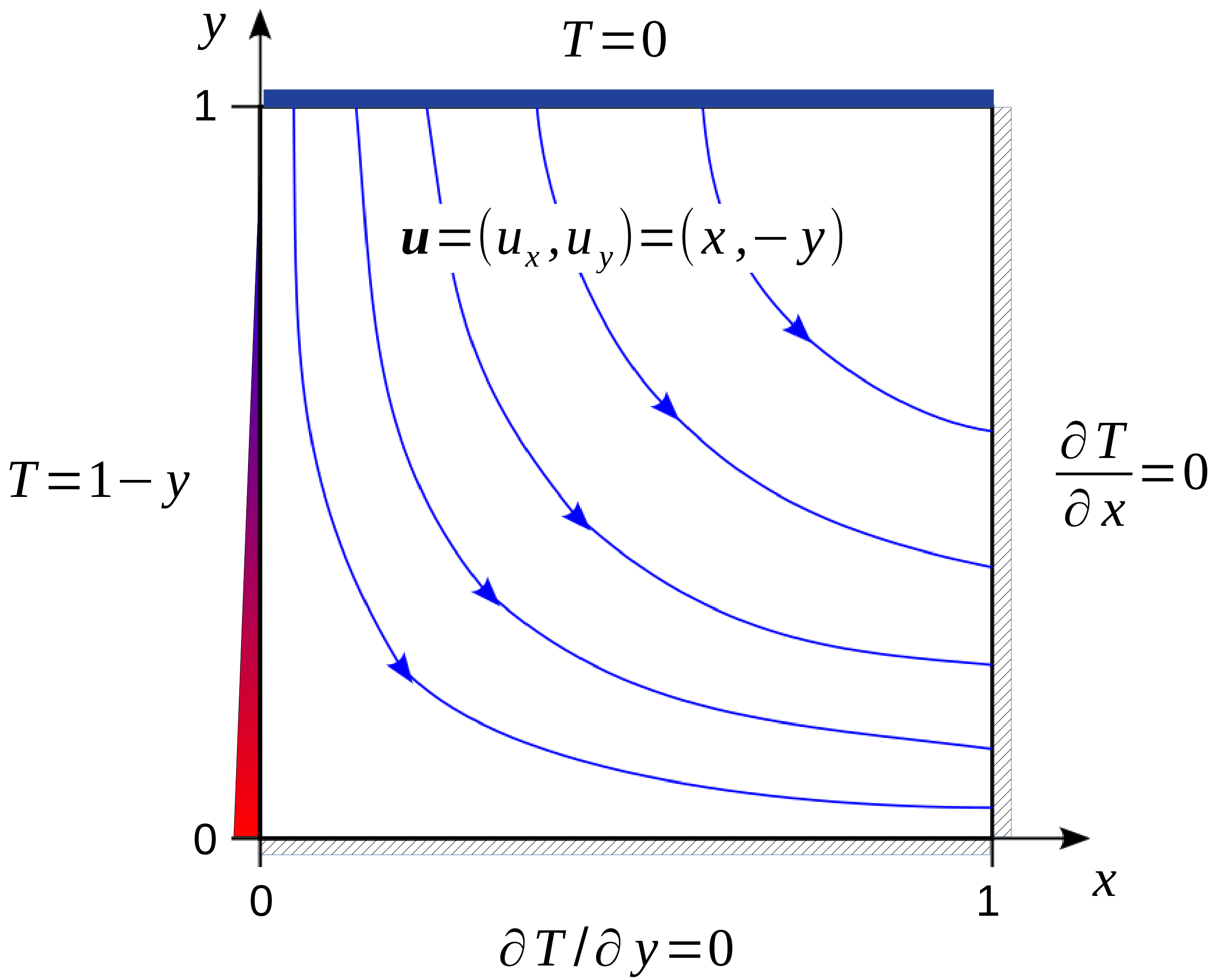
Finite volume method (FVM) for scalar transport

