Electromagnetism Notes

When considering forces which do not entirely support or oppose eachother, one must split the forces in to their horizontal and vertical components, by trigonometry. The force is still calculated over the total distance however.

Coulomb's Law: $\vec{F}=\frac{q_1 q_2}{4\pi\epsilon_0 r^2}\hat{r}$

Electric Field: $\vec{E}=\frac{\vec{F}}{q}=\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$ i.e. the Electric Field is a representation of potential forces, and so is present even if only one charge exists.

The Electric Field follows the same principle of superposition as the force does. i.e.:

$$\vec{E}_{total}=\sum_i^n \frac{1}{4\pi\epsilon_0}\cdot\frac{q}{r^2}\vec{r}$$

For continuous problems use integration, i.e. consider a small charge dQ and then work out the electric field dE due to it, then use the charge density to substitute dQ with some function of dy or dx and integrate.

i.e. the charge density: $\frac{Q}{L}=\frac{dQ}{dx}$ dx may be dy or dr depending on the situation.

Applying this to a (negative and positive) Dipole Electric field obtains the result:

$$\vec{E}=\frac{2\mathrm{qL}}{4\pi\epsilon_0\left(x^2+y^2\right)^{\frac{3}{2}}}\hat{y}$$

Then using the far-field approximation:

$\vec{E}=\frac{-2\mathrm{QL}}{4\pi\epsilon_0 x^3}\hat{y}$ for the electric field across the x-axis (in the case that the dipoles are positioned on the y-axis), and so one sees that the field strength falls off more rapidly for a dipole than for a point charge.

Applying this to a line charge positioned along the y-axis, to find the electric field along the x-axis, one obtains the result:

$$\vec{E}=\frac{Q}{4\pi\epsilon_0}\cdot\frac{1}{x\sqrt{x^2+L^2}}\hat{x}$$

Gauss's Law:

The surface vector is perpendicular to the surface area.

When the surface vector is parallel to the electric field the Electric Flux (rate of volume flow per unit area) is maximum, when the surface vector is perpendicular to the electric field vector the electric flux is 0 (this is used in simplifying problems).

The dot product: $\vec{A}\cdot\vec{B}=|\vec{A}||\vec{B}|\cos(\theta)$

Gauss's Law: $\Phi_E = \oint \vec{E} \cdot d\vec{A}$
In words: "*The total electric flux through a closed surface is proportional to the total net electric charge inside the surface*".

So: $\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$ The surface integral must be taken over a closed loop.

Calculating the Electric Flux of a Point Charge:
$\vec{A} = 4\pi r^2 \hat{r}$
$\Phi_E = \int \vec{E} \cdot d\vec{A} \therefore \int |\vec{E}| \cos(\theta) dA$ Since $\vec{E} \| \vec{A} \therefore \cos(\theta) = 1$ And: $|\vec{E}| = \frac{q}{4\pi\epsilon_0 r^2}$

So: $\Phi_E = \int \frac{q}{4\pi\epsilon_0 r^2} dA$ since r is constant: $\Phi_E = \frac{q}{4\pi\epsilon_0 r^2} \int dA = \frac{q}{4\pi\epsilon_0 r^2}(4\pi r^2)$

Simplifying: $\Phi_E = \frac{q}{\epsilon_0}$.

In a conductor, the charges gather on the surface of the object, whereas in an insulator they are uniformly spread out throughout the volume. This means that the charge density in a spherical insulator is given by $\rho = \frac{3Q}{4\pi r^3}$, whereas in a similar conductor the charge density is: $\rho = \frac{Q}{4\pi r^2}$. **The electric field inside any conductor is zero.** The charges on the surface(s) of the conductor will always be such that the field inside is zero by Gauss' law. It must be zero because if it was not zero then the charges would move to balance the field and it would become zero, electrostatics considers stationary charges and hence it is zero.

For different charge distributions different Gaussian surfaces must be used. As used above, the Gaussian surface used for a Sphere is also a Sphere, but it is not always this simple:

Charge Distribution	Sphere	Sheet	Rod (i.e. wire)
Electric Field Distribution	Radial	x-direction	Radial
Gaussian Surface	Sphere	Cylinder (only caps matter)	Cylinder (only the tube matters)
Surface Area of Gaussian Surface	$4\pi r^2$	$2\pi r^2$ (2 since there are 2 caps to the cylinder)	$2\pi r L$

If the surface considered is complex, such as a hemisphere, then the integrals may be split up for each surface and its electric field.

The Electric Field, at a distance r, from a uniformly charged shell (i.e. a conducting sphere) of radius R:
$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$ $\therefore$since r is constant $E = \frac{Q_{enc}}{\epsilon_0 A}$ $\therefore$ $\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$

The same result is found for a uniformly charged sphere with uniformly distributed charges throughout the volume (i.e. an insulator) since the enclosed charge and surface area are the same. However, an important difference is that in the shell's case there is no electric field within the sphere (i.e. at radii less than the shell) since there is no enclosed charge, whereas in the sphere's case there is an electric field inside the sphere since there is still enclosed charge.

Finding the Electric Field of an Infinite Sheet with charge density σ:

$A=\pi r^2$ so: $\vec{E}=\frac{Q_{enc}}{2A\epsilon_0}$ but since charge density= σ: $\frac{Q_{enc}}{A}=\sigma$ so:

$\vec{E}=\frac{\sigma}{2\epsilon_0}$ so the Electric Field is independent of distance (this is only true for a truly infinite sheet (which doesn't exist in reality)).

Finding the Electric Field of an infinite line charge with charge density λ:

$A=2\pi r L \quad \therefore \quad E=\frac{Q_{enc}}{2\pi r L\epsilon_0}$ But since charge density $=\lambda$ Then: $Q_{enc}=\lambda L$

And so: $E=\frac{\lambda L}{2\pi r L\epsilon_0}=\frac{\lambda}{2\pi r\epsilon_0}$.

Potential Energy:

The Electric Field is a conservative field, this means that the work done is independent of the path length.

Deriving the expression for a point charge:

$$W=U_a-U_b=\int_a^b \vec{F}\, d\hat{r}=\int_a^b \frac{Qq}{4\pi\epsilon_0 r^2}\, d\hat{r}=\left[\frac{-Qq}{4\pi\epsilon_0 r^2}\right]_a^b dr=\frac{qQ}{4\pi\epsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

So the absolute potential energy (b tends to infinity) is given by:

$$U=\frac{Qq}{4\pi\epsilon_0 r}$$

The absolute potential energy also follows the principal of superposition, i.e.:

$$U=\frac{q_0}{4\pi\epsilon_0}\sum_1^N \frac{q_i}{r_i}$$

So potential energy requires at least two charges to exist.

