:

$$x' = \gamma(x - v_0 t)$$
$$t' = \gamma\left(t - v_0 \frac{x}{c^2}\right)$$
$$y' = y$$
$$z' = z$$

Proper time, τ:

$$\tau = \frac{t}{\gamma}$$

Spacetime line element $(dS)^2$:

$$(dS)^2 = \eta_{\alpha\beta} dx'^{\alpha} dx'^{\beta}$$

Where Minkowski metric, η is:

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

So for flat space:

$$(dS)^2 = c^2(dt)^2 - (dx)^2 - (dy)^2 - (dz)^2$$

Four vectors:

$$X = (ct, x, y, z), \quad \text{scalar contraction: } (dS)^2$$
$$U = (\gamma c, \gamma\vec{v}), \quad \text{scalar contraction: } c^2$$
$$P = (m\gamma c, m\gamma\vec{v}) = \left(\frac{E}{c}, m\gamma\vec{v}\right), \quad \text{scalar contraction: } m^2c^2$$

$$F = \left(\frac{\gamma}{c}\vec{f}\cdot\vec{v}, \gamma\vec{f}\right)$$

Derivation of U^α, note proper time in first equation:

$$U^\alpha = \frac{dX^\alpha}{d\tau} = \frac{\gamma dX^\alpha}{dt}$$

$$U^0 = \frac{\gamma c dt}{dt} = c\gamma$$

$$U^1 = \gamma\frac{dx}{dt} = \gamma\vec{v}$$

$$U = (\gamma c, \gamma\vec{v})$$

Scalar contractions are done by:

$$\text{Scalar} = \eta_{\alpha\beta}X^\alpha X^\beta$$

So for U:

$$U^0U^0 - U^1U^1 - U^2U^2 - U^3U^3 = \gamma^2c^2 - \gamma^2v^2 = c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right) = c^2$$

Derivation of Energy-Momentum relation using scalar contraction of P:

$$\frac{E^2}{c^2} - \vec{p}^2 = m^2c^2$$

$$E^2 = m^2c^4 + p^2c^2$$

General Relativity

An inertial reference frame is a non-accelerating, non-rotating frame, such that Newton's second law is true.

Equivalence Principle: Inertial mass and gravitational mass are equivalent, so motion of a body under gravitational forces cannot be distinguished from a body under appropriate acceleration.

Principle of covariance: Physical laws are the same in every reference frame.

Principle of consistency: Any new theory must also be consistent with the successful results of the previous theory. I.e. General relativity is consistent with special relativity, Newtonian gravity and classical electromagnetism.

Spacetime line element $(dS)^2$:

$$(dS)^2 = \eta_{\alpha\beta}dx'^\alpha dx'^\beta$$

Where Minkowski metric, η is:

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Transformation from local inertial frame x' to accelerated frame X in terms of partial derivatives (used in derivation of metric tensor):

$$x' = \Lambda^{\mu}_{\nu} X^{\nu} = \frac{\partial x'^{\mu}}{\partial X^{\nu}} X^{\nu}$$

Deriving metric tensor $g_{\alpha\beta}$ from $(dS)^2$:

$$(dS)^2 = \eta_{\alpha\beta} dx'^{\alpha} dx'^{\beta}$$

$$dx'^{\alpha} = \frac{\partial x'^{\alpha}}{\partial X^{\mu}} dX^{\mu}$$

$$(dS)^2 = \eta_{\alpha\beta} \left(\frac{\partial x'^{\alpha}}{\partial X^{\mu}} dX^{\mu} \right) \left(\frac{\partial x'^{\beta}}{\partial X^{\nu}} dX^{\nu} \right) = \eta_{\alpha\beta} \left(\frac{\partial x'^{\alpha}}{\partial X^{\mu}} \frac{\partial x'^{\beta}}{\partial X^{\nu}} \right) dX^{\mu} dX^{\nu} = g_{\mu\nu} dX^{\mu} dX^{\nu}$$

$$g_{\mu\nu} = \eta_{\alpha\beta} \left(\frac{dx'^{\alpha}}{dX^{\mu}} \right) \left(\frac{dx'^{\beta}}{dX^{\nu}} \right)$$

Getting diagonal $g_{\alpha\alpha}$ from $(dS)^2$: Take the coefficient to each element, and put it in the diagonal. Remember to take the inverse to get $g^{\alpha\beta}$ for the connection coefficients.

Geodesic equation:

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \sum_{\nu,\rho} \Gamma^{\mu}{}_{\nu,\rho} \frac{dx^{\nu}}{d\lambda} \frac{dx^{\rho}}{d\lambda} = 0$$

Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\kappa T_{\mu\nu}$$

where κ is the Einstein constant:

$$\kappa = \frac{8\pi G}{c^4}$$

$R_{\mu\nu}$ is the Ricci Tensor. R is the Ricci scalar:

$$R = g^{\alpha\beta} R_{\alpha\beta}$$

$R^{l}{}_{ijk}$ is the Riemann curvature tensor.
The Ricci scalar is obtained by contracting over the Ricci tensor, which is obtained by contracting over the Riemann curvature tensor.

