PHY1106: Waves and Oscillations

Oscillations:

Simple Harmonic Motion:

The acceleration of the body is proportional to the displacement of the body and acts in the opposite direction to the displacement.

Deriving the wave equation for a mass-spring system:

$-kx = ma \quad \therefore -kx = m\ddot{x}$ Rearranging: $\ddot{x} = \frac{-k}{m}x$

Trying $x(t) = A\cos\omega t$ as a wave solution:

$\dot{x} = -A\omega\sin\omega t \quad \therefore \quad \ddot{x} = -A\omega^2\cos\omega t$ Left hand side

$\ddot{x} = \frac{-k}{m}A\cos\omega t \; = -\omega^2 A\cos\omega t$ Right hand side

Both are equivalent so $x(t) = A\cos\omega t$ is a solution as long as ω is constant and $\omega = \sqrt{\frac{k}{m}}$.

A more versatile solution to the equation of motion is $Asin(\omega t + \phi)$. This can be shown to be a solution in the same method as above. It is more versatile because the phase angle allows many starting points.

Given that $x = Asin(\omega t + \phi)$, one can derive expressions for velocity ($\dot{x} = A\omega\cos(\omega t + \phi)$) and acceleration ($\ddot{x} = -A\omega^2\sin(\omega t + \phi)$) by differentiating with respect to t. The main result here is the velocity and acceleration amplitudes, $v_{max} = A\omega$ and $a_{max} = A\omega^2$ respectively. By plotting graphs of these expressions, it is clear that the velocity is 90º out of phase with the displacement and the acceleration is 180º out of phase with the displacement.

Energy of SHM:

In SHM there is a conversion between Kinetic Energy and Potential Energy. In undamped motion this transfer is perfect as no energy is lost to surroundings.

The kinetic energy can be expressed as $\frac{1}{2}m\dot{x}^2$. The potential energy in a mass-spring system can be expressed as $\frac{1}{2}kx^2$. Given that $x = Asin(\omega t + \phi)$ and so $\dot{x} = A\omega\cos(\omega t + \phi)$ .Then one can derive an expression for the total energy:

$E = \frac{1}{2}mA^2\omega^2\cos^2(\omega t + \phi) + \frac{1}{2}kA^2\sin^2(\omega t + \phi)$, using $\omega = \sqrt{\frac{k}{m}}$:

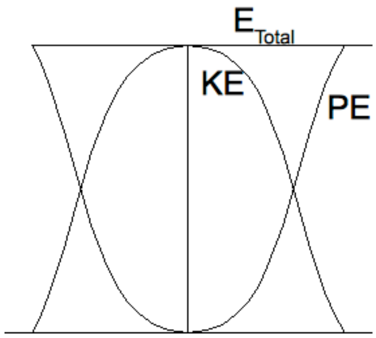
$E = \frac{1}{2}mA^2\frac{k}{m}\cos^2(\omega t + \phi) + \frac{1}{2}kA^2\sin^2(\omega t + \phi)$ so: $E = \frac{1}{2}kA^2\left(\cos^2(\omega t + \phi) + \sin^2(\omega t + \phi)\right)$

Using the trigonometric identity: $\cos^2(\theta) + \sin^2(\theta) = 1$:

$$E=\frac{1}{2}kA^2$$

Given that $\frac{1}{2}kA^2=\frac{1}{2}m\dot{x}^2+\frac{1}{2}kx^2$, one can re-arrange for velocity:

$\dot{x}=\pm\sqrt{\frac{k}{m}}\ (A^2-x^2)^{\frac{1}{2}}$ and so the kinetic energy is a parabolic function of displacement. The PE is the inverse of this such that the total energy is constant (in an undamped system).



When KE=PE: $\frac{1}{2}m\dot{x}^2=\frac{1}{2}kx^2$ so $\frac{1}{2}kA^2=kx^2$ so $x=\frac{A}{\sqrt{2}}$

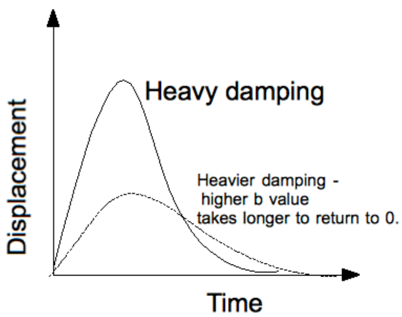
Damping:

In a damped system the equation of motion becomes $m\ddot{x}=-(kx+b\dot{x})$ where b is the coefficient of retardation.

The solution to this equation of motion is: $x=Ce^{\alpha t}$.

Using this we derive the equation: $Ce^{\alpha t}\left(m\alpha^2+b\alpha+k\right)=0$, the solution C=0 is trivial, the other solution is: $\alpha=\frac{-b}{2\mathrm{m}}\pm\sqrt{\frac{b^2}{4\mathrm{m}^2}-\frac{k}{m}}$.

Heavy damping (overdamping):

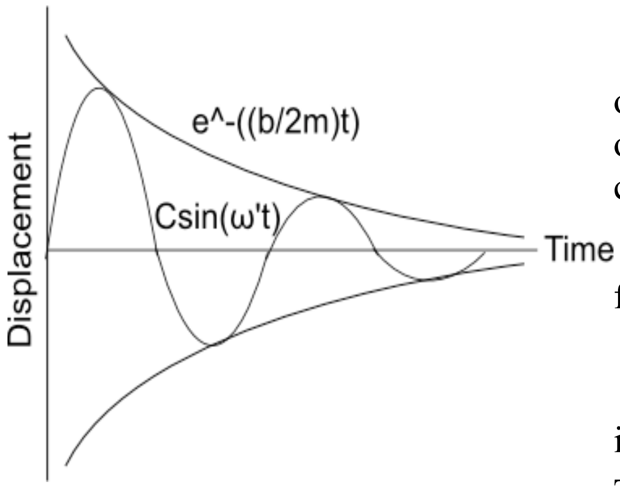


$\frac{b^2}{4\mathrm{m}^2}>\frac{k}{m}$, the system slowly returns to the equilibrium position and there is no consequent oscillatory motion.

Critical damping:

$\frac{b^2}{4\mathrm{m}^2}=\frac{k}{m}$, The system returns to the equilibrium position in the quickest possible time without producing consequent oscillatory motion.

Light damping (underdamping):



$\frac{k}{m} > \frac{b^2}{4\mathrm{m}^2}$, in this case the system overshoots the equilibrium positions and4 oscillatory motion follows. This is the most common type of damping.

$x = Ce^{\frac{-bt}{2\mathrm{m}}}(\sin(\omega' t + \phi))$, this becomes sin from cos due to the phase angle. z

$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4\mathrm{m}^2}}$.

i.e. the amplitude is given by $Ce^{\frac{-bt}{2\mathrm{m}}}$

This comes from expressing the exponential complex number in terms of cos and sin, and then taking the real part.

Since $Energy \propto Amplitude^2$ Then: $Energy \propto e^{-\frac{b}{m}t}$

Logarithmic Decrement:

$\delta = \ln\left(\frac{x_0}{x_1}\right) = \frac{b}{2\mathrm{m}} T'$ by definition, can be proven by substituting in t and (t+T) for x_0 and x_1. $T' = \frac{2\pi}{\omega'} = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4\mathrm{m}^2}}}$.

Q-factor:

$Q = 2\pi \times \frac{\text{Energy stored in the system}}{\text{Energy lost per cycle}} = 2\pi \frac{E}{dE}$, previously: $E \propto A^2 \propto \left(e^{-\frac{b}{2\mathrm{m}}t}\right)^2$ and so: $E = E_0 e^{-\frac{b}{m}t}$

Solving for Q:

$E = E_0 e^{-\frac{b}{m}t} \quad \therefore \frac{dE}{dt} = -\frac{b}{m} E_0 e^{-\frac{b}{m}t} \quad \therefore dE = -\frac{b}{m} E \cdot dt$ let dt = T' since dE is the energy lost per cycle

$$\therefore dE = -\frac{b}{m} E \cdot T' \quad \therefore Q = 2\pi \frac{E}{dE} = -2\pi \frac{m}{bT'} = \frac{\omega' m}{b}$$

The minus sign disappears because we take the Q-factor to be a positive value by definition.

It can also be shown that: $Q = \frac{\pi}{\delta}$ since $\delta = \frac{bT'}{2\mathrm{m}}$ and $\omega' = \frac{2\pi}{T'}$ and $Q = \frac{\omega' m}{b}$

The Forced Oscillator:

Driving force: $F_0 \cos(\omega t)$

Equation of motion: $m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t) = F_0 e^{i\omega t}$

The solution: $\hat{x}=\hat{A}e^{i\omega t}$ where x and A are complex numbers.