The Electric Potential (no to be confused with potential energy) is to the potential energy what the electric field is to the force. It is a scalar field that describes the potential energies at different points, i.e. $V=\frac{U}{q_0}$

So for a system of point charges: $V=\frac{U}{q_0}=\frac{1}{4\pi\epsilon_0}\sum_i \frac{Q_i}{r_i}$

If the charge distribution is not a point source, but the electric field is known, the Electric Potential can be calculated from the Electric Field:

$$V_a - V_b = \int_a^b \vec{E} \cdot d\vec{L}$$

Electric potential of a charged sphere

We found before that inside a conducting sphere the electric field is zero and outside of it:

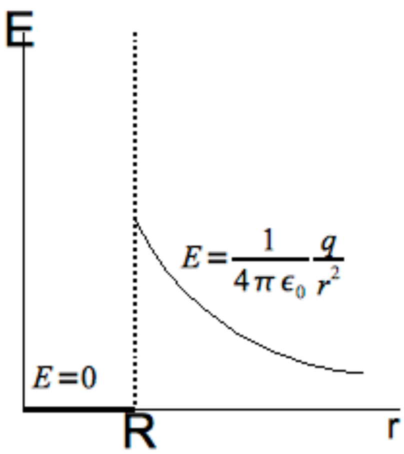
$$\vec{E} = \frac{q}{\epsilon_0 4\pi r^2}\hat{r}$$

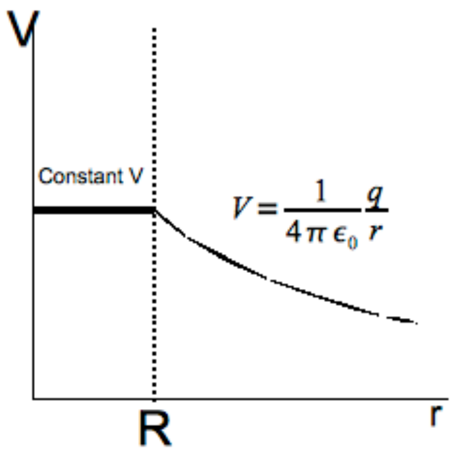
Therefore the electric potential outside of the sphere may be represented as:

$$V = \int_r^\infty E\,dr = \int_r^\infty \frac{q}{4\pi\epsilon_0 r^2}dr = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r} - \frac{1}{\infty}\right) = \frac{q}{4\pi\epsilon_0 r}$$

At the surface r is simply equal to R and thus $V = \frac{q}{4\pi\epsilon_0 R}$

Inside the sphere, the electric field is 0 and hence the potential remains constant at the value it has at the surface.





When the sphere is an insulator the charge is evenly distributed throughout its volume. The electric field and potential outside the sphere are simply the same as for outside of the conductor.

To determine the potential inside the insulating sphere we must first solve for the Electric Field.

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

$$\frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0}\frac{q}{\frac{4}{3}\pi R^3}\left(\frac{4}{3}\pi r^3\right) = \frac{q}{\epsilon_0}\left(\frac{r}{R}\right)^3$$

$$\frac{q}{\epsilon_0}\left(\frac{r}{R}\right)^3 = \oint \vec{E}\cdot d\vec{A} = E\int dA = E\,4\pi r^2$$
$$E = \frac{q}{\epsilon_0}\left(\frac{r}{R}\right)^3 \frac{1}{4\pi r^2} = \frac{qr}{4\pi\epsilon_0 R^3}$$

Then using the definition of electric potential,

$$V_a - V_b = \int_a^b \vec{E}\cdot d\vec{L} \quad \text{and} \quad \vec{E}\cdot d\vec{l} = E\cos(\theta)\,dl = E\,dr$$

We can substitute in our expression for E:

$$V_a - V_b = \int_{r=a}^{b} \frac{qr}{4\pi\epsilon_0 R^3}\,dr = \left[\frac{qr^2}{8\pi\epsilon_0 R^3}\right]_{r=a}^{b}$$

Therefore if a=r and b=R we obtain that:

$$V_a - \frac{q}{4\pi\epsilon_0 R} = -\left[\frac{qr^2}{8\pi\epsilon_0 R^3} - \frac{qR^2}{8\pi\epsilon_0 R^3}\right]$$

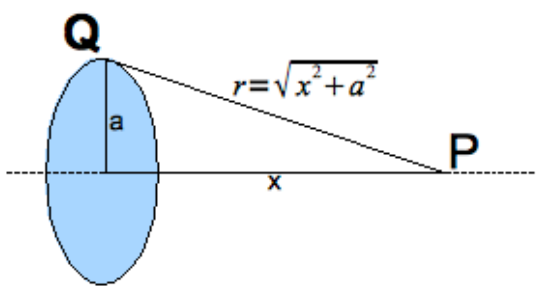
Hence:

$$V = \frac{q}{8\pi\epsilon_0 R}\left[3 - \frac{r^2}{R^2}\right]$$ is the potential inside the sphere.

Calculating the electric potential of a continuous charge distribution

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} \quad \text{so} \quad V = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r}$$

Example of a ring of charge



P is just a point some distance from the ring, Q is the charge which is uniformly distributed throughout the ring.

Plugging the expression for r into the above equation for V we obtain:

$$V = \frac{1}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2+a^2}}\int dq = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2+a^2}}$$

Which is the electric potential for a ring of charge.

Equipotentials

Equipotentials are lines where the electric potential has a constant value, they are analogous to contour lines for gravitational potential on a map.

As the electric potential is constant, the electric potential energy of any charge on an equipotential is constant and hence no work is done by the electric field on a charge moving on an equipotential.

As a point can only have one value for its electric potential, equipotential lines of different values do not cross or touch.

The surface of a conductor is an equipotential.

Electric field lines are always perpendicular to the equipotential lines and always point in the direction of decreasing electric potential.

For a radial field: $E_r = \frac{-dV}{dr}$

and more generally: $\vec{\boldsymbol{E}} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) = -\vec{\boldsymbol{\nabla}} V$

Capacitors

Definition: $C = \frac{Q}{V}$ where Q is the charge (in the parallel plates there is +Q on one plate and -Q on the other) and V is the potential difference.

Parallel plate capacitors

The electric field between the plates is given by: $\vec{\boldsymbol{E}} = \frac{\sigma}{\epsilon_0}\hat{x} = \frac{Q}{A\epsilon_0}\hat{x}$ via Gauss' Law

Therefore the potential:

$$V = \int_{l=a}^{l=b} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{l}} = E\int_{l=a}^{l=b} dl = \frac{Qd}{A\epsilon_0}$$

And hence:

$$C = \frac{Q}{V} = \frac{A\epsilon_0}{d}$$

Cylindrical Capacitors

Inner conductor has radius r_a and charge density λ, outer conductor has a radius r_b and is a cylindrical shell.

As before we aim to find E, then V then C.

From Gauss' Law $\vec{\boldsymbol{E}} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{r}$

And hence the potential difference:

$$V_a - V_b = \int_{r_a}^{r_b} \vec{E} \cdot d\vec{l} = \int_{r_a}^{r_b} E\, dr = \int_{r_a}^{r_b} \frac{1}{2\pi\epsilon_0} \frac{\lambda}{r} dr = \frac{\lambda}{2\pi\epsilon_0} [\ln r]_{r_a}^{r_b} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_b}{r_a}\right)$$

Finally the capacitance:

$$C = \frac{Q}{V} = \frac{\lambda l}{V} = \frac{\lambda l}{\frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{r_b}{r_a}\right)} = \frac{2\pi\epsilon_0 l}{\ln\left(\frac{r_b}{r_a}\right)}$$

dividing by L gets the charge density per unit length:

$$\frac{C}{L} = \frac{2\pi\epsilon_0}{\ln\left(\frac{r_b}{r_a}\right)}$$

A dielectric material can be placed between the conducting layers of the capacitor, the dielectric is a non-conductor and lowers the potential thus resulting in a greater capacitance.

The dielectric constant, K , is defined as the ratio of the capacitance with the dielectric to the capacitance without it (i.e. free space) $K = \frac{C}{C_0}$

Due to the dielectric the permittivity is no longer the permittivity of free space but is instead: $\epsilon = K\epsilon_0$

The dielectric constant may also be related to the Electric Field change through the capacitor by the equation: $K = \frac{E_0}{E}$

The Electric field is related to the charge density on the capacitor plates, σ, by the equation: $E_0 = \frac{\sigma}{\epsilon_0}$ (and so $\sigma = \epsilon_0 E_0$)when there is no dielectric, when a dielectric is present the electric field is related to the charge density on the surface of the dielectric, σ_i by the equation:

$E = \frac{\sigma - \sigma_i}{\epsilon_0}$, and so solving for the charge density on the surface of the dielectric, σ_i:

$$\sigma_i = \sigma - E\epsilon_0 = E_0\epsilon_0 - E\epsilon_0 = \epsilon_0(E_0 - E)$$

Capacitors may be combined in parallel or series.

