# Mathematics with Physical Applications

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## 1 Ordinary Differential Equations

**Ordinary** - One independent variable, i.e. $y = f(x)$ not $z = f(x, y)$
**Order** - The highest order of the differential.
**Degree** - The power to which the highest order differential is raised i.e $\frac{dy}{dx} = 3y$ is first-order, first degree, $\frac{d^2y}{dx^2} + kx = 0$ is second-order, first-degree, $(\frac{d^3y}{dx^3})^2 - (\frac{dy}{dx})^3 = 0$ is third-order, second-degree.
**Linear** - No powers of the dependent variable (usually $y$) above 1.

An nth order differential equation has a general solution containing $n$ arbitrary constants. When initial conditions are specified the particular solution can be determined.

### 1.1 Method 1: Separable Variables

Applicable when $f(x, y) = g(x)h(y)$ so $\frac{dy}{dx} = g(x)h(y) \rightarrow \frac{dy}{h(y)} = g(x)dx$ and so solution is of the form: $\int \frac{dy}{dx} = xy \quad \therefore \frac{dy}{y} = xdx \quad \therefore ln(y) = \frac{x^2}{2} + c \quad \therefore y = Ae^{\frac{x^2}{2}}$ where $A = e^c$

### 1.2 Method 2: Homogeneous Equation

Applicable if $f(x, y) = g(\frac{y}{x})$ i.e. the function can be expressed entirely in multiples of $\frac{y}{x}$.
Let:

$$v = \frac{y}{x} \quad \therefore y = vx \quad \therefore \frac{dy}{dx} = x\frac{dv}{dx} + v \;\; \therefore \;\; x\frac{dv}{dx} + v = g(v)$$

which is separable to solve for $v$. Then multiply solution for v by x to obtain final solution for $y$.

Example: $\frac{dy}{dx} = \frac{x^2+y^2}{2x^2}$ so $\frac{dy}{dx} = \frac{1}{32} + \frac{1}{2}(\frac{y}{x})^2$ ,letting $v = \frac{y}{x} \quad \therefore y = vx \quad \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$ so $v + x\frac{dv}{dx} = \frac{1}{2} + \frac{v^2}{2} \quad \therefore x\frac{dv}{dx} = \frac{v^2}{2} - v + \frac{1}{2} = \frac{1}{2}(1 - v^2)$ so: $\frac{2dv}{(1-v)^2} = \frac{dx}{x}$ Integrating: $\frac{-2}{1-v} = ln(x) + c$ let $c = ln(A), 1 - v = \frac{2}{ln(Ax)} \quad \therefore v = 1 - \frac{2}{ln(Ax)}$ so: $y = x(1 - \frac{2}{ln(Ax)})$

### 1.3 Method 3: The exact equation

Equation is of form $P(x, y)dx + Q(x, y)dy = 0$
Solution is of the form: $F(x, y) = C$ where $C$ is a constant.
Can calculate F from properties of exact differential, that is: $\frac{\partial F}{\partial x} = P(x, y)$ and $\frac{\partial F}{\partial y} = Q(x, y)$
From this expression for F, can solve for y by using quadratic formula etc.
**Example:**
$\frac{dy}{dx} = \frac{2y+x}{5y-2x}$ so: $(2y + x)dx = (5y - 2x)dy \quad \therefore (2y + x)dx + (2x - 5y) = 0$
i.e. $P = 2y + x$ and $Q = 2x - 5y$. Verifying exact differential condition: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2$
So is exact and this method is applicable.
So:$\frac{\partial F}{\partial x} = 2y + x$ , $\frac{\partial F}{\partial y} = 2x - 5y$ From deduction: $F = \frac{x^2}{2} + 2yx - \frac{5y^2}{2} = constant$
This is a quadratic in y which can be solved using the quadratic formula to yield the final solution.

### 1.4 Method 4: The integrating factor

Applicable if $f(x, y) = Q(x) - P(x)y$ i.e. if the DE is in the form $\frac{dy}{dx} + P(x)y = Q(x)$
i.e. it is a linear first-order differential equation.
Multiply the equation by an integrating factor $R(x) = e^{\int P(x)dx}$
Then:

$$y = \frac{\int Q(x)e^{\int P(x)dx}dx + c}{e^{\int P(x)dx}}$$

**Example:**

$$\frac{dy}{dx}+\frac{y}{x}=4x^2 \;\therefore\; P(x)=\frac{1}{x} \quad \therefore \int P(x)dx=\int\frac{dx}{x}=lnx$$
$$\therefore\; y=\frac{\int 4x^3dx+c}{x}=\frac{x^4+c}{x}=x^3+\frac{c}{x}$$

Remember that any operation with the dependent variable cannot merely be absorbed in to the constant. The constant in the equation for y is the only integration constant considered, adding others during the integration is wrong.

# 2 Higher Order Differential Equations

To solve any any nth order linear ordinary differential equation:

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_1(x)\frac{dy}{dx}+a_0(x)y=f(x)$$

We must first find the solution to the *complementary equation* where $f(x)=0$ i.e. the homogeneous equation:

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+...+a_1(x)\frac{dy}{dx}+a_0(x)y=0$$

To determine a solution to the complementary equation we must find $n$ linearly independent functions that satisfy it. Then the general solution may be given as a linear superposition of these $n$ functions i.e.:

$$y_c(x)=c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)$$

where $y_c$ is the solution to the complementary equation (known as the *complementary function* and $y_n$ are the linearly independent functions.
A set of $n$ functions are linearly independent if there must not exist *any* set of constants $c_1,c_2,...,c_n$ such that: $c_1y_1(x)+c_2y_2(x)+...+c_ny_n(x)=0$ except for the trivial case where $c_n=0$
From these considerations (by differentiating (n-1) times to obtain a set simultaneous equations) we may obtain the test of the Vanishing Wronskian for linear independence of a set of functions. If:

$$W(y_1,y_2,...,y_n)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & \vdots \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & \cdots & y_n^{(n-1)} \end{vmatrix}\neq 0 \qquad (1)$$

Although a non-zero Wronskian ensures that the functions are linearly independent, a vanishing Wronskian does not necessarily guarantee that the functions are linearly dependent.

If the original equation is homogeneous then the complementary function $y_c$ is also the general solution but if not then the general solution is given by $y(x)=y_c(x)+y_p(x)$ where $y_p$ is the /emphparticular integral which is simply /emphany function which satisfies the original equation directly, provided it is linearly independent of $y_c(x)$. All of this discussion only concerns linear differential equations (no powers of y above 1) as non-linear are as hard as a priest in a preschool and often do not possess closed-form solutions so we need not worry our pretty little heads about them.

## 2.1 Linear equations with constant coefficients

These are similar to the more general form discussed above with the simplification that the coefficients are constants rather than functions of x, i.e.:

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+...+a_1\frac{dy}{dx}+a_0y=f(x)$$

To solve them we first find the complementary function, $y_c(x)$ and then find the particular integral $y_p(x)$. If the equation is homogenous (i.e. $f(x) = 0$) then we do not have to find a particular integral as the complementary function alone is the solution.
$y_c(x)$ must satisfy the complementary equation $a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1}y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = 0$ and contain $n$ arbitrary constants.
To find it we simply try a solution of the form $y = Ae^{\lambda x}$ which upon substituting it into the complementary equation and dividing through by $Ae^{\lambda x}$ we are left with a polynomial of $\lambda$ of order n, this is the *auxillary equation*:

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

We then solve the auxillary equation to obtain the n roots $\lambda_1, \lambda_2, ..., \lambda_n$.
If all these roots are real and distinct then the complementary function is given by:

$$y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + ... + c_n e^{\lambda_n x} \tag{2}$$

If some of the roots are complex then if the coefficients in the complementary function are real, which they usually are, then if any complex number is a root its complex conjugate must also be a root) then

$$y_c(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} = e^{\alpha x}(d_1 cos(\beta x) + d_2 sin(\beta x)) = Ae^{\alpha x} \left\{ \begin{array}{c} sin \\ cos \end{array} \right\} (\beta x + \phi) \tag{3}$$

where $\alpha$ and $\beta$ are from the complex roots $\alpha \pm i\beta$
If one root of the auxillary equation is repeated (i.e. let the root $\lambda_1$ occur $k$ times, where $k > 1$) then the complementary function is given by

$$y_c(x) = (c_1 + c_2 x + ... + c_k x^{k-1})e^{\lambda_1 x} + c_{k+1}e^{\lambda_{k+1}x} + c_{k+2}e^{\lambda_{k+2}x} + ... + c_n e^{\lambda_n x} \tag{4}$$

If more than one root is repeated then we simply extend the above equation. I.e. if $\lambda_1$ is a $k$ -fold root of the auxillary equation but $\lambda_2$ is also an $l$ -fold root where $k > 1, l > 1$ then:

$$\begin{aligned} y_c(x) = \ & (c_1 + c_2 x + ... + c_k x^{k-1})e^{\lambda_1 x} \\ + \ & (c_{k+1} + c_{k+2}x + ... + c_{k+l}x^{l-1})e^{\lambda_2 x} \\ + \ & c_{k+l+1}e^{\lambda_{k+l+1}x} + c_{k+l+2}e^{\lambda_{k+l+2}x} + ... + c_n e^{\lambda_n x} \end{aligned} \tag{5}$$

In our course we will most often be dealing only with second-order equations in this limiting case where n=2 the forms of the complementary function are as follows:
**MEMORISE THESE!**
When the roots are real and distinct: $y_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$
If the roots are complex:

$$y_c(x) = c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} = e^{\alpha x}(d_1 cos(\beta x) + d_2 sin(\beta x)) = Ae^{\alpha x} \left\{ \begin{array}{c} sin \\ cos \end{array} \right\} (\beta x + \phi)$$

where $\alpha$ and $\beta$ are from complex roots $\alpha \pm i\beta$ as above
If the roots are repeated (it is 2nd order so only two roots so if there is any repetition then it can only be that a single root appears twice) then $y_c(x) = (c_1 + c_2 x)e^{\lambda x}$

## 2.2 Examples:

Find the complementary function of the equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$
Set the RHS to zero to obtain the complementary function: $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$
Substitute $y = Ae^{\lambda x}$, $\frac{dy}{dx} = \lambda Ae^{\lambda x}$, $\frac{d^2y}{dx^2} = \lambda^2 Ae^{\lambda x}$ therefore $\lambda^2 Ae^{\lambda x} - 2\lambda Ae^{\lambda x} + Ae^{\lambda}x = 0$ and divide through by $Ae^{\lambda x}$ to obtain the auxillary equation:

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)(\lambda - 1) = 0 \tag{6}$$

which thus has a repeated root $\lambda = 1$ therefore from (4) with $n = 2, k = 2, \lambda_1 = 1$ we may see that the solution is: $y_c(x) = (c_1 + c_2 x)e^x$.

### 2.3 Finding the particular integral, $y_p$, by the *method of undetermined coefficients* :

If the RHS of the differential equation i.e. f(x) has of several forms then there are standard trial functions which can be tried for the particular integral , $y_p$, and the unknown parameters deduced. However **this method fails if any term in the assumed trial function is also contained within the complementary function**, $y_c$, if this is the case then the trial function must be multiplied by the smallest integer power of $x$ such that it will contain no term that is already in $y_c$.

**MEMORISE THESE!**

| $f(x)$ | $y_p$ |
|---|---|
| $ae^{rx}$ | $be^{rx}$ |
| $a_1 sin(rx) + a_2 cos(rx)$ ($a_1$ or $a_2$ may be zero) | $b_1 sin(rx) + b_2 cos(rx)$ |
| $a_0 + a_1 x + ... + a_N x^N$ (some $a_m$ may be zero) | $b_0 + b_1 x + ... + b_N x^N$ |

### 2.4 Example continued:

So to continue our earlier example where:
$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x$ and $y_c(x) = (c_1 + c_2 x)e^x$ we can see from the table that the trial solution is to use $be^{rx}$ however, this is already contained within $y_c$ as $c_1 e^x$ and as the complementary function also includes $c_2 xe^x$ we must multiply the given function by $x^2$ therefore $y_p(x) = bx^2e^x$ and $\frac{dy}{dx} = 2bxe^x + bx^2e^x$ and $\frac{d^2y}{dx^2} = 2be^x + 2bxe^x + 2bxe^x + bx^2e^x$ so substituting it into the differential equation (which the particular integral must solve directly):

$$\begin{aligned}(2b + 4bx + bx^2)e^x + (-4bx - 2bx^2)e^x + bx^2e^x &= e^x\\ (2b + 4bx + bx^2) + (-4bx - 2bx^2) + bx^2 &= 1\\ 2b + (4bx - 4bx) + (bx^2 - bx^2 + bx^2) &= 1\\ 2b &= 1\\ b &= \frac{1}{2}\\ \therefore y_p &= \frac{x^2e^x}{2}\end{aligned}$$

### 2.5 Constructing the general solution, $y(x) = y_c(x) + y_p(x)$

It is easiest to learn via example and so we will now consider the new equation:

$$\frac{d^2y}{dx^2} + 4y = x^2 sin(2x) \tag{7}$$

Let us set the RHS to zero in order to obtain the complementary equation:

$$\frac{d^2y}{dx^2} + 4y = 0 \tag{8}$$

Substitute in $y = Ae^{\lambda x}$, $\frac{dy}{dx} = \lambda Ae^{\lambda x}$, $\frac{d^2y}{dx^2} = \lambda^2 Ae^{\lambda x}$ to obtain:

$$\begin{aligned}\lambda^2 Ae^{\lambda x} + 4Ae^{\lambda x} &= 0\\ \lambda^2 + 4 &= 0\\ \lambda^2 &= -4\\ \lambda &= \pm 2i\end{aligned}$$

which are complex so from (3) with $\alpha = 0$, $\beta = 2$ we obtain $y_c = d_1 cos(2x) + d_2 sin(2x)$
Let us now try to find $y_p$ we see that $f(x) = x^2 sin(2x)$ which is the same as two cases in the table multiplied together so our trial function will simply be those trial functions multiplied together:
$(b_1 sin(2x) + b_2 cos(2x))(c_0 + c_1 x + c_2 x^2)$ multiplying these together and simplifying the arbitrary constants by representing their products as new arbitrary constants a,b,c... we obtain:

$(ax^2+bx+c)sin(2x)+(dx^2+ex+f)cos(2x)$ as the trial function. But this function has terms in sin(2x) and cos(2x) which also appear in the complementary function and therefore we must multiply the trial function through by x in order to obtain:

$$\begin{aligned}
y_p &= (ax^3+bx^2+cx)sin(2x)+(dx^3+ex^2+fx)cos(2x) \\
\tfrac{dy}{dx} &= (3ax^2+2bx+c)sin(2x)+2(ax^3+bx^2+cx)cos(2x)+(3dx^2+2ex+f)cos(2x)-2(dx^3+ex^2+fx)sin(2x) \\
&= (3ax^2+2bx+c-2dx^3-2ex^2-2fx)sin(2x)+(2ax^3+2bx^2+2cx+3dx^2+2ex+f)cos(2x) \\
\tfrac{d^2y}{dx^2} &= (6ax+2b-6dx^2-4ex-2f)sin(2x)+(6ax^2+4bx+2c-4dx^3-4ex^2-4fx)cos(2x)+ \\
&\quad (6ax^2+4bx+2c+6dx+2e)cos(2x)+(-4ax^3-4bx^2-4cx-6dx^2-4ex-2f)sin(2x) \\
&= (-4ax^3+(-4b-12d)x^2+(6a-8e-4c)x+(2b-4f))sin(2x)+ \\
&\quad (-4dx^3+(12a-4e)x^2+(8b+6d-4f)x+(4c+2e))cos(2x)
\end{aligned}$$

Subbing these into (7):

$$(0x^3-12dx^2+(6a-8e)x+(2b-4f))sin(2x)+(0x^3+12ax^2+(8b+6d)x+(4c+2e))cos2x = x^2sin(2x)$$

Comparing coefficients of LHS and RHS:

$$\begin{aligned}
-12d = 1 \quad &\therefore d = \frac{-1}{12} \\
12a = 0 \quad &\therefore a = 0 \\
6a-8e = 0 \quad \therefore -8e = 0 \quad &\therefore e = 0 \\
8b+6d = 0 \quad \therefore 8b = \frac{6}{12} \quad &\therefore b = \frac{1}{16} \\
2b-4f = 0 \quad \therefore 4f = \frac{1}{8} \quad &\therefore f = \frac{1}{32} \\
4c+2e = 0 \quad \therefore 4c = 0 \quad &\therefore c = 0
\end{aligned}$$

So putting these values of a,b,c... into the equation for $y_p$ we get: $y_p = \frac{-x^3}{12}cos(2x)+\frac{x^2}{16}sin(2x)+\frac{1}{32}cos(2x)$ so combining this with the complementary function we obtain the general solution to (7):
$y = y_c + y_p = d_1cos(2x)+d_2sin(2x)-\frac{x^3}{12}cos(2x)+\frac{x^2}{16}sin(2x)+\frac{1}{32}cos(2x)$
where $d_1$ and $d_2$ are arbitrary constants whose values are determined by boundary conditions - here we have two constants and thus would require two boundary conditions to determine their value and obtain a specific solution.

# 3 Vector Analysis

$$\vec{\nabla} = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$\vec{\nabla}^2\phi = \vec{\nabla}\cdot\vec{\nabla}\phi = \frac{\partial^2\phi_x}{\partial x^2}+\frac{\partial^2\phi_y}{\partial y^2}+\frac{\partial^2\phi_z}{\partial z^2}$$

In cylindrical and polar:

$$\vec{\nabla}f = \frac{\partial f}{\partial\rho}\hat{\rho}+\frac{1}{\rho}\frac{\partial f}{\partial\theta}\hat{\theta}+\frac{\partial f}{\partial z}\hat{z}$$

$$\vec{\nabla}f = \frac{\partial f}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$$

$$\vec{\nabla}^2u \text{ in cylindrical: } \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial u}{\partial\rho})+\frac{1}{\rho^2}\frac{\partial^2u}{\partial\phi^2}+\frac{\partial^2u}{\partial z^2}$$

$$\vec{\nabla}^2u \text{ in spherical: } \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

# 4 Fourier Series

Fourier series are analogous to the Taylor Power Series in that they approximate a function via an infinite series of terms, however rather than using polynomial terms as in the Taylor series, sinusoidal terms are used (i.e. sin and cos).
As every function may require both a set of sin and cosine terms of differing frequency and amplitude in order to completely approximate the function, the sine and cosine terms are often referred to as orthogonal, it helps to think of this in terms of x and y which are the cartesian unit vectors, every point may require both x and y co-ordinates to completely describe it and thus the x and y vectors are orthogonal.
However in the cartesian system there are some points such as (n,0) or (0,n) which require only an x or a y co-ordinate to completely describe the point. Likewise, in Fourier series any function which is odd, is described entirely by sine terms and any function which is even is described entirely by cosine terms. General functions which are neither even nor odd must be described by a combination of both sine and cosine terms. The sine and cosine components are known as the basis functions.
For a function to be able to be approximated by a Fourier expansion it must satisfy the Dirichlet conditions: The function must be periodic (Non-periodic functions may be approximated by extending them outside of their domain as if they were periodic, see $x^2$ example.) It must be single-valued and continuous(The function may contain a finite number of finite discontinuities such as the square wave which has two finite discontinuities per cycle) It must have only a finite number of maxima and minima within one period The integral over one period of $|f(x)|$ must converge.

## 4.1 Orthogonal Properties

Because the sine and cosine terms are orthogonal there are certain properties of integrals of their products which may be derived from the properties of the trigonometrical functions although their derivation is not important to the application of Fourier analysis and will not be detailed here. It is not even necessary to know these properties, it is sufficient to simply memorise the results for the Fourier coefficients that they lead to however one may wish to use them to check the formulas for the Fourier coefficients and thus the properties are detailed here.
$\int_{x_0}^{x_0+L} \sin(\frac{2\pi rx}{L})\cos(\frac{2\pi px}{L}) = 0$ for all values of r and p

$$\int_{x_0}^{x_0+L} \cos(\frac{2\pi rx}{L})\cos(\frac{2\pi px}{L}) = \begin{cases} L \text{ if } r = p = 0 \\ \frac{L}{2} \text{ if } r = p > 0 \\ 0 \text{ if } r \neq p \end{cases}$$

$$\int_{x_0}^{x_0+L} \sin(\frac{2\pi rx}{L})\sin(\frac{2\pi px}{L}) = \begin{cases} L \text{ if } r = p = 0 \\ \frac{L}{2} \text{ if } r = p > 0 \\ 0 \text{ if } r \neq p \end{cases}$$

## 4.2 The Fourier Expansion

The general form of the Fourier expansion in sine and cosine form (as opposed to complex exponential form is:

$$f(x) \approx \frac{a_0}{2} + \sum_{r=1}^{\infty}\left[a_r \cos\left(\frac{2\pi rx}{L}\right) + b_r \sin\left(\frac{2\pi rx}{L}\right)\right]$$

where L is the period. (You could also write this as $\cos(\omega rx)$ where omega is the angular frequency $\omega = 2\pi f = \frac{2\pi}{T}$ As with the Taylor series the most difficult part is in determining the coefficients of each of the terms.
The equations for coefficients are:

$$a_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\cos\left(\frac{2\pi rx}{L}\right)dx$$

$$b_r = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)\sin\left(\frac{2\pi rx}{L}\right)dx$$

Thus: $a_0 = \frac{2}{L}\int_{x_0}^{x_0+L} f(x)dx$ and hence the first term in the Fourier expansion $\frac{a_0}{2}$ is simply the average value of the function over a cycle.

### 4.3 Symmetry considerations

If f(x) multiplied by sine/cosine is even then we can simply evaluate half of the period and double it, therefore it is useful to remember the following: The product of two even functions is an even function. The product of two odd functions is an even function. The product of an even function and an odd function is an odd function.
Aside from the obvious considerations of whether f(x) is symmetrical (even) or anti-symmetrical (odd) about the origin, there is also an interesting symmetry that occurs about the quarter period (i.e. if we are going from 0 to T, then the quarter period is the instant T/4). The derivation of this is not important and can be found elsewhere, but the result is as follows:
If the function is even about the quarter period, then there are no odd components of cosine and no even components of sine. (By this I mean that for all odd values of r the value of a is 0, and for all even values of r the value of b is 0.)
Similarly, If the function is odd about the quarter period then there are no even components of cosine and no odd components of sine.

### 4.4 Discontinuities and Non-periodic functions

Functions which have a discontinuity such as the square wave may still be approximated by the Fourier series, at the discontinuity the Fourier expansion will have the value halfway between the two values either side of the discontinuity for example if you have a square wave that goes from -1 to 1 at x=0 then the Fourier series will pass through the point (0,0). Either side of the discontinuity the expansion will overshoot the function it is describing, this overshoot continues to exist no matter how many harmonics are used to approximate the function, the magnitude of the overshoot is proportional to the size of the discontinuity (i.e. the difference between the values either side of the discontinuity). This behaviour is known as Gibbs' phenomenon.
Functions which are not periodic can still be approximated by the Fourier series within a given region by extending them beyond that region as if they were periodic, it often makes sense to extend them in such a way that the function is odd or even such that the amount of work required may be reduced as odd functions have no cosine components and sine functions have no sine components.

### 4.5 Integrating and differentiating Fourier Series

Sometimes we can find the Fourier series of a function by integrating or differentiating the Fourier series of another function. When integrating we must determine the value of the constant of integration. If f(x) satisfies the Dirichlet conditions then the Fourier series obtained by differentiating the Fourier series of f(x) term by term will correctly approximate f'(x) provided that f'(x) also obeys the Dirichlet conditions. Therefore we may obtain complicated Fourier series by the differentiation and integration of simpler series.

### 4.6 Exponential representation

$$f(x) \approx \sum_{r=-\infty}^{\infty} c_r exp\left(\frac{2\pi irx}{L}\right)$$

with coefficients:

$$C_r = \frac{1}{L}\int_{x_0}^{x_0+L} f(x) exp\left(\frac{2\pi irx}{L}\right)dx$$

Note that the trigonometric and exponential coefficients may be related by $C_r = \frac{1}{2}(a_r - \ ib_r)$ and $C_{-r} = \frac{1}{2}(a_r + ib_r)$ thus $C_{-r} = C_r^*$ $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$ and $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

## 5 Partial Differential Equations

A partial differential equation is of the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial dxdy} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

These equations occur in many physical phonomena such as Laplace's equation, the wave equation, the heat transport equation and Schrödinger's equation.

## 5.1 Separation of variables

here we assume that the function $u(x, y)$ can be written as the product of two separate functions in $x$ and $y$, i.e.:

$$u(x, y) = X(x)Y(y)$$

### 5.1.1 A first-order example:

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \text{ with the boundary condition: } u(0, y) = 6e^{-3y}$$

$$\therefore\ Y\frac{dX}{dx} = 4X\frac{dY}{dy}\ \therefore\ \frac{1}{X}\frac{dX}{dx} = \frac{4}{Y}\frac{dY}{dy} = c$$

This yields two ODEs for $X$ and $Y$, for $X$:

$$\frac{1}{X}\frac{dX}{dx} = c\ \therefore\ \frac{dX}{dx} = cX\ \therefore\ \frac{dX}{dx} - cX = 0$$

and so the characteristic equation is:

$$m - c = 0\ \therefore\ m = c\ \therefore\ X = ae^{cx}$$

Similarly for $Y$:

$$\frac{4}{Y}\frac{dY}{dy} = c\ \therefore\ \frac{dY}{dy} - \frac{cY}{4} = 0\ \therefore\ m - \frac{c}{4} = 0\ \therefore\ m = \frac{c}{4}\ \therefore\ Y = be^{\frac{c}{4}y}$$

Recall that $u = X(x)Y(y)$ so:

$$u(x, y) = ae^{cx} \cdot be^{\frac{c}{4}y} = de^{c\left(x+\frac{y}{4}\right)}$$

Applying the boundary condition $u(0, y) = 8e^{-3y}$, we see that $d = 8$ and $c = -12$, and so the final solution is:

$$u(x, y) = 8e^{-12\left(x+\frac{y}{4}\right)}$$

### 5.1.2 A second-order example:

Taking a specific example of heat transport in one-dimension:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \text{ in the range: } 0 < x < 3$$

with boundary conditions:

$$u(0, t) = u(3, t) = 0 \text{ and } u(x, 0) = 5sin(4\pi x) - 3sin(8\pi x) + 2sin(10\pi x)$$

so assuming the separation of variables:

$$\frac{1}{2T}dTdt = \frac{1}{X}d^2Xdx^2 = -\lambda^2$$

We use $-\lambda^2$ instead of $c$ because this provides us with an oscillatory solution. It is standard for solving second-order PDEs. So as previously we solve the two ODEs separately, for $T$:

$$\frac{dT}{dt} = -2\lambda^2 T\ \therefore\ \frac{dT}{dt} + 2\lambda^2 T = 0\ \therefore\ m + 2\lambda^2 = 0\ \therefore\ m = -2\lambda^2\ \therefore\ T = ae^{-2\lambda^2 t}$$

and for $X$:

$$\frac{d^2X}{dx^2} = -\lambda^2 x\ \therefore\ \frac{d^2X}{dx^2} + \lambda^2 x = 0\ \therefore\ m^2 + \lambda^2 = 0\ \therefore m^2 = -\lambda^2\ \therefore m = \pm\lambda i$$

$$\therefore X = c_1 e^{\lambda i x} + c_2 e^{-\lambda i x} = bsin(\lambda x) + ccos(\lambda x)$$

Note that the constant b may be imaginary.

$$\therefore u = ae^{-2\lambda^2 t}\left(bsin(\lambda x) + ccos(\lambda x)\right) \text{ combining constants: } \quad = e^{-2\lambda^2 t}\left(bsin(\lambda x) + ccos(\lambda x)\right)$$

Applying BC $u(0,t) = 0$ we see that $c = 0$. Applying BC that $u(3,t) = 0$ we see that:

$$e^{-2\lambda^2 t} \cdot dsin(3\lambda) = 0 \ \therefore \ \lambda = \frac{n\pi}{3}$$

However this $n$ condition gives us a series, so we can write that:

$$u = \sum_{n}^{N?} e^{-2\lambda_n{}^2 t} \cdot b_n \cdot sin\left(\frac{n\pi}{3} x\right)$$

But we know that $u(x,0) = 5sin(4\pi x) - 3sin(8\pi x) + 2sin(10\pi x)$ and so:

$$5sin(4\pi x) - 3sin(8\pi x) + 2sin(10\pi x) = b_1 sin\left(\frac{\pi n_1}{3} x\right) + b_2 sin\left(\frac{\pi n_2}{3} x\right) + b_3 sin\left(\frac{\pi n_3}{3} x\right)$$

So we see that:

$$n_1 = 12 \ n_2 = 24 \ n_3 = 30 \ b_1 = 5 \ b_2 = -3 \ b_3 = 2$$

We must also substitute for $\lambda$ in the exponential term, noting that $\lambda_i = \frac{n_i \pi}{3}$ so $\lambda_i^2 = \frac{n_i^2 \pi^2}{9}$, and so we obtain the final solution:

$$u(x,t) = 5e^{-32\pi^2 t} sin(4\pi x) - 3e^{-128\pi^2 t} sin(8\pi x) + 2e^{-200\pi^2 t} sin(10\pi x)$$