Solving the equation:

$$\hat{A}=\frac{F_0}{ib\omega+k-m\omega^2}$$

Multiplying top and bottom by -i: $\hat{A}=\frac{-iF_0}{\omega\left[b+i\left(\omega m-\frac{k}{\omega}\right)\right]}$

Mechanical Impedence: $\hat{Z}_m=b+i\left(\omega m-\frac{k}{\omega}\right)$

Substituting this in: $\hat{A}=\frac{-iF_0}{\omega\hat{Z}_m}$

Substituting this expression in to: $\hat{x}=\hat{A}e^{i\omega t}$

The simplified solution: $\hat{x}=-i\frac{F_0e^{i\omega t}}{\omega\hat{Z}_m}$

From an Argand diagram we know: $|\hat{Z}_m|=\sqrt{b^2+\left(\omega m-\frac{k}{\omega}\right)^2}$

And so: $\hat{x}=-i\frac{F_0e^{i\omega t}}{\omega|\hat{Z}_m|e^{i\phi}}=-i\frac{F_0e^{i(\omega t-\phi)}}{\omega|\hat{Z}_m|}$ this is the equation for the displacement with respect to time and the driving force.

The maximum displacement is given when $\omega m=\frac{k}{\omega}$ $\therefore$ $\omega=\sqrt{\frac{k}{m}}$ where ω is the angular frequency of the driving force, and so $|Z_m|=b$.

A phase difference, Φ, exists between x and the Driving Force, F_0.

There is an additional phase difference introduced by the -i term, such that even if Φ=0 then the Force would lead the displacement by 90˚.

Finding the real part of x: $\hat{x}=-i\frac{F_0}{\omega|\hat{Z}_m|}[\cos(\omega t-\phi)+i\sin(\omega t-\phi)]$

So: $\hat{x}=-i\frac{F_0}{\omega|\hat{Z}_m|}[-i\cos(\omega t-\phi)+\sin(\omega t-\phi)]$

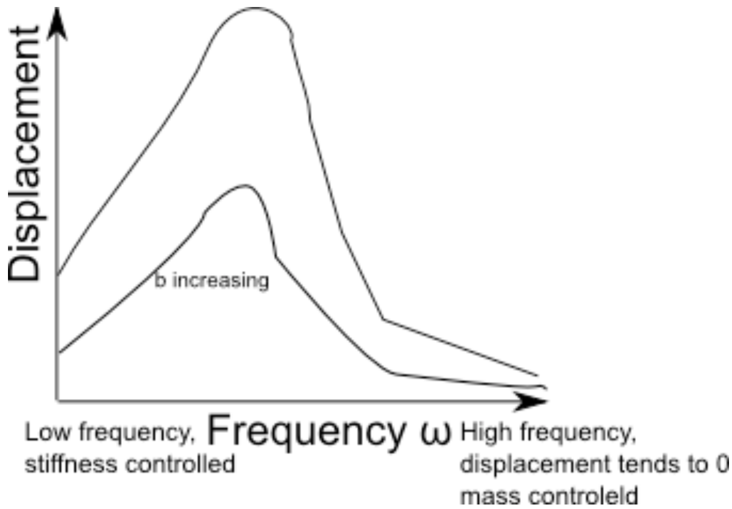
Therefore: $\Re(x)=\frac{F_0}{\omega|Z_m|}\sin(\omega t-\phi)$.

So the maximum value for the displacement is, Displacement Amplitude:

$$x_{max}=\frac{F_0}{\omega|Z_m|}$$

At low frequencies: $|Z_m| \to \frac{k}{\omega}$, so $x_{max} \to \frac{F_0}{k}$. So at low frequencies the motion is *stiffness controlled.*

At high frequencies: $|Z_m| \to \omega m$, so $x_{max} \to \frac{F_0}{\omega^2 m} \to 0$, taking ω to reasonably small values (i.e. Z_m doesn't actually reach 0) then the motion is described as *mass*



controlled.

Velocity: $x = -i\frac{F_0 e^{i(\omega t - \phi)}}{\omega|\hat{Z}_m|}$ $\therefore$ $\dot{x} = \frac{F_0 e^{i(\omega t - \phi)}}{|Z_m|}$ Note the missing i factor means that the velocity is 90° leading the displacement. The velocity is in phase with the driving force except for the phase angle Φ.

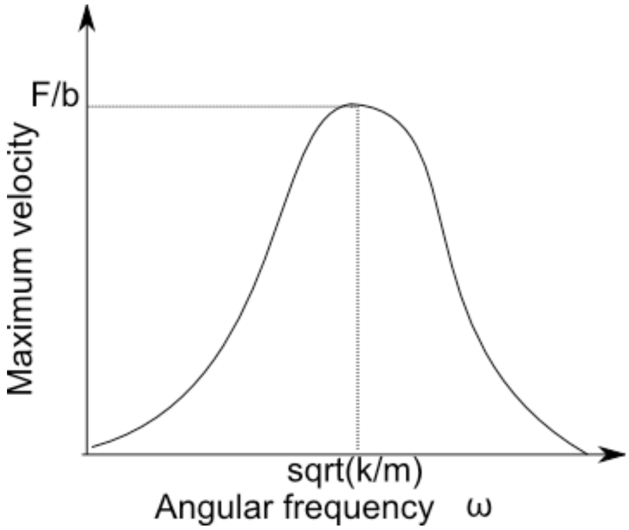
Velocity resonance occurs when velocity is a maximum, and so Z_m is a minimum. So Velocity resonance at: $\omega = \sqrt{\frac{k}{m}}$.

At low frequencies: $|Z_m| \to \frac{k}{\omega}$ so $\dot{x} \to 0$

At high frequencies: $|Z_m| \to \omega m$ so $\dot{x} \to 0$

Taking the real part of the velocity: $\Re(\dot{x}) = \frac{F_0}{|Z_m|}\cos(\omega t - \phi)$

At resonance: $A = v_{max} = \frac{F_0}{b}$ (since at some time and phase angle $\omega t = \phi$)



<u>Power:</u>

$$P=\Re(F)\times v=F_0\cos(\omega t)\times\frac{F_0}{|Z_m|}\cos(\omega t-\phi)=\frac{F_0{}^2}{|Z_m|}\cos(\omega t)\cos(\omega t-\phi)$$

Average Power: $P_{av}=\frac{\text{Total work per oscillation}}{\text{Period of oscillation}}$

$$P_{av}=\int_0^T\frac{P}{T}dt=\frac{F_0^2}{|Z_m|T}\int_0^T\cos(\omega t)\cos(\omega t-\phi)dt$$

Using: $\cos(a-b)=\cos(a)\cos(b)+\sin(a)\sin(b)$

$$P_{av}=\frac{F_0^2}{|Z_m|T}\int_0^T\cos^2(\omega t)\cos(\phi)+\cos(\omega t)\sin(\omega t)\sin(\phi)dt$$

Using: $\cos^2(x)=\frac{\cos(2\text{x})+1}{2}$, $\sin(x)\cos(x)=\sin\frac{(2\text{x})}{2}$

$$P_{av}=\frac{F_0^2}{|Z_m|T}\int_0^T\left(\left(\frac{1+\cos(2\omega t)}{2}\right)\cos(\phi)+\left(\frac{\sin(2\omega t)}{2}\right)\sin(\phi)\right)dt$$

$$P_{av}=\frac{F_0^2}{|Z_m|2\text{T}}\left[\left(t+\frac{1}{2\omega}\sin(2\omega t)\right)\cos(\phi)+\left(\frac{-1}{2\omega}\cos(2\omega t)\right)\sin(\phi)\right]_0^T$$

But $\sin(2\omega T)=\sin(0)$ and $\cos(2\omega T)=\sin(0)$ since both are periodic functions.

$$P_{av}=\frac{F_0^2}{|Z_m|2\text{T}}((T+0)-(0+0))\cos(\phi)+(0-0)\sin(\phi)=\frac{F_0^2}{|Z_m|2\text{T}}T\cos(\phi)$$

Finally: $P_{av}=\frac{F_0^2}{2|Z_m|}\cos(\phi)$

Where cos(ϕ) is the *Power Factor*.

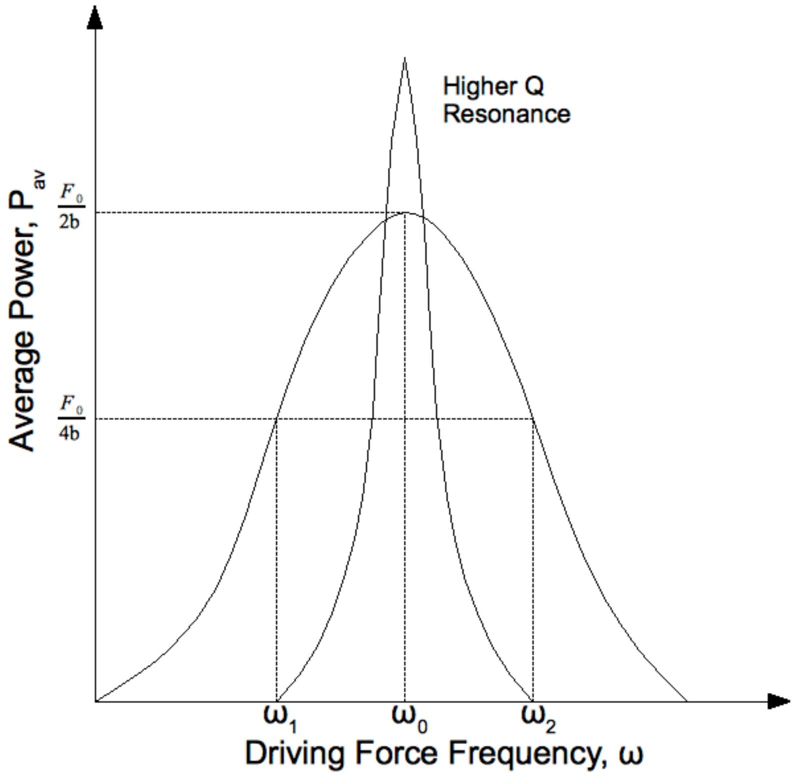
From Argand diagram of Z_m: $\cos(\phi)=\frac{b}{|Z_m|}$, so at velocity resonance: $\cos(\phi)=1$

So: $P_{av}=\frac{bF_0^2}{2|Z_m{}^2|}$ so at maximum velocity, maximum power= $P_{avmax}=\frac{F_0^2}{2|Z_m|}$

Maximum power occurs in the velocity resonance when:

- Force and velocity are in phase (i.e. $\phi=0$)
- The driving frequency, ω, equals the natural frequency of the system: $\sqrt{\frac{k}{m}}$
- Z_m is a minimum value, equal to the damping constant b

Q-factor:



The Q-value describes how sharp the resonance is, the sharper the resonance, the higher the Q-value.

