PHY1116: Mathematics For Physicists:

Trigonometric Identities:

$\sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b)$ $\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b)$

$1 + \tan^2(\theta) = sec^2(\theta)$ $1 + \cot^2(\theta) = cosec^2(\theta)$ $\sin(2x) = 2\sin(x)\cos(x)$

$\frac{d}{dx}(\tan(\theta)) = sec^2(\theta)$ $\cos^2(x) = \frac{1}{2}\cos(2x) + \frac{1}{2}$ $\cos(2x) = \cos^2(x) - \sin^2(x)$

Differentiation:

Product Rule:

$(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$ or $\frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + \frac{dv}{dx} \cdot u$

Quotient Rule:

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$ or $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx} \cdot v - \frac{dv}{dx} \cdot u}{v^2}$

Chain Rule:

$f(g(x))' = g'(x) \cdot f'(g(x))$ or $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Integration:

Integration by parts:

$\int f(x)g'(x) \ dx = f(x)g(x) - \int g(x)f'(x) \ dx$

or

$\int u \frac{dv}{dx} \ dx = uv - \int v \frac{du}{dx} \ dx$

Integration by substitution:

$\int_a^b f(g(t))g'(t)\,dt = \int_{g(a)}^{g(b)} f(x)\,dx$ where $dx = g'(t) \cdot dt$

Changing Logarithmic base:

$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ when a, x $\neq$ 1.

Logarithmic differentiation:

In general, $y = f(x)$ so $\ln(y) = \ln(f(x))$ hence after implicit differentiation

$\frac{1}{y}\frac{dy}{dx} = \frac{f'(x)}{f(x)}$ therefore $\frac{dy}{dx} = \frac{y \times f'(x)}{f(x)} = f'(x)$

For example,

$y=e^{2x+3}$ therefore ln(y) = 2x+3, therefore 1/y*dy/dx = 2 therefore dy/dx = (2)y therefore dy/dx = $2e^{2x+3}$, as expected.

Partial fractions are best illustrated by an example:

$\frac{x+3}{x^2 - 3x - 40} = \frac{x+3}{(x-8)(x+5)} = \frac{A}{x-8} + \frac{B}{x+5}$ Multiplying by the denominator (x+8)(x-5) we get $x+3=A(x+5)+B(x-8)$ by substituting x = 8, $13\text{A}=11 \Rightarrow A=\frac{11}{13}$ and by substituting x=-5 we obtain $-13\text{B}=-2 \Rightarrow B=\frac{2}{13}$ Therefore:

$$\frac{x+3}{x^2-3\text{x}-40}=\frac{11}{13(x-8)}+\frac{2}{13(x+5)}$$

Series:

Taylor Series: $f(x)=\sum_{n=0}^{\infty}\frac{(x-a)^n}{n!}f^{(n)}(a)$ approximates the function as a polynomial about a. The Maclaurin series is a special case of the Taylor series taken about 0, such that a=0

Series approximations are only valid if all the derivatives exist and the series converges to a finite value. Convergence may be tested first with the D'alemberts ratio test. For a power series $a_0+a_1x+a_2x^2 ...$ the series converges if:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<\frac{1}{|x|}$$

L'Hopitals Rule: For a function $f(a)=\frac{g(a)}{h(a)}$ where g(a) and h(a) are both zero or ±infinity, i.e. the limit is indeterminate, $f(a)=\frac{g^{(1)}(a)}{h^{(1)}(a)}$ if g'(a) and h'(a) are still both zero or both infinity then $f(a)=\frac{g^{(2)}(a)}{h^{(2)}(a)}$ and so on.

Complex Numbers:

When adding/subtracting complex numbers, simply deal with the real and imaginary parts separately. When multiplying, simply multiply out like brackets, or convert to modulus-argument form and add the arguments and multiply the moduli.

The conjugate of a complex number is the same complex number but with the sign of i reversed. A complex fraction may have its denominator made real by multiplying by the complex conjugate e.g. $\frac{2+i}{1+i}=\frac{2+i}{1+i}\frac{1-i}{1-i}=\frac{3-i}{2}$

For modulus-argument form, $a=rcos(\theta)$, $b=rsin(\theta)$, $z=r(\cos\theta+isin\theta)$

$r=\sqrt{a^2+b^2}=\sqrt{zz^*}=|z|$, $\theta=\arctan\left(\frac{b}{a}\right)$ The argument is not unique as $\theta=\theta+2\text{k}\pi$ k=0,1,2,n etc. The principal value of the argument is when:

$-\pi<\theta\leq\pi$

De Moivre's theorem: $z^n=(\cos\theta+isin\theta)^n=\cos(n\theta)+i\sin(n\theta)$

Exponential representation: $z=re^{i\theta}$ hence $z^*=re^{-i\theta}$

Trig and Hyperbolic identities:

$$\cos\theta=\frac{1}{2}(e^{i\theta}+e^{-i\theta}),\quad \sin\theta=\frac{1}{2\mathrm{i}}(e^{i\theta}-e^{-i\theta})$$
$$\cosh\theta=\frac{1}{2}(e^{\theta}+e^{-\theta}),\quad \sinh\theta=\frac{1}{2}(e^{\theta}-e^{-\theta})$$
$$\tanh\theta=\frac{\sinh\theta}{\cosh\theta}$$

Matrices:

A matrix whose elements are all equal to zero is known as a null matrix.
A matrix whose elements are all zero except for the leading diagonals is known as a diagonal matrix.

Matrix multiplication is conducted row by column for each element, and is not commutative.

The matrix with all elements set to zero except its leading diagonals which is set to ones is called the Unit or Identity Matrix. In general, $(\boldsymbol{I})_{ij}=\delta_{ij}$ where δ_{ij} is the Kronecker Delta which is equal to 1 when i=j and 0 elsewhere.

Any matrix multiplied by its inverse results in the identity matrix.