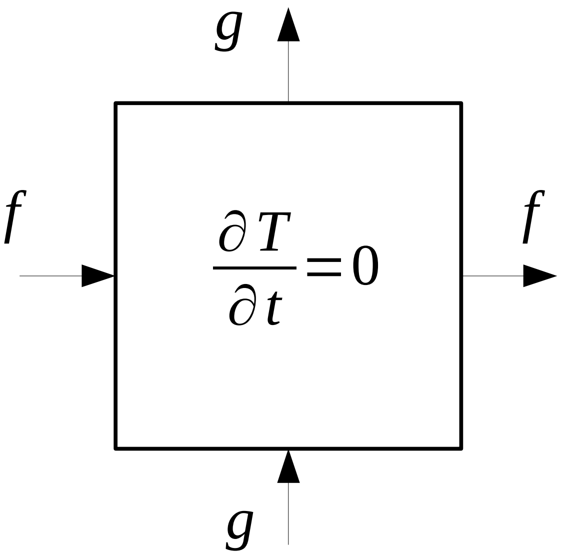
- Consider the transport of a scalar quantity in a given velocity field by convection and diffusion:



Integral conservation law

- Since there are no source terms and we are looking for the steady state solution, the conservation law in integral form reduces to

$$\oint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = 0$$



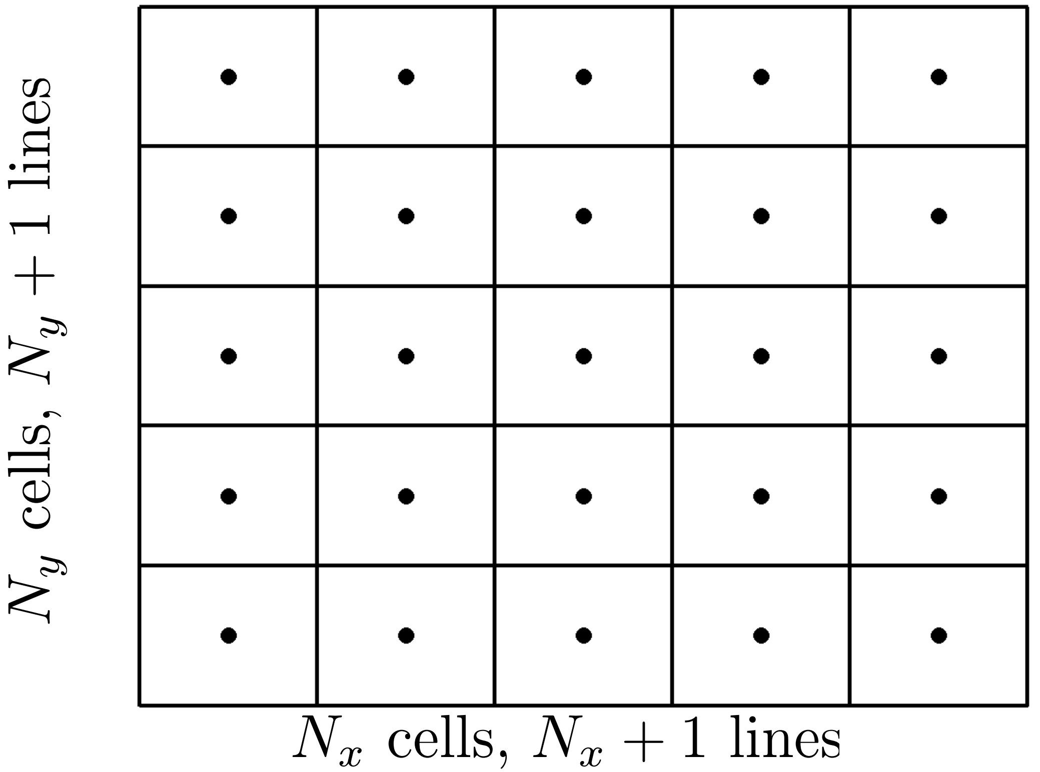
- The flux is given by the sum of convective and diffusive flux

