Incompressible flow

Consider the incompressible flow in a lid driven cavity (fig. 1). The flow on the domain $x \in \langle 0, 1\rangle \times y \in \langle 0, 1\rangle$ is governed by the Navier–Stokes equations for a Newtonian incompressible fluid

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \mathrm{Re}^{-1} \Delta \boldsymbol{u}, \tag{1}$$
$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

with the following boundary conditions

$$\boldsymbol{u} = 0 \qquad \text{at } x = 0 \cup x = 1 \cup y = 0 \tag{3}$$
$$\boldsymbol{u} = (1, 0) \qquad \text{at } y = 1. \tag{4}$$

We want to compute the evolution of the flow starting from an initial condition

$$\boldsymbol{u}(\boldsymbol{x}, t = 0) = 0. \tag{5}$$

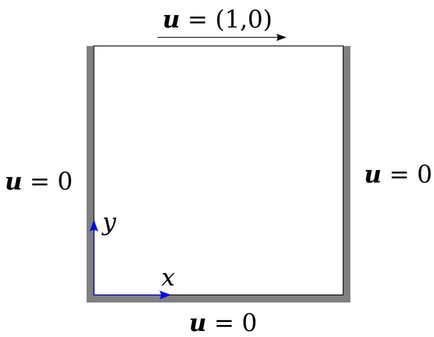

Figure 1: Lid-driven cavity

Discretization in time

We discretize the problem semi-implicitly in time, computing the non-linear convective flux explicitly and the linear diffusive flux implicitly

$$\frac{\boldsymbol{u}^{n+1} - \boldsymbol{u}^n}{\Delta t} + \boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n = -\nabla p^{n+1} + \mathrm{Re}^{-1} \Delta \boldsymbol{u}^{n+1}, \tag{6}$$
$$\nabla \cdot \boldsymbol{u}^{n+1} = 0, \tag{7}$$

and solve the problem with the fractional step method

$$\frac{\boldsymbol{u}^* - \boldsymbol{u}^n}{\Delta t} - \mathrm{Re}^{-1} \Delta \boldsymbol{u}^{n+1} = -\boldsymbol{u}^n \cdot \nabla \boldsymbol{u}^n \tag{8}$$
$$\Delta p^{n+1} = \frac{1}{\Delta t} \nabla \cdot \boldsymbol{u}^* \tag{9}$$
$$\boldsymbol{u}' = -\Delta t \nabla p^{n+1} \tag{10}$$
$$\boldsymbol{u}^{n+1} = \boldsymbol{u}^* + \boldsymbol{u}'. \tag{11}$$

The Poisson equation (9) for pressure will be solved with homogeneous Neumann BC

$$\frac{\partial p^{n+1}}{\partial n} \equiv \boldsymbol{n} \cdot \nabla p^{n+1} = 0 \qquad \text{at all boundaries,} \tag{12}$$

where $\boldsymbol{n}$ is an outward pointing normal to the boundary. Note that p is a relative pressure. If p^{n+1} is the solution of (9), then $p^{n+1} + c$ is also a solution. Thus, we should relate the pressure to some reference. A quick-and-dirty way is to specify some value of the pressure at a single (arbitrary) point on the boundary.

Discretization in space

We can discretize the problem in space, e.g., with the finite volume method on a staggered grid (fig. 2). Let us first discretize the <u>convective</u> term on the right-hand side of (8). We will employ the identity

$$u \partial_x u + v \partial_y u \overset{\nabla \cdot \boldsymbol{u} = 0}{=\!=} \partial_x \left(u^2\right) + \partial_y (uv) \tag{13}$$
$$u \partial_x v + v \partial_y v \overset{\nabla \cdot \boldsymbol{u} = 0}{=\!=} \partial_x (uv) + \partial_y \left(v^2\right) \tag{14}$$

to compute the convection in <u>conservative</u> form. Integrating the convection of x-momentum (13) over its control volume (red square in fig. 2) we obtain

$$\iint\limits_{\Omega_u} \left[\partial_x \left(u^2\right) + \partial_y (uv)\right] \mathrm{d}x\mathrm{d}y = \left[\int u^2 \mathrm{d}y\right]_w^e + \left[\int uv \mathrm{d}x\right]_s^n$$
$$\approx \left[u^2 \Delta y\right]_w^e + \left[uv \Delta x\right]_s^n. \tag{15}$$

The values of u at the cell faces u_e, u_w, u_n and u_s need to be obtained by interpolation. The choice of the interpolation scheme affects both the stability and accuracy of the overall solution. Let us try a simple linear interpolation

$$u_e \approx \frac{U_E + U_P}{2}, \qquad u_w \approx \frac{U_W + U_P}{2}, \qquad u_n \approx \frac{U_N + U_P}{2}, \qquad u_s \approx \frac{U_S + U_P}{2}.$$

For y-momentum, the discretization is analogous.

Tasks:

1. Write MATLAB functions to evaluate the right-hand side of (8) for x and y momentum.
2. Discretize the diffusive fluxes.
3. Discretize equation (8). Write the left hand side in the form

$$A^u_P u_P + A^u_E u_E + A^u_W u_W + A^u_N u_N + A^u_S u_S$$

   for u and analogously for v and express the coefficients.
4. Write the discrete equations for the control volumes on the boundaries and in the corners. Express the coefficients.
5. Discretize equations (9) and (10).
6. Compute the solution in MATLAB.

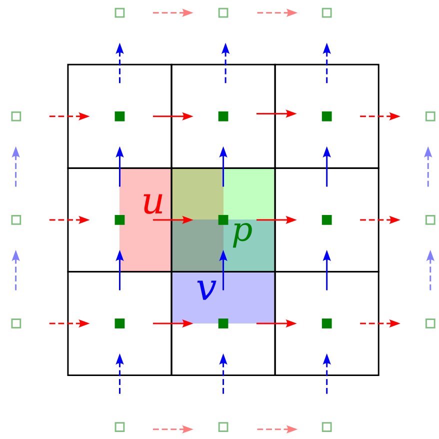

Figure 2: Staggered grid