<u>Capacitors in parallel</u>

In parallel the voltage is constant(i.e. The same for each capacitor). Let us first find the total charge:

$$Q_T = \sum_i Q_i = \sum_i C_i V = V \sum_i C_i$$

Hence the equivalent capacitance:

$$C_T = \frac{Q_T}{V} = \frac{V \sum_i C_i}{V} = \sum_i C_i$$

Capacitors in series

In series the charge on each capacitor is constant (i.e. the same). Therefore finding the total potential drop across the capacitors:

$$V_T = \sum_i V_i = \sum_i \frac{Q}{C_i} = Q \sum_i \frac{1}{C_i}$$

therefore the equivalent capacitance:

$$\frac{1}{C_t} = \frac{V_t}{Q} = \sum_i \frac{1}{C_i}$$

Energy storage in a capacitor

The potential energy, U, of a capacitor is the work done to charge the capacitor.

The electric potential is given by $V = \frac{U}{Q}$

When charging a capacitor it must pass through an intermediate stage before it is fully charged, the intermediate potential can be given by:

$$C = \frac{q}{V} \quad \therefore v = \frac{q}{C}$$

And hence the work done to transfer a charge dq may be expressed as:

$$dq = C\,dv \quad \therefore dW = v\,dq = \frac{q\,dq}{C}$$

$$W = \int_{v=0}^{v=V} v\,dq = \int_{v=0}^{v=V} vC\,dv = \frac{1}{2} C V^2$$

We may also represent the potential energy stored (i.e. the work done in charging the capacitor) as:

$$U = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} Q V$$

Field Energy Density

Now let us solve for the Energy Density, u , which is the energy per unit volume. $u = \frac{U}{Vol}$

$$Vol = Ad$$

$$\therefore \frac{U}{Vol} = \frac{1}{2} C V^2 \cdot \frac{1}{Ad} \quad \text{Energy Density}$$

$$\frac{U}{Vol} = \frac{1}{2} \left(\frac{A \epsilon_0}{d} \right) V^2 \cdot \frac{1}{Ad} = \frac{1}{2} \epsilon_0 \frac{V^2}{d^2} \quad \text{Substitute capacitance}$$

$$V = Ed \quad \text{Electric potential}$$

Therefore we obtain that the energy density is:

$$u = \frac{1}{2} \epsilon_0 E^2$$

This is independent of the particular configuration of the parallel plate capacitor and is a general result.

Dielectrics:

Use of Dielectrics allow for simpler construction, a higher breakdown voltage and a larger capacitance.

With no Dielectric: $|\vec{\boldsymbol{E}}_0| = \frac{\sigma}{\epsilon_0}$ and capacitance of C_0.

With Dielectric: $|\vec{E}|=\frac{\sigma-\sigma_i}{\epsilon_0}$ and capacitance of C.

Note $E<E_0$ and $C>C_0$.

Definition of Dielectric constant: $K=\frac{C}{C_0}$ (Dimensionless constant)

Electric field with a dielectric: $E=\frac{E_0}{K}$

Permittivity: $\epsilon=K\epsilon_0$

Parallel Plate Capacitor: $C=KC_0=K\epsilon_0\frac{A}{d}=\epsilon\frac{A}{d}$

Field Energy Density: $u=\frac{1}{2}K\epsilon_0 E^2=\frac{1}{2}\epsilon E^2$

Gauss's Law in a dielectric: $\oint K\vec{E}\cdot d\vec{A}=\frac{Q_{enc-free}}{\epsilon_0}$

Dielectric Breakdown:
If field (so voltage) is too powerful then electrons are ripped off the atoms in the dielectric and accelerates by the electric field, the dielectric becomes a conductor in a chain reaction.

Current

$I=\frac{dQ}{dt}$ it has units of C/s or Amperes, it is a scalar quantity.
Rate of flow of electric charge through a cross-sectional area
Direction of flow is that of a positive charge (by convention)

When no electric field is applied the free electrons move in a random direction with speeds $\sim 10^6$ ms^{-1} , since the velocity vectors are randomly oriented the average velocity is zero so no current flows.

When an E field is applied the electrons experience a force $F=qE$, electrons have additional velocity in a direction opposite to the applied field, the drift velocity, convention positive charges not electrons.

Kinetic energy is dissipated via collisions, but there is still a net drift $\sim 10^{-4}$ ms^{-1}
Field does work on the charges, which goes into KE and hence to material via inelastic collisions between electrons and ionic lattice. This is Joule heating.

The number of particles in a given volume: $nAv_d dt$
where n is the concentration of charges in m^{-3} , A is a cross-sectional area in m^2 , and q is the individual charge units in C, V_d is the drift velocity in ms^{-1} and dt is the time interval in s
Total charge: $dQ=qnAv_d dt$
Current: $\frac{dQ}{dt}=nqAv_d$

Current Density

$|\vec{J}| = \frac{I}{A} = n|q|v_d$ This is the magnitude and has units Am^{-2}

$\vec{J} = nq\vec{v}_d$ Current density is a vector

If q is positive, v_d is in the direction of E and hence v_d is positive and qv_d is positive
If q is negative, v_d is in the direction opposite to E and hence v_d negative but qv_d is +ve
Notice that whilst the current is scalar, the current density is a vector quantity.
The magnitude of the current density is **NOT** the current.

Resistivity

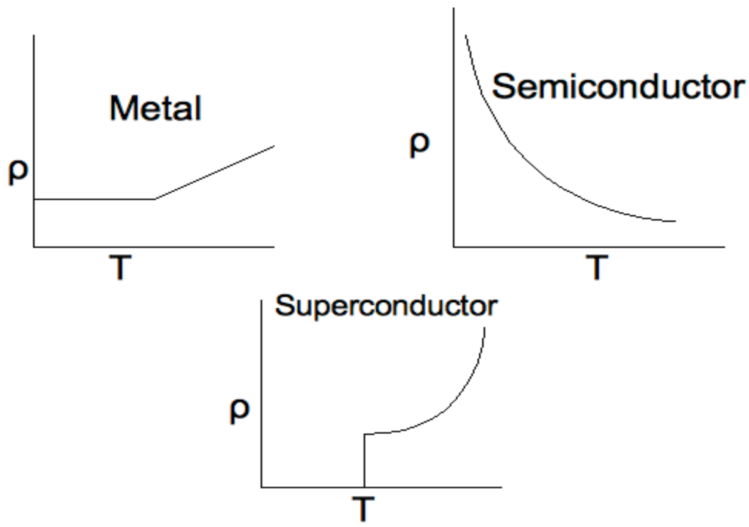
$\rho = \frac{E}{J}$ units of Ωm

Conductors have low resistivity whilst insulators have resistivity.
Typical values are $1.72 \times 10^{-8} \Omega m$ for copper (conductor), $0.6 \Omega m$ for germanium (semiconductor) and $10^{13} \Omega m$ for teflon (insulator)

Conductivity

$\sigma = \frac{1}{\rho} = \frac{J}{E}$ so $\vec{J} = \sigma \vec{E} = \frac{1}{\rho} \vec{E}$

The resistivity of a metallic conductor varies with the temperature, near room temperature:
$\rho_T = \rho_0 [1 + \alpha (T - T_0)]$ where α is the coefficient of resistivity



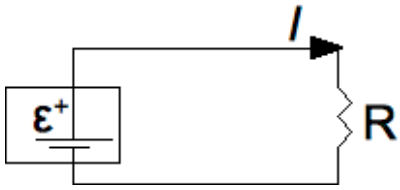
Ohm's Law $V = IR$

Power $P = \frac{dW}{dt} = VI = I^2 R = \frac{V^2}{R}$ has units Js^{-1} or watts.

