# PHY2007 - Relativity II and Mechanics

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# Vector Manipulation

## Multiplication

**Scalar Product:**

$$\vec{A} \cdot \vec{B} = |A||B| \cos(\theta)$$

This is true in all co-ordinate systems.
In orthogonal co-ordinates systems i.e. where $A = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$, $B = B_1\hat{e}_1 + B_2\hat{e}_2 + B_3\hat{e}_3$ and $\hat{e}_1$, $\hat{e}_2$, $\hat{e}_3$ are orthogonal unit vectors, then the scalar product can also be defined as:

$$\vec{A} \cdot \vec{B} = A_1B_1 + A_2B_2 + A_3B_3$$

The scalar product gives the projection of $\vec{B}$ along $\vec{A}$ multiplied by $|\vec{A}|$. It takes two vectors and returns a scalar quantity.

**Vector (Cross) Product:**
Where $\hat{n}$ is a unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$, this is defined as:

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \sin(\theta)\hat{n}$$

This gives the area of the rhombus created by the vectors $\vec{A}$ and $\vec{B}$. $\hat{n}$ is given by the right hand rule and so:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

i.e. it is not commutative.

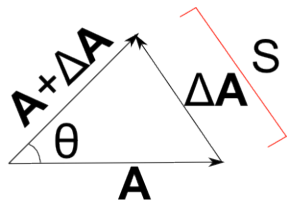
**Vector Triple Product:**
This is different from the Scalar Triple Product which returns a scalar quantity. It is defined as:

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{C} \cdot \vec{A})\vec{B} - (\vec{B} \cdot \vec{A})\vec{C}$$

Note that:

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

### Vector Differentiation




The distance of $\Delta\vec{A}$ is $\Delta S$. The definition of the scalar derivative is:

$$\frac{d\vec{A}}{dS} = \lim_{\Delta\to 0}\frac{\Delta\vec{A}}{\Delta S}$$

## Velocity and Acceleration

### Rectangular co-ordinates (e.g. Cartesian)

Here $\vec{r}$ is defined by a set of co-ordinates along three static, orthogonal axes, $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$.

$$\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$$

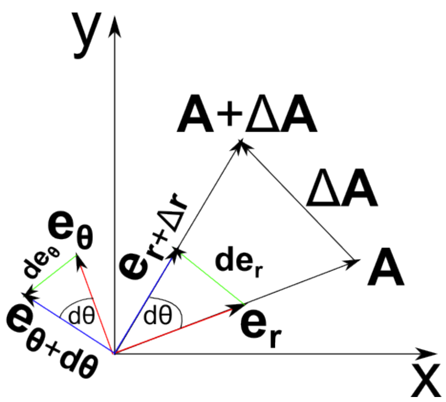
$$\vec{v} = \dot{\mathbf{r}} = \frac{d\vec{r}}{dt} = \frac{dx_1}{dt}\hat{e}_1 + \frac{dx_2}{dt}\hat{e}_2 + \frac{dx_3}{dt}\hat{e}_3$$

$$\vec{a} = \ddot{\mathbf{r}} = \frac{d\vec{v}}{dt} = \frac{d^2x_1}{dt^2}\hat{e}_1 + \frac{d^2x_2}{dt^2}\hat{e}_2 + \frac{d^2x_3}{dt^2}\hat{e}_3$$

This is simple since the position of the unit vectors does not change with the movement of the vectors, and so they are static. This is not the case for polar co-ordinates.

### Plane Polar co-ordinates

Here a vector is defined by an angle $\theta$ to the vertical axis and the distance $r$ from the origin. As the vector changes, the position of the unit vectors also change. We need to work out




*how* the unit vectors change with respect to the motion. The magnitude of the change is given by: $|d\vec{e}_r| = |\vec{e}_r|d\theta = d\theta$. $d\vec{e}_r$ is perpendicular to $\vec{e}_r$ and so parallel to $\vec{e}_\theta$. So:

$$d\vec{e}_r = d\theta\vec{e}_\theta \text{ similarly } d\vec{e}_\theta = -d\theta\vec{e}_r$$

So:

$$\vec{r} = r\vec{e}_r$$

$$v = \frac{d\vec{r}}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$

So the velocity has radial and transverse components To calculate the acceleration we simply apply the product rule again:

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt} + r\frac{d^2\theta}{dt^2}\vec{e}_\theta$$

$$\vec{a} = \ddot{r}\vec{e}_r + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + \dot{r}\dot{\theta}\vec{e}_\theta - r\dot{\theta}^2\vec{e}_r + r\ddot{\theta}\vec{e}_\theta = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta$$

## General Recipe

1. Write down position vectors in chosen co-ordinates.
2. Write down the unit vectors in Cartesians.
3. Differntiate the unit vectors in Cartesians to get the time derivatives.
4. Substitute the time derivatives for the unit vectors in to the position vectors in the chosen co-ordinates.

## Cylindrical Polar Co-ordinates

Here the co-ordinates are defined by $(r, \phi, z)$. Remember that (since $\phi$ is a function of $t$):

$$\mathbf{e}_r = (\cos(\phi), \sin(\phi), 0),\ \dot{\mathbf{e_r}} = (-\dot{\phi}\sin(\phi), \dot{\phi}\cos(\phi), 0) = \dot{\phi}\mathbf{e}_\phi$$

$$\mathbf{e}_\phi = (-\sin(\phi), \cos(\phi), 0)$$

So:

$$\mathbf{r} = \mathbf{re_r} + \mathbf{ze_z}$$

$$\mathbf{v} = \dot{\mathbf{r}}\mathbf{e_r} + r\dot{\mathbf{e_r}} + \dot{z}\mathbf{e_z}$$

$$\mathbf{a} = (\ddot{\mathbf{r}} - \mathbf{r}\dot{\phi}^2)\mathbf{e_r} + (\mathbf{r}\ddot{\phi} + 2\dot{\mathbf{r}}\dot{\phi})\mathbf{e}_\phi + \ddot{\mathbf{z}}\mathbf{e_z}$$

## Spherical Polar Co-ordinates

Here the co-ordinates are defined by $(r, \theta, \phi)$. Remember that:

$$\mathbf{e_r} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$$

$$\mathbf{e}_\phi = (\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), \sin(\theta))$$

$$\mathbf{e}_\phi = (-\sin(\phi), \cos(\phi), 0)$$

So:

$$\mathbf{v} = \dot{\mathbf{r}}\mathbf{e_r} + \mathbf{r}\dot{\theta}\mathbf{e}_\theta - r\sin(\theta)\dot{\phi}\mathbf{e}_\phi$$

$$\mathbf{a} = (\ddot{\mathbf{r}} - \mathbf{r}\dot{\theta}^2 - \mathbf{r}\dot{\phi}^2\sin^2(\theta))\mathbf{e_r} + (\mathbf{r}\ddot{\theta} + 2\dot{\mathbf{r}}\dot{\theta} - \mathbf{r}\dot{\phi}^2\sin(\theta)\cos(\theta))\mathbf{e}_\theta + (\mathbf{r}\ddot{\phi}\sin(\theta) + 2\dot{\mathbf{r}}\dot{\phi}\sin(\theta) + 2\mathbf{r}\dot{\theta}\dot{\phi}\cos(\theta))\mathbf{e}_\phi$$

## Newton's Laws

1. Bodies continue at rest or in uniform motion unless acted on by a force.

2. The force is equal to the rate of change of momentum (where momentum is the product of the mass and velocity).

3. The forces of action and reaction are equal and opposite.

Rephrasing the Third Law, for isolated bodies, the accelerations of interaction are:

1. In opposite directions.

2. In constant ratio.

3. Where the ratio is given by:

$$\text{Ratio} = \frac{1}{\text{ratio of masses}}$$

This leads to a definition of mass but leaves the possibility that mass depends on the nature of the force. There were thought to be two distinct types of mass:

**Gravitational mass:** Mass under gravitation.

**Inertial mass:** Mass under any other force.

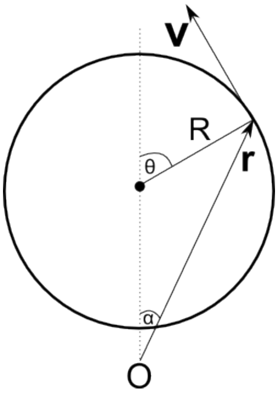
However, these masses were observed to be equivalent and General Relativity assumes their equivalence.

# Motion of a particle under a central force

## Circular Motion

At any instant , any motion can be described as circular motion about a particular axis (i.e. a circle of different radius and position).
If we define the scalar angular speed as:

$$\omega = \frac{d\theta}{dt}$$




Linear velocity, $\mathbf{v}$:

$$|\mathbf{v}| = R\frac{d\theta}{dt}$$

Then if we define the vector, angular velocity, $\omega$ in the plane perpendicular to the circle then:

$$|\mathbf{v}| = |\mathbf{r}|\sin(\alpha)\frac{d\theta}{dt} = |\mathbf{r}|\sin(\alpha)\omega$$

So:

$$\mathbf{v} = \omega \times \mathbf{r}$$

This is the implicit definition of angular velocity. It is important to note that $\omega$ acts in a direction perpendicular to the linear velocity, like a corkscrew.

## Kinematic Relationships

$$\mathbf{v} = \mathbf{v_0} + \mathbf{a}t \ , \ \omega = \omega_\mathbf{0} + \dot{\omega}t$$

## Angular Momentum

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

note that $\mathbf{L}$ is perpendicular to the $\mathbf{r} \cdot \mathbf{p}$ plane.
To derive the basic scalar formula $L = mr^2\omega$:

$$\mathbf{p} = m\mathbf{v} = m(\vec{\omega} \times \mathbf{r})$$

$$\mathbf{L} = m\mathbf{r} \times (\vec{\omega} \times \mathbf{r}) = m(\mathbf{r} \cdot \mathbf{r})\vec{\omega} - m(\vec{\omega} \cdot \mathbf{r})\mathbf{r} = mr^2\vec{\omega}$$

As $\vec{\omega} \cdot \mathbf{r} = 0$ as the vectors are perpendicular to one another.