The bandwidth is the band of frequency values whereby the amplitude does not drop to below $\frac{1}{2}$ of its maximum.

$Q=\frac{\omega_0}{\omega_2-\omega_1}$ where at ω_1, ω_2: $P_{av}=\frac{1}{2}P_{avmax}$

This simplifies to: $Q=\frac{\omega_0 m}{b}$

Electronics and AC Circuits:

Voltage: AC voltages cause currents to flow in circuits, however due to the nature of the components there may be a phase difference between the voltage and the current.

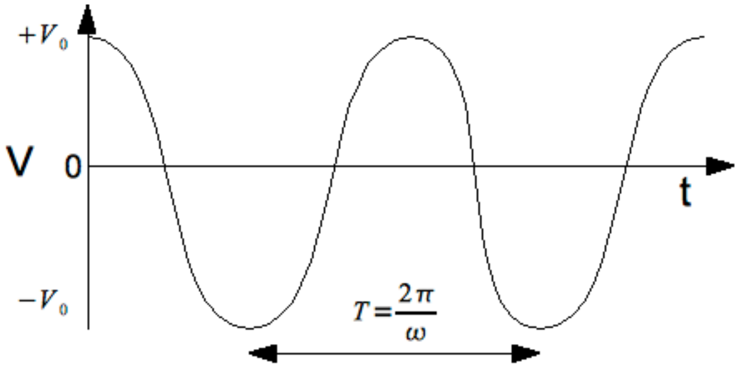
Current: To calculate the current flowing in circuits, we use a version of Ohm's Law. By using exponential notation, we can see the effect of the phase difference.

Resistor: A resistor in an AC circuit, works the same way as for a DC circuit. It's impedance is real-only.

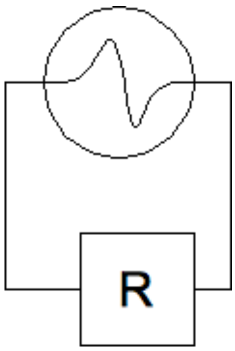
Capacitor: The frequency of driving voltage affects the size of the current (and its phase) flowing through the capacitor. Its impedance is Complex (Imaginary).

Inductor: The frequency of driving voltage also affects the current, and its phase, through an inductor. Its impedance is Complex (Imaginary).

$$V = V_0 \cos(\omega t)$$
$$V = V_0 e^{-j\omega t}$$

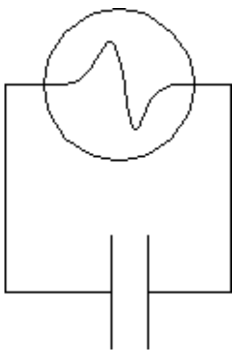


Resistance:



The generating voltage is: $V = V_0 e^{j\omega t}$
Ohm's Law is obeyed: $V = IR$

Capacitance:

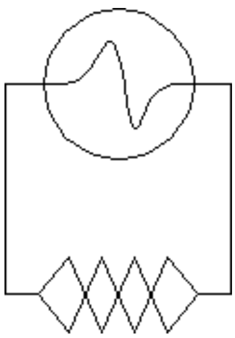


The charge stored depends upon the value of C and the voltage across it. $q = CV \;\therefore\; \frac{dq}{dt} = j\omega C V_0 e^{j\omega t}$

$$I = C \cdot \frac{dV}{dt} \;\therefore\; I = C\, j\omega V \;\therefore\; V = I \cdot \frac{1}{C\, j\omega} \;\therefore\; V = I Z_c$$

where $\frac{1}{C\, j\omega} = Z_c$ is the impedance of the capacitor.

Inductance:



An inductor is a device that opposes the changes in current. This opposition is frequency dependent.

$$V = L \frac{dI}{dt}$$

The driving voltage is $\hat{V} = V_0 e^{j\omega t} = V_0(\cos(\omega t) + jsin(\omega t))$

Resistor

We can write $\hat{I}=\frac{\hat{V}}{R}=V_0\frac{e^{j\omega t}}{R}=I_0 e^{j\omega t}$

so the current is $I_0 e^{j\omega t}$ and the voltage is $V_0 e^{j\omega t}$ so they are in phase.

Capacitance

$$\hat{I}=C\cdot\frac{dV_0 e^{j\omega t}}{dt}=j\omega C V_0 e^{j\omega t}=j\omega C\hat{V}$$

So expressing the equation with voltage as the subject: $\hat{V}=-j\frac{1}{\omega C}\hat{I}$ the -j is a phase term, whilst the 1/ωC is the impedance term.

Inductance

Start with $V=L\frac{dI}{dt}$ so integrate w.r.t. Time

$$\int\hat{v}\,dt=L\int dI$$

$$V_0\int e^{j\omega t}dt=L\int dI$$

$$\frac{-jV_0}{\omega}e^{j\omega t}=LI$$

$$V_0 e^{j\omega t}=j\omega L\hat{I}$$

$\hat{V}=j\omega L\hat{I}$ j is the phase term so V is 90 degrees ahead.

In summary the equations are:

Resistor: $V_R=IR$

Capacitor: $V_C=-j\frac{1}{\omega C}I$

Inductor: $V_L=j\omega L I$

These all have the form $V=IZ$ Z is the impedance.

$$Z_R=R$$

$$Z_C=-j\frac{1}{\omega C}$$

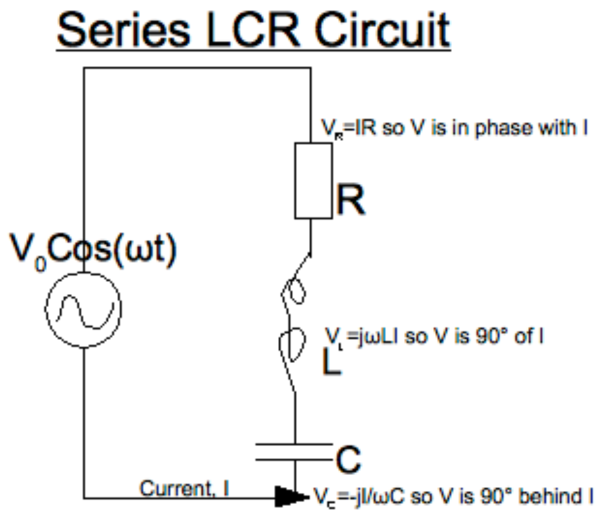
$$Z_L=J\omega L$$

The real part of the complex impedance is the resistance part and the imaginary part is the reactance part.

So the respective reactances are:

X_C is the capacitive resistance i.e. $\frac{-1}{\omega C}$

X_L is the inductive reactance i.e. ωL



In a series circuit the applied driving voltage is divided between the components:
$V=V_R+V_L+V_C$ and $Z_{total}=Z_R+Z_L+Z_C$
Applying Kirchoff's Laws to this AC circuit therefore the same current flows through all components. So by substitution:
$$V=(RI)+(j\omega LI)-(j\frac{1}{\omega C}I)=Z_{tot}I$$
Simplifying:
$$\hat{Z}\cdot\hat{I}=(R+j\omega L-j\frac{1}{\omega C})\cdot\hat{I}$$
Therefore the expression for the impedance: $\hat{Z}=(R+j\omega L-j\frac{1}{\omega C})$

Therefore the electrical impedance in a series LCR circuit: $\hat{Z}=R+j(\omega L-\frac{1}{\omega C})$

We can compare this to the expression for the mechanical impedance:
$\hat{Z}_m=b+j(\omega m-\frac{k}{\omega})$, i.e. L is analogous to m, and 1/C is analogous to k, and b is analogous to R.