<u>The determinant of a 2x2 Matrix.</u>

$$det\begin{pmatrix} a\, d \\ b\, c \end{pmatrix}=\begin{vmatrix} a\, d \\ b\, c \end{vmatrix}=ad-bc$$

<u>Cramer's Rule</u>

If $\boldsymbol{A}x=b$ then $x_i=\frac{|\boldsymbol{C}(i)|}{|\boldsymbol{A}|}$ where **C**(i) is the matrix formed by replacing column *I* of **A** with **b**

<u>The determinant of a 3x3 Matrix.</u>

The minor of an element of a 3x3 matrix is the determinant of the 2x2 matrix formed by crossing out the row and column of that element.

The co-factor is the signed minor, where the signs used follow the convention $(-1)^{i+j}$ or $\begin{pmatrix} + - + \\ - + - \\ + - + \end{pmatrix}$ NOTE THAT THE CO-FACTOR IS JUST SIGNED AND IS **NOT** MULTIPLIED BY THE ELEMENT ITSELF AS IN THE DETERMINANT
The determinant of the matrix is then equal to the the elements in a row or column multiplied by their respective cofactors and summed together.

i.e. $A=\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ then

$$det(A)=(a_1\cdot((b_2\cdot c_3)-(b_3\cdot c_2)))+(a_2\cdot((b_1\cdot c_3)-(b_3\cdot c_1)))+(a_3\cdot((b_1\cdot c_2)-(b_2\cdot c_1)))$$

Determinant Properties

1)Exchanging rows for columns leaves the determinant unchanged. $|\boldsymbol{A}^T|=|\boldsymbol{A}|$
2)Multiplying any row or column by a constant increases the determinant to be increased by the same factor, therefore if any row or column is zero, the determinant is zero
3)Exchanging any two rows or columns reverses the sign of the determinant. Therefore if any two rows or columns are equal, the determinant is zero.
4)If the elements of any row or column are the sums or differences of one or more terms, then the determinant can be written as the sum of difference of two or more determinants.
5)If any row or column is added to any other row/column the determinant is unchanged.
6) If an row/column can be expressed as a linear combination of any other two rows/columns the determinant is zero
7)The sum of the products of any row/column with the cofactors of any other row is zero.

The adjoint matrix is the transpose of the matrix of cofactors, remember that a cofactor of an element is it's signed minor and the minor is the 2x2 determinant obtained by crossing out the row and column of the element.

The inverse of a matrix is the adjoint matrix divided by the determinant of the original matrix. $\boldsymbol{A}^{-1}=\frac{adj(\boldsymbol{A})}{|\boldsymbol{A}|}$

If a set of simultaneous equations is written as **A**x = b then if we know the inverse, $\mathbf{A}^{-1}$, and hence find the solution x = $\mathbf{A}^{-1}$ **b**

If the determinant of **A** is zero then there is no unique solution. If the equations are homogenous then there must be infinitely many solutions, if the equations are not homogenous then there must be either infinitely many or no solutions(inconsistent).

A homogenous system of equations is one that may be written as **A**x=0

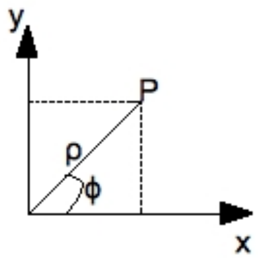
Eigenvectors and Eigenvalues

These have the form $\mathbf{Ax} = \boldsymbol{\lambda}\mathbf{x}$ therefore $(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})\mathbf{x} = \mathbf{0}$ and hence for non-trivial, i.e. non-zero, solutions to exist, $|(\boldsymbol{a}-\lambda\boldsymbol{I})|=0$ we can calculate this to get a polynomial in lambda and solve it to obtain the values of lambda, i.e. the eigenvalues.

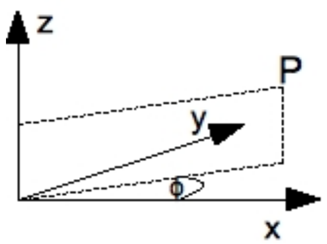
We can then substitute in the eigenvalues into the previous equation of **(A – λI)x = 0** in order to obtain the eigenvectors for each eigenvalue respectively. In general, any scalar multiple of an eigenvector is also an eigenvector, however normalized eigenvectors satisfy the property that $\mathbf{x}(\mathbf{x}^*)^T = \mathbf{1}$.

<u>Partial Differentiation and co-ordinate systems</u>
<u>Co-ordinate systems</u>
Cartesian Co-ordinates: The point P in 3D space has co-ordinates (x,y,z) and its distance from the origin is $r=\sqrt{x^2+y^2+z^2}$

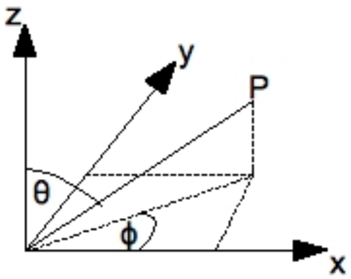
2D Polar Co-ordinates: The point P has co-ordinates (ρ,ϕ) , it is related to 2D cartesian co-ordinates by $x=\rho\cos\phi,\ y=\rho\sin\phi$ so $\rho=\sqrt{x^2+y^2},\ \phi=\arctan\left(\frac{y}{x}\right)$



Cylindrical Polar Co-ordinates: Each point has co-ordinates (ρ,ϕ,z) where
$x=\rho\cos\phi,\ y=\rho\sin\phi,\ z=z,\ \rho=\sqrt{x^2+y^2},\ \phi=\arctan\left(\frac{y}{x}\right)$



Spherical Polar Co-ordinates: Each point has co-ordinates (r,θ,ϕ) with cartesian relations $x=r\sin\theta\cos\phi,\ y=r\sin\theta\sin\phi,\ z=r\cos\theta$ and $r=\sqrt{x^2+y^2+z^2},\ \phi=\arctan\left(\frac{y}{x}\right)$
$\theta=\arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$



Direction cosines

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=\lambda$ where lambda can take any value. They can also be calculated using the rule:

$\cos\alpha=\frac{A.B}{|A||B|}$ where B is the unit vector.