### Torque

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}$$

### Relationship between torque and angular momentum:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$
$$\frac{d\mathbf{L}}{dt} = \frac{d\mathbf{r}}{dt} \times \mathbf{p} + \mathbf{r} \times \frac{d\mathbf{p}}{dt} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F} = \mathbf{N}$$

So if angular momentum is conserved then there is no torque.

### Central Force

Acts purely in the radial direction, i.e. the are no transverse components. So:

$$\mathbf{F}(r) = F(r)\mathbf{e_r}$$

e.g. gravity and electromagnetism.
Central forces have the properties:

1. Exert no torque as $\mathbf{r} \times F(r)\hat{r} = 0$
2. Therefore conserve angular momentum, $\frac{d\mathbf{L}}{dt} = 0$

# Orbital Mechanics

For a small mass $m$ orbiting a large, fixed mass $M$, Newton's Law of Gravitation states that:

$$\vec{F} = \frac{-GmM}{r^2}\hat{e_r} = m\vec{a}$$

Putting this in Plane Polar co-ordinates:

$$\frac{-GmM}{r^2}\hat{e_r} = m\left[(\ddot{r} - r\dot{\theta}^2)\hat{e_r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e_\theta}\right]$$

Each component of this equation must balance:
The radial equation becomes:

$$\frac{-GM}{r^2} = \ddot{r} - r\dot{\theta}^2$$

The tangential/transverse equation becomes:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

### Special Case when the radius, r, is constant

In this case the transverse equation simplifies to:

$$r\ddot{\theta} = 0 \;\therefore\; \dot{\theta} = \omega = \text{ constant}$$

The radial equation simplifies to:

$$\frac{-GM}{r^2} = -r\omega^2 \;\therefore \omega = \sqrt{\frac{GM}{r^3}}$$

And so the angular frequency is independent of the small mass, $m$.

## If the radius, r, is not constant

Here the transverse equation simplifies to:

$$\frac{\partial}{\partial t}\left(r^2\dot{\theta}\right) = 0 \;\therefore\; r^2\dot{\theta} = \text{ constant } = \omega r^2$$

For a circular orbit (r is constant), the area swept out by an angle $d\theta$ is $dA = r \cdot (rd\theta)$, and so the area swept out in a unit time is:

$$\frac{dA}{dt} = r^2\frac{d\theta}{dt} = r^2\dot{\theta} = \text{ constant}$$

So the area swept out in a unit time is independent of the small mass $m$. This led to Kepler's Laws.
The radial equation cannot be simplified much, and is difficult to solve:

$$\ddot{r} = \frac{-GM}{r^2} + r\dot{\theta}^2$$

Instead we use energy considerations.

## Energy Consideration

$$T = \frac{1}{2}m(\vec{v}\cdot\vec{v}) = \frac{1}{2}m\left[\dot{r}^2 + r^2\dot{\theta}^2\right]$$

$$\vec{L} = mr^2\vec{\omega} \;\therefore |\vec{L}| = mr^2\dot{\theta} = \text{ constant for central force}$$

$$\therefore\; T = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2}$$

Potential Energy, $U$:

$$U = -\int \vec{F}\cdot d\vec{r} = \frac{-GMm}{r}$$

Total Energy, $E$:

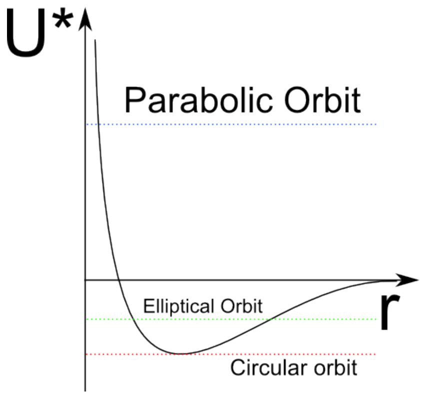
$$E = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GmM}{r} = \text{ constant}$$

This equation gives the relation between $\dot{r}$ and $r$. So the total energy is the sum of the radial kinetic energy, the transverse kinetic energy and the (gravitational) potential energy. But the transverse kinetic energy looks like a potential energy and so can be associated with a pseudoforce called the **centrifugal force**.
We can write the function for the pseudo-potential as:

$$U^{\star}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r}$$

The parabolic orbit is an orbit which approaches to the distance of closest approach to the central mass and then returns to a great distance away. The elliptical orbit is bound by two points when $\dot{r} = 0$ of minimum and maximum radii. The circular orbit exists only with an orbit at the minimum of the function and so is defined by:

$$r_{\text{circular}} = \frac{L^2}{GM}$$




# Dynamics of Particle arrays

In this case we must define a mean property and the properties of the individual particles with respect to that mean.
We define the centre of mass so that the moments of the system are the same as if all the mass were centred at that position. So $M = \sum_i m_i$, choose $\vec{R}$ such that:

$$\vec{R}M = \sum_i m_i \vec{r}_i = \int \vec{r} dm$$

where $dm$ is the mass at position $\vec{r}$.

## Newton's Laws

$$\vec{F}_i = \text{ external force on the } i\text{th particle}$$
$$\vec{F}_{ij} = \text{ internal force between particles } i \text{ and } j$$

Newton's Third Law states that $\vec{F}_{ij} = -\vec{F}_{ji}$, Newton's Second Law states that for particle $i$:

$$m\mathbf{\ddot{r}_i} = \vec{F}_i + \sum_{i \neq j} \vec{F}_{ij}$$

Summing this over all particles:

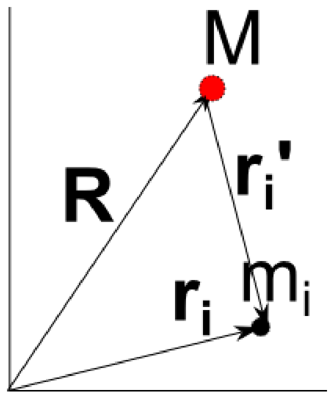
$$\sum_i m\mathbf{\ddot{r}_i} = \sum_i \vec{F}_i + \sum_i \sum_{j \neq i} \vec{F}_{ij}$$

The sum of the internal forces cancels out due to the third law, and the other term is simply the total external force on the particles. So:

$$M\mathbf{\ddot{R}} = \vec{F}$$

i.e. the centre of mass is independent of internal forces. Similarly:

$$\vec{P} = M\mathbf{\dot{R}}$$




## Angular Momentum

We define the positions with respect to the centre of mass:

$$\vec{r_i} = \vec{R} + \vec{r_i'}\ \vec{L} = \sum_i \vec{L_i} = \sum_i \vec{r_i} \times \vec{p_i}$$

But from definition of $\vec{R}$ we know that $\sum_i m_i \vec{r'}_i = 0$, so:

$$\vec{L} = \vec{R} \times \vec{P} + \sum_i \vec{r'}_i \times \vec{p'}_i$$

Where $\vec{R} \times \vec{P}$ is the angular momentum of the centre of mass about the origin and $\sum_i \vec{r'}_i \times \vec{p'}_i$ is the angular momentum of the particles about the centre of mass.

## Dynamics Of Rigid Bodies (if they exist)

Collection of points who relative positions are fixed, any motion is equivalent to a translation of any point in the body and rotation about the body. I.e.

$$\vec{v}(x) = \text{ velocity of point x} = \vec{v}_T + \vec{\omega} \times \vec{r}(x)$$

Where $\omega$ is the same for all points, the algebra is easier if the point considered is the centre of mass.
Kinetic Energy:

$$T = \frac{1}{2} \sum_i m_i (\vec{v_T} + \vec{\omega} \times \vec{r_i}) \cdot (\vec{v_T} + \vec{\omega} \times \vec{r_i})$$

But for the special case of plane rotation then $\vec{v_T} = 0$

$$\therefore\ T = \frac{1}{2} \sum_{a,b} I_{ab} \omega_a \omega_b$$

Where $I_{ab}$ is the inertia tensor. $I_{ij} = -I_{ji}$ for $i \neq j$ are the products of inertia, the diagnol parts of the matrix $I_{ii}$ are the moments of inertia.

Special case: Planar body rotating about an axis perpendicular to the plane:
So $\vec{\omega} \perp \vec{r_i}$ for all i, and so:

$$T = \frac{1}{2}\sum_i m_i(\vec{v_T} + \vec{\omega} \times \vec{r_i}) \cdot (\vec{v_T} + \vec{\omega} \times \vec{r_i}) = \frac{1}{2}\sum_i m_i {r_i}^2 \omega^2 = \frac{1}{2} I\omega^2$$

Where $I = \sum_i m_i {r_i}^2$ and the dimensions are mass $\times$ length$^2$.

$$I = \int_{\text{body}} r^2 dm$$

where $dm$ is the mass of an element at a distance $r$ from the axis of rotation.
Comparing to $T = \frac{1}{2}mv^2$ then $I$ plays the role of mass in rotational motion. So:

$$\vec{p} = m\vec{v} \ \rightarrow \ \vec{L} = I\vec{\omega}$$

$$\vec{L} = \sum_i L_i = \sum m_i \vec{r_i} \times (\vec{\omega} \times \vec{r_i}) = \sum_i m_i {r_i}^2 \omega$$

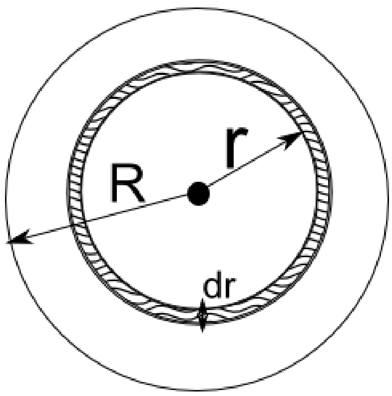
This also applies to the kinematic equations so $Power = N\omega$, $\omega^2 = 2\alpha\theta + \omega_0^2$.