$$\boldsymbol{F}=(f,g)$$

$$f=f^c+f^d=T u_x-\kappa\frac{\partial T}{\partial x}$$

$$g=g^c+g^d=T u_y-\kappa\frac{\partial T}{\partial y}$$

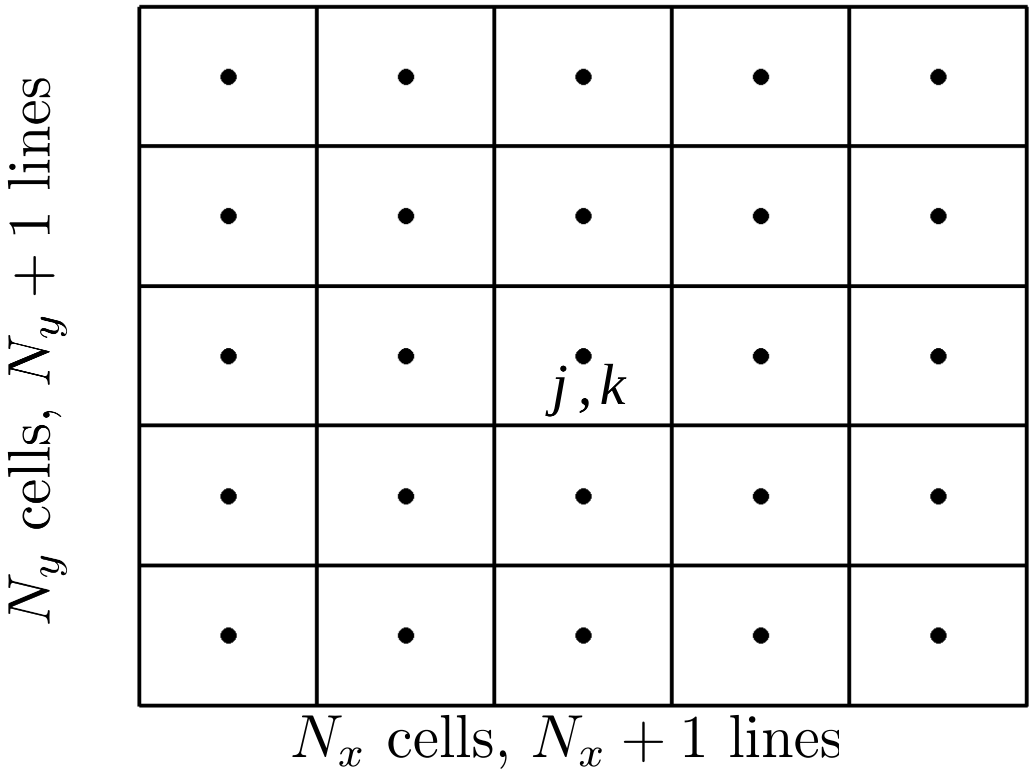
Finite volume mesh



$$\Delta x = \frac{1}{N_x}$$

$$\Delta y = \frac{1}{N_y}$$