In series: $Z_{total}=Z_R+Z_L+Z_C$
In parallel: $\frac{1}{Z_{total}}=\frac{1}{Z_R}+\frac{1}{Z_L}+\frac{1}{Z_C}$

Power in AC Circuits:

In a mechanical system: $P_{av}=\frac{F_0^2}{2|Z_m|}\cos(\phi)$

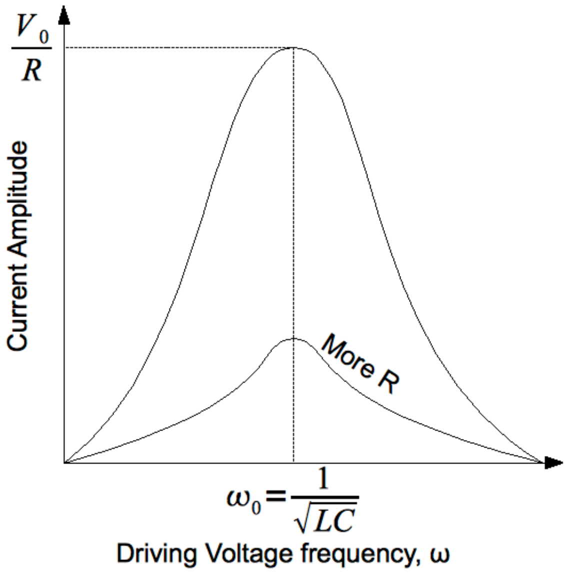
In an electrical system it is defined the same way: $P_{av}=\frac{V_0^2}{2|Z_e|}\cos(\phi)$

But since $\frac{V}{|Z_e|}=I$ so: $P_{av}=\frac{V_0 I_0}{2}\cos(\phi)$
and: $P_{av}=V_{RMS}I_{RMS}\cos(\phi)$

Resonance in AC LCR series Circuits:

$I=\frac{V}{|Z_e|}$ then I is a maximum when Z is a minimum, so if $\hat{Z}=R+j(\omega L-\frac{1}{\omega C})$

Then the Z is a minimum when $\omega L=\frac{1}{\omega C}$ *so* $\omega=\frac{1}{\sqrt{LC}}$ this is the resonant frequency ω_0.



Q-Factor:

$Q=\frac{\omega_0}{\omega_2-\omega_1}$ same definition as for mechanical systems.

For mechanical system: $Q=\frac{\omega_0 m}{b}$ using the analogous relations:

$k\Rightarrow\frac{1}{C}$ $m\Rightarrow L$ $b\Rightarrow R$ and so $Q=\frac{\omega_0 L}{R}$

Problem-solving:

To find the phase difference between the driving voltage and the current in a circuit, find an expression for the total electrical impedance of the circuit, and then calculate the angle ϕ as when plotted on an Argand diagram.

When determining whether the voltage lags or leads the current, use the fact that in a resistor, V is in phase with I, in an inductor, V is 90º ahead of I, and in a capacitor, V is 90º behind I.

When considering the RMS potential differences in each element of a circuit, remember not to take the real part of the voltage until the end. (i.e. multiply the current and the impedance in their complex forms, then take the real part, you do **NOT** just multiply the moduli together).

Waves:

Waves vs. Oscillations:

An Oscillation is a disturbance that varies with time, such as mass-spring system.

A wave is a time-varying disturbance that propagates in space. e.g. ripples on a pond. Waves transfer energy from one place to another, but without any permanent transfer of the medium. This is an important distinction between waves and oscillations.

Oscillation equation: $A(t)=A_0 \sin(\omega t)=A_0 e^{j\omega t}$
Wave equation for a point on the wave: $A(x,t)=A_0 \sin(\omega t-kx)=A_0 e^{j(\omega t-kx)}$

where k is the wavevector, $k=\dfrac{2\pi}{\lambda}$

A **Longitudinal Wave** (such as sound in a gas) consists of a series of compressions and rarefactions. The motion of the medium is parallel (i.e. in the same direction) to the direction of wave propagation. A(x,t) describes the displacement of the molecules at (x,t) (or pressure).

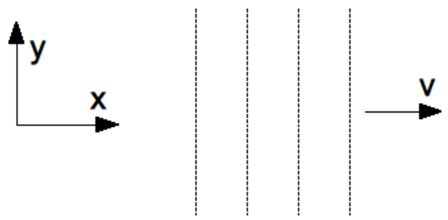
A **Transverse Wave** (such as a wave on a string) has the motion of the medium perpendicular to the direction of wave propagation.

It is important to note that the speed of wave propagation is **not** the same as the speed of the individual particles in the medium.

Sea waves are a combination of both longitudinal and transverse motions. The particles travel in clockwise circles, the radius of which decreases as the depth of the water increases.

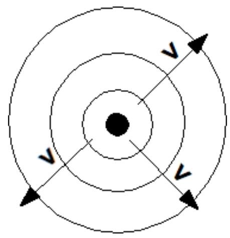
For a transverse wave there are two possible linear polarizations which are directly perpendicular to the wave propagation.

Plane Wave



In a plane wave the surface of constant disturbance is a plane, i.e. peaks and troughs form parallel planes.

Circular wave



Surface of constant phase (displacement) is a sphere, with the wave source at the centre.

An important thing to note about wave equations is the direction of the wave. A wave with the equation $A(x,t)=A_0 \sin(\omega t-kx)$ will be travelling in the **positive x** direction. Whereas a wave with the equation $A(x,t)=A_0 \sin(\omega t+kx)$ will travel in the $-\hat{x}$ direction.

The displacement of a particle at x=0 is given by $A=A_0 \sin(\omega t)$, the motion at any x position is the same as the motion at x=0 at the earlier time of $t-\frac{kx}{\omega}$.

Phase velocity:
Consider the motion of a wave crest, the point has a specific phase φ, $\phi=\omega t-kx$.
The point travels a distance dx in a time dt.
For φ is constant, differentiating with respect to t gives:
$\frac{d\phi}{dt}=\frac{d}{dt}(\omega t-kx) \quad \therefore 0=\omega-k\frac{dx}{dt} \quad \therefore \frac{dx}{dt}=\frac{\omega}{k}$ so the phase velocity, v_{phase}, is $\frac{\omega}{k}$, this is the same as $v_{phase}=f\lambda$ by substituting in the definition of the wavevector and angular frequency.

So for a wave travelling in the x-direction, the phase velocity: $\vec{\boldsymbol{v}}_{phase}=\frac{\omega}{k}\hat{\boldsymbol{x}}=f\lambda\hat{\boldsymbol{x}}$.

Particle Velocity:
Remember that the phase velocity (the velocity of the wave propagation) and the particle velocity (the velocity of individual particles) are **not the same**.
Considering a transverse wave: $y(x,t)=y_0\sin(\omega t-kx)$, x is constant so the particle velocity is given by $\frac{dy}{dt}=v_{particle}=\omega y_0\cos(\omega t-kx)$.

So the amplitude of the particle velocity is given by ωy_0 this is **different** to the phase velocity, $v_{phase}=\frac{\omega}{k}$.

The Wave Equation:
For simple harmonic oscillators the equation of motion was of the form:

$$\frac{d^2x}{dt^2}=-\omega^2x$$

Waves are coupled oscillators, so the wave differential equation will involve both time and space, so it will be a partial differential equation.

In One Dimension:

$$\frac{\partial^2 A}{\partial x^2}=\frac{1}{v_{ph}^2}\frac{\partial^2 A}{\partial t^2}$$

In Three Dimensions:

$$\nabla^2 A=\frac{1}{v_{ph}^2}\frac{\partial^2 A}{\partial t^2}$$

$$\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\frac{\partial^2 A}{\partial z^2}=\frac{1}{v_{ph}^2}\frac{\partial^2 A}{\partial t^2}$$

Solutions to the wave equation:
Any function of the form $A(x,t)=f(vt\pm x)$ is a solution.

Let $vt\pm x=s$ so $A(x,t)=f(s)$

Differentiating with respect to x:

$$\frac{\partial A}{\partial x}=\frac{df}{ds}\cdot\frac{\partial s}{\partial x}=\pm\frac{df}{ds}$$

and $\dfrac{\partial^2 A}{\partial x^2} = \pm \dfrac{d^2 f}{ds^2} \cdot \dfrac{\partial s}{\partial x} = \pm \dfrac{d^2 f}{ds^2}$

Differentiating with respect to t:

$$\frac{\partial A}{\partial t} = \frac{df}{ds} \cdot \frac{\partial s}{\partial t} = \frac{df}{ds} v$$

and $\dfrac{\partial^2 A}{\partial t^2} = v \dfrac{d^2 f}{ds^2} \cdot \dfrac{\partial s}{\partial t} = v^2 \dfrac{d^2 f}{ds^2}$

And so: $\dfrac{\partial^2 A}{\partial x^2} = \dfrac{1}{v^2} \dfrac{\partial^2 A}{\partial t^2}$

The function f can be any function as long as x and t appearing it as vt±x.

E.g. the following harmonic solutions are valid:

$A = A_0 \sin(k(vt \pm x)) = A_0 \sin(\omega t \pm kx)$ and $A = A_0 \cos(\omega t \pm kx)$

with $v = \dfrac{\omega}{k}$ are special cases from which we see that v is the phase velocity.

<u>Waves on a string:</u>

It is a transverse wave, we assume the tension T is the same everywhere, that the oscillations are small (and so θ can be approximated as a small angle), gravity is neglected and the string has a uniform linear density $\rho = \dfrac{dm}{dl} = constant$.