# Calculations of I:

## Thin Disc rotating about axis through its centre:

$$I = \int r^2 dm$$

Where $dm$ is a circle at radius $r$, thickness $dr$, and density per unit area $\rho$.




$$dm = 2\pi r dr \rho, \ I = \int_0^R r^2 \cdot (2\pi\rho r) dr = 2\pi\rho \int_0^R r^3 dr$$

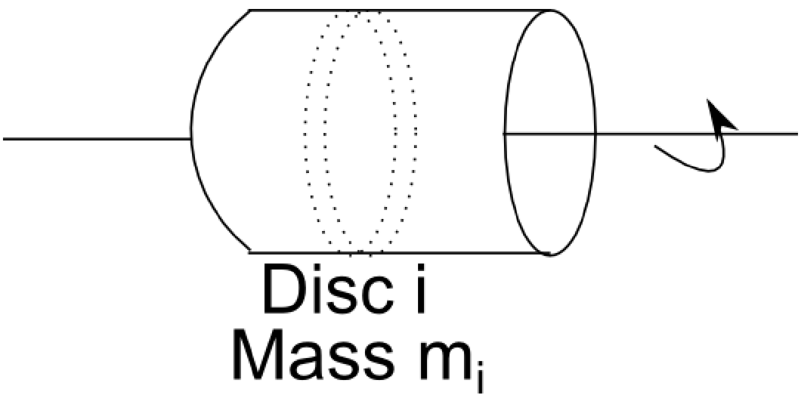
$$I = \pi\rho\frac{R^4}{2}, \quad \text{total mass of disc } = M = \pi\rho R^2$$

$$\therefore \ I = \frac{MR^2}{2}$$

So for a disc of an inner radius $R_1$ and an outer radius $R_2$ (i.e. a CD shape):

$$I = \int_{R_1}^{R_2} 2\pi\rho r^3 dr = \frac{\pi\rho}{2}\left(R_2^4 - R_1^4\right), \quad \text{but } M = \pi\rho\left(R_2^2 - R_1^2\right) \ \therefore \ I = \frac{M}{2}\left(R_1^2 + R_2^2\right)$$

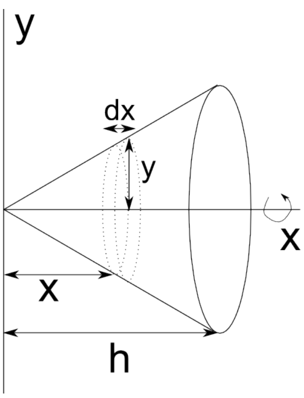
## Cylinder rotating about long-axis




$$I_i = \frac{m_i R^2}{2}, \quad \text{for stack: } I = \sum I_i = \sum_i \frac{m_i R^2}{2} = \frac{MR^2}{2}$$

where M is the total mass of the cylinder.

## Cone rotating about axis through vertex




Consider disc of thickness $dx$ at a distance $x$ from the vertex. Note that $\rho$ is now a volume density.

$$dI = \frac{dm y^2}{2}\ , \ dm = \text{ mass of disc } = \pi y^2 \rho dx$$

So the mass of inertia of the whole cone is given by:

$$I = \int dI = \frac{\pi\rho}{2}\int_0^h y^4 dx$$

But $y$ depends on $x$:

$$y = \frac{R}{h}x, \ I = \frac{\pi\rho}{10}R^4 h, \quad \text{where the volume of the cone } = \frac{1}{3}\pi R^2 h$$

### Sphere rotating about axis through centre

A sphere of radius $R$ and volume density $\rho$ is rotating about an axis through it's centre (like a billiard ball).

$$I = r^2 dm$$

Treat as stack of discs of varying radius , so:

$$dm = \pi a^2 \rho dx, \quad \text{where} \ \ a^2 = R^2 - x^2 \ dm = \pi(R^2 - x^2)\rho dx$$

$$dI = \frac{a^2 dm}{2} = \frac{\pi(r^2 - x^2)^2 \rho dx}{2}$$

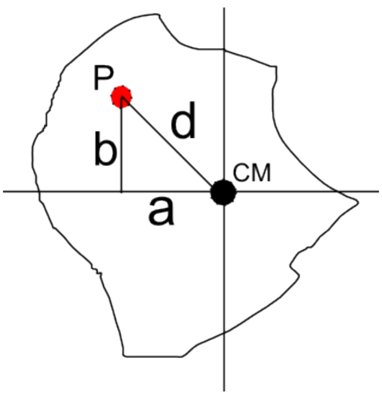
$$I = \int dI = \frac{\pi\rho}{2}\int (R^4 - 2x^2R^2 + x^4)dx = \frac{\pi\rho}{2}\left[R^4 x - \frac{2x^3R^2}{3} + \frac{x^5}{5}\right]_{-R}^{R}$$

$$\therefore I = \frac{\pi\rho}{2}\left(\left(R^5 - \frac{2R^5}{3} + \frac{R^5}{5}\right) - \left(-R^5 + \frac{2R^5}{3} - \frac{R^5}{5}\right)\right)$$

$$\therefore \ I = \frac{8\pi\rho R^5}{15} = \frac{2mR^2}{5}$$

## Parallel axis theorem:

If we centre the $(x, y)$ co-ordinates on the centre of mass.




$I_{CM}$ = Moment of inertia about the centre of mass $= \sum_i m_i \left(x_i^2 + y_i^2\right)$.

$I_P$ = Moment of inertia about the axis passing through point P at a distance $d$ from the centre of mass - i.e. with co-ordinates $(a, b)$.

Point $i$ is at distance $(x_i - a, y_i - b)$ from P.

$$I_P = \sum_i m_i \left((x_i - a)^2 + (y_i - b)^2\right)$$

$$I_P = \sum M_i \left(x_i^2 + y_i^2\right) - 2a \sum m_i x_i - 2b \sum m_i y_i + \sum m_i \left(a^2 + b^2\right)$$

The first term is just the moment of inertia about the centre of mass, whilst the inner terms are both zero because the origin is the centre of mass and moments about the centre of mass are zero by definition.
So:

$$I_p = I_{CM} + d^2 M$$

And so the moment of inertia is at a minimum about the centre of mass.

# Perpendicular Axis Theroem

For a 2-D body in the x-y plane.

$$I_x = \sum_i m_i \left(y_i^2 + z_i^2\right)$$

$$I_y = \sum_i m_i \left(x_i^2 + z_i^2\right)$$

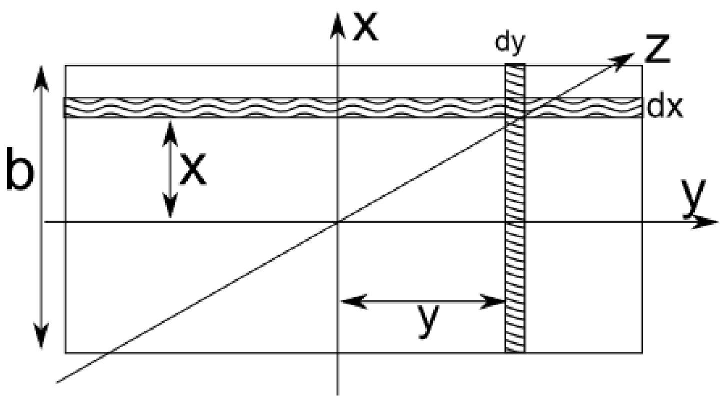
But the $z_i$ terms equal zero as the body lies in the $x$-$y$ plane.

$$I_z = \sum m_i \left(x_i^2 + y_i^2\right)$$

And so:

$$I_z = I_x + I_y$$

## Application: MI of thin rectangular sheet lying in the x-y plane:




The z-axis is in to the plane of the paper. $\rho$ is the density per unit area. Calculating $I_x$: $dm = \rho b dy$

$$I_x = \int_{\frac{-l}{2}}^{\frac{l}{2}} \rho b y^2 dy = \frac{\rho b l^3}{12} = \frac{M l^2}{12}$$

Where $M$ is the total mass. Similarly, for $I_y$: $dm = \rho l dx$

$$I_y = \int_{\frac{-l}{2}}^{\frac{l}{2}} \rho l x^2 dx = \frac{Mb^2}{12}$$

For the perpendicular axis, $I_z$:

$$I_z = I_x + I_y = \frac{M}{12}\left(l^2 + b^2\right)$$

## Radius of Gyration

$$I = \text{ numerical factor, } k \times M \times (\text{length})^2$$

$$k = \sqrt{\frac{I}{M}}$$

so $k$ has dimensions of length, and is called the **radius of gyration**.
For a circle:

$$k = \frac{R}{\sqrt{2}}$$

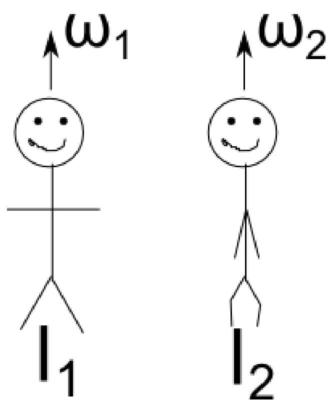
This is useful when we do not know the shape of a body, and so cannot calculate $I$ directly.

## Conservation of Angular Momentum and Energy

If there is no torque then angular momentum is conserved.

$$|\vec{L}| = I_1|\vec{\omega_1}| = I_2|\vec{\omega_2}|, \quad I_1 > I_2 \ \therefore \ |\vec{\omega_2}| > |\vec{\omega_1}|$$

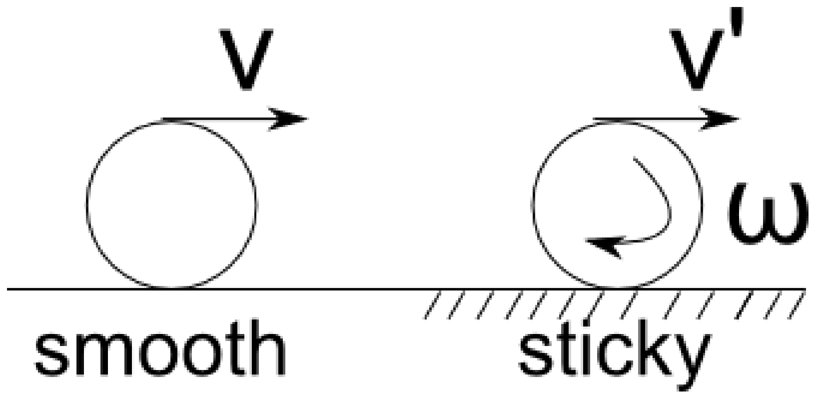
i.e. can change speed of rotation by changing the mass distribution.