Finite volume mesh



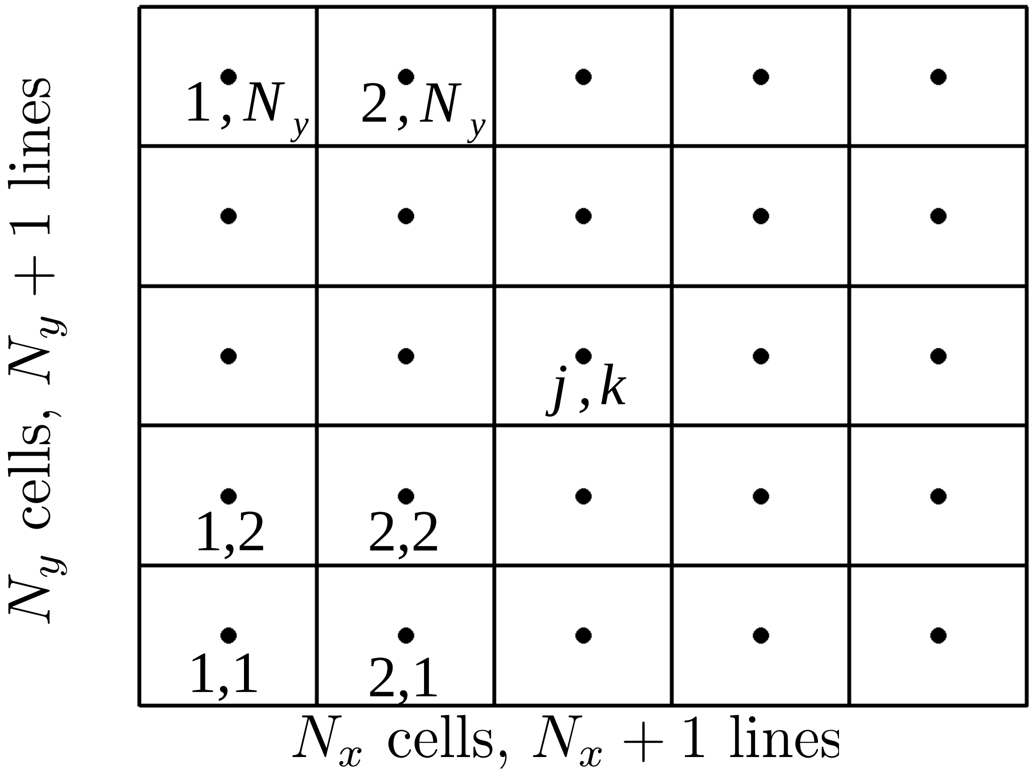
$$\Delta x = \frac{1}{N_x}$$

$$\Delta y = \frac{1}{N_y}$$

$$x_j = j \Delta x$$

$$y_k = k \Delta y$$

Finite volume mesh



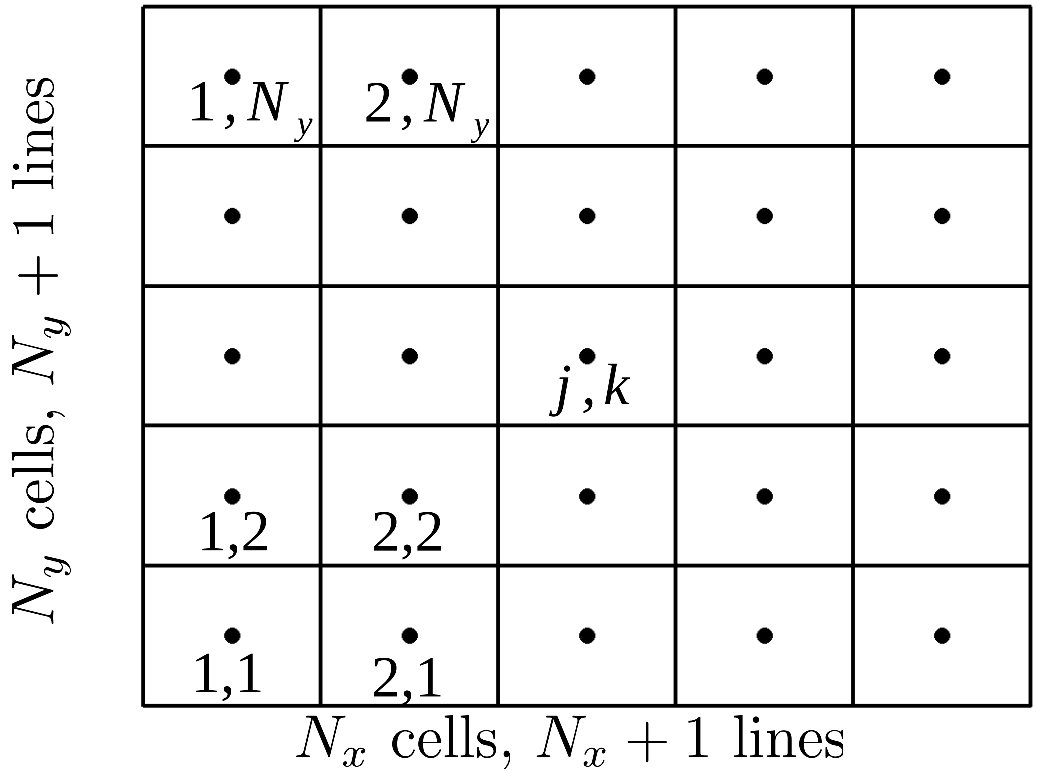
$$\Delta x = \frac{1}{N_x}$$

$$\Delta y = \frac{1}{N_y}$$