The Electro-Motive Force ε is **not a true force**, it has units of Volts, it acts to move a charge from low to high potential.

Circuits require a source of EMF for charge to flow, batteries are the most common source of EMF.

Batteries cause charges to move from negative to positive terminal with non-Coulomb forces.



An ideal battery is a source of emf, and it maintains a potential difference between two points. Wires are assumed to have negligible resistance.

The power output of the emf source is matched by the power dissipated in the resistor, so the emf source acts like a 'charge pump'

Real batteries have an internal resistance and hence the emf is not ε but ($\varepsilon - Ir_i$) there is a voltage drop Ir_i across the internal resistance r_i

In series the current is identical: $R_{eq} = R_1 + R_2 ... = \sum_i R_i$

In parallel the voltage is identical: $\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} ... = \sum_i \frac{1}{R_i}$

<u>Kirchoff's rules</u>
Conservation of charge: the algebraic sum of currents at a junction at zero
$\sum I = 0$

Conservation of energy implies $\sum V = 0$ around a closed path (include emfs)

Sign conventions must be used with potential differences.

Across the battery from -ve terminal to +ve terminal there is a +ve change in potential. Therefore across the battery from +ve to -ve there is a -ve change in potential.

When travelling across a resistor there is a positive change in potential if the travel direction is opposite to the current. When travelling in the same direction of the current the change in potential is negative.

Across capacitors the voltage change is positive from the negative to positive plate and negative from the positive to negative plate.

To apply Kirchoff's rules you should decide the current directions arbitrarily and use the junction rule to find the relationships between the currents. Then use the loop rule, being careful with the signs, IR is positive if the current has the same direction as the loop travel direction, ε is positive if the direction is the same as the direction the battery drives the current.

RC Circuit – Conditions at time t:
Potential difference across Resistor: $V=iR$ (Where lower case indicates quantities that vary with time)

Potential difference across Capacitor: $V=\frac{q}{C}$

Applying Kirchoff's loop rule: $\epsilon-iR-\frac{q}{C}=0 \quad \therefore \quad i=\frac{\epsilon}{R}-\frac{q}{RC}$

So arranging in to differential equation: $i=\frac{dq}{dt}=\frac{1}{RC}(q-C\epsilon)$

Solving $\int_0^q \frac{dq}{q-C\epsilon}=-\int_0^t \frac{dt}{RC}$ so $\ln\left(\frac{q-C\epsilon}{-C\epsilon}\right)=\frac{-t}{RC}$

Finally: $q=C\epsilon\left(1-e^{\frac{-t}{RC}}\right)$

So at time t=RC, $q=Q_{final}\left(1-\frac{1}{e}\right)$

Solving for current as a function of time:

$$i=\frac{dq}{dt}=-C\epsilon\left(\frac{-1}{RC}\right)e^{\frac{-t}{RC}}=\frac{\epsilon}{R}e^{\frac{-t}{RC}}$$

so when t=RC, current has dropped to 1/e of the initial current.

Magnetism:
Differs from electric field in that magnetic monopoles do not exist, and the charges must be in motion to create a (magnetic) B field.

Electric force: $\vec{\boldsymbol{F}}=q\vec{\boldsymbol{E}}$
Magnetic force: $\vec{\boldsymbol{F}}=q\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}}$

So if Electric and magnetic forces are present then the total force is given by:
$\vec{\boldsymbol{F}}_{total}=q(\vec{\boldsymbol{E}}+\vec{\boldsymbol{v}}\times\vec{\boldsymbol{B}})$

Cross Product:
To find the direction of u x v, stick your thumb in the direction of u, your index finger in the direction of v, and then stick your middle finger out so that it's perpendicular to both u and v - that direction is that of the result vector.

To find the magnitude of the cross-product in the determinant of the matrix of the vectors with the top row being i, j and k unit vectors.

Motion in a Uniform B-Field

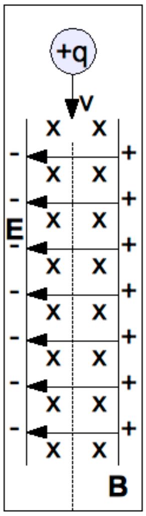
When the velocity is parallel to the B-field direction, then **v**X**B**=0 since there is no perpendicular component, and so the motion continues as if no there was no B-field present.

When the velocity is perpendicular to the B-Field direction, then the direction of the velocity changes with position and so it creates a circular path, defined by:

$$|q|vB=\frac{mv^2}{r} \quad \text{, so:} \quad r=\frac{mv}{|q|B}, \quad \omega=\frac{v}{r}=v\frac{qb}{mv}=\frac{qb}{m}$$

When there are combined perpendicular and parallel components then a helical path is produced. The pitch is defined as the distance between two common points on the helical path.

Combined E and B fields: Velocity Selector



The electric field acts in the opposite direction to the magnetic field. Only when they are both balanced will the charge pass through, this is at a specific velocity.

$$q|E|=qvB \quad \therefore \quad v=\frac{E}{B}$$

Thomson's experiment:
First accelerates the electrons:

$$\frac{1}{2}mv^2=eV \quad \rightarrow \quad v=\sqrt{\frac{2\text{eV}}{m}}$$

Plates act as velocity selectors:

$$\frac{E}{B}=\sqrt{\frac{2\text{eV}}{m}} \quad \therefore \quad \frac{e}{m}=\frac{E^2}{2\text{VB}^2}$$

This e to m ratio is a fixed value regardless of the material.

Current Carrying Conductor:

Number of particles, in given volume: $Number=nAv_d\,dt$
Total Charge: $dQ=qnAv_d\,dt$
Current: $I=\frac{dQ}{dt}=nqAv_d$

Differential force: $d\vec{F}=q\vec{v}_d\times\vec{B}$
Total force combines all charges q:

$$\vec{F}=n(AL)q\,\boldsymbol{v}_d\times\vec{B}$$

For Current: $I=\frac{dQ}{dt}=nqAv_d$

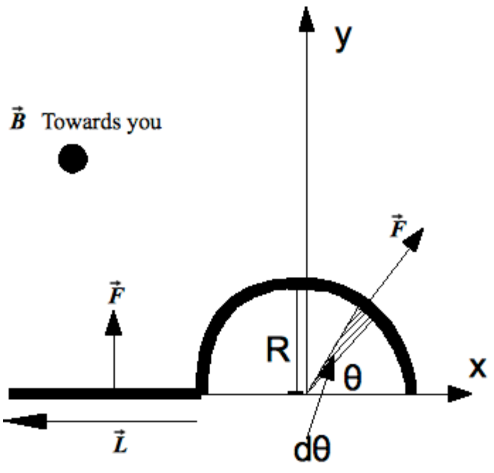
Total Force: $\vec{F}=I\vec{L}\times\vec{B}$

For variable magnetic fields, or curved wires, integrate the differential form:

$$d\vec{F}=I\,d\vec{L}\times\vec{B}$$

Forces on a Curved/Straight wire:

Wire is of the shape:



So on linear part, $d\vec{l} \perp \vec{B}$, force direction is $+\hat{y}$.
Total Force magnitude = $|F| = BIL$.