Kinetic Energy:

$$T_1 = \frac{1}{2}I|\vec{\omega_1}|^2, \ T_2 = \frac{1}{2}I_2|\vec{\omega_2}|^2 \ \therefore \ T^2 = \frac{|\vec{\omega_2}|}{|\vec{\omega_1}|}T_1$$

So Kinetic energy is not conserved (as the potential energy is lower in 2 than in 1). The moment of inertia can be lowered by raising the centre of gravity (i.e. raising hands) but this requires work.




### Transition from sliding to rolling:

When sliding:

$$T_{\text{slide}} = \frac{1}{2}mv^2$$

When rolling there is also the rotational component:

$$T_{\text{roll}} = \frac{1}{2}mv'^2 + \frac{1}{2}I\omega^2$$

Conservation of energy:

$$mv^2 = mv'^2 + I\omega^2 \quad \text{but} \quad I = \frac{mR^2}{2}$$

$$v' = R\omega \ \therefore \ v^2 = v'^2 + \frac{R^2}{2} \cdot \frac{v'^2}{R^2} = \frac{3}{2}v'^2$$

$$\therefore \ v' = \sqrt{\frac{2}{3}}v$$

so $v'$ is independent of $R$ and $m$.

## Physical Examples

### Body rolling down an inclined plane

$$\text{Kinetic Energy, } T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Recall that $I = mk^2$ where $k$ is the radius of gyration, and $v = r\omega$ if there is no slipping. The body starts from rest, rolls a vertical distance $h$ and reaches a speed $v$.

$$\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = mgh \ \therefore \ \frac{1}{2}mv^2 + \frac{mk^2v^2}{2r^2} = mgh$$

$$\therefore \ v^2 = \frac{2ghr^2}{k^2 + r^2}$$

Note that $v$ depends on $k$ but not $m$.
For a fluid filled body the behaviour is complicated and depends on the viscosity.

## Swinging bar

A bar of mass $m$ and length $L$ is pivoted on a frictionless bearing on one side. It is initially horizontal and then released from the free end.
About one end:

$$I = \frac{mL^2}{3}$$

Ignoring vectors:

$$\text{Torque, } N = \frac{dL}{dt} = I\dot{\omega} \quad \text{and} \quad N = \vec{r} \times \vec{F} = \frac{L}{2}mg$$

$$\dot{\omega} = \frac{N}{I} = \frac{3mgL}{2mL^2} = \frac{3g}{2L}$$

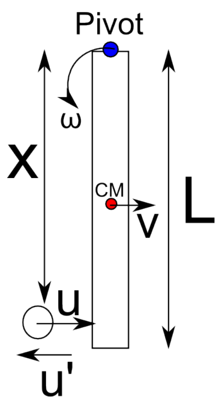
Force on rod when rod is horizontal: Force balance: $F_0 - mg = ma$ where $a$ is the vertical component of the linear acceleration.

$$a = \frac{L}{2}\dot{\omega} \ \therefore F_0 = \frac{7}{4}mg$$

What is the force when the rod is vertical?

## The centre of percussion of a cricket bat

In this case we ignore gravity, and so assume that the bat is lying on a horizontal, frictionless surface. Bat is of mass $M$ and length $L$, the ball is of mass $m$.
In this system energy is not conserved (due to the loss in sound, etc.) and so we must use the conseration of linear and angular momentum.




The velocity of the bat after the colission is $v$, so:

$$mu = mu' + Mv$$

From conservation of angular momentum about the centre of mass:

$$mu\left(x - \frac{L}{2}\right) = mu'\left(x - \frac{L}{2}\right) + I\omega$$

where $I = \frac{ML^2}{12}$ about the centre.

If rotation of top of bat to left balances the translation to the right then there is no movement at the pivot and so no reaction force. i.e. $x$ is the "sweet spot" or the centre of percussion.

$$v = \omega \frac{L}{2}$$

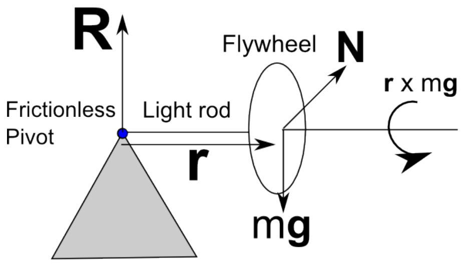
And so by substitution we obtain:

$$x = \frac{2L}{3}$$

## The Falling Cat

There is no external torque, and so the net change in angular momentum must be zero, however the cat can generate internal angular momentum through the use of work which summed with the whole body rotation results in no net change in angular momentum.

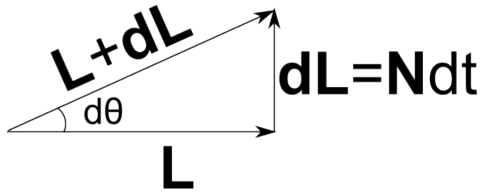
## The Gyroscope




If there is no spin then we expect the flywheel to fall:

$$\vec{N} = \vec{r} \times m\vec{g}$$

which is in the plane of the paper. Initially $\vec{L} = 0$ and so $d\vec{L} = \vec{N}dt$ and so there is downwards rotation (falling).
If there is rapid spin then $\vec{L} = I\vec{\omega}$ along the axis of the rod, where $\omega$ is the angular velocity of the flywheel. So looking from above:




Note that rotation must be rapid such that this diagram can be produced in one plane. In a time $dt$ the rod rotates through an angle $d\theta$.

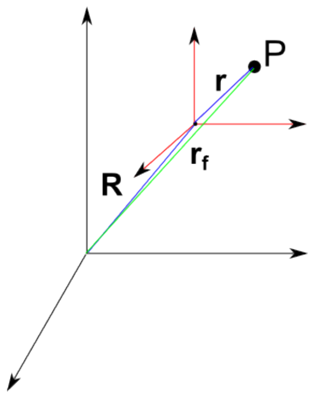
$$d\theta = \frac{|\vec{N}|dt}{|\vec{L}|} = \frac{mgr}{I\omega}dt$$

$$\therefore \ \frac{d\theta}{dt} = \frac{mgr}{I\omega} = \Omega$$

Where $\Omega$ is the rotation frequency of the rod in to the plane of the paper. This rotation of the axis of rotation is known as precession and so $\Omega$ is called the **precession frequency**. Note that $\Omega \propto \frac{1}{\omega}$ and so the rotation of the rod is faster when the spin of the flywheel is slower.
This derivation is only possible for rapid spin (large $\Omega$) so that the diagram can be constructed in one plane, for small $\Omega$ the situation is more complicated and there are higher order terms which give rise to wobbling known as **nutation**.

# Rotating frames of reference

Newton's Laws are not valid in non-inertial reference frames, this is fixed by introducing fictitious forces.




The rotating frame rotates with angular velocity $\vec{\omega}$. $\vec{R}$ is the position vector of the origin of the rotating frame, $\vec{r}$ is the position vector of the point P in the rotating frame, while $\vec{r}_f$ is the position vector of point P in the fixed frame.
If P is stationary in the rotating frame then it is rotating at angular speed $\omega$ in the fixed frame, i.e.:

$$\left(\frac{dr}{dt}\right)_f = \vec{\omega} \times \vec{r}$$

If P is moving in the rotating frame with velocity $\left(\frac{dr}{dt}\right)_r$ then in the fixed frame:

$$\left(\frac{dr}{dt}\right)_f = \left(\frac{dr}{dt}\right)_r + \vec{\omega} \times \vec{r}$$

and

$$\vec{r}_f = \vec{R} + \vec{r}$$

Substituting the latter in to the former:

$$\left(\frac{d\vec{r}_f}{dt}\right)_f = \left(\frac{d\vec{r}}{dt}\right)_r + \left(\frac{d\vec{R}}{dt}\right)_r + \vec{\omega} \times \vec{r}$$

Letting $\left(\frac{d\vec{r}_f}{dt}\right)_f = v_f$, $\left(\frac{d\vec{r}}{dt}\right)_r = v_r$ and $\left(\frac{d\vec{R}}{dt}\right)_r = V_R$.

$$\left(\frac{d\vec{v}_f}{dt}\right) = \left(\frac{d\vec{V}_R}{dt}\right)_f + \left(\frac{d\vec{v}_r}{dt}\right)_f + [\vec{\omega} \times \vec{r} + \vec{\omega} \times \vec{r}]_f$$

But in general:

$$\left(\frac{df}{dt}\right)_f = \left(\frac{df}{dt}\right)_r + \vec{\omega} \times \vec{r}$$

So:

$$\vec{a}_f = \ddot{\mathbf{R}}_\mathbf{f} + \vec{a}_r + \dot{\omega} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + 2\vec{\omega} \times \vec{v}_r$$

Where $\vec{a}_r$ is the acceleration in the rotating frame, $\ddot{\mathbf{R}}$ is the acceleration of moving origin, $\dot{\omega}$ is the angular acceleration of the moving origin.
From Newton's Second Law, in an inertial reference frame: $\vec{F} = m\vec{a}_f$.
In rotating frame we define $\vec{F}_{\text{eff}} = m\vec{a}_r = m\vec{a} +$ non-inertial terms

$$\vec{F}_{\text{eff}} = m\vec{a}_f - M\ddot{\mathbf{R}}_\mathbf{f} - m(\dot{\omega} \times \vec{r}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m(\vec{\omega} \times \vec{v}_r)$$

Where the first term is the inertial force, the second term is due to the linear acceleration of restoring force, the third term is due to the angular acceleration of the rotating frame, the 4th term is the centrifugal force and is independent of the motion in the rotating frame and the last term is the Coriolis force and depends on $\vec{\omega}$ and $\vec{v}_f$.
The centrifugal force:

$$\vec{F}_{\text{cent}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

note that this force acts radially outward.