Length of a string between x and x+dx is:

$dl = \sqrt{dx^2 + dy^2} = dx \sqrt{1 \left(\dfrac{dy}{dx}\right)^2}$ so $dl \approx dx$ for small oscillations.

So the mass of a single string element is $\rho\, dx$.

Considering the net force on the element in the y-direction:

$$F = T\sin(\theta + d\theta) - T\sin(\theta) \approx T[\tan(\theta + d\theta) - \tan(\theta)]$$

since for small angles, $\sin(\theta) \approx \tan(\theta) \left(= \dfrac{dy}{dx}\right)$

And so $Force = T\left(\dfrac{dy}{dx}_{x+dx} - \dfrac{dy}{dx}_{x}\right) = T \dfrac{d^2 y}{dx^2} dx$

Knowing the force and the mass, the equation of motion is given by:

$$T \frac{\partial^2 y}{\partial x^2} dx = \rho\, dx \frac{\partial^2 y}{\partial t^2}$$

and so: $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\rho}{T} \dfrac{\partial^2 y}{\partial t^2}$

Comparing this to the general wave equation: $\dfrac{\partial^2 A}{\partial x^2} = \dfrac{1}{v^2} \dfrac{\partial^2 A}{dt^2}$

Solving for the phase velocity: $\frac{\partial t^2}{\partial x^2}=\frac{T}{\rho}\quad \therefore\ v^2=\frac{T}{\rho}\quad \therefore v=\sqrt{\frac{T}{\rho}}$

The wave equation $\frac{\partial^2 y}{\partial x^2}=\frac{\rho}{T}\frac{\partial^2 y}{\partial t^2}$ is typical for mechanical waves.

The phase velocity is always of the form: $v_{ph}=\sqrt{\frac{\text{Elasticity}}{\text{Inertia of medium}}}\quad \left(=\sqrt{\frac{T}{\rho}}\text{ for the string.}\right)$

<u>Energy transfer in the waves on a string:</u>
Energy = Force X Distance and Power = Force X Velocity. $y(x,t)=y_0\sin(\omega t-kx)$

$P(x,t)=-T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}=T\,\omega\,k\,y_0^{\,2}\cos^2(\omega t-kx)$ This is the instantaneous power.

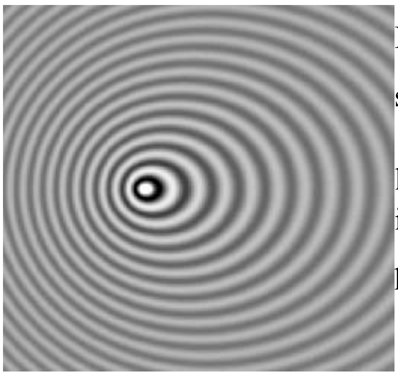
As long as k is positive, then P is always positive (i.e. energy will flow from left to right).

The time-average of P is given by $P_{av}=\frac{1}{2}T\,\omega\,k\,y_0^{\,2}$ because the time average of $\cos^2(x)=\frac{1}{2}$.

<u>Doppler Shift:</u>

Doppler shift: $f'=f\left(\frac{c-v_{observer}}{c-v_{source}}\right)$ when observer and source move in the same direction.

<u>Derivation:</u>



If the source velocity is v_s then in the time $\tau_0=\frac{1}{f_0}$ between successive crests the source will have moved to the left a distance $v_s\tau_0$ whilst the previously emitted crest will have moved to the left a distance λ and so the actual distance between the two crests is $\lambda'=\lambda-v_s\tau_0$ and so an observer to the left of the source will hear a frequency of $f'=\frac{c}{\lambda'}$

Therefore the observed frequency for a stationary observer with an *approaching* source is

$$f'=\frac{c}{\lambda'}=\frac{c}{\lambda-v_s\tau_0}=\frac{c}{\lambda}\left(\frac{1}{1-v_s\frac{\tau_0}{\lambda}}\right)=f_0\left(\frac{1}{1-\frac{v_s}{c}}\right)$$

And by a parallel argument the observed frequency for a stationary observer with a *receding* source is $f'=f_0\left(\frac{1}{1+\frac{v_s}{c}}\right)$ i.e. flip the sign of v_s

Now consider a stationary source and a moving observer, if the observer moves with a speed v_{obs} towards a stationary frequency f_0 source then they will meet the oncoming wave crests. The wave crests are λ apart in the air and moving at a speed *c*. If the time between the observer meeting successive crests is τ' then during this time the observer travels a distance $v_{obs}\tau'$ whilst the crest travels a distance $c\tau'$ to meet the observer and these two distances combined make up the distance λ between the crests. Therefore $\lambda = c\tau' + v_{obs}\tau' = \tau'(c+v_{obs})$ and therefore $\tau' = \frac{\lambda}{(c+v_{obs})}$ and so the sound frequency measured by the observer is given by:

$$f' = \frac{1}{\tau'} = \frac{v_{obs}+c}{\lambda} = \frac{c}{\lambda}\left(1+\frac{v_{obs}}{c}\right) = f_0\left(1+\frac{v_{obs}}{c}\right)$$

If both the source and observer are moving towards one another we can combine the above arguments to obtain:

$$f' = f_0\left(\frac{1+\frac{v_{obs}}{c}}{1-\frac{v_s}{c}}\right)$$

Both motions act to increase the observed frequency. If either observer or source is moving in the opposite direction then we can find the observed frequency by switching the sign of the corresponding *v*.

For a source approaching stationary observer: $\lambda' = (c-v_s)T \quad \therefore\ f' = \frac{c}{(c-v_s)T} = \frac{c}{(c-v_s)} f_{source}$

For a source receding from stationary observer: $\lambda' = (c+v_s)T \quad \therefore\ f' = \frac{c}{(c+v_s)T} = \frac{c}{(c+v_s)} f_{source}$

Wave superposition: Standing Waves

If we add two waves moving in opposite directions with equal amplitudes then we get a standing wave i.e.: $y = A_{tot}(x,t) = A\sin(\omega t - kx) + A\sin(\omega t + kx) = 2\mathrm{A}\sin(\omega t)\cos(kx)$ This standing wave has amplitude $2\mathrm{A}_0 \sin\omega t$ and therefore it changes with time. The $\cos kx$ part means that the amplitude changes with position.

Nodes are points of no disturbance at any time, these occur when cos(kx) = 0 and hence when $kx = (n+\frac{1}{2})\pi$

Anti-nodes have a maximum disturbance equal to the sum of the amplitudes i.e. cos(kx) = 1 and hence occur when $kx = n\pi$

Energy transfer in standing wave

Remember: $P_{inst}(x,t) = -T\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$ so for the standing wave we obtain that:

$$P_{inst} = T\omega k A^2 \sin(2\omega t)\sin(2\mathrm{kx})$$

To get the average power we need to take the time average of the time dependent function i.e. $\sin(2\omega t)$ this has a time-average of zero and therefore the **standing wave transmits no energy** (i.e. the energy carried by the two travelling waves cancel)

Normal modes of a stretched string

The analysis that follows is for vibrations of a string with fixed ends, since the ends are fixed, the boundary conditions for this problem are that the wave displacement is zero at x=0 and x=L where L is the string's length.

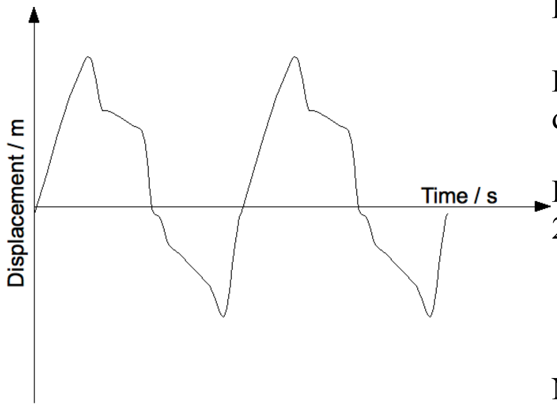
So; Nodes at x=0 and L imply standing waves therefore the equation for the wave is something like:
$\Psi(x,t)=2\mathrm{A}\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\omega t\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}kx$ i.e. there are four different possible wave equations depending on whether sin or cos is used. But due the node at x=0, the solution must use sin(kx) therefore we are restricted to a choice between $\Psi(x,t)=2\mathrm{A}\sin\omega t\sin kx$ or $\Psi(x,t)=2\mathrm{A}\cos\omega t\sin kx$.
Soft boundaries use cos, hard boundaries use sin.

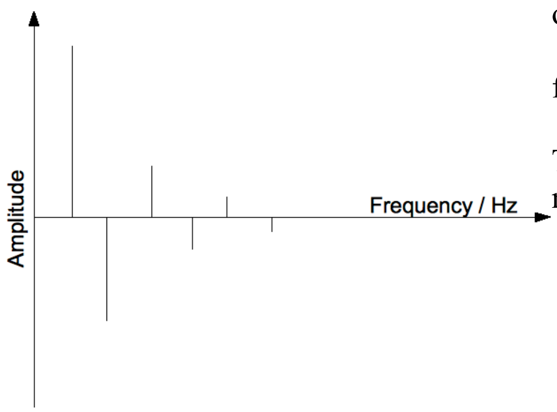
To satisfy the boundary condition at x=L, we find that k can only take certain values.
$$\lambda=\frac{2\mathrm{L}}{n}=\frac{2\pi}{k}\quad k=\frac{n\pi}{L}$$
Taking an example of a standing wave:
$$\Phi(x,t)=2\mathrm{A}\sin(\omega_n t)\sin\left(\frac{n\pi}{L}x\right)$$

This is actually a series of functions with different n values, called the normal modes, or eigenfunctions of the system. The eigenmodes are the stable states of the system (if the system is set vibrating in these modes it will continue for eternity if undissipated and unperturbed).