$$x_j = j\,\Delta x$$

$$y_k = k\,\Delta y$$

Finite volume mesh



$$\Delta x = \frac{1}{N_x}$$

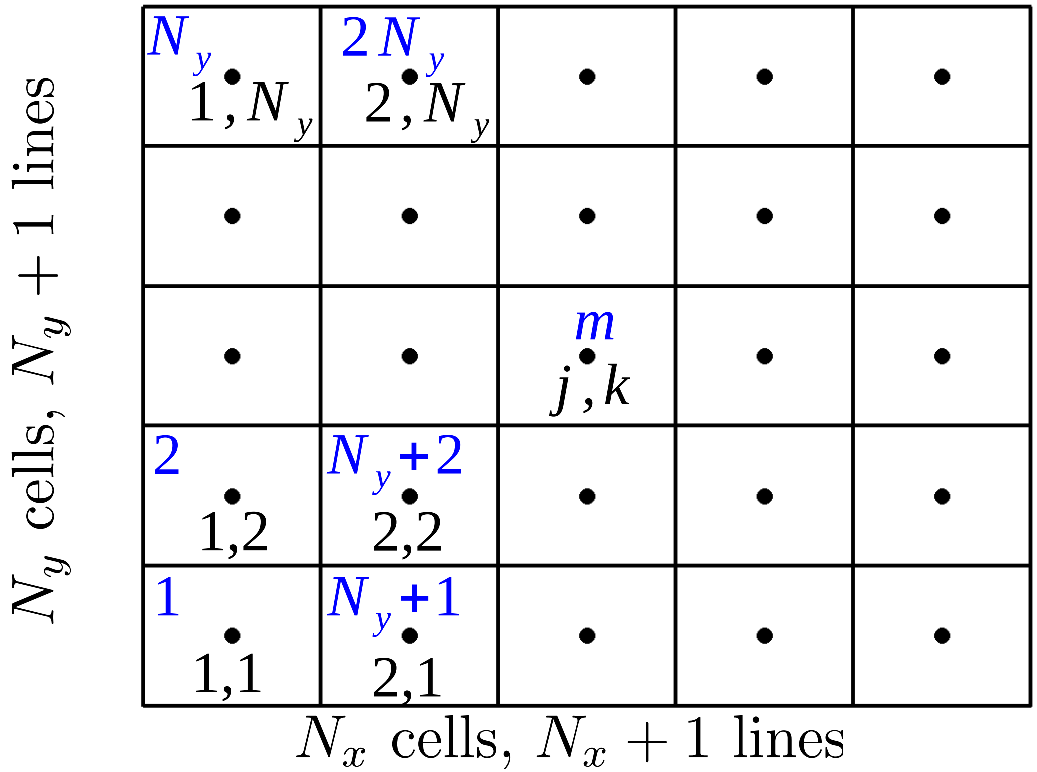
$$\Delta y = \frac{1}{N_y}$$

$$x_j = j\,\Delta x$$

$$y_k = k\,\Delta y$$

j, k matrix ordering $\longrightarrow$ m vector ordering