For standard differentiation: $\frac{dy}{dx}=\lim_{x\to 0}\frac{\Delta y}{\Delta x}=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$

In standard differentiation, the differential may be defined as:

$dy=\frac{dy}{dx}dx=f'(x)\,dx$

Partial Differentiation

$\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$ and $\frac{\partial f}{\partial y}=\lim_{k\to 0}\frac{f(x+k,y)-f(x,y)}{k}$

$\frac{\partial f}{\partial x}$ can also be written as $\left(\frac{\partial f}{\partial x}\right)_y$ where the subscript denotes the terms being held constant or f_x where the subscript denotes the variable it has been differentiated with respect to.

To take the partial derivative with respect to a variable you simply differentiate the function treating the other variables as constants.

For partial differentiation the differential becomes:

$df=\left(\frac{\partial f}{\partial x}\right)_y dx+\left(\frac{\partial f}{\partial y}\right)_x dy$

The Reciprocal Theorem: $\left(\frac{\partial x}{\partial z}\right)_y=\frac{1}{\left(\frac{\partial z}{\partial x}\right)_y}$

The Reciprocity Theorem: $\left(\frac{\partial x}{\partial y}\right)_z\left(\frac{\partial y}{\partial z}\right)_x\left(\frac{\partial z}{\partial x}\right)_y=-1$

The Chain Rule for Partial Derivatives: $\frac{df}{dt}=\left(\frac{\partial f}{\partial x}\right)_y\frac{dx}{dt}+\left(\frac{\partial f}{\partial y}\right)_x\frac{dy}{dt}$ Total derivative of *f* wrt. t

If we have the situation where f(x, y(x)) where f is a function f and y but y is itself a function of x then we can adapt the above formula but take x to be the parameter upon which both x and y depend i.e.

$$\frac{df}{dx}=\left(\frac{\partial f}{\partial x}\right)_y+\left(\frac{\partial f}{\partial y}\right)_x\frac{dy}{dx}$$

For f(x,y,z(x,y)):

$$\left(\frac{\partial f}{\partial x}\right)_y=\left(\frac{\partial f}{\partial x}\right)_{y,z}+\left(\frac{\partial f}{\partial z}\right)_{x,y}\left(\frac{\partial z}{\partial x}\right)_y$$

Differentiating implicit functions

If we can write a function f(x,y) = 0, then we can show from the differential that:

$$\frac{dy}{dx}=-\frac{\left(\frac{\partial f}{\partial x}\right)_y}{\left(\frac{\partial f}{\partial y}\right)_x}$$

Higher order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}=\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)_y\right)_y=f_{xx} \qquad \frac{\partial^2 f}{\partial y^2}=\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)_x\right)_x=f_{yy}$$

$$\frac{\partial^2 f}{\partial y\partial x}=\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)_y\right)_x=f_{yx} \qquad \frac{\partial^2 f}{\partial x\partial y}=\left(\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)_x\right)_y=f_{xy}$$

Most functions have an exact differential such that $f_{xy}=f_{yx}$

This is important in Physics as a function with an exact differential is independent of path and therefore a conservative force i.e. like gravity rather than friction.

This provides us with the test for a conservative force:

$f_{xy}=f_{yx}$ therefore $\left(\frac{\partial F_x}{\partial y}\right)_x=\left(\frac{\partial F_y}{\partial x}\right)_y$

It may be shown by subbing dy/dx into the expression for the total derivative that:

$$\frac{d^2y}{dx^2}=-\frac{(f_x^2 f_{yy}-2f_x f_y f_{xy}+f_y^2 f_{xx})}{f_y^3}$$

Previously for the case where *f* is a function of *u* which is itself a function of *x* we found that $\frac{df}{dx}=\frac{df}{du}\frac{du}{dx}$

If *f* is a function of (*u,v*) each of which are functions of (*x,y*) then:

$$\left(\frac{\partial f}{\partial x}\right)_y = \left(\frac{\partial f}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial f}{\partial v}\right)_u \left(\frac{\partial v}{\partial x}\right)_y \quad \text{and} \quad \left(\frac{\partial f}{\partial y}\right)_x = \left(\frac{\partial f}{\partial u}\right)_v \left(\frac{\partial u}{\partial y}\right)_x + \left(\frac{\partial f}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$$

Multiple Integration:

Follow the integrals from the inside-out, if the limits of the integrals are completely independent from one another then the two integrals may be evaluated separately and multiplied. If there is a relation (i.e. the upper limit for y is a function of x or the functions are not separable I.e f(x) = sin(xy), then they must be done in order, complete with limits.

If one variable has limits which are functions of other variables then this variable must be integrated first.

Using alternative co-ordinate systems

In Cartesian co-ordinates, dA = dxdy

In 2D polars, $dA=\rho\, d\rho\, d\phi$

In cylindrical polars, $dV=\rho\, d\rho\, d\phi\, dz$

In spherical polars $dV=r^2\sin(\theta)\, dr\, d\theta\, d\phi$

If you are given asked to find an volume then generally you just have to integrate dV in the easiest co-ordinate system whilst thinking carefully about limits. If you are given a density function and want the mass then you must integrate the density function multiplied by dV.

Line integrals

Consider a small element of the line ds, therefore $ds=\sqrt{(dx)^2+(dy)^2}$

Consider the line integral $I=\int_C F(x,y)\,ds$

If the curve is described by the function y=y(x) then we can express dS as

$ds=\sqrt{1+(\frac{dy}{dx})^2}\,dx$ therefore $I=\int_{a1}^{a2} F(x,y(x))\sqrt{1+(\frac{dy}{dx})^2}\,dx$ which is an ordinary integral wrt. x.