## Example: Ball rolling on a frictionless turntable

Assume the lab is an inertial frame, then the rotating rame is fixed on the turntable.

$$\vec{a}_f = 0,\ \ddot{\mathbf{R}}_\mathbf{f} = 0,\ \dot{\omega} = 0$$

$$\vec{F}_{\text{eff}} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - 2m\vec{\omega} \times \vec{v}_r$$

This can be integrated twice to get the trajectory however it is difficult to do and must be done analytically.

## Apparent Gravity on The Earth

The rotation of the earth on its axis has a much larger effect than the motion around the Sun which has a much larger effect than the motion of the galaxy, etc.
So we choose a fixed co-ordinate at the centre of the Earth.
In the inertial (fixed) frame:

$$\vec{F} = m\vec{g_0}\ ,\ \vec{g_0} = \frac{-GM}{|R^3|}\vec{R}$$

In our co-ordinates we measure:

$$\vec{g_{\text{eff}}} = \vec{g_0} - \ddot{\mathbf{R}} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Recall that:

$$\left(\frac{d\vec{X}}{dt}\right)_f = \left(\frac{d\vec{X}}{dt}\right)_r + (\vec{\omega} \times \vec{X})$$

$$\left(\frac{dR}{dt}\right)_r = 0 \text{ as we move with it}$$

$$\therefore\ \dot{\mathbf{R}} = \vec{\omega} \times \vec{r}\ \therefore\ \ddot{\mathbf{R}} = \vec{\omega} \times (\vec{\omega} \times \vec{R})$$

$$\therefore\ \vec{g_{\text{eff}}} = \vec{g}_0 - \vec{\omega} \times (\vec{\omega} \times (\vec{R} + \vec{r})) = \vec{g}_0 - \vec{\omega} \times (\vec{\omega} \times (\vec{R}))\ \text{, assuming } \vec{R} >> \vec{r}$$

The rotation of the Earth affects both the gravitational magnitude and direction and depends upon the latitude. There is no change at the pole and maximum change at the Equator. The change effects the shape of the Earth, shifting it from a sphere in to a slight egg-shape.

$$g_{\text{pole}} - g_{\text{equator}} = 0.052 \text{ m s}^{-2}$$

## Coriolis Force on Earth

$$\vec{F_{\text{cor}}} = -2m(\vec{\omega} \times \vec{v}_r)$$

In the Northern Hemisphere $\vec{\omega}$ has component in the $+\hat{z}$ direction, while in the southern hemisphere it acts in the $-\hat{z}$ direction.
The force acts to right the direction of motion. It affects the motion of projectiles and the air flow (creating spirals around areas of low pressure).

## Foucault's Pendulum

This is an observation of the effect of the rotation of the Earth. The equation of motion for the long pendulum is:

$$\vec{F}_{\text{eff}} = \vec{T} + m\vec{g}_{\text{eff}} - 2m(\vec{\omega} \times \vec{v}_r)$$

This makes it an elliptical oscilattor as there is a force perpendicular to the direction of motion which is largest during the centre of the motion.

# Lagrangian Mechanics

Newton's laws are hard to solve if there is interaction between components (as in the 3-body problem) or if the geometry is complicated or components are constrained (i.e. rolling around a bowl).
Minimal Principles:

1. Fermat's Principle: Light rays travel along paths of minimum transit time.

2. Hamilton (1834): A dynamical system follows a path which subject to constraint, minimises the integral $I = \int_{t_1}^{t_2} (T - U)dt$

Kinetic Energy: $T(x_i, \dot{x}_i$ , Potential Energy: $U(x_i, \dot{x}_i)$.
Velocity: $\dot{x}_i = \frac{dx_i}{dt}$. Note that $x_i$ and $\dot{x}_i$ are generalised co-ordinates considered mathematically as independent.
The calculus of variations is the same as solving the equations:

$$\frac{\partial L}{\partial x_i} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$$

Where $L(x_i, \dot{x}_i) = T(x_i, \dot{x}_i) - U(x_i, \dot{x}_i)$, this is the Lagrangian equation and there is one for each degree of freedom (dimension).
The recipe:

1. Choose co-ordinate system $x_i$
2. Write down $T(x_i, \dot{x}_i)$ and $U(x_i, \dot{x}_i)$
3. Calculate L = T-U
4. Calculate derivatives $\frac{\partial L}{\partial x_i}$ and $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right)$
5. Write and solve the Lagrangian equation: $\frac{\partial L}{\partial x_i} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = 0$

## 1-D Harmonic Oscillator

A spring lying on a frictionless horizontal surface. Applying the recipe:

1. Co-ordinate: Displacement of mass $x$
2. $T = \frac{1}{2}m\dot{x}^2$ and $U = \frac{1}{2}kx^2$
3. $L = \frac{1}{2}\left(m\dot{x}^2 - kx^2\right)$
4. $\frac{\partial L}{\partial x} = -kx$ and $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$ $\therefore$ $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$
5. So $m\ddot{x} + kx = 0$ as we would have obtained from the application of Newton's Laws.

## The Plane Pendulum

1. Co-ordinate is angle $\theta$
2. $T = \frac{1}{2}I\omega^2 = \frac{1}{2}mL^2\dot{\theta}^2$ and $U = mgL(1 - \cos\theta)$ where $U = 0$ at $\theta = 0$
3. $L = \frac{1}{2}mL^2\dot{\theta}^2 - mgL(1 - \cos\theta)$
4. $\frac{\partial L}{\partial \theta} = -mgL\sin\theta$, $\frac{\partial L}{\partial \dot{\theta}} = mL^2\dot{\theta}$, $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mL^2\ddot{\theta}$
5. $\ddot{\theta} + \frac{g}{L}\sin\theta = 0$

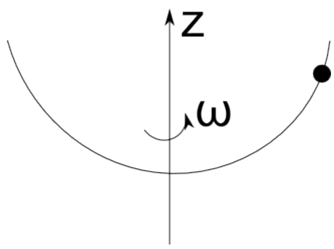
### Demonstrating the equivalence of Newton and Lagrange

In a conservative system: $T = \frac{1}{2}m\dot{x}^2$ and $U(x)$ is independent of $\dot{x}$.
Lagrange:

$$\frac{\partial}{\partial x}\left[\frac{1}{2}m\dot{x}^2 - U(x)\right] - \frac{\partial}{\partial t}\left(\frac{\partial}{\partial \dot{x}}\left[\frac{1}{2}m\dot{x}^2 - U(x)\right]\right) = 0$$

$-\frac{\partial U}{\partial x} - m\ddot{x} = 0$ but $F = -\frac{dU}{dx}$ and $m\ddot{x} = \frac{dP}{dt}$ and so $F = \frac{dP}{dt}$ which demonstrates Newton's Laws.

### A bead on a frictionless parabolic rotating wire




$z = cr^2$

1. Choose co-ordinate system: Cylindrical polars
2. $T = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{1}{2}m(\dot{r}^2 + \dot{z}^2 + (r\dot{\theta})^2)$ and $U = mgz$. But $z = cr^2$ so $\frac{dz}{dt} = \dot{z} = 2cr\dot{r}$ and $\theta = \omega t$ so $\dot{\theta} = \omega$.
3. $L = T - U = \frac{m}{2}(\dot{r}^2 + 4c^2r^2\dot{r}^2 + r^2\omega^2) - mgcr^2$ only depends on $r$ and so there is only one Lagrangian equation.
4. $\ddot{r}(1 + 4c^2r^2) + \dot{r}^2(4c^2r) + r(2gc - \omega^2) = 0$

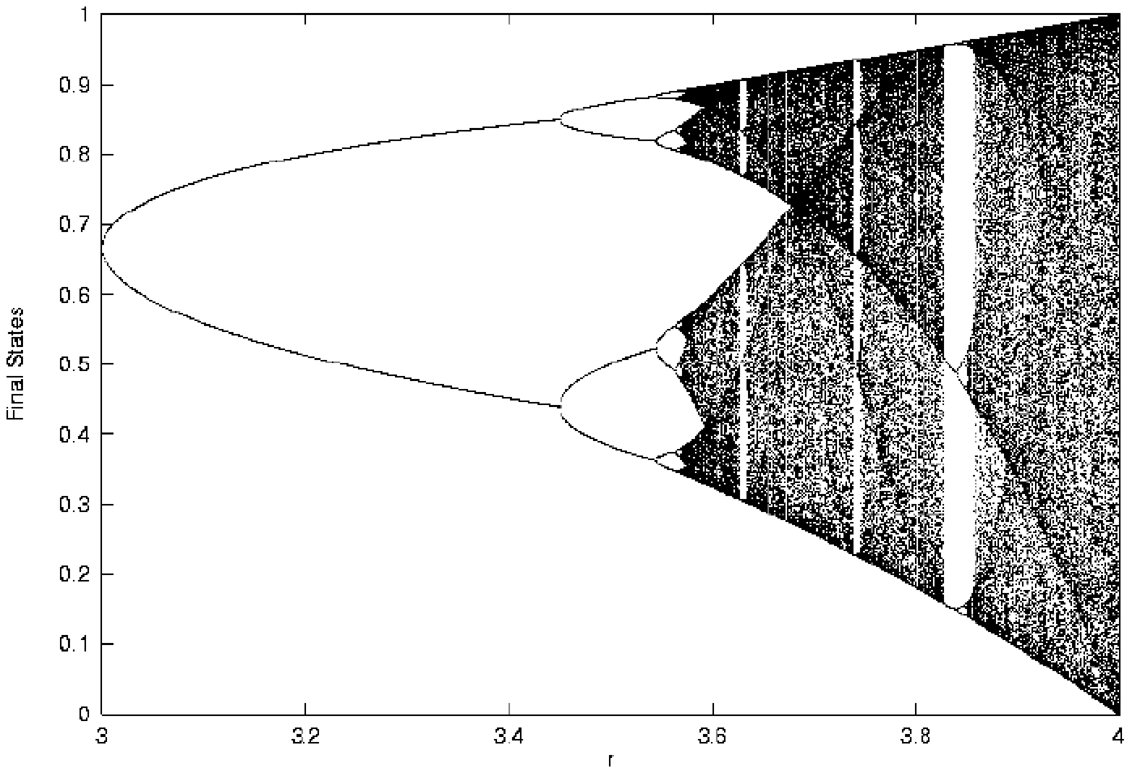
At steady state $\ddot{r} = 0, \dot{r} = 0$ and so $2gc = \omega^2$ so $\omega = \sqrt{2gc}$ for a steady height.