Finite volume mesh



$$\Delta x = \frac{1}{N_x}$$

$$\Delta y = \frac{1}{N_y}$$

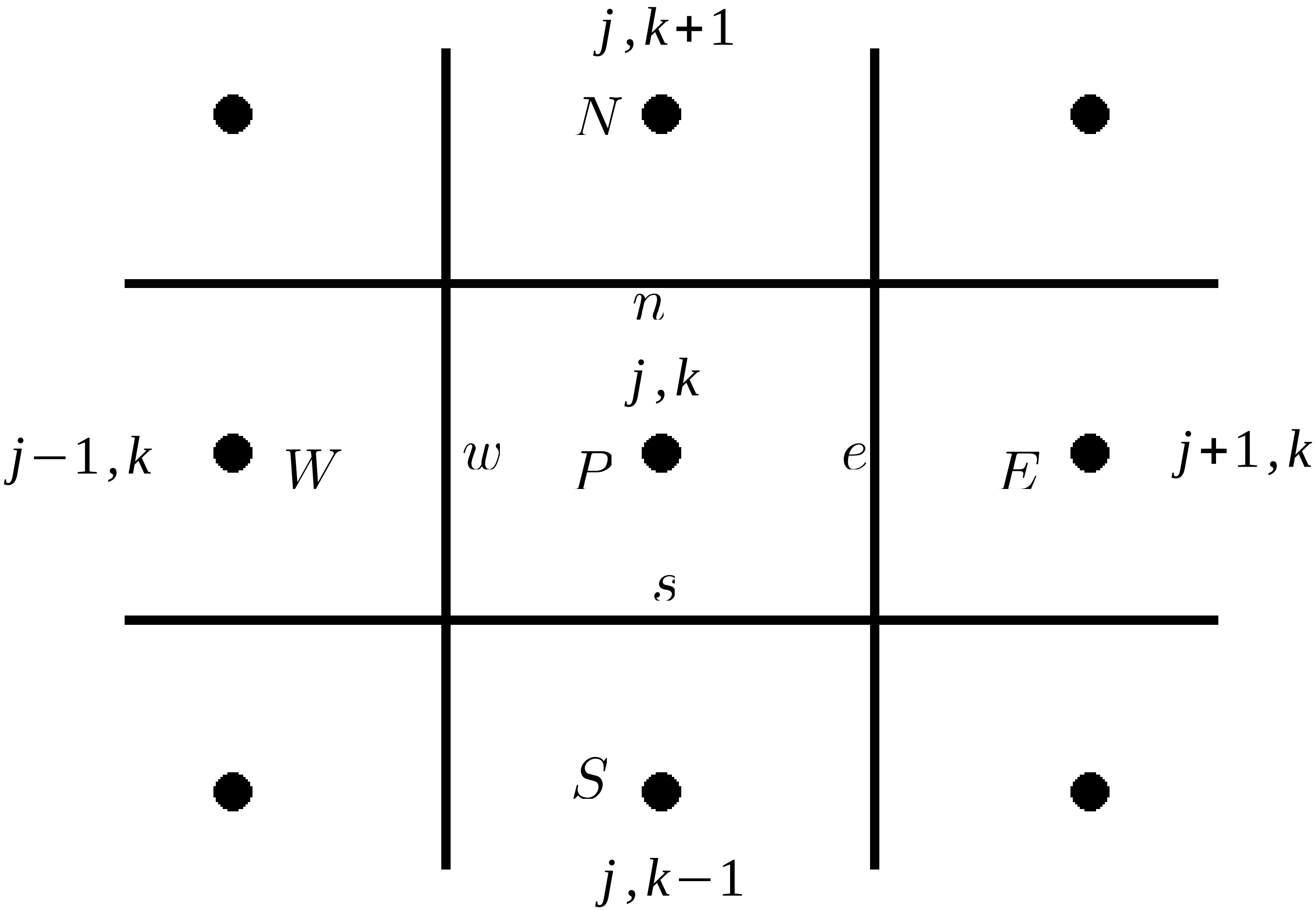
$$x_j = j\,\Delta x$$

$$y_k = k\,\Delta y$$

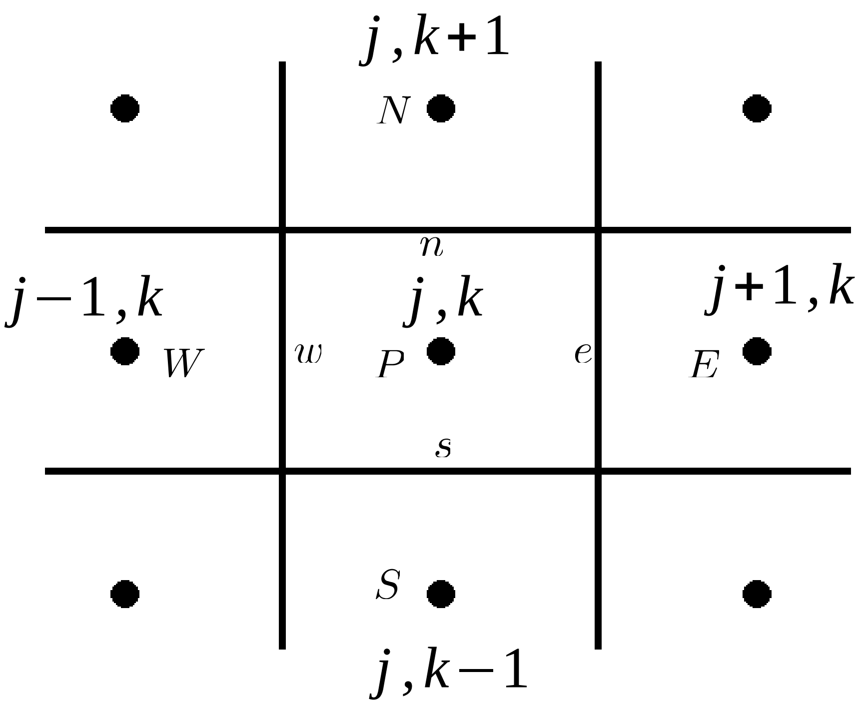
j, k matrix ordering ⟶ m vector ordering