If the curve is described by x=x(y) I.e as a function of y, then we can instead use

$ds=\sqrt{(\frac{dx}{dy})^2+1}\,dy$ so I becomes an ordinary integral wrt to y, i.e.

$$I=\int_{b1}^{b2} F(x(y),y)\sqrt{(\frac{dx}{dy})^2+1}\,dy$$

Alternatively if both x and y are functions of a parameter t then we may parameterise the line integral with respect to t as follows:

$$ds=\sqrt{(dx)^2+(dy)^2}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

If the function is best described by 2D polar co-ordinates, then parameterise it with respect to ϕ, i.e.:

$$ds=\sqrt{\left(\frac{dx}{d\phi}\right)^2+\left(\frac{dy}{d\phi}\right)^2}\,d\phi \quad \text{where} \quad x=\rho\cos(\phi) \quad , \quad y=\rho\sin(\phi)$$

<u>Properties of line integrals:</u>

$$\int_C [P(x,y)dx+Q(x,y)dy]=\int_C P(x,y)dx+\int_C Q(x,y)dy$$

Reversing the direction of integration along C reverses the sign of the integral:

$$\int_{C(a_1,b_1)}^{a_2,b_2}[Pdx+Qdy]=-\int_{C(a_2,b_2)}^{a_1,b_1}[Pdx+Qdy]$$

Complicated curves can be split up into several line integrals and summed using the additive property.

A line integral about a closed curve has a sense, anti-clockwise is +ve. Clockwise is -ve.

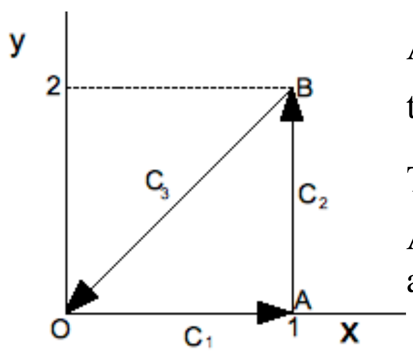
<u>Green's Theorem</u>

$$\iint_{\Re}\frac{\partial Q}{\partial x}dx\,dy=\oint_C Q\,dy\,, \quad \iint_{\Re}\frac{\partial P}{\partial y}dx\,dy=-\oint_C P\,dx$$

$$\oint_C (P\,dx+Q\,dy)=\iint_{\Re}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\,dy$$

An example of Green's Theorem:

Evaluate $I=\oint_C[(2x+y)dx+(3x-2y)dy]$ taken in an anti-clockwise manner round the triangle with vertices at O(0,0), A(1,0) and B(1,2).



Applying Green's Theorem we see that P=2x+y therefore $\frac{\partial P}{\partial y}=1$ and Q =3x-2y so $\frac{\partial Q}{\partial x}=3$

Therefore $I=-\iint_{\Re}2\,dx\,dy=2\iint_{\Re}dx\,dy=2\mathrm{A}$

A is just the area of the triangle i.e. (0.5*1*2*=1) and hence *I* = (2*1) = 2

If $\frac{\partial Q}{\partial x}=\frac{\partial P}{\partial y}$ this means that $\oint_C P\,dx+Q\,dy=0$ which can be expressed as $dZ=P\,dx+Q\,dy=\left(\frac{\partial Z}{\partial x}\right)_y dx+\left(\frac{\partial Z}{\partial y}\right)_x dy$, and so dZ is an exact differential. Then the path length is independent of the path taken, this is a property of conservative forces.

This question may be split into three line integrals C_1, C_2 and C_3. Therefore it is not always necessary to use Green's Theorem and it may not always be the easiest method to use, but in many cases it will be and it should be learnt for the examination in any case as it may be explicitly examined.

Properties of the ellipse.

The ellipse has the equation: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ It may be considered as a deformation of the unit circle, stretching it a factor *a* in the x-direction and *b* in the y-direction. Therefore as the unit circle has area $A=\pi(r^2)=\pi(1)^2=\pi$ the ellipse simply has the area $A=\pi ab$ as it may be considered a deformation of the unit circle.

Surface Integrals:

In Cartesian co-ordinates: $dS=\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$

In Cylindrical polars: $dS=\rho\, d\phi\, dz$

In Spherical polars: $dS=r^2\sin(\theta)\,d\theta\, d\phi$

The Dirac Delta-Function:

The function is infinitely high at x=0, and is zero elsewhere, it has an area of 1.

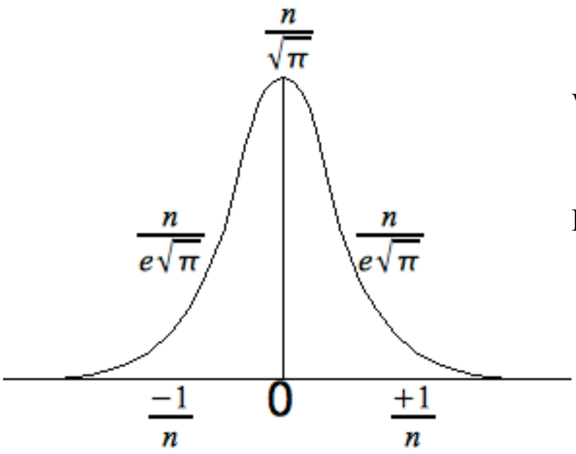
$\int_{-\infty}^{+\infty} f(x)\delta(x-X)dx=f(X)$ i.e. so to evaluate this integral one evaluates the function, f, at x value X.

Properties:

1) $\delta(-x)=\delta(x)$ i.e. the function is symmetric about x=0
2) $\delta[a(x-X)]=\frac{1}{|a|}\delta(x-X)$
3) $\int_{-\infty}^{+\infty}\delta'(x)f(x)dx=-f'(0)$
4) $\frac{d}{dx}\delta(x)=-\frac{1}{x}\delta(x)$

Generating functions:
Top-Hat function: Has area of 1, at half maximum, height = 1/2w, width = 2w.