Any other state can be written as a linear combination of the eigenmodes. i.e. any function can be made up of a Fourier series of the eigenfunctions.

Fourier Series:





Both descriptions contain exactly the same information.

Fourier Analysis transforms between the two descriptions.

For a function f(x) in the interval (-L, L0 that has period 2L.
$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L}+b_n\sin\frac{n\pi x}{L}\right)$$

Note that odd functions only need sin, even functions only need cos. A general function needs both.

f(x) is periodic.

The initial term provides a constant vertical offset if necessary.

Wave Packets:

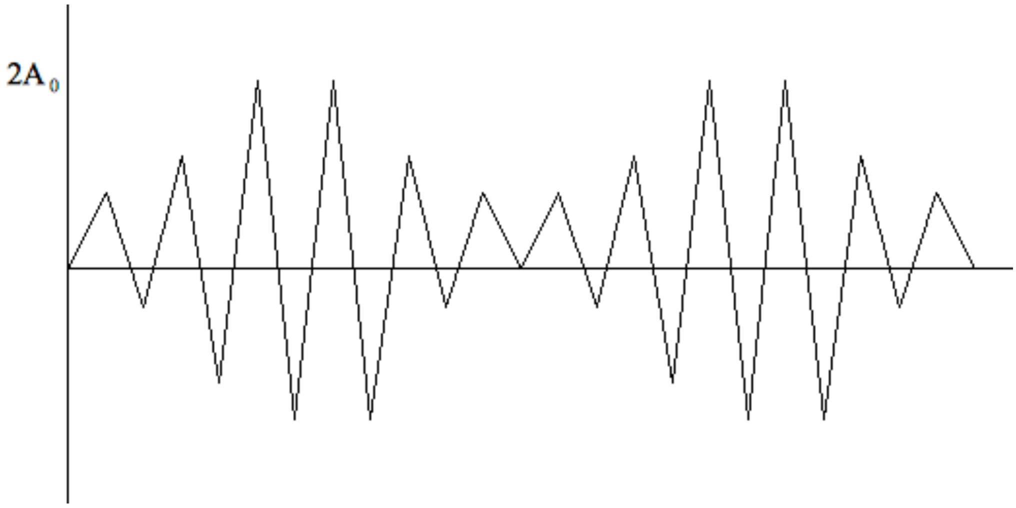
In real situations, waves travel as packets since they are made up of many different waves with different values for ω and k.

Superpose two waves: $A_1 = A_0 \cos(\omega t - kx)$ and $A_2 = A_0 \cos[(\omega + \Delta\omega)t - (k + \Delta k)x]$

$$A_1 + A_2 = 2\mathrm{A}_0 \cos\left[\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right]\cos\left[\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)k\right]$$

(Using the identity $\cos(A) + \cos(B) = 2\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$

Diagram:



The velocity of a specific point, in a wave packet, is called the phase velocity.

$$v_{phase} = \frac{\omega}{k}$$

The velocity of the whole wave, or the enveloping function, is called the group velocity.

$$v_{group} = \frac{d\omega}{dk}$$

The group velocity may be equal to, less than, or greater than the phase velocity. The group velocity is the velocity of energy/information transfer.

The dispersion relation describes the dependance of the angular frequency, ω, on the wavevector, k.

The group velocity is equal to the phase velocity only if there angular frequency is independent of the wavevector. These are called non-dispersive waves.

The dispersion is called normal when $v_{group} < v_{phase}$ and anomalous when $v_{group} > v_{phase}$, there is no dispersion when $v_{group} = v_{phase}$.

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(k v_p)\text{ , since } \omega = v_p k \text{ , } v_g = \frac{d}{dk}(k v_p) = v_p + k\frac{d v_p}{dk}$$ substituting $k = \frac{2\pi}{\lambda}$

and using the chain rule $\frac{d v_p}{dk} = \frac{d v_p}{d\lambda}\frac{d\lambda}{dk}$

so $\frac{d\lambda}{dk}=\frac{d}{dk}\left(2\pi k^{-1}\right)=-2\pi k^{-2}=\frac{-2\pi}{k}\frac{1}{k}=\frac{-\lambda}{k}$ so: $v_g=v_p-\lambda\frac{d v_p}{dk}$.

Dispersion originates from the fact that when light is incident on a material, such as a dielectric like glass, its electric field interacts with the charges of the constituent atoms. The nature of these interactions depends on the polarisability of the material, $\vec{\boldsymbol{p}}=\chi_e\epsilon_0\vec{\boldsymbol{E}}$ where $\chi_e=susceptibility=\epsilon_r-1$.

Polarisability – *Relative tendency of a charge distribution, like the electron cloud of an atom or molecule, to be distorted from its normal shape by an external electric field.*

The molecules can be modelled as mechanical oscillators, whereby the electrons are connected to the nucleus by springs. All the springs are identical and undergo damping.

$$Restoring\ Force,\ F_R=-kx=-m\omega^2 x,\ since\ resonant\ frequency,\ \omega_0=\sqrt{\frac{k}{m}}$$

$$Driving\ Force,\ F_D=q_e E\quad,\quad Damping\ Force,\ F_{damping}=-m\gamma v$$

So the resulting equation of motion: $q_e E=m\ddot{x}+\gamma m\dot{x}+m\omega^2 x$

So if there is no E-field and no damping (E=0, γ=0): $\ddot{x}=-\omega_0^2 x$

So using the complex representation for E: $E=E_0 e^{j\omega t}$

and the forced oscillation solution for displacement: $x=x_0 e^{j\omega t}$

where $x_0=\frac{q_e E}{m(\omega_0^2-\omega^2-j\gamma\omega)}$

This leads to a polarisability for the material that depends on the frequency, *ω*: $p=q_e x$

And a refractive index, *n*, which also depends on the frequency, *ω:* $n^2=\left[1+\frac{P}{\epsilon_o E}\right]$

so phase velocity depends on *ω* and *λ*: $v_{phase}=\frac{c}{n}$ so $n=\frac{c}{v_p}$

so $\frac{d v_p}{d\lambda}$ is the origin of dispersion.

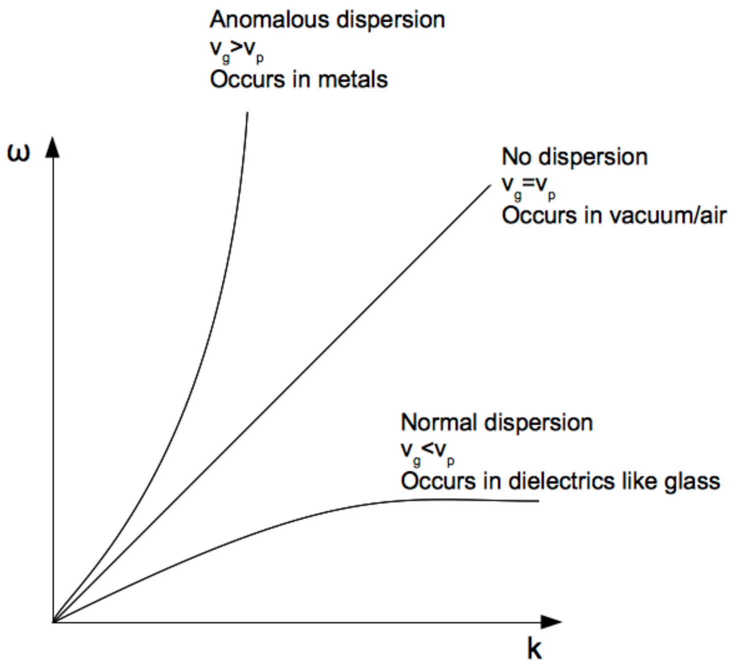
Normal dispersion: v_p increases with increasing *λ*, $\frac{d v_p}{d\lambda}>0$, $v_g<v_p$.

Anomalous dispersion: v_p decreases with increasing *λ*, $\frac{d v_p}{d\lambda}<0$, $v_g>v_p$.

No dispersion: v_p does not depend on *λ*, $\frac{d v_p}{d\lambda}=0$, $v_g=v_p$.

The dispersion relation (the relation between *ω* and *k*) has the same role as the energy-momentum relationship for particles.

Dispersion plot:



Anomalous dispersion is typical of systems with damping of friction. Normal dispersion is typical for light travelling in dielectric media.

Snell's Law: *Amount of light that bends in a prism is proportional to the refractive index, n, and* $n=\frac{c}{v}$ .Blue light travels slower than red and so bends more.

Wave packets travelling in dispersive media generally change shape as they propagate. Components of different frequency travel at different speeds and this leads to distortion in optical fibres, etc. - so the wavelengths are chosen such that the dispersion is 0.