In curved part: Force on segment: $dF = I(R\,d\theta)B$
Force components:

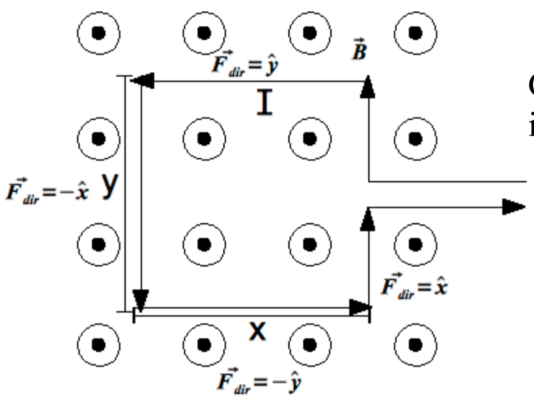
$dF_x = IRd\theta\, B\cos(\theta)$, $dF_y = IRd\theta\, B\sin(\theta)$

$$F_x = IRB\int_0^{\pi}\cos(\theta)\,d\theta = IRB\left[\sin(\theta)\right]_0^{\pi} = 0$$

$$F_y = IRB\int_0^{\pi}\sin(\theta)\,d\theta = -IRB\left[\cos(\theta)\right]_0^{\pi} = 2\mathrm{IRB}$$

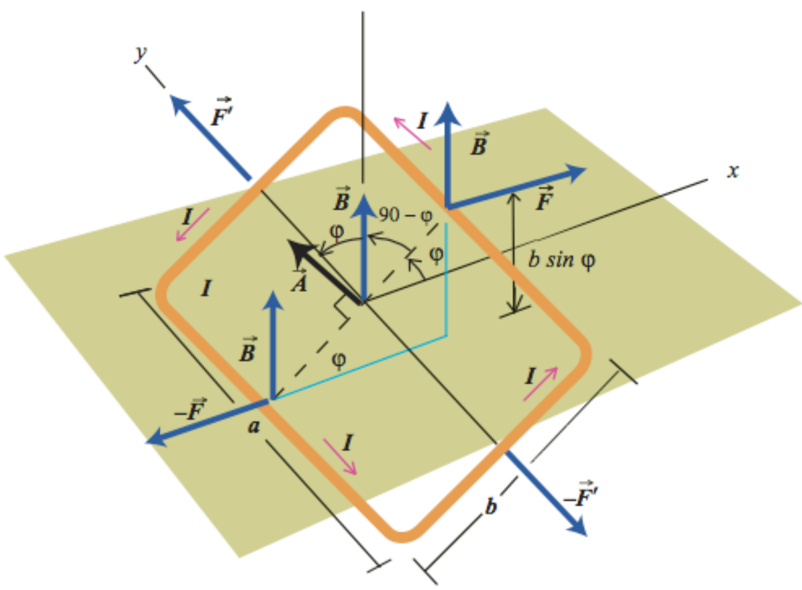
so cancels in x-dir, net force is in y-dir.

<u>Forces on a Current Loop:</u>
In each segment $\vec{F} = I\,\vec{l} \times \vec{B}$
First case: loop plane $\perp$ $\vec{B}$-field direction so loop plane vector || $\vec{B}$-field direction



Current directions oppose each other so overall there is no net current.

Second case:
Loop area vector at angle ϕ to the **B**-field vector.



Sides with length a:
$\vec{l} \perp \vec{B} \Rightarrow \vec{l} \times \vec{B} = aB$
$F = |I\,\vec{L} \times \vec{B}| = IaB$

Sides with b: angle between $\vec{l}, \vec{B}$ is $90 - \phi$

$F = |I\,\vec{l} \times \vec{B}| = IbBsin(90 - \phi) = IbBcos(\phi)$

Definition of Magnetic Moment:
$\vec{\mu} = I\,\vec{A}$ Units: Am^2.
Definition of Magnetic Torque:
$\vec{\tau} = \vec{\mu} \times \vec{B}$

When **μ** and **B** are parallel:
$|\vec{\tau}| = 0$ i.e. torque is at a minimum

When **μ** and **B** are perpendicular:
$|\vec{\tau}| = \mu\,B\sin(90^{\circ}) = \mu\,B$ Torque is at a maximum.

In the general case at angle **ϕ** between **μ** and **B** the torque magnitude is given by:
$|\vec{\tau}| = \mu\,B\sin(\phi)$

The torque direction is found by using the right-hand rule.

<u>The Hall effect:</u>
The Hall effect is used to determine the sign of the charges and measure the density of the charges.

A conductor is used where either positive charges are moving right (the classical view) or negative charges are moving left (the actual system). A uniform B-field exists perpendicular to the conductor and exerts a force on the charges.

Measuring the sign:
Force from B-field: $\vec{F} = q\,\vec{v}_d \times \vec{B}$, moves positive charges upward

Hall voltage: Charge separation causes E-field. E-field causes a voltage across the conductor and the sign of the voltage indicates the sign of the charges.

Measuring the density of charge carriers:
Forces from the E-field and B-field balance in steady state: $\vec{B} = \vec{E}$
$q\,v_d\,B = qE$, the current density, J: $J = nq\,v_d$. Note the current density is **NOT** the same as the density of charge carriers.

So density of charge carriers, nq, $nq = \frac{JB}{E}$

<u>Magnetic field of a moving charge</u>

$$\vec{B}=\frac{\mu_0}{4\pi}\frac{q\vec{V}\times\hat{r}}{r^2}$$

$\mu_0=4\pi\times10^{-7}\,T\,m/A$ Permeability of free space

B-field of a current element

$dQ=(nq)(Adl)$ n is the charge density not the number of charges, therefore nq is the charge per unit volume and Adl is the volume.

Substituting into field of a moving charge as a current element consists of many moving charges and the principle of superposition may be used:

$d\vec{B}=\frac{\mu_0}{4\pi}\frac{(nqAdl)\vec{v}_d\times\hat{r}}{r^2}$ but $I=nqv_dA$ so $d\vec{B}=\frac{\mu_0}{4\pi}\frac{I\vec{dl}\times\hat{r}}{r^2}$

Note that the *r* unit vector points in the direction from the charge to the field point.

Biot-Savart Law

$$d\vec{B}=\frac{\mu_0}{4\pi}\frac{I\vec{dl}\times\hat{r}}{r^2}$$

Draw a diagram, the dl vector points in the current direction, and the r vector points from dl to the field point. Integrate over the whole current carrying wire to solve for total B-field.

B-field at a current loop centre

Each current element is perpendicular to radius vector, B-field at the centre is directed along axis of loop.