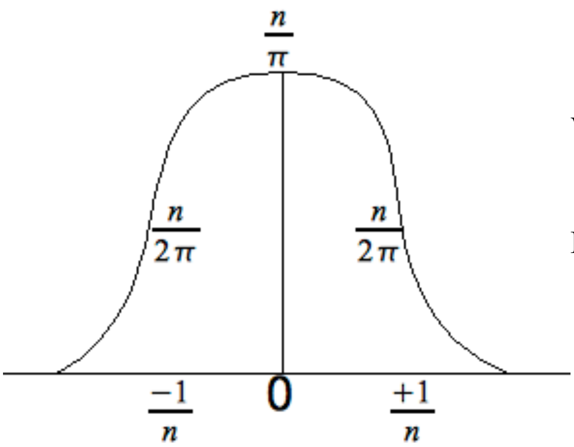
Gaussian: $\delta(x) = \lim_{n\to\infty} \frac{n}{\sqrt{\pi}} e^{-n^2x^2}$



Width at 1/e of peak is $\pm\frac{1}{n}$

Peak is at $\frac{n}{\sqrt{\pi}}$

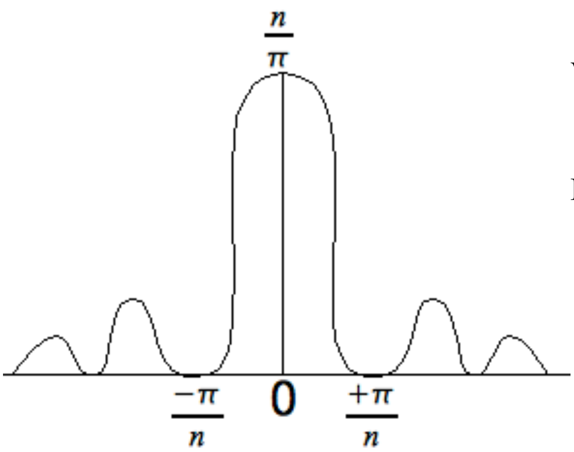
Lorentzian: $\delta(x) = \lim_{n\to\infty} \frac{n}{\pi(1+n^2x^2)}$



Width at half maximum is $\pm\frac{1}{n}$

Peak is at $\frac{n}{\pi}$

Sinc: $\delta(x) = \lim_{n\to\infty} \frac{\sin(nx)}{\pi x}$



Width at first 0 is $\pm\frac{\pi}{n}$

Peak is at $\frac{n}{\pi}$

Vectors

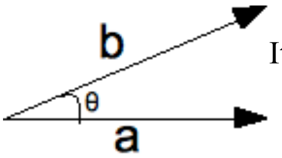
A general unit vector: $\hat{\boldsymbol{a}} = \frac{\vec{\boldsymbol{a}}}{|\boldsymbol{a}|}$

If $\hat{\boldsymbol{a}} = \hat{a}_x \boldsymbol{i} + \hat{a}_y \boldsymbol{j} + \hat{a}_x \boldsymbol{k}$ then it's cartesian components are just the direction cosines of a line in the direction of $\hat{\boldsymbol{a}}$

$$\hat{a}_x = \cos\alpha \qquad \hat{a}_y = \cos\beta \qquad \hat{a}_z = \cos\gamma$$

When adding/subtracting vectors simply add/subtract the elements.

Multiplication – Scalar(dot) product



$a \cdot b = a_x b_x + a_y b_y + a_z b_z = |\boldsymbol{a}||\boldsymbol{b}|\cos(\theta)$

It is commutative so $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$

Projection of a along the direction of b, i.e. the component of a along b: $|\boldsymbol{a}|\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{b}|} = \boldsymbol{a}\cdot\boldsymbol{b}$

Angle between a and b: $\cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}$ hence for non-zero vectors $\boldsymbol{a}\cdot\boldsymbol{b} = 0 \Rightarrow \boldsymbol{a} \perp \boldsymbol{b}$

Magnitude of a: $|\boldsymbol{a}|^2 = \boldsymbol{a}\cdot\boldsymbol{a}$

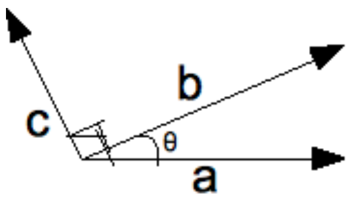
Cartesian unit vectors: $\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = \hat{z}\cdot\hat{z} = 1$ and $\hat{x}\cdot\hat{y} = \hat{x}\cdot\hat{z} = \hat{y}\cdot\hat{z} = 0$

Dot product may be considered as matrix multiplication of row vector with column vector.

Vector Cross product

Easiest to work out as determinant of 3x3 matrix i.e.:

$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$



c = **a** x **b**, c is perpendicular to both **a** and **b** and has magnitude, $|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}|\sin\theta$

Note: **a** x **b** = - **b** x **a** so the cross product is not commutative

Vector normal to a a plane

If a and b are two vectors in the plane, then $\hat{\boldsymbol{c}} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$ is the unit vector normal to the plane, $|\boldsymbol{a}\times\boldsymbol{b}|$ is the area of the parallelogram made by the two vectors.

For cartesian unit vectors:

$$\hat{x}\times\hat{y} = \hat{z} \quad \hat{y}\times\hat{z} = \hat{x} \quad \hat{z}\times\hat{x} = \hat{y}$$
$$\hat{y}\times\hat{x} = -\hat{z} \quad \hat{z}\times\hat{y} = -\hat{x} \quad \hat{x}\times\hat{z} = -\hat{y}$$
$$\hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = 0$$

Scalar triple product

$$\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\begin{vmatrix} a_x & a_y & a_z \\ b_a & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \qquad \begin{array}{c} \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a})=\boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) \\ \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})=-\boldsymbol{b}\cdot(\boldsymbol{a}\times\boldsymbol{c})\ \ etc. \\ \boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{a})=0 \end{array}$$

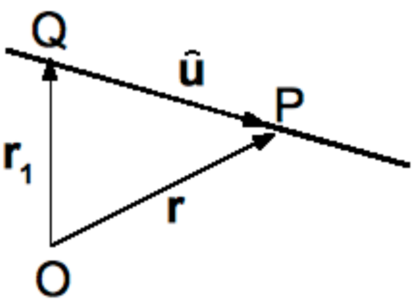
The scalar triple product $\boldsymbol{a}\cdot(\boldsymbol{b}\times\boldsymbol{c})$ finds the volume of the parallelepiped formed by the three vectors where c and b form the base and a forms the height.

Vector triple product

$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ is a vector which is in the plane containing b and c. It is easiest to calculate by simply working out **b** x **c** and then crossing that with **a**

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})\neq(\boldsymbol{a}\times\boldsymbol{b})\times\boldsymbol{c}$$
$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=\boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c})-\boldsymbol{c}(\boldsymbol{a}\cdot\boldsymbol{b})$$
$$(\boldsymbol{a}\times\boldsymbol{b})\cdot(\boldsymbol{c}\times\boldsymbol{d})=\boldsymbol{a}\cdot(\boldsymbol{b}\times(\boldsymbol{c}\times\boldsymbol{d}))$$

Vector equation of a straight line



The line is defined by its direction (along **u**) and the position vector of a point Q on the line ($\mathbf{r}_1$)

An expression for the vector **r** from the origin O to any point P on the line is given by:

$$\boldsymbol{r}=\boldsymbol{r}_1+\lambda\hat{\boldsymbol{u}}$$

The cartesian equation of a straight line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=\lambda$$ where l, m, n are the direction cosines through the points x_1, y_1, z_1

Proof from vectors:
Cartesian components of a unit vector are just its direction cosines so:

$$\hat{\boldsymbol{u}}=l\hat{\boldsymbol{x}}+m\hat{\boldsymbol{y}}+n\hat{\boldsymbol{z}} \quad \boldsymbol{r}_1=x_1\hat{\boldsymbol{x}}+y_1\hat{\boldsymbol{y}}+z_1\hat{\boldsymbol{z}} \quad \boldsymbol{r}=x\hat{\boldsymbol{x}}+y\hat{\boldsymbol{y}}+z\hat{\boldsymbol{z}}$$

$$\boldsymbol{r}-\boldsymbol{r}_1=\lambda\hat{\boldsymbol{u}}\rightarrow(\boldsymbol{r}-\boldsymbol{r}_1)\cdot\hat{\boldsymbol{u}}=\lambda$$
$$(x-x_1)\hat{\boldsymbol{x}}+(y-y_1)\hat{\boldsymbol{y}}+(z-z_1)\hat{\boldsymbol{z}}=\lambda(l\hat{\boldsymbol{x}}+m\hat{\boldsymbol{y}}+n\hat{\boldsymbol{z}})$$

Equate components along x,y,z directions on LH and RH:

$$(x-x_1)=\lambda l,\ (y-y_1)=\lambda m,\ (z-z_1)=\lambda n$$

Therefore: $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}=\lambda$

Vector equation of a plane

The plane is defined by the unit vector **n** normal to the plane and the perpendicular distance p of the plane from the origin O.

For any position vector **r** describing a point on the plane the projection of **r** along **n** is p:
i.e. $\boldsymbol{r}\cdot\hat{\boldsymbol{n}}=p$

$\hat{\boldsymbol{n}}=l\,\hat{\boldsymbol{x}}+m\,\hat{\boldsymbol{y}}+n\,\hat{\boldsymbol{z}}$ then (l,m,n) are the direction cosines of the normal to the plane and
$\boldsymbol{r}\cdot\hat{\boldsymbol{n}}=lx+my+nz=p$ (cartesian equation of plane)

Scalar and vector fields

Scalar fields consist of a scalar quantity which has different values at different co-ordinates. (It is just a function of three variables) . Examples: Temperature in a room, air pressure etc.

Vector fields consist of a 3D vector quantity which varies at different co-ordinates (it is a set of three functions of three variables one for each vector component)
Examples: rate of heat flow in room, magnetic field, electric field

$\nabla=\hat{\boldsymbol{x}}\frac{\partial}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial}{\partial y}+\hat{\boldsymbol{z}}\frac{\partial}{\partial z}$ ∇T is the gradient of T or grad T

Moving in the direction of grad T results in the largest change in T for a given magnitude of d**r**. The normal to the gradient is the surface in which there value remains constant i.e. dr = 0
Moving in an arbitrary direction by d**r** then the change in T is:
$dT=\nabla T\cdot d\,\boldsymbol{r}=(\nabla T\cdot d\,\hat{\boldsymbol{r}})|d\,\boldsymbol{r}|$
therefore $\frac{dT}{|d\,\boldsymbol{r}|}=(\nabla T\cdot d\,\hat{\boldsymbol{r}})$ is the directional gradient in the direction of unit vector d**r**

Inverse operation

If $\boldsymbol{A}=\nabla\phi$ then what is phi in terms of **A**?
$d\,\phi=\nabla\phi\cdot d\,\boldsymbol{r}$
$\phi=\int\nabla\phi\cdot d\,\boldsymbol{r}=\int\boldsymbol{A}\cdot d\,\boldsymbol{r}$

Grad in other co-ordinate systems

In cylindrical polars: $\nabla f=\frac{\partial f}{\partial\rho}\hat{\boldsymbol{\rho}}+\frac{1}{\rho}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}+\frac{\partial f}{\partial z}\hat{\boldsymbol{z}}$

In spherical polars: $\nabla f=\frac{\partial f}{\partial r}\hat{\boldsymbol{r}}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\boldsymbol{\phi}}$

This can be proved by working out the cartesian partial derivatives in the other co-ordinate systems, and substitute these into the cartesian expression for grad.

Divergence

Definition: $\nabla\cdot\boldsymbol{a}=\left(\hat{\boldsymbol{x}}\frac{\partial}{\partial x}+\hat{\boldsymbol{y}}\frac{\partial}{\partial y}+\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)\cdot(a_x\hat{\boldsymbol{x}}+a_y\hat{\boldsymbol{y}}+a_z\hat{\boldsymbol{z}})=\frac{\partial a_x}{\partial x}+\frac{\partial a_y}{\partial y}+\frac{\partial a_z}{\partial z}$

Describes the net flow out of a microscopic region i.e. the extent to which the vector field flow behaves like a source or sink at that point. Positive divergence means there is a net flow outwards i.e. it acts like a source, whereas negative divergence means there is a net flow inwards, i.e. it acts like a sink.

When dealing with vector questions involving **r** in cartesian co-ordinates it is important to remember: $\boldsymbol{r}=x\,\hat{\boldsymbol{x}}+y\,\hat{\boldsymbol{y}}+z\,\hat{\boldsymbol{z}}$, $|\boldsymbol{r}|=\sqrt{x^2+y^2+z^2}$, $\hat{\boldsymbol{r}}=\frac{\boldsymbol{r}}{|\boldsymbol{r}|}=\frac{x\,\hat{\boldsymbol{x}}+y\,\hat{\boldsymbol{y}}+z\,\hat{\boldsymbol{z}}}{\sqrt{x^2+y^2+z^2}}$

The Divergence Theorem

First define the divergence as the net flux of a vector **a** per unit volume out of an infinitesimal volume. $\nabla\cdot\boldsymbol{a}=\lim_{dV\to 0}\frac{\oint\!\!\oint_S \boldsymbol{a}\cdot\boldsymbol{dS}}{dV}$ Then by considering the flow through opposite faces of an infinitesimal volume. We can derive that the **divergence is in spherical polars is**

$$\nabla\cdot\boldsymbol{a}=\frac{1}{r^2}\frac{\partial(r^2 a_r)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(a_\theta\sin\theta)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial(a_\phi)}{\partial\phi}$$

Divergence in cylindrical polars:

$$\nabla\cdot\boldsymbol{a}=\frac{1}{\rho}\frac{\partial(\rho\, a_\rho)}{\partial\rho}+\frac{1}{\rho}\frac{\partial a_\phi}{\partial\phi}+\frac{\partial a_z}{\partial z}$$

A vector field whose divergence is zero everywhere is termed solenoidal.

By considering the flux out of a finite volume we can derive **the divergence theorem:**

$$\iiint_V \nabla\cdot\boldsymbol{a}\,dV=\oint\!\!\oint_S \boldsymbol{a}\cdot\boldsymbol{dS}$$

Curl

The curl is the result of the cross product between the del operator and the vector i.e.

$$\nabla\times\boldsymbol{a}=\begin{vmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ a_x & a_y & a_z\end{vmatrix}$$

The curl has a physical interpretation as the amount and direction of rotation that is induced in an infinitesimally small flywheel placed in the vector field. Positive rotation (i.e. +ve curl) is anti-clockwise and negative rotation is clockwise. In other words the curl describes the rotation, shear or vorticity of the vector field.

Stokes' Theorem

If we first generalise our definition of the curl to an infinitesimal loop with a surface normal $\hat{\boldsymbol{n}}$ and enclosed area dS

$(\nabla\times\boldsymbol{a})\cdot\hat{\boldsymbol{n}}=\lim_{dS\to 0}\frac{\oint_{dC}\boldsymbol{a}\cdot d\boldsymbol{l}}{dS}$ This can be used to derive curl in other co-ordinate systems which is covered in supplementary web handout, and also for deriving Stokes' Theorem.

By summing over a finite surface we obtain **Stokes' Theorem:**

$$\iint_S (\nabla\times\boldsymbol{a})\cdot d\boldsymbol{S}=\oint_C \boldsymbol{a}\cdot d\boldsymbol{l}$$

For a sphere, dS acts in the $\hat{\boldsymbol{r}}$ direction.

Irrotational vector force fields (I.e those with a curl of zero everywhere) describe conservative forces (note that these include ANY central forces – those with only an **r**-component)

Note that:

$\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{r}}=\cos(\theta)$

$\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{\phi}}=-\sin(\phi)$ $\qquad$ $\hat{\boldsymbol{y}}\cdot\hat{\boldsymbol{\phi}}=\cos(\phi)$ $\qquad$ $\hat{\boldsymbol{z}}\cdot\hat{\boldsymbol{\phi}}=0$

Ordinary Differential Equations:

Ordinary – One independent variable (i.e. y=f(x), not z=f(x,y))

Order – The highest order of the differential.

Degree – The power to which the highest order differential is raised. i.e. i.e. $\frac{dy}{dx}=3y$ is first-order, first-degree $\frac{d^2y}{dx^2}+kx=0$ is second-order, first-degree, $\left(\frac{d^3y}{dx^3}\right)^2-\left(\frac{dy}{dx}\right)^3=0$ is third-order, second-degree.

Linear – No powers of the dependent variable (y usually) above 1.

An *n*th order differential equation has a general solution containing *n* arbitrary constants. When initial conditions are specified the particular solution can be determined.

Method 1: Separable variables

Applicable when $f(x,y)=g(x)h(y)$, so $\frac{dy}{dx}=g(x)h(y)\rightarrow\frac{dy}{h(y)}=g(x)dx$

so solution is of the form: $\int\frac{dy}{h(y)}=\int g(x)dx+c$.

Example: $\frac{dy}{dx}=xy \therefore \frac{dy}{y}=x\,dx \therefore \ln(y)=\frac{x^2}{2}+c \therefore y=Ae^{\frac{x^2}{2}}$ $where\ A=e^c$

Method 2: Homogeneous equation:

Applicable if $f(x,y)=g\left(\frac{y}{x}\right)$ i.e. the function can be expressed entirely in terms of multiples of $\frac{y}{x}$.

Let $v=\frac{y}{x} \therefore y=vx \therefore \frac{dy}{dx}=x\frac{dv}{dx}+v$ And so: $x\frac{dv}{dx}+v=g(v)$ which is separable to solve for v. Then multiply solution for v by x to obtain final solution for y.

Example: $\frac{dy}{dx}=\frac{x^2+y^2}{2x^2}$ so $\frac{dy}{dx}=\frac{1}{2}+\frac{1}{2}\left(\frac{y}{x}\right)^2$, letting $v=\frac{y}{x} \therefore y=vx \therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$

so: $v+x\frac{dv}{dx}=\frac{1}{2}+\frac{v^2}{2}$ $\therefore$ $x\frac{dv}{dx}=\frac{v^2}{2}-v+\frac{1}{2}=\frac{1}{2}(1-v^2)$ so: $\frac{2\,dv}{(1-v)^2}=\frac{dx}{x}$ Integrating:

$\frac{-2}{1-v}=\ln(x)+c$, $let\ c=\ln(A)$, $1-v=\frac{2}{\ln(Ax)}$ $\therefore$ $v=1-\frac{2}{\ln(Ax)}$ so: $y=x\left(1-\frac{2}{\ln(Ax)}\right)$

<u>Method 3: The exact equation:</u>

Equation is of form: $P(x,y)dx+Q(x,y)dy=0$, method is applicable when $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$.

So is reduced to form: $dF(x,y)=P(x,y)dx+Q(x,y)dy=0$
Solution is of the form: $F(x,y)=constant$

Can calculate F from properties of exact differential, that is: $\frac{\partial F}{\partial x}=P(x,y)$ and $\frac{\partial F}{\partial y}=Q(x,y)$.

From this expression for F, can solve for y by using quadratic formula, etc.

Example: $\frac{dy}{dx}=\frac{2y+x}{5y-2x}$ so: $(2y+x)dx=(5y-2x)dy$ $\therefore$ $(2y+x)dx+(2x-5y)dy=0$

i.e. $P=2y+x$ and $Q=2x-5y$. Verifying exact differential condition: $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}=2$
so is exact differential and this method is applicable.
So: $\frac{\partial F}{\partial x}=2y+x$, $\frac{\partial F}{\partial y}=2x-5y$. From deduction: $F=\frac{x^2}{2}+2yx-\frac{5y^2}{2}=constant$.
This is a quadratic in y, which can be solved using the quadratic formula to yield the final solution.

<u>Method 4: The integrating factor:</u>

Applicable if $f(x,y)=Q(x)-yP(x)$ i.e. the DE is in the form: $\frac{dy}{dx}+P(x)y=Q(x)$
i.e. it is a linear first-order differential equation

Multiply the equation by an integrating factor, R(x): $R(x)\frac{dy}{dx}+R(x)P(x)y=R(x)Q(x)$

R is chosen such that the left-hand side is equal to $\frac{d}{dx}[R(x)y]$

so ODE simplifies to: $\frac{d}{dx}[R(x)y]=R(x)Q(x)$ whose solution is: $R(x)y=\int R(x)Q(x)dx$

so: $\frac{d}{dx}[R(x)y]=R(x)\frac{dy}{dx}+y\frac{dR(x)}{dx}=R(x)\frac{dy}{dx}+R(x)P(x)y$ $\therefore$ $\frac{dR(x)}{dx}=R(x)P(x)$

$\int\frac{dR}{R}=\int P(x)dx\rightarrow R(x)=e^{(\int P(x)dx)}$ so $y=\frac{\int Q(x)e^{(\int P(x)dx)}dx+c}{e^{(\int P(x)dx)}}$

Learn this solution: $y=\frac{\int Q(x)e^{(\int P(x)dx)}dx+c}{e^{(\int P(x)dx)}}$

Example: $\frac{dy}{dx}+\frac{y}{x}=4x^2$ so $P(x)=\frac{1}{x}$ $\therefore$ $\int P(x)dx=\int\frac{dx}{x}=\ln(x)$

so $y=\frac{\int 4x^3 dx+c}{x}=\frac{x^4+c}{x}=x^3+\frac{c}{x}$

Remember that any operation with the dependent variable cannot merely be absorbed in to the constant. The constant in the equation for y is the only integration constant considered, adding others during the integration is wrong.

Second Order Differential Equations (Linear, with constant coefficients)

These have the form of the equation for the damped harmonic oscillator:

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega^2 x=0$$

Solution has the form: $x(t)=e^{\beta t}$ this yields the 'Auxillary Equation': $\beta^2+2\gamma\beta+\omega^2=0$ which tells us the values of beta which allow $e^{\beta t}$ to be a solution.

When $\gamma>\omega$, β is real (strongly damped), $x(t)=Ae^{\beta_1 t}+Be^{\beta_2 t}$

When $\gamma<\omega$, β is complex (weakly damped), $x(t)=E\,e^{-\gamma t}\sin(\sqrt{\omega^2-\gamma^2}\,t+\phi)$

When $\gamma=\omega$, $\beta_1=\beta_2=-\gamma=-\omega$ and it is critically damped, such that $x(t)=Ae^{-\omega t}+Be^{-\omega t}=(A+B)e^{-\omega t}$ but this is not the general solution as it only has one arbitrary constant as (A+B)=C.

However by expressing the original equation for the damped harmonic oscillator as: $\left(\frac{d}{dt}+\omega\right)\left(\frac{d}{dt}+\omega\right)x=0$ we notice that $x(t)=Ce^{-\omega t}$ is a solution to $\left(\frac{d}{dt}+\omega\right)x=0$ and thus the general solution is any solution of

$$\left(\frac{d}{dt}+\omega\right)x=C\,e^{-\omega t}$$

Solving this via integrating factors we obtain the general solution: $x(t)=(Ct+D)e^{-\omega t}$