# Non-Linear Dynamics (Chaos)

Chaotic behaviour is when a small change in initial conditions dramatically changes the outcome. Note it is different from random behaviour since it is reproducible with the same initial conditions.
The logistic equation is:

$$x_{n+1} = \lambda x_n(1 - x_n)$$

where $\lambda$ is the feedback parameter. This equation is widely applicable in population dynamics.




The Bifurcation diagram is a plot of $x_n$ against $\lambda$ as $n \to \infty$. The original constant value soon splits in to two different values by a **pitchfork bifurcation**. Note the diagram has some gaps where it returns to determinism, and that there is a finite range of final values. This is the **period doubling** route to chaos, and it is found for equations with a quadratic maximum. Note that the structure of the bifurcation diagram is self-similar. To investigate the $\lambda$ dependence let $lambda_n$ be the value at which the $n$th period doubling occurs. So:

$$\Delta\lambda_n = \lambda_n - \lambda_{n-1} \;\therefore\; \delta_n = \frac{\Delta\lambda_n}{\Delta\lambda_{n+1}}$$

Note that:

$$\lim_{n\to\infty} \delta_n = 4.66902 \quad \text{Feigenboum's number}$$

This is constant for all mappings with a quadratic number.

## Feigenbaum size scaling

Let $h_n$ be the difference between values of $x_n$ at bifurcation.

$$\alpha = \lim_{n\to\infty} \frac{h_n}{h_{n+1}} = 2.5029$$

But the geometrical structure is self-similar and so there is no inherent length scale in the problem.
If we take trajectories starting at $x_0$ and $x_0 + \epsilon$ (for small $\epsilon$). $d_n$ is the difference between the trajectories after $n$ iterations.
Assume $d_n$ grows exponentially on average:

$$d_n = \epsilon e^{n\kappa}$$

Where $\kappa$ is the Lyapunov exponent. For $\kappa < 0$ the trajectories will converge, while for $\kappa > 0$ there will be chaos.

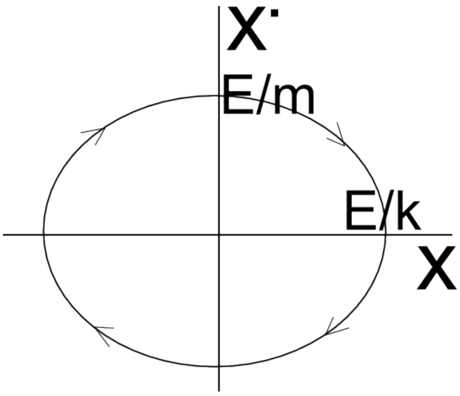
# Mechanics in phase space

Phase space has co-ordinates $(x_i, \dot{x}_i)$ where $\dot{x}_i = \frac{\partial x_i}{dt}$. So it has $2n$ dimensions for a system with $n$ degrees of freedom.

## Example 1: Simple Harmonic Oscillator

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

Where $E$ is the total energy and $k$ is the spring constant. The usual solution is $x(t) = A\cos(\omega t) + B\sin(\omega t)$ but this is not very useful instead we want to exmaine the system in phase space.

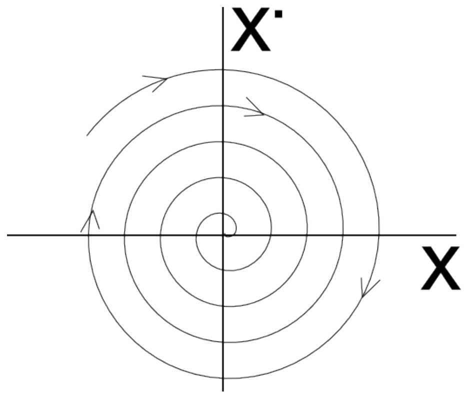



Note the direction as $E$ is constant and so as $x$ increases, $\dot{x}$ must decrease.

## Example 2: Damped Harmonic Oscillator

$$\ddot{x} + 2\beta\dot{x} + {\omega_0}^2 x = 0 \; x(t) = Ae^{-\beta t}\cos(\omega t - \delta)$$

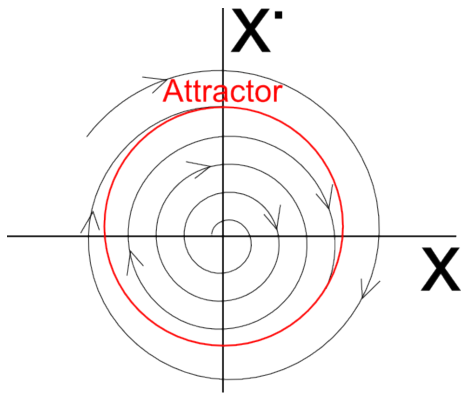
The damped oscillator spirals to the centre as it dissipates energy. The centre is called the **attractor** and it is the structure of the attractors that dictates long-term behaviour.




**Example 3: Van der Pol Oscillator**

$$\ddot{x} - \mu(x_0^2 - x^2)\dot{x} + \omega_0^2 x = 0$$

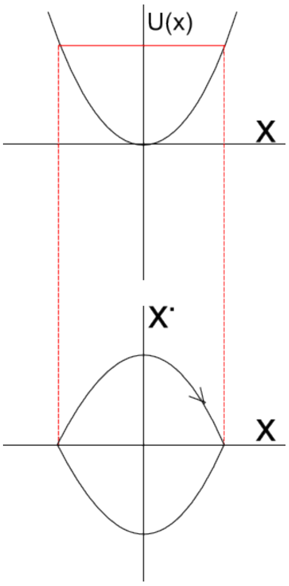
This is a damped oscillator if the bracketed term is negative, ie. if $x^2 > x_0^2$ so x is large, whilst it is a driven oscillator if the bracketed term is positive i.e. $x_0^2 > x^2$ so x is small.




The attractor is a circle, the trajectories never cross the attractor.

## Constructing phase-space trajectories

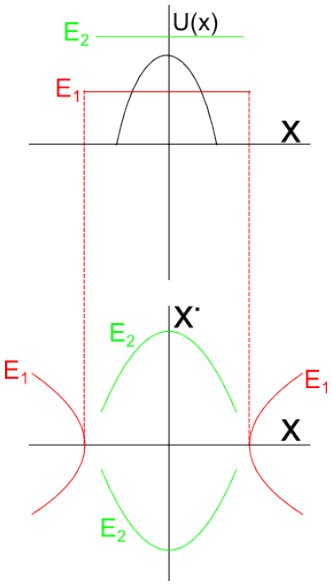
For conservative systems $E = \frac{1}{2}m\dot{x}^2 + U(x)$, so $\dot{x} = \sqrt{\frac{2(E-U)}{m}}$.




The phase space diagram is produced by plotting the potential diagram and realising that, for a fixed energy, at the maximum potential there is no velocity.

## Repulsive Potential

The diagram can also be constructed for repulsive potentials:




The centre acts as a negative attractor, in that all particles move away from it. So it represents a point of unstable equilibrium.

## Attractors in chaotic systems

1. Trajectories end on attractors.
2. Trajectories cannot cross intersects as the equations have unique solutions for unique starting conditions.
3. Trajectories in chaotic systems must continually diverge.

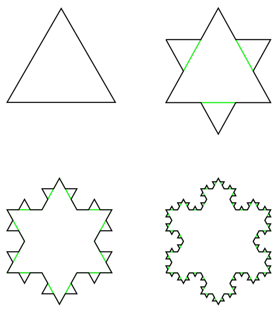
This last point means that attractors in a chaotic system must have fractal structures, as they must be capable of constant divergence to prevent the trajectories crossing intersections.

## Fractals

Fractals are structures with non-integer dimensions, and have self-similarity.

$$\text{Dimension, } D = \frac{\log n}{\log \epsilon}$$

where $N$ is the number of self-similar copies and $\epsilon$ is the length of each copy.
An example is the **Koch snowflake**:



**Initial:** Side length $= 3L$ and perimeter $= 9L$

**i=1:** Side length $= L$ and perimeter $= \frac{4}{3} \cdot 9L$

**i=2:** Side length $= \frac{L}{3}$ and perimeter $= \left(\frac{4}{3}\right)^2 \cdot 9L$

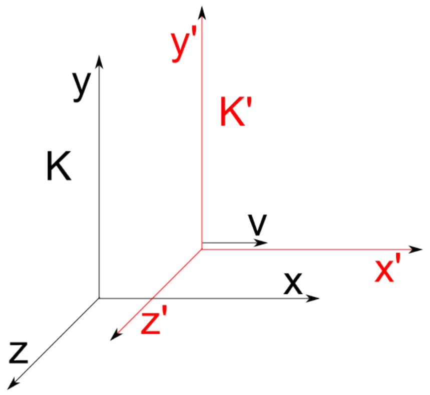
**Infinite iterations:** Infinite perimeter but finite area

So this could be an attracor in a chaotic system. An example of chaos is a damped pendulum on an oscillating sphere, note that nondimensionalisation is necessary to reduce the equations of motion to solvable forms.

## The Poincare Section

Here we plot the phase space diagram but with time on the z-axis and then take a series of snapshots of where the particle is in time and plot these on the phase space diagram (rather than considering all times). This helps to expose hidden structure in chaotic systems.

# Relativity




$K'$ is moving relative to $K$ in the $\hat{x}$ direction with constant velocity $v$.