Reflection and Transmission of waves:

Characteristic Impedance, z – Property of the medium, describes how hard it is to set up the wave in the medium.

For all mechanical waves: $z=\frac{driving\ \ force}{appropriate\ \ particle\ \ medium\ \ velocity}$ i.e. $z=\frac{F}{v_p}$ for transverse waves.

So for transverse waves on a string, considering the wave: $y=y_0 e^{j(\omega t-kx)}$.

From wave equation: $z=\frac{transverse\ driving\ \ force}{transerse\ \ velocity}=\frac{-T\frac{dy}{dx}}{\frac{dy}{dt}}$ so substituting expressions:

$$z=\frac{jTky_0 e^{j(\omega t-kx)}}{j\omega y_0 e^{j(\omega t-kx)}}=\frac{Tk}{\omega}=\frac{T}{v_{ph}}=\frac{T}{\sqrt{\frac{T}{\rho}}}=\sqrt{T\rho}$$

Calculating the energy flow in terms of z:

$$P_{av}=\frac{1}{2}T\omega k\, y_0^2=\frac{1}{2}\frac{t}{v_p}v_p\omega k\, y_0^2=\frac{1}{2}z\omega^2 y_0^2 \quad \text{using} \quad z=\sqrt{T\rho}=\frac{Tk}{\omega} \quad \text{and} \quad \omega=v_p k$$

so $P_{av}=\frac{1}{2}z\omega^2 y_0^2$ in terms of z.

Reflection and Transmission at a boundary:

One side of boundary has density ρ_1, other has density ρ_2.

Incident wave: $A_1 e^{j(\omega t-k_1 x)}$, Transmitted wave: $A_2 e^{j(\omega_2 t-k_2 x)}$, reflected wave: $Be^{j(\omega_1 t+k_1 x)}$.

Amplitude Reflection Coefficient: $R=\frac{Amplitude\ of\ reflected\ wave}{Amplitude\ of\ incident\ wave}=\frac{B}{A_1}$

Amplitude Transmission Coefficient: $T=\frac{Amplitude\ of\ transmitted\ wave}{Amplitude\ of\ incident\ wave}=\frac{A_2}{A_1}$

Calculating R and T in terms of z:

Applying the boundary conditions at x=0:

1) **The wave function must be continuous at all times** or the string would break
2) **The first derivative with respect to x of the wave function must be continuous** (since the transverse force is proportional to this derivative, and if the force were not continuous then there would be a finite net force on an infinitesimal segment of the string at x=0, which would cause infinite acceleration).

Applying condition 1 at x=0:

$A_1 e^{j(\omega_1 t)}+B e^{j(\omega_1 t)}=A_2 e^{j(\omega_2 t)}$, this can only be true at all times if $\omega_1=\omega_2=\omega$ and so: $A_1+B=A_2$.

Using condition 2: $\frac{dy_I}{dx}+\frac{dy_r}{dx}=\frac{dy_t}{dx}$ at x=0, at all times:

$A_1 jke^{j\omega t}-Bjk_1 e^{j\omega t}=A_2 jk_2 e^{j\omega t}$ and so: $(A_1-B)k_1=A_2 k_2$

but $z=\frac{Tk}{\omega}$ so for fixed ω and T, then z is proportional to k, And so the second boundary condition becomes: $(A_1-B)z_1=A_2 z_2$

So solving the simultaneous equations $A_1+B=A_2$ and $(A_1-B)z_1=A_2 z_2$: Using that $R=\frac{B}{A_1}$ and $T=\frac{A_2}{A_1}$: Solving for R:

$\frac{(A_1-B)z_1}{A_1+B}=z_2 \therefore (A_1-B)z_1=(A_1+B)z_2(A_1-B)z_1 \therefore A_1 z_1-Bz_1=A_1 z_2+B z_2$

$$z_1-Rz_1=z_2+R z_2 \therefore R=\frac{z_1-z_2}{z_1+z_2}$$

Solving for T: $A_1+B+A_2 \therefore 1+\frac{B}{A_1}=\frac{A_2}{A_1} \therefore T=1+R=\frac{2z_1}{z_1+z_2}$

Case 1: Wave goes from less dense to more dense region, so $z_1<z_2$, R is negative so the reflected wave is upside down compared to the incident wave. The transmitted amplitude is less than the reflected amplitude since T<1.

Case 2: Wave goes from more dense region to less dense region, so $z_1>z_2$. R is positive, so reflected wave is similar to incident wave (but smaller), transmitted amplitude is greater than the incident amplitude since T>1. Note this doesn't violate the conservation of energy since the greater amplitude is produced in a less dense material and so has the same kinetic energy.

Case 3: String rigidly fixed at x=0. $z_2\to\infty$. R=-1, T=0, the wave is reflected back exactly as it was incident, except upside-down.

Case 4: String has free end at x=0. $z_2 \to 0$. R=1, T=2. The reflected pulse is exactly the same as the incident pulse. No real transmission occurs but the free end oscillates up to 2 times the incident amplitude.

Transmitted and Reflected Power:

$T = \frac{2z_1}{z_1 + z_2}$ and $R = \frac{z_1 - z_2}{z_1 + z_2}$ are the **Amplitude** transmission and reflection coefficients.

The rate of energy flow in a wave is related to the amplitude by: $P = \frac{1}{2} z \omega^2 y_0^2$

The power coefficients are defined by: $T_p = \frac{transmitted\ \ power}{incident\ \ power}$ and $R_p = \frac{reflected\ \ power}{incident\ \ power}$

$$R_p = \frac{reflected\ \ power}{incident\ \ power} = \frac{\frac{1}{2} z_1 \omega^2 B^2}{\frac{1}{2} z_1 \omega^2 A_1^2} = \frac{B^2}{A_1^2} = \left(\frac{B}{A_1}\right)^2 = R^2 \quad \text{and}$$

$$T_p = \frac{transmitted\ \ power}{incident\ \ power} = \frac{\frac{1}{2} z_2 \omega^2 A_2^2}{\frac{1}{2} z_1 \omega^2 A_1^2} = \frac{z_2}{z_1}\left(\frac{A_2}{A_1}\right)^2 = \frac{z_2}{z_1} T^2 = \frac{4 z_1 z_2}{(z_1 + z_2)^2}$$

Note that the conservation of energy is held: $R_p + T_p = 1$

Impedance Matching:

The condition for maximum transmitted power (and so no reflection) is $z_1 = z_2$ so R=0.
In order to get complete transmission between two media of different impedances (useful for glasses, etc.), a trick called impedance matching is employed.

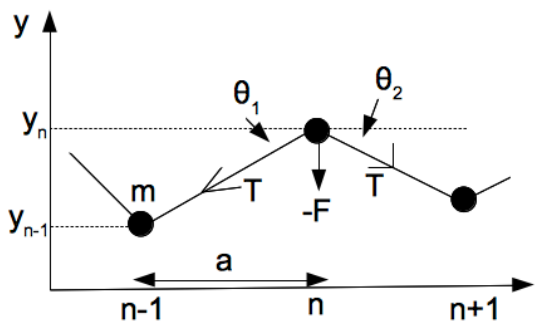
The quarter-wave transformer:

A wave of wavelength λ must pass from a medium of z_1 to a medium of z_3 with no reflected power.
To do this we give the middle material an impedance of $z_2 = \sqrt{z_1 z_3}$, with a width of $\frac{\lambda_2}{4}$.

The width of $\frac{\lambda_2}{4}$ ensures that the two reflections are exactly out of phase, and so annihilate eachother perfectly.

The equation of motion of a string of finite elements:

Consider an array of mass m particles attached to a string with tension *T,* with the particles spaced a distance *a* apart.



This is analogous to a crystal – the atoms are regularly spaced and interacting via interatomic forces rather than string.

Setting up the equation of motion:

$F = -(T\sin(\theta_1) + T\sin(\theta_2)) \approx -(T\tan(\theta_1) + T\tan(\theta_2))$ as θ is small and $\tan(\theta) = \frac{dy}{dx}$.

$$F=-T\left(\frac{y_n-y_{n-1}}{a}+\frac{y_n-y_{n+1}}{a}\right)$$

Newton's 2nd Law: $ma=F_{net}$ so $\frac{md^2y_n}{dt}=\frac{T}{a}(y_{n-1}-2\mathrm{y}_n=y_{n+1})$ so similar to an oscillator in that $acceleration \propto displacement$.

Note the acceleration of the *n*th particle depends on both the displacement of that particle and it's 2 nearest neighbours.

Coupled harmonic oscillators become waves, $y_n=Ae^{j(\omega t-kx)}=Ae^{j(\omega t-kna)}$
Origin of x is take to be at the zeroth particle, so the *n*th particle is at x=na.
Adjacent particles oscillate with the same frequency but are successively shifted in phase by -ka.