Simplification of Einstein Field equation:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{-8\pi G}{c^4} T_{\alpha\beta}$$

Contracting with $g^{\alpha\beta}$:

$$g^{\alpha\beta} R_{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} g_{\alpha\beta} R = \frac{-8\pi G}{c^4} g^{\alpha\beta} T_{\alpha\beta}$$

$$R - 2R = \frac{-8\pi G}{c^4} T \;\therefore\; R = \frac{8\pi G}{c^4} T$$

Rearranging first equation:

$$R_{\alpha\beta} = \frac{-8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{c^4}{8\pi G}\frac{1}{2}g_{\alpha\beta}R\right)$$

Substiuting for R:

$$R_{\alpha\beta} = \frac{-8\pi G}{c^4}\left(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T\right)$$

Schwarzchild Metric

Schwarzchild metric:

$$(dS)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}} - r^2 d\theta^2 - r^2\sin^2(\theta)d\phi^2$$

Schwarzchild radius, R_s:

$$R_s = \frac{2GM}{c^2}$$

Proper time:

$$d\tau_{\mathrm{ob}} = \sqrt{1 - \frac{2GM}{c^2 r_{\mathrm{ob}}}}dt_{\mathrm{ob}}$$

Recall that: $\tau = \frac{t}{\gamma}$ usually, so one can define an effective new γ:

$$\gamma' = \sqrt{1 - \frac{2GM}{c^2 r}}$$

To derive the new lengths, etc.

Derivation of observed frequency from Schwarzchild metric:

$$(dS)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}} - r^2 d\theta^2 - r^2\sin^2(\theta)d\phi^2$$

Assume same position for emitter:

$$(dS_{\mathrm{em}})^2 = \left(1 - \frac{2GM}{c^2 r}\right)(cdt_{\mathrm{em}})^2$$

$$d\tau_{\mathrm{em}} = \frac{dS_{\mathrm{em}}}{c} = \sqrt{1 - \frac{2GM}{c^2 r_{\mathrm{em}}}}dt_{\mathrm{em}}$$

$$(dS_{\mathrm{ob}})^2 = \sqrt{1 - \frac{2GM}{c^2 r_{\mathrm{ob}}}}dt_{\mathrm{ob}} = \left(1 - \frac{2GM}{c^2 r_{\mathrm{ob}}}\right)c^2(dt_{\mathrm{em}})^2$$

$$d\tau_{\mathrm{ob}} = \frac{d\tau_{\mathrm{em}}}{\sqrt{1 - \frac{2GM}{c^2 r_{\mathrm{em}}}}} \;\therefore\; f_{\mathrm{ob}} = f_{\mathrm{em}}\sqrt{1 - \frac{2GM}{c^2 r_{\mathrm{em}}}}$$

Derivation of time for light to travel outward from radius r_1 to r_2 in the vicinity of a black hole:

$$(dS)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}} - r^2 d\theta^2 - r^2 \sin^2(\theta) d\phi^2$$

Assume same radial direction:

$$(dS)^2 = \left(1 - \frac{2GM}{c^2 r}\right)(cdt)^2 - \frac{dr^2}{1 - \frac{2GM}{c^2 r}}$$

For light: $dS^2 = 0$

$$c^2 dt^2 = \frac{-dr^2}{\left(1 - \frac{2GM}{c^2 r}\right)^2} \quad \therefore \; dt = \pm \frac{dr}{c\left(1 - \frac{R_s}{r}\right)}$$

Take positive root for outward light:

$$\int dt = \int \frac{dr}{c\left(1 - \frac{R_s}{r}\right)}$$

$$t_2 - t_1 = \frac{1}{c}\int_{r_1}^{r_2} \left(\frac{dr}{1 - \frac{R_s}{r}}\right) = \frac{1}{c}\int_{r_1}^{r_2} \left(\frac{r}{r - R_s}\right) dr = \frac{1}{c}\int_{r_1}^{r_2} \left(\frac{r + R_s - R_s}{r - R_s}\right) dr$$

$$t_2 - t_1 = \frac{1}{c}\int_{r_1}^{r_2} \left(1 + \frac{R_s}{R - R_s}\right) dr = \frac{1}{c}\left((r_2 - r_1) + R_s \ln\left(R - R_s\right)_{r_1}^{r_2}\right) = \frac{1}{c}\int_{r_1}^{r_2} \left(\frac{r + R - R_s}{r - R_s}\right) dr$$

$$t_2 - t_1 = \frac{r_2 - r_1}{c} + \frac{R_s}{c} \ln\left(\frac{r_2 - R_s}{r_1 - R_s}\right)$$