$|\vec{B}|=\oint\frac{\mu_0}{4\pi}\frac{I\vec{dl}\times\hat{r}}{r^2}$ Integrating over loop

$|\vec{B}|=\oint\frac{\mu_0}{4\pi}\frac{I\vec{dl}}{r^2}$ $\quad\vec{dl}\perp\hat{r}$

$|\vec{B}|=\frac{\mu_0}{4\pi}\frac{I}{r^2}\oint dl$ r=R, constant

$|\vec{B}|=\frac{\mu_0}{4\pi}\frac{I}{r^2}2\pi R$ Circumference is $2\pi R$

$|\vec{B}|=\frac{\mu_0 I}{2\mathrm{R}}\hat{x}$ Multiply by N if N loops

B-field on a current loop axis

Calculate B field at a point P, distance x from centre

Consider element I dl which is tangent to loop and thus perpendicular to r

From symmetry: y-components cancel therefore B-field is in x-direction only

Field from a current element:

$$|\vec{dB}|=\frac{\mu_0}{4\pi}\frac{I\,\vec{dl}\times\hat{r}}{r^2}=\frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}$$

From diagram:

$$r^2=x^2+R^2 \qquad \sin\theta=\frac{R}{\sqrt{x^2+R^2}}$$

Subbing in:

$$|\vec{B}|=\oint\frac{\mu_0}{4\pi}\frac{I\,dl\sin\theta}{r^2}=\oint\frac{\mu_0}{4\pi}\frac{I\,dl}{x^2+R^2}\frac{R}{\sqrt{x^2+R^2}}$$

Taking out constant R,x

$$|\vec{B}|=\frac{\mu_0}{4\pi}\frac{RI}{(x^2+R^2)^{\frac{3}{2}}}\oint\vec{dl}$$

$$\oint dl=2\pi R$$

so $\vec{B}=\frac{\mu_0}{2}\frac{R^2 I}{(x^2+r^2)^{\frac{3}{2}}}\hat{x}$ multiply by N if N loops

If we go far away from the loop i.e. x>>R then the expression simplifies to

$\vec{B}=\frac{\mu_0}{2}\frac{R^2 I}{x^3}\hat{x}$ multiply by N if there are N loops

Straight wire

Calculate B field a distance x from the wire:

Wire length = 2a

$$r^2=x^2+y^2 \qquad \sin\theta=\frac{x}{\sqrt{x^2+y^2}}$$

From Biot Savart

$$|\vec{B}|=\int_{-a}^{a}\frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r^2}=\int_{-a}^{a}\frac{\mu_0}{4\pi}\frac{I\,dy}{(x^2+y^2)}\frac{x}{\sqrt{x^2+y^2}}=\frac{\mu_0 I}{4\pi}\int_{-a}^{a}\frac{xdy}{(x^2+y^2)^{\frac{3}{2}}}$$

This integrates to

$|\vec{B}|=\frac{\mu_0 I}{4\pi}\frac{2a}{x\sqrt{x^2+a^2}}$ via trig substitutions or integral tables

For a long wire a>>x

$$|\vec{B}|=\frac{\mu_0 I}{2\pi x}$$

and therefore at a radius r from the wire

$$|\vec{B}|=\frac{\mu_0 I}{2\pi r}$$

Force on parallel wires

We can calculate the B field of one wire to be:

$|\vec{B}_1|=\frac{\mu_0 I_1}{2\pi r}$ in magnitude, direction tangent to circle.

Therefore the force it exerts on the 2nd wire is $d\vec{F}_2 = I_2 d\vec{l}_2 \times \vec{B}_1$
so $|d\vec{F}_2| = I_2 |d\vec{l}_2||\vec{B}_1|$
and the force per unit length of wire:

$$\frac{dF_2}{dl_2} = I_2 B_1 = \frac{\mu_0 I_2 I_1}{2\pi r} \quad \text{since} \quad B_1 = \frac{\mu_0 I_1}{2\pi r}$$

One Ampere is empirically defined as the amount of current in two parallel conductors such that there is a force of 2×10^{-7} Nm^{-1} if the conductors are 1m apart in free space.

Ampere's Law:

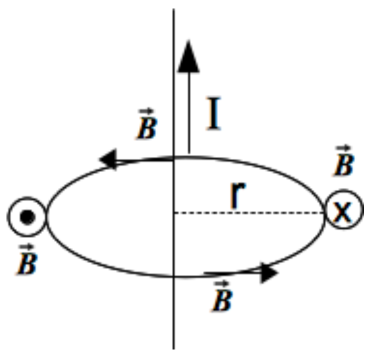
$\oint \vec{\boldsymbol{B}} \cdot d\,\vec{\boldsymbol{l}} = \mu_0 I_{enc}$ where the magnetic constant, $\mu_0 = 4\pi \times 10^{-7}\ T\ m\ A^{-1}$

Note that only the currents enclosed by the integration path are considered.
Applying Ampere's Law:

1) Choose integration path
2) Use symmetry to simplify the dot product
3) Sum the enclosed currents with correct direction
4) Evaluate path integral

Use the right-hand rule to determine the sign convention, curl your right hand in the direction of the path and the direction that your thumb points in the direction of positive current flow.

B-Field outside a straight wire:



1) Choose integration path: Circle with radius r, so path length = 2πr acting in a counterclockwise direction.
2) Simplify dot product: B-field is tangential to path so $\oint \vec{\boldsymbol{B}} \cdot d\,\vec{\boldsymbol{l}} = B \oint d\,\vec{\boldsymbol{l}} = B 2\pi r$
3) Sum enclosed currentsL Using right-hand rule current is +I.
4) Evaluate path integral: $2\pi r B = \mu_o I \ \therefore\ B = \dfrac{I \mu_0}{2\pi r}$

   direction is as shown in diagram.

Note this is much simpler than solving the same problem using the Biot-Savart Law.

B-Field of a Cylindrical Wire:

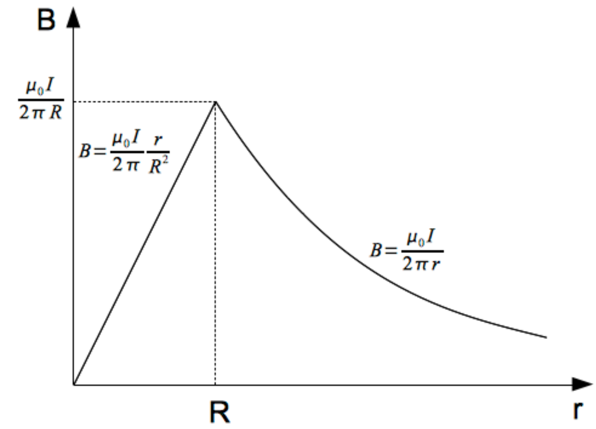
Want to calculate B-field as a function of r from a wire of radius R and current I uniformly distributed over the cross-sectional area.

Case 1: (r>R)
Same as example of outside a straight wire, since conditions are identical.

So $B = \dfrac{I \mu_0}{2\pi r}$

Case 2: (r<R)

$$I_{enc} = \frac{I}{\pi R^2} \pi r^2 = I\left(\frac{r^2}{R^2}\right) \quad , \quad \oint \vec{\boldsymbol{B}} \cdot d\,\vec{\boldsymbol{l}} = 2\pi r B \quad \text{so} \quad B = \frac{I r^2 \mu_0}{2\pi r R^2} = \frac{I r \mu_0}{2\pi R^2}$$

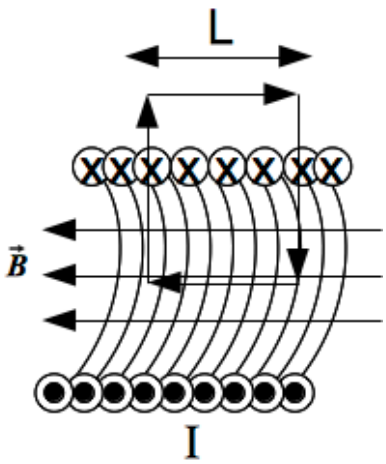


Inside the conductor (r<R) the B-field increases with r.

Outside the conductor (r>R) the B-field decreases with 1/r.

B-Field of a Solenoid:
Want to calculate B-Field at the centre of the solenoid.
n – Number of turns **per unit length**.
I – current in the wire.

The B-Field lines are nearly parallel at the central region, outside the solenoid the field is much weaker (B~0 outside solenoid).



Integration Path: Rectangle with top and bottom length L. Direction is counterclockwise.

Symmetry: B-field is perpendicular to bottom path, parallel to right/left path. There is no B-field at the top path.

$$\oint \vec{B}\cdot d\vec{l}=\int_{bottom}\vec{B}\cdot d\vec{l}+\int_{left}\vec{B}\cdot d\vec{l}+\int_{top}\vec{B}\cdot d\vec{l}+\int_{right}\vec{B}\cdot d\vec{l}$$
$$\oint \vec{B}\cdot d\vec{l}=\int_{bottom}\vec{B}\cdot d\vec{l}=BL \quad , \quad I_{enc}=NI=nLI$$
so $B=\frac{N\,I\,\mu_0}{2\pi r}=n\,I\,\mu_0$

Induction:
A time-varying B-field can induce a current, when there is **changing magnetic flux**. The induced EMF is the corresponding EMF that causes the current, the induced EMF has different properties compared to an EMF caused by charges.

Magnetic Flux: $\Phi_B=\int \vec{B}\cdot d\vec{A}=\int B\,dA\cos(\phi)$
Gauss's Law for Magnetism: $\oint \vec{B}\cdot d\vec{A}=0$

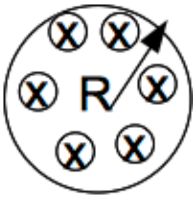
Faraday's Law: $\epsilon=\frac{-d\,\Phi_B}{dt}$: *The induced EMF in a closed loop equals the negative time rate of change of magnetic flux through the loop.*

Sign convention: Use the right hand rule with your thumb pointed in the direction of the area vector, the fingers curl in the direction of positive EMF.

An EMF is produced if $\frac{d\Phi_B}{dt} \neq 0$ and so if there is:

1) A changing B-Field
2) A changing Area
3) A changing angle

Changing B-Field:



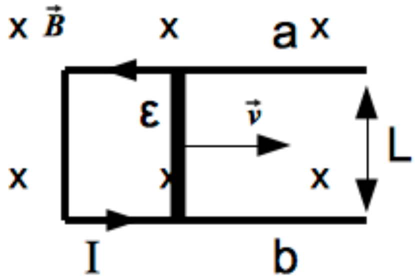
A B-field is perpendicular to the page and uniform within the circular region of radius R. Outside the region the B-field is zero. The rate of change of B within R is $\frac{dB}{dt}$.

1) Choose Area vector direction: Out of the page – chosen arbitrarily decides ε direction later.
2) Flux definition: $\Phi_B = \int \vec{B} \cdot d\vec{A} = -BA$ since B, A vectors are antiparallel (so ϕ=180°).
3) Induced EMF: $\epsilon = \frac{-d\Phi_B}{dt} = \frac{d}{dt}(BA)$ consider which terms involved in the flux vary.

When (r<R): Area is constant, $\epsilon = \frac{d}{dt}(BA) = A\frac{dB}{dt} = \pi r^2 \frac{dB}{dt}$

When (r>R): B-field is zero and constant outside R. Flux=0 outside R, only flux inside R contributes. $\epsilon = \pi R^2 \frac{dB}{dt}$ note that only the area that intercepts a change of flux is considered.

Changing Area: Slidewire Generator:

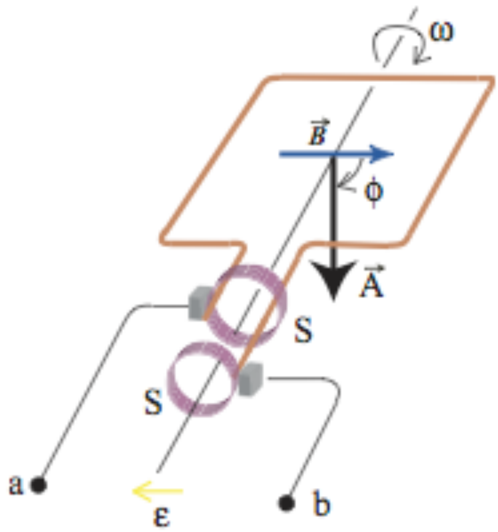


Choose area vector in the same direction as the B-field (into the page).

$\epsilon = \frac{-d\Phi_B}{dt} = -B\frac{dA}{dt}$, $dA = L v\, dt$

$\epsilon = -BLv$ acts in anti-clockwise direction due to the negative sign on ε.

Changing Area: Alternator:



Want to find induced EMF in Alternator/

Rectangular coil of N turns (1 in diagram).

Angular velocity, $\omega=\frac{d\phi}{dt}$

Angle as a function of time: $\phi=\omega t$

Magnetic Flux: $\Phi_B=N\int\vec{B}\cdot d\vec{A}=NBA\cos(\phi)$

Induced EMF:

$$\epsilon=\frac{-d\Phi_B}{dt}=-NBA\frac{d\cos(\phi)}{dt}=-NBA\frac{d\cos(\omega t)}{dt}$$

Only angle varies: $\epsilon=\omega NBA\sin(\phi)$

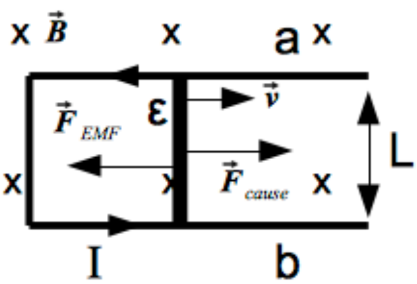
The EMF and the flux are out of phase by 90°.

Lenz's Law:

The direction of any magnetic induction effect is such as to oppose the cause of the effect.

Alternative formulation of Faraday's Law to determine the direction of an induced EMF. Required by conservation of energy.

The Slidewire Generator:



Cause of the effect: $\vec{F}_{cause}$ acting right.
Direction of EMF creates an opposing force.
Force from induced current: $\vec{F}_{EMF}=I\vec{L}\times\vec{B}$ points to the left.
Note that if Lenz's Law weren't true then a small push would quickly accelerate the slidewire and violate the conservation of energy.

Motional EMF:

EMF from a conductor element: $d\epsilon=(\vec{v}\times\vec{B})\cdot d\vec{l}$

Total EMF along a closed loop conductor: $\epsilon=\oint(\vec{v}\times\vec{B})\cdot d\vec{l}$

Equivalent to Faraday's Law but simpler to apply. Only appropriate for **moving conductors**. Remember: *The direction of any magnetic induction effect is such as to oppose the cause of the effect.*

Use the right-hand rule point your thumb in the direction of the B-field, if the B-field is going in to the page then the induced current is clockwise.

By Lenz's Law, the induced current must cause a B-field opposing the increase, so by the right hand rule, the induced current is clockwise.

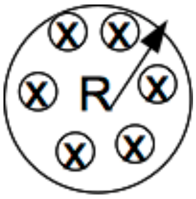
Induced Electric Field:

Induced EMF: $\oint\vec{E}\cdot d\vec{l}=\epsilon$

This is a special case of Faraday's Law: $\oint \vec{E} \cdot d\vec{l} = \frac{-d\Phi_B}{dt}$

Comparison with Electrostatic E-Fields:

Induced:	Electrostatic:
$\oint \vec{E} \cdot d\vec{l} = \epsilon$ Non-Conservative	$\oint \vec{E} \cdot d\vec{l} = 0$ Conservative
$\oint \vec{E} \cdot d\vec{A} = 0$ Field lines are loops	$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$ Field lines start/end.



Find the induced E-field at a distance r which is <R and then >R.
A B-field is perpendicular to the page and uniform in a circular region of radius R.
Outside the region the B field is 0.
The rate of change of the B field within R is $\frac{dB}{dt}$.