## Einstein's Postulates

1. Physical laws are identical in all inertial frames, and so we can only observe relative motion.

2. Speed of light in free space is a universal constant. This appears to be experimentally true.

## Co-ordinate transformations

### Pre-relativity: Galilean transformation

Time is absolute i.e. $t = t'$ Spatial co-ordinates: $x' = x + vt$, $y' = y$, $z' = z$ as $v$ is only in the $x$ direction.
Length Transformation:
$dL^2 = dx^2 + dy^2 + d^2$ , but $dx = x_a - x_b$, $dx' = x'_a - x'_b$. $dL'^2 = dx'^2 + dy'^2 + dz'^2$ so $dx' = x_a - vt - (x_b - vt)$ so $dx' = dx$ i.e. length is invariant.

Newton's Laws:
$F_x = m\ddot{x}$, $F'_x = m\ddot{x}'$. $x' = x - vt$ so $\frac{d}{dt}(x') = \frac{dx}{dt} - v$ so $\frac{d^2}{dt^2}(x') = \ddot{x}$ so $F' = F$ i.e. force is invariant (assuming the mass is constant).

However the speed of light is not invariant under Galilean transformations as if light speed is $c$ in reference frame $K$ then in $K'$ light speed would be $c - v$ (in the x-direction).

## The Lorentz Transform

For a photon travelling along the x axis $K$ sees $c = \frac{x}{t}$ where x is the distance travelled in time t. $K'$ sees $c = \frac{x'}{t'}$ so time cannot be invariant to maintain invariant $c$.

Let us assume that: $x' = \gamma(x - vt)$ where $\gamma$ is some fiddle factor. Then we expect that $x = \gamma'(x' + vt')$ but if the physical laws are identical in both reference frames then $\gamma' = \gamma$. Substituting the second equation in to the first obtains: $x = \gamma\left[\gamma\left(x - vt\right) + vt'\right]$. Which gives the relation between $t$ and $t'$.

$$t' = \gamma t + \frac{x}{\gamma v}\left(1-\gamma\right)^2$$

Light pulse satisfies $x = ct$ and $x' = ct'$. Substituting the expression for $t'$ in to $x' = ct'$ obtains:

$$\gamma\left(x - vt\right) = c\left[\gamma t + \frac{x}{\gamma v}\left(1-\gamma^2\right)\right]$$

Substituting $x = ct$ obtains:

$$\gamma^2 = \frac{1}{1 - \frac{v^2}{c^2}}$$

So in summary:

$$x' = \gamma(x - vt)\ y' = y\ z' = z\ t' = \gamma\left(t = \frac{vx}{c^2}\right)$$

Note that if $v << c$ then $x' \approx x$ and $t' \approx t$ so the classical Galilean transformations are recovered.

The transform can be written in matrix notation (the Minkowski representation) as:

$$\left(\begin{array}{c} x' \\ y' \\ z' \\ ict' \end{array}\right) = \left(\begin{array}{cccc} \gamma & 0 & 0 & i\frac{v}{c}\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\frac{v}{c}\gamma & 0 & 0 & \gamma \end{array}\right)\left(\begin{array}{c} x \\ y \\ z \\ ict \end{array}\right)$$

Or in the summing form as:

$$x'_\mu = \sum_\nu \lambda_{\mu\nu} x_\nu$$

where $\lambda_{00} = \gamma$ and $\lambda_{01} = 0$, etc. and
*lambda* is the Lorentz matrix.

Note that there are two conventions that $\nu$ runs from 0 to 3 and 0 is $ict$ or that $\nu$ runs from 1 to 4 and 4 is $ict$.

However for $\left(\begin{array}{c} x \\ y \\ z \\ ict \end{array}\right)$ to qualify as a vector, we need a definition of length.

Let $ds^2 = c^2dt^2 - d\vec{x}^2 = \sum_\mu \vec{x}_\mu \cdot \vec{x}_\mu$, this is invariant under the Lorentz transform. i.e. $ds'^2 = c^2dt'^2 - dx'^2 = ds^2$.

Where $ds$ is called the **interval** and is the length in 4-Dimensional space.

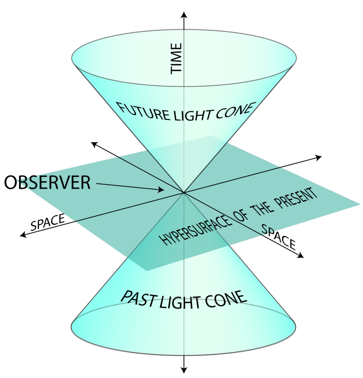
## Metric Tensor

We define the interval by: $ds^2 = \sum g_{\mu\nu} dx_\mu dx_\nu$ where $dx_\nu = \begin{pmatrix} dx \\ dy \\ dz \\ cdt \end{pmatrix}$ and:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is true for Euclidean geometry (empty space-time), but General Relativity states that the curvature of space-time (i.e. the form of $g_{\mu\nu}$ is determined by the energy content.

## 4-D Space-Time




World lines are trajectories plotted on a diagram of space-time, note that a shallower gradient means a faster object as it will cover more space in a given time. The speed of light is the limit and so no communication is possible between spaces until the light cones intersect.

## Proper Time

Proper time is the time measured by the observer moving with the event, i.e. in the unique reference frame where the event is at rest. So in $K'$ the observer times the interval between 2 events occuring at the same place (i.e. $dx' = 0$) then observer $K$ sees the events occuring a different time intervals ($dt$) and at different places ($dx$).

Recall that the time-space interval is invariant so $ds^2 = c^2dt^2 - d\vec{x}^2 = c^2dt'^2$, so:

$$dx' = \gamma(dx - vdt) = 0 \;\therefore\; dx = vdt \;\therefore\; dt' = \frac{dt}{\gamma}$$

$dt'$ measured in the rest frame of the event is generally given a new symbolt $d\tau$ to indicate proper time, so $d\tau = \frac{dt}{\gamma}$. I.e. moving clocks appear to tick more slowly.

This is a testable prediction of the Lorentz transformation, and muons generated in the outer atmosphere with short half-lives that shouldn't reach Earth are observed to do so, so it appears to be correct.

### Twin Paradox

One twin leaves the Earth in a spaceship whilst one remains on the Earth. The observer on the Earth says that he spaceship is mobing and so the astronaut's time will pass more slowly and he will return younger than the twin. However the twin on the sapceship says the Earth is mobing away and so expects the Earth observer to be younger. At first this would seem like a paradox due to the relativism of the situation, however, the astronaut is wrong because his frame must accelerate to leave and return to the Earth and so the effects of General Relativity must be considered and SR alone cannot be used.

### Length Measurement: Lorentz-Fitzgerald contraction

If in $K$, $L = x_1 - x_2$. Then the length measured in $K'$ is $L' = x'_1 - x'_2 = \gamma\left[(x_1 - vt_1) - (x_2 - vt_2)\right]$.

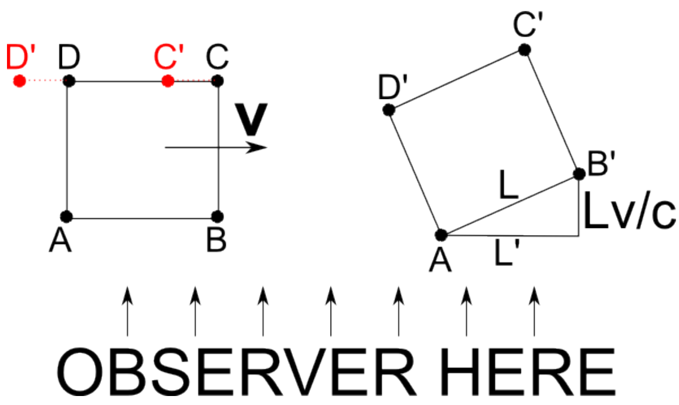
If the measurements are simultaneous in $K'$ then $t_2{}' = t_1{}'$ so:

$$t_1{}' = \gamma\left(t_1 - \frac{vx_1}{c^2}\right) - \gamma\left(t_2 - \frac{vx_2}{c^2}\right)$$

Substituting the latter equation in to the former obtains the equation fot the Lorentz-Fitzgerald contraction:

$$L' = \frac{L}{\gamma}$$

### Apparent rotation




To the stationary observer the sides AB and CD undergo length contraction, so their apparent length $L' = \frac{L}{\gamma}$. But sides AD and BC do not undergo Lorentz contraction, but light from D has to travel an additional distance L as it is emitted $\frac{L}{c}$ earlier than the light at A to arrive at the same time. i.e. the observed D is at a horizontal distance $\frac{vL}{c}$ to the left. This is a problem in astrophysics measurements.

# Lorentz Velocity Transform

## "Classical" derivation

In K: $u_i = \frac{dx_i}{dt}$, in K': $u_i' = \frac{dx_i'}{dt'}$. $u_i$ is the particle velocity and $v$ is the relative frame velocity. Substituting the Lorentz transform in to the equation for K':

$$u_i' = \left(\frac{dx_1 - vdt}{dt - \frac{v}{c^2}dx_1}\right)\frac{\gamma}{\gamma}$$

So:

$$u_1' = \frac{u_1 - v}{1 - \frac{vu_1}{c^2}}$$

$$u_2' = \frac{dx_2'}{dt'} = \frac{dx}{\gamma\left(dt - \frac{v}{c^2}dx_1\right)} = \frac{u_2}{\gamma\left(1 - \frac{v^2}{c^2}u_1\right)}$$

Same for $u_3' = \frac{u_3}{\gamma\left(1 - \frac{vu_1^2}{c^2}\right)}$.

Note that $u_1'$ is independent of $\gamma$ as the length contraction and time dilation cancel. While there is no such cancellation for $U_2'$ and $u_3'$.

## 4-Vector velocity

$x\mu = (\vec{x}, ct)$ is a 4-vector. Can divide by the proper time (invariant) to form another 4-vector:

$$U_p = \frac{dx_p}{d\tau} = \frac{d}{d\tau}(\vec{x}, ct)$$

Where $d\tau = \frac{dt}{\gamma(u)}$, $U_\mu = \gamma(u)(\vec{u}, c)$ where $\gamma(u)$ is defined with respect to the particle velocity not the frame velocity. One could check that the transforms are the same for homework.