Cosmology

Cosmological Principle: At a sufficiently large scale, the properties of the Universe are the same for all observers - homogenous and isotropic. i.e. we don't have a privileged position.
Robertson-Walker metric:

$$(dS)^2 = c^2 dt^2 - R^2(t)\left[\frac{(dr)^2}{1 - kr^2} + r^2 (d\theta)^2 + r^2 \sin^2(\theta)(d\phi)^2\right]$$

Friedmann equations:
Curvature:

$$\left[\frac{1}{R}\frac{dR}{dt}\right]^2 = \frac{8\pi G}{3}\left[\rho_m \left(\frac{R_0}{R}\right)^3 + \rho_r \left(\frac{R_0}{R}\right)^4 + \rho_\Lambda\right] - \frac{kc^2}{R^2}$$

Acceleration:

$$\frac{1}{R}\frac{d^2 R}{dt^2} = \frac{-4\pi G}{3}\left[\rho_m \left(\frac{R_0}{R}\right)^3 + 2\rho_r \left(\frac{R_0}{R}\right)^4 - 2\rho_\Lambda\right]$$

Derivation of Friedmann equations:
Fluid equation:

$$d(\Delta E) = -pd(\Delta V)$$

Total energy density is ρc^2. $\rho c^2 \Delta V$ is energy density in element of volume.

$$\Delta E = \rho c^2 \Delta V,\ \Delta V = \frac{4}{3}\pi R^3(t)\ \therefore \Delta E = c^2\rho(t)\frac{4}{3}\pi R^3(t)$$

$$\frac{d}{dt}\left(c^2\rho(t)\frac{4}{3}\pi R^3(t)\right) = -p\frac{d}{dt}\left(\frac{4}{3}\pi R^3(t)\right)$$

$$c^2\frac{d}{dt}\left(\rho(t)R^3(t)\right) = -p\frac{d}{dt}\left(R^3(t)\right)$$

$$c^2\frac{d\rho}{dt}R^3(t) + c^2\rho(t)3R^2\frac{dR}{Dt} = -3pR^2\frac{dR}{dt}$$

$$\frac{d\rho}{dt} = \frac{-3}{R}\frac{dR}{dt}\left(\rho + \frac{p}{c^2}\right)$$

Energy equation:

$$T = \frac{1}{2}m\dot{R}^2,\ M = \frac{4}{3}\pi\rho R^3,\ V = \frac{-GM}{R} = -G\frac{4}{3}\pi\rho\frac{R^3}{R}$$

Let m=1, unit mass.

$$T + V = \frac{1}{2}\dot{R}^2 - \frac{4}{3}\pi\rho G R^2 = \text{const.}$$

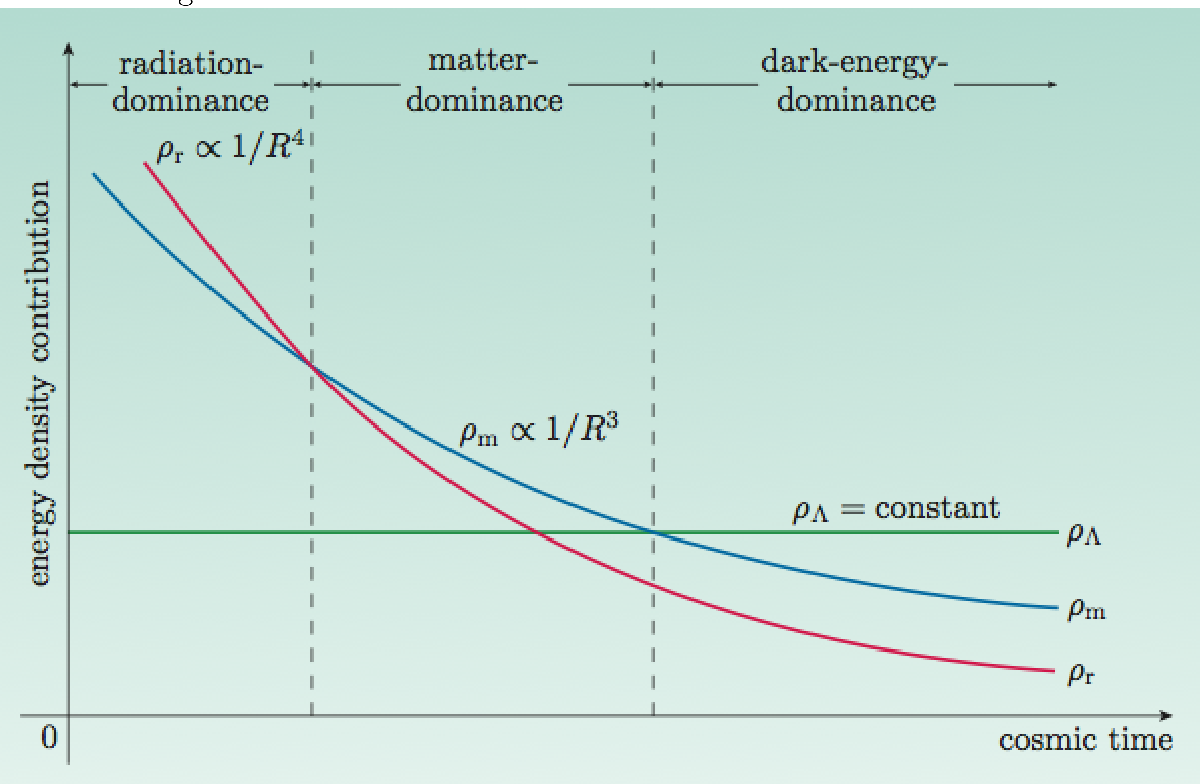
Let constant be $\frac{-kc^2}{2}$:

$$\frac{1}{2}\dot{R}^2 - \frac{4}{3}\pi\rho G R^2 = \frac{-kc^2}{2}\ \therefore\ \dot{R}^2 - \frac{8}{3}\pi\rho G R^2 = -kc^2$$

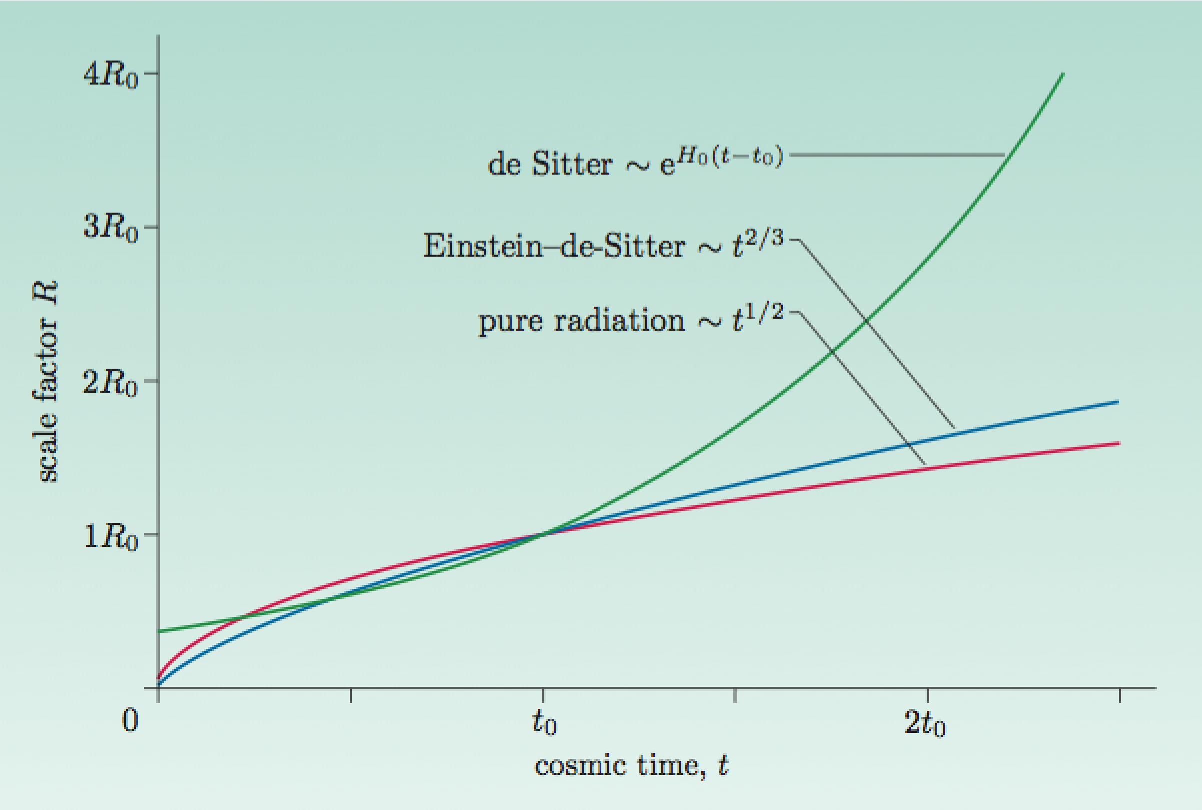
$$\left[\frac{1}{R}\frac{dR}{dt}\right]^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2}$$

Diagrams:

Dominance against time:



Expansion against time:



Different model behaviours: