## 8.2: diffusion of $x$-momentum

Consider the diffusion of $x$-momentum, expressed by the term $\mathrm{Re}^{-1}\Delta u^*$. We can again integrate the Laplace operator $\Delta = \partial_{xx} + \partial_{yy}$ over a control volume

$$\begin{aligned}
\iint\limits_{\Omega_u} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}x\mathrm{d}y &= \left[\int \frac{\partial u}{\partial x}\mathrm{d}y\right]_w^e + \left[\int \frac{\partial u}{\partial y}\mathrm{d}x\right]_s^n \\
&\approx \left[\frac{\partial u}{\partial x}\Delta y\right]_w^e + \left[\frac{\partial u}{\partial y}\Delta x\right]_s^n \\
&\approx \left(\frac{u_E - u_P}{\Delta x} - \frac{u_P - u_W}{\Delta x}\right)\Delta y + \left(\frac{u_N - u_P}{\Delta y} - \frac{u_P - u_S}{\Delta y}\right)\Delta x \\
&= \frac{u_E - 2u_P + u_W}{\Delta x}\Delta y + \frac{u_N - 2u_P + u_S}{\Delta y}\Delta x.
\end{aligned} \tag{1}$$

The value of the Laplacian at the center of a cell is then approximated by a mean value

$$\begin{aligned}
\Delta u &\approx \frac{1}{\Delta x \Delta y} \iint\limits_{\Omega_u} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \mathrm{d}x\mathrm{d}y \\
&\approx \frac{u_E - 2u_P + u_W}{\Delta x^2} + \frac{u_N - 2u_P + u_S}{\Delta y^2} \\
&= \underbrace{\frac{1}{\Delta x^2}}_{A_E^u} u_E + \underbrace{\frac{1}{\Delta x^2}}_{A_W^u} u_W + \underbrace{\frac{1}{\Delta y^2}}_{A_N^u} u_N + \underbrace{\frac{1}{\Delta y^2}}_{A_S^u} u_S \underbrace{-2\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right)}_{A_P^u} u_P.
\end{aligned} \tag{2}$$

For $y$-momentum, the coefficients are the same. Recall that the numbers of cells for $u$ and $v$ are $N_y \times (N_x - 1)$ and $(N_y - 1) \times N_x$, respectively. Thus, the vector indexing for $u$ and $v$ is

$$m_u = k_u + (j_u - 1)N_y, \qquad m_v = k_v + (j_v - 1)(N_y - 1), \tag{3}$$

and

$$W_u = m - N_y, \qquad W_v = m - (N_y - 1), \qquad \text{etc.}$$

The discrete Laplacian matrices can then be constructed as in `Laplacian_u.m` and `Laplacian_v.m`.

On the cells for $u$ adjacent to the left boundary, the values of $u_W$ are prescribed directly by the boundary conditions. Thus, we put $-A_W^u u_W$ into $\boldsymbol{b}_u$ for $m = 1, \ldots, N_y$. Similarly, for the cells adjacent to the right boundary we put $-A_E^u u_E$ into $\boldsymbol{b}_u$ for $m = (N_x - 2)N_y + 1, \ldots, (N_x - 1)N_y$.

The situation is different at the top and at the bottom boundary. Take the top boundary as an example. The boundary condition $u_n = 1$ is prescribed at the north edge of control volumes adjacent to the top boundary. But to compute the diffusive flux through the north face with centered finite differences, we would need the nodal values at the opposite sides of the north edge

$$\partial_y u_n \approx \frac{u_N - u_P}{\Delta x}.$$

We can use $u_N$ as a *ghost point* and prescribe some expression to it. We will relate the known boundary value $u_n$ to the linear interpolation of $u_N$ and $u_P$

$$u_n \approx \frac{u_N + u_P}{2}.$$

By re-arrangement we obtain the expression for the ghost-point value

$$u_N = 2u_n - u_P.$$

Substituting into the discretization

$$\Delta u \approx A_E u_E + A_W u_W + A_S u_S + A_N(2u_n - u_P) + A_P u_P$$

we obtain the coefficients $\tilde{A}_P^u = A_P - A_N$ and $-b_u = 2A_N u_n$ such that

$$\Delta u \approx A_E u_E + A_W u_W + A_S u_S + \tilde{A}_P^u u_P - \boldsymbol{b}_u$$

at the top cells for $u$.