Remember: $y_n=Ae^{j(\omega t-kna)}$, $\frac{md^2y_n}{dt}=\frac{T}{a}(y_{n-1}-2\mathrm{y}_n=y_{n+1})$
Differentiating and substituting back in to equation of motion:

$$-m\omega^2y_n=\frac{T}{a}\left(Ae^{j(\omega t-k(n-1)a)}-2\mathrm{Ae}^{j(\omega t-kna)}+Ae^{j(\omega t-k(n+1)a)}\right)$$

$$-m\omega^2y_n=\frac{T}{a}Ae^{j(\omega t-kna)}\left(e^{jka}-2+e^{-jka}\right)\quad\therefore\quad -m\omega^2=\frac{2\mathrm{T}}{a}\left(\frac{e^{jka}+e^{-jka}}{2}-1\right)$$

Using the identity: $\cos(\alpha)=\frac{e^{j\alpha}+e^{-j\alpha}}{2}$: $-m\omega^2=\frac{2\mathrm{T}}{a}(\cos(ka)-1)$ but $\cos(2\mathrm{x})=1-2\sin^2(x)$ so: $-m\omega^2=\frac{-4\mathrm{T}}{a}\sin^2\left(\frac{ka}{2}\right)$

And so the Dispersion Relation for the particles on a string is given by:

$$\omega^2=\frac{4\mathrm{T}}{ma}\sin^2\left(\frac{ka}{2}\right)\quad\therefore\quad\omega=2\sqrt{\frac{T}{ma}}\left|\sin\left(\frac{ka}{2}\right)\right|$$

Phase velocity: $v_p=\frac{\omega}{k}=\frac{2}{k}\sqrt{\frac{T}{ma}}\left|\sin\left(\frac{ka}{2}\right)\right|$

Group velocity: $v_g=\frac{d\omega}{dk}=\sqrt{\frac{Ta}{m}}\left|\cos\left(\frac{ka}{2}\right)\right|$ where a is the distance between adjacent particles.

The dispersion relation is periodic in *k* with a period of $\frac{2\pi}{a}$ therefore the motions of the particles are fully described by *k*'s in the range $\frac{-\pi}{a}<k\leq\frac{\pi}{a}$. This region is called the 'First Brillouin Zone'

The dispersion relation has a maximum $\omega_{max}=2\left(\frac{T}{ma}\right)^{\frac{1}{2}}$ This is the cut-off frequency above which no modes exist.

When $k=\pm\frac{\pi}{a},\ \lambda=2\frac{\pi}{k}=2\mathrm{a}$ This represents the situation when adjacent particles oscillate in anti-phase. i.e. a standing wave with anti-nodes at particle positions and nodes halfway between.

At the anti-nodes the slope of the graph is zero and hence the group velocity is zero which means that there is no energy transfer between particles.

This is the region of the first Brillouin zone, values of k outside this region have wavelengths smaller than 2a but these lead to exactly the same particle motion as that described by the first Brillouin Zone.

If we take a system of a string of length 4a fixed at both ends with 3 particles positioned at a, 2a and 3a along it then we know that the normal modes of the string on its own are just standing waves with wavelengths $\lambda = 2\mathrm{L}, \frac{2\mathrm{L}}{2}, \frac{2\mathrm{L}}{3}$ i.e. $k = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L} \ldots$ where L = 4a

We need only consider the modes within the first Brillouin zone i.e. $k = \frac{\pi}{4\mathrm{a}}, \frac{\pi}{2\mathrm{a}}, \frac{3\pi}{4\mathrm{a}}$ although the next mode $k = \frac{\pi}{a}$ is at the edge of the zone, all of the particles are positioned at nodes so we don't need to consider this.

The modes are standing waves consisting of equal amplitude travelling waves travelling in opposite directions with $\lambda = \frac{2\pi}{k}$

<u>Sound waves in a bar:</u>

At x_0 the displacement is ξ. At x_0 + dx, the displacement is ξ + dξ. Where $d\xi = \frac{\partial \xi}{\partial x} dx$

Young's Modulus: $E = \frac{FL}{Ax}$ is a measure of material's resistance to being stretched.

Wave is stretching the material between x_0 and x_0 + dx by an amount dξ. So: $E = \frac{F}{A} \frac{\partial \xi}{\partial x}$

So force exerted on the element form the bar to the left is: $F_{x=x_0} = -EA \frac{\partial \xi}{\partial x}_{x=x_0}$

At the other side $F_{x=x_0+dx} = EA \frac{\partial \xi}{\partial x}_{x=x_0+dx}$ note the change in sign due to the direction of the force.

But $\frac{\partial \xi}{\partial x}$ is different at the two ends of the element and so they do not simply cancel.

The rate of change of force with distance is $\frac{\partial F}{\partial x} = EA \frac{\partial^2 \xi}{\partial x^2}$

So the force at x_0 + dx is: $F_{x=x_0+dx} = EA \frac{\partial \xi}{\partial x}_{x=x_0} + \frac{\partial F}{\partial x} dx$ and so substituting the above expression: $F_{x=x_0+dx} = EA \frac{\partial \xi}{\partial x}_{x=x_0} + EA \frac{\partial^2 \xi}{\partial x^2} dx$

So the Net Force on the element dx is: $F = EA \frac{\partial^2 \xi}{\partial x^2} dx$

The mass of the element is $\rho A\, dx$ so applying Newton's Second Law for the element dx:

$EA \frac{\partial^2 \xi}{\partial x^2} dx = \rho A\, dx \frac{\partial^2 \xi}{\partial t^2}$ but remember the wave equation is of the form: $\frac{\partial^2 A}{dx^2} = \frac{1}{v_{ph}^2} \frac{\partial^2 A}{\partial t^2}$

and so for sound waves in a bar: $v_{ph}=\sqrt{\frac{E}{\rho}}$

This is analogous to the equation for transverse waves on a string: $v_{ph}=\sqrt{\frac{T}{\rho}}$

Both are dispersionless wave equations.
Note that in both cases the phase velocity is larger in a medium which is mechanically stronger (greater Young's Modulus, can maintain higher tension) and is smaller in a medium with large inertia (large mass per unit length/volume).

<u>Characteristic Impedance:</u>

Remember that z is of the form: $z=\frac{\text{driving force}}{\text{appropriate particle/medium velocity}}$

$z=\frac{-E\frac{\partial\xi}{\partial x}}{\frac{\partial\xi}{\partial t}}$, so for a harmonic wave of the form: $\xi=\xi_0 e^{j(\omega t-kx)}$ differentiating and substituting: $z=\frac{Ek}{\omega}=\frac{E}{v_{ph}}=\sqrt{E\rho}$

<u>Sound waves in a Gas:</u>

Similar derivation to longitudinal waves in a solid bar, but this time the displacement, ξ, at x_0 causes a change in pressure, dP.

The measure of a gas's resistance to being compressed is called its **Bulk Modulus** (=1/compressibility): $B=\frac{-dP}{dV/V}=-V\frac{dP}{dV}$

Assuming an adiabatic process during the wave transmission: $PV^{\gamma}=constant$

Differentiating with respect to V: $V^{\gamma}\frac{dP}{dV}=\gamma PV^{\gamma-1}=0$ so $\frac{dP}{dV}=-\gamma\frac{P}{V}$

Substituting definition of B: $B=\gamma P$

From a similar derivation to the solid bar, the et force on the element comes from the difference in pressure, and is given by: $F_{net}=\frac{-\partial P}{\partial x}dx$

Using Newton's Second Law as before: $-\frac{\partial P}{\partial x}dx=\rho\, dx\frac{\partial^2\xi}{\partial t^2}$

To get a wave equation we first need a relation between P and ξ. From the definition of the Bulk Modulus: $B=-V\frac{dP}{dV}$ so $dP=-B\frac{dV}{V}$ and so: $V=A\,dx\,,\ dV=A\,d\xi$ so $dP=-B\frac{d\xi}{dx}$

The total pressure is the ambient pressure plus the change in pressure due to the wave:
$P=P_0+dP$

But since only dP changes as the wave passes through the medium then:

$$\frac{\partial P}{\partial x}=\frac{\partial(dP)}{\partial x}=-B\frac{\partial^2\xi}{\partial x^2}$$

By substitution in to our previous expression and simplifying:

$$\frac{\partial P}{\partial x}=-B\frac{\partial^2\xi}{\partial x^2}\ ,\quad \frac{-\partial P}{\partial x}dx=Pdx\frac{\partial^2\xi}{\partial t^2}\quad \text{so}\quad \frac{\partial^2\xi}{\partial x^2}=\frac{\rho}{B}\frac{\partial^2\xi}{\partial t^2}$$

And so the phase velocity: $v_{ph} = \sqrt{\frac{B}{\rho}}$ for sound waves in a gas

The pressure through the gas acts as a type of wave, using the harmonic solution to the wave equation for ξ: $\xi = \xi_0 e^{j(\omega t - kx)}$ so. remembering that: $dP = -B\frac{d\xi}{dx}$ and so $dP = j\,k\,B\,\xi$

i.e. The excess pressure dP is itself a wave which oscillates π/2 out of phase with the displacement.

Acoustic Impedance:

In this case: $z = \frac{dP}{d\xi/dt}$ so for a harmonic wave of the form: $\xi = \xi_0 e^{j(\omega t - kx)}$

$$z = \frac{kB}{\omega} = \frac{B}{v_{ph}} = \sqrt{\gamma P \rho} = \rho v_{ph}$$