Case 1: r>R:
Choose integration path circle at radius r:
$\oint \vec{E} \cdot d\vec{l} = \oint E\, dl = E \oint dl$, $\vec{E} \| d\vec{l}$ and constant , $E \oint dl = 2\pi r E$
Magnetic Flux: $\Phi_B = \vec{B} \cdot \vec{A} = BA = B\pi R^2$ since only the area which intercepts the B lines is considered.
Change in Flux: $\frac{d\Phi_B}{dt} = \pi R^2 \frac{dB}{dt}$
Induced E-Field: $E = \frac{-R^2}{2\text{r}} \frac{dB}{dt}$

Case 2: r<R:
Choose integration path circle at radius r:
$\oint \vec{E} \cdot d\vec{l} = \oint E\, dl = E \oint dl$, $\vec{E} \| d\vec{l}$ and constant , $E \oint dl = 2\pi r E$
Magnetic Flux: $\Phi_B = \vec{B} \cdot \vec{A} = BA = B\pi r^2$
Change in flux: $\frac{d\Phi_B}{dt} = \pi r^2 \frac{dB}{dt}$
Induced E-field: $E = \frac{-r}{2} \frac{dB}{dt}$

Eddy Currents:
Currents can be induced within metal objects. As a disk rotates, EMF induced, this EMF causes currents which have a swirling pattern.

Inductance

Inductance is not the same as induction. If there is a coil with changing current then the magnetic flux through the coil changes and thus a current and emf is induced in the coil.

Self-Inductance occurs when there is only one circuit. Mutual inductance occurs when there are two nearby circuits.

Self-Inductance

$L=\frac{N\phi_B}{i}$ where L is inductance in Henrys (Tm^2A^{-1}) N is the number of coils and phi is the magnetic flux whilst *i* is the varying current (it is lowercase as it varies). Therefore the inductance is simply the magnetic flux divided by the current.

Inductors

Inductors are circuit elements with particular L value. They help to maintain a steady current flow as the self-induced emf opposed a change in the current. $\epsilon=-L\frac{di}{dt}$

Note: does not necessarily oppose current as if the current is decreasing then it will act with the current.

Can relate EMF to potential difference: $\epsilon=\int\vec{E}\cdot\vec{dl}=\Delta V$

Self-Inductance (cont.)

As the current varies: $L=\frac{N\phi_B}{i}$ $\rightarrow$ $Li=N\phi_B$ and taking time derivative $L\frac{di}{dt}=N\frac{d\phi_b}{dt}$

From Faraday's Law for N coils: $\epsilon=-N\frac{d\phi_B}{dt}$ we can see that the self-induced can be expressed as $\epsilon=-L\frac{di}{dt}$ Therefore the self-induced EMF opposes the change in current.

Example Inductor – Solenoid

For a solenoid with *n* turns per unit length, a coil radius of *a* and *I* current in the wire.

The B-field inside a solenoid: $B=\mu_0 n I$

$$\phi_B=\int\vec{B}\cdot\vec{dA}=B\int dA=B\pi a^2=\pi a^2\mu_0 n i$$

$$L=\frac{N\phi_B}{i}=N\pi a^2\mu_0=\mu_0 n^2(\pi a^2 l)$$

Mutual Inductance

For 2 nearby circuits, the flux through circuit 1 depends on the current in both circuit 1 and 2. If you change the current in coil 1 then the B-field changes and the flux in coil 2 changes and thus an EMF/Current is induced in coil 2.

B-field of coil 1 proportional to current in coil 1.

Flux in coil 2 is proportional to B-field of coil 1 and thus current of coil 1, the proportionality constant M_{21}, $N_2\phi_{B2}=M_{21}i_1$ where ϕ_{B2} is the flux in one loop.

The definition of mutual inductance $M_{21}=M_{12}=M$ and $M=\frac{N_2\phi_{B2}}{i_1}=\frac{N_1\phi_{B1}}{i_2}$

For a changing current i_1:

$N_2\frac{d\phi_{B2}}{dt}=M_{21}\frac{di_1}{dt}$ First term in Faraday's Law

$\epsilon_2=-M_{21}\frac{di_1}{dt}$ EMF in coil 2, induced by current 1

Therefore the mutually induced EMFs: $\epsilon_2=-M\frac{di_1}{dt}$ $\epsilon_1=-M\frac{di_2}{dt}$

Example – Nested solenoids

Flux in outer solenoid due to inner field B-field: $\phi_{B2} = \int \vec{B} \cdot \vec{dA} = B_1 \pi r_1^2$ Use r_1 as the B-field from coil 1 is zero outside of 1. (Coil 1 is the inner coil)

B-field of inner solenoid is $B_1 = \frac{\mu_0 N_1 i_1}{L}$ from Ampere's Law.

Therefore: $\phi_{B2} = \frac{\mu_0 N_1 i_1}{L}(\pi r_1^2)$ so $M = \frac{N_2 \phi_{B2}}{i_1} = \frac{\mu_0 N_1}{L}(\pi r_1^2)$

Energy in an inductor

Potential: $V = L\frac{di}{dt}$

Power: $P = Vi = Li\frac{di}{dt}$

Energy: $U = \int_0^t P\,dt = \int_0^I Li\,di = \frac{1}{2}LI^2$

The Magnetic Energy Density is $u = \frac{B^2}{2\mu_0}$ therefore if not in free space use μ not μ_0

Compare this with the density for the electric field: $u = \frac{1}{2}\epsilon_0 E^2$

Sign conventions for inductors

When the emf is zero, the current is constant and hence the potential difference across the inductor is zero.

If the emf opposes the current then $\frac{di}{dt} > 0$ so the current increases and thus as $\Delta V = L\frac{di}{dt} > 0$ so the voltage is positive. If the EMF is in the same direction as the current then the current decreases and the voltage across the inductor is negative.

LR circuits - Charging

Applying Kirchoffs Loop rule: $\epsilon - iR - L\frac{di}{dt} = 0$

Solving for rate of current change and solving differential equation: $i = \frac{\epsilon}{R}\left(1 - e^{\frac{-R}{L}t}\right)$

Thus the current starts at t=0, i=0 and rises exponentially to $I = \frac{\epsilon}{R}$

Time constant $\tau = \frac{L}{R}$ is the time at which the current equals $(1 - \frac{1}{e})$ times the final value.

LR Circuits – decay

The current decays if the battery is disconnected. Via Kirchoffs Loop: $0 = iR + L\frac{di}{dt}$

Solving diff eq: $i = I_0 e^{\frac{R}{L}t}$ so starts at initial current I_0 and has time constant $\tau = \frac{L}{R}$ which is the time at which the current equals $\frac{1}{e}$ times the initial value.

RC circuits

Time-varying current charge and voltage.
At t=0, no charge or current, at t=t, current = i and charge = q

Applying kirchoffs loop and subbing in dq/dt for I
$\frac{dq}{dt} = \frac{-1}{RC}(q - C\epsilon)$ solving for charge $q = C\epsilon(1 - e^{\frac{-t}{RC}})$
Therefore the charge starts at 0 and exponentially rises to $q_f = C\epsilon$ has time constant $t = RC$ where $w = q_f(1 - \frac{1}{e})$

Similarly $i = \frac{dq}{dt} = \frac{\epsilon}{R} e^{\frac{-t}{RC}}$ has time constant $\tau = RC$ at which current drops to $\frac{1}{e}$ of the initial current. As the current decreases exponentially with time as the capacitor charges.

RC circuit stored energy

Power: $P = i^2 R = \frac{\epsilon^2}{R} e^{\frac{-2t}{RC}}$

Energy: $\int_0^\infty P\,dt = \frac{\int_0^\infty \epsilon^2}{R} e^{\frac{-2t}{RC}}\,dt = \left[\frac{-\epsilon^2}{R} \cdot \frac{RC}{2} e^{\frac{-2t}{RC}}\right]_0^\infty = \frac{\epsilon^2 C}{2}$