## Does the velocity transform work?

**Is c invariant?** :
Let $u_1 = c$ then to $K'$':

$$u_1' = \frac{u_1 - v}{1 - \frac{vu_1}{c^2}} = \frac{c - v}{1 - \frac{v}{c}} = c$$

So $c$ is invariant as required.
**Addition of velocities:**

Let $K'$ move at velocity $v_1$ relative to $K$ and $K''$ move at velocity $v_2$ relative to $K'$. Classically the speed of $K''$ relative to $K$ is $v_1 + v_2$ however this cannot be possible if they are near $c$. The Lorentz transform from $K'$ to $K$:

$$u_k = \sum \lambda_{K' \to K} u_{k'}$$

$$u_{k'} = \sum \lambda_{k'' \to k'} u_{k''}$$

$$\therefore\ u_{k''} = \sum\sum \lambda_{k' \to k}\lambda_{k'' \to k} u_k$$

$\lambda_{k''\to k}$ is formed by multiplying (not adding) two transform matrices.

$$\lambda_{K''\to K} = \begin{pmatrix} \gamma_1\gamma_2\left(1+\frac{v_1}{c}\frac{v_2}{c}\right) & 0 & 0 & \gamma_1\gamma_2\left(\frac{v_1}{c}+\frac{v_2}{c}\right) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma_1\gamma_2\left(\frac{v_1}{c}+\frac{v_2}{c}\right) & 0 & 0 & \gamma_1\gamma_2\left(\frac{v_1}{c}+\frac{v_2}{c}\right) \end{pmatrix}$$

i.e. the added velocity is:

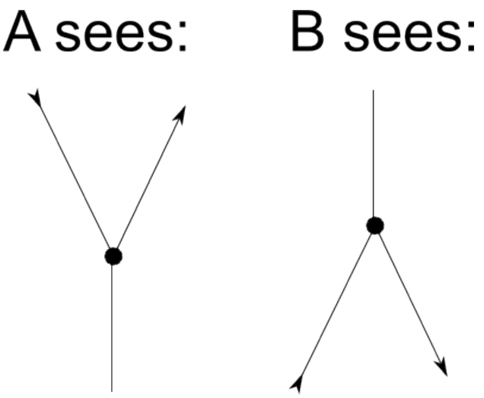
$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$$

so the velocity cannot exceed c.

# Relativistic Momentum and Energy

## The problem with momentum

If an observe A is stationary and and observer B is travelling at velocity $v$, and they both roll billiard balls towards eachother then they see:




But now classically $\Delta p_A \neq \Delta p_B$ after Lorentz transform. So we need a new definition of momentum:

$$\vec{p} = k(u)m\vec{u}$$

But what factor k is necessary to conserve momentum? From space maths it is:

$$k(u) - \sqrt{1-\frac{u^2}{c^2}} = \gamma(u)$$

So relativistic momentum: $\vec{p} = m\gamma(u)\vec{u}$, there are two conventions as to the definition:

1. Mass is a function of velocity $\vec{p} = (\gamma m)\vec{u}$.
2. Mass is relativistically invariant and we define a 4-vector mometum $\vec{p} = m(\gamma\vec{u})$.

Where the 4-vector momentum is:

$$P_\mu = mU_\mu = (\gamma m\vec{u}, \gamma mc)$$

## Relativistic Kinetic Energy

Define Force: $F = \frac{d}{dt}(\gamma m\vec{u})$.

$$\text{Work}_{1\to 2} = \int_{\text{path}} \vec{F}\cdot d\vec{L} = T_2 - T_1 \ \therefore\ w_{12} = \int \frac{d}{dt}(\gamma m\vec{u})\cdot d\vec{r}$$

But $d\vec{r} = \vec{u}dt$ so:

$$w_1 2 = \int_1^2 d(\gamma m\vec{u})\cdot\vec{u} = \left[\gamma m u^2\right]_1^2 - \int_1^2 m\gamma u\cdot du$$

Note that the last term is equal to $\int_1^2 \frac{udu}{\sqrt{1-\frac{u^2}{c^2}}}$.

The boundary conditions are $u_1 = 0$, $T_1 = 0$, $u_2 = u$, $T_2 = T$.

$$\therefore w_1 2 = \gamma m u^2 + mc^2\sqrt{1-\frac{u^2}{c^2}} - mc^2 = T \ \therefore w_1 2 = \gamma mc^2 - mc^2 = T$$

Note that for small $\frac{v}{c}$:

$$\gamma = \frac{1}{sqrt1 - \frac{v^2}{c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \ \therefore\ T = \frac{1}{2}mv^2$$

$$T = \gamma mc^2 - mc^2$$

so $\gamma mc^2$ is the total energy and $mc^2$ is the rest energy.

And $\vec{P} = \gamma m\vec{u}$ so $u = c\sqrt{1-\frac{1}{\gamma^2}}$.

$$P^2c^2 = \gamma^2 m^2 c^4\left(1 - \frac{1}{\gamma^2}\right) = E^2 - E_0^2$$

this follows from the definition of "length" for the 4-vector.

So $E^2 = p^2c^2 + E_0^2$ so for a photon $E = pc$ as there is no rest energy.

So the momentum 4-vector is $\vec{p}_\mu = \left(\vec{p}, \frac{E}{c}\right)$.

From this we know the Lorentz transform immediately:

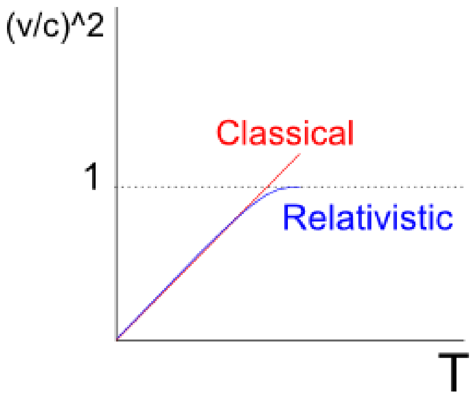
$$P_1' = \frac{P_1 - \frac{EV}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}\ ,\ P_2' = P_2\ ,\ P_3' = P_3\ ,\ E' = \frac{E - VP_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

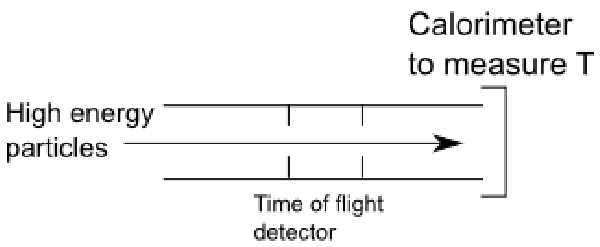
Note the mixing of energy and momentums. So what is conserved in inelastic collisions?

## Experimental test of prediction of T-equation

## Relativistic Dynamics: Perfectly inelastic collision

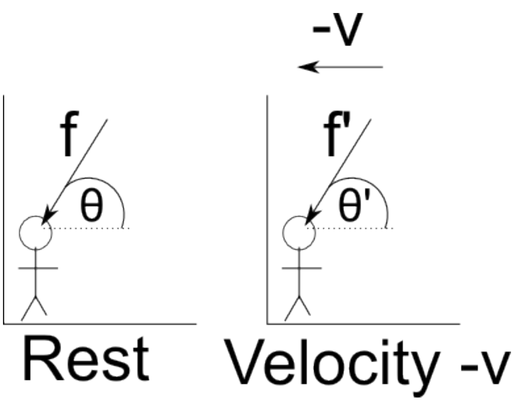
A large particle of mass $M$, momentum $\vec{P}$ and energy $E$ coalesces with a small particle of mass $m$ at rest. The combined particle then has mass $m'$, momentum $p_f$ and energy $E_f$.




From the conservation of total energy: $E_f = E + mc^2$
Momentum is also conserved so $p_f = p$.
But $E^2 - p^2c^2 = m^2c^4$, so for the small mass $E_0 = mc^2$
so $E_f - p_f^2c^2 = m_f^2c^4$, solve for the kinetic energy of the composite after the collision $T$.
$T = E_f - m'c^2$, and from complicated maths: $m'^2 - (M+m)^2 = \frac{2mT}{c^2}$. But $T$ must be positive so $m' > M + m$ i.e. there is mass gain. This demonstrates the interconvertability of mass and energy.

## 0.1 The Relativistic Doppler Effect: The photon approach




$$p_1 = \gamma\left(p_1' + \frac{vE'}{c^2}\right) \ , \ p_2 = p_2' \ , \ E = \gamma\left(E' + vp_1'\right)$$

$$p_1 = p\cos(\theta) \ , \ p_1' = -p'\cos(\theta') \ , \ p_2 = -p\sin(\theta) \ , \ p_2 = -p'\sin(\theta')$$

For photon:

$$E = pc = hf \ \therefore \ p = \frac{hf}{c} \ , \ E = (E' - vp_1'\cos(\theta'))\gamma$$

Rewriting this:

$$f = \gamma\left(f' - \frac{vf'}{c}\cos(\theta')\right) \ \therefore \ f = \gamma\left(1 - \frac{v}{c}\cos(\theta')\right)f'$$

**Special cases**:
Transverse shift, $\theta' = 90°$, $f = \gamma f'$
Radial shift, $\theta' = 0$, $f = \gamma \left(1 - \frac{v}{c}\right) f' = sqrt\frac{1-\frac{v}{c}}{1+\frac{v}{c}} f'$
Can derive the classical results from the binomial expansion for $\frac{v}{c} << 1$. Classically there is no first-order transverse shift. Experimental tests from the observation of light emitted from gases at low pressure and high temperature. Affects astrophysical observations.

## Aberration: Change in apparent direction

$$-f\cos(\theta)\left[-f'\cos(\theta') + \frac{vf'}{c}\right]\gamma$$

Use frequency relation to replace $f$ with $f'$.

$$\therefore \cos(\theta) = \frac{\cos(\theta') - \frac{v}{c}}{1 - \frac{v}{c}\cos(\theta')}$$

This maps $\theta'$ in to small values of $\theta$ and gives rise to the **Searchlight effect**. The observer sees uniformly emitted rays as focussed towards the axis.