$$m = k + (j-1)N_y$$

Finite volume mesh



Finite volume mesh



$$m = k + (j-1)N_y$$

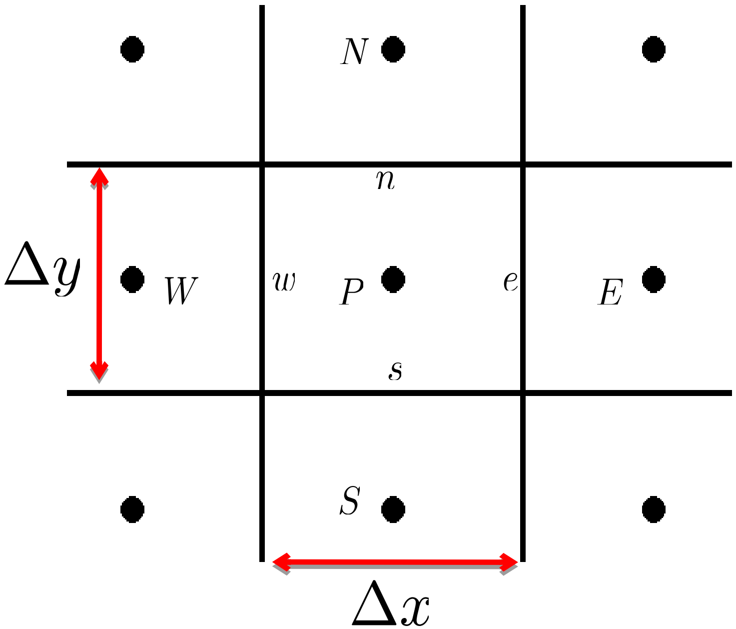
matrix ordering   vector ordering

point	(j,k)	m
P	j,k	m
W	$j-1,k$	$m-N_y$
E	$j+1,k$	$m+N_y$
S	$j,k-1$	$m-1$
N	$j,k+1$	$m+1$

Discretization

- The equation to be solved

$$\oint_S \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{S} = 0 \qquad \boldsymbol{F} = (f, g)$$



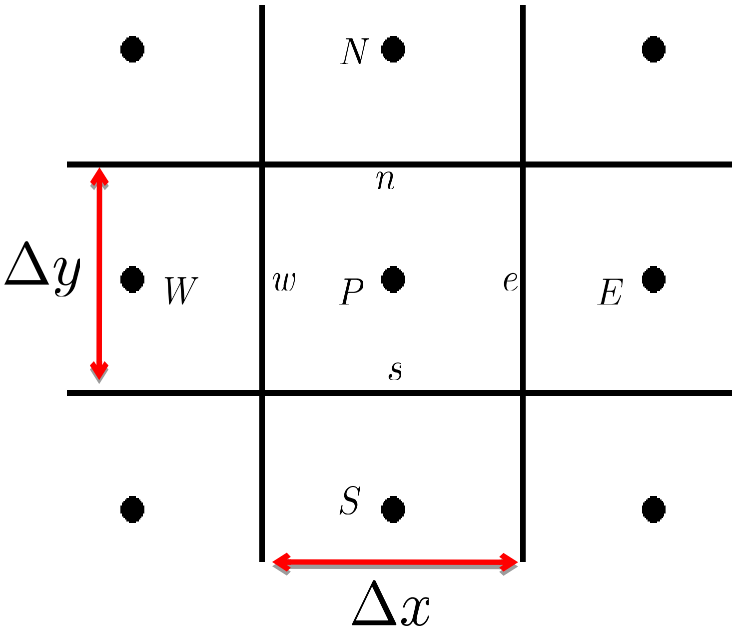
$$(f_e - f_w)\Delta y + (g_n - g_s)\Delta x = 0$$

$$f = f^c + f^d = T u_x - \kappa \frac{\partial T}{\partial x}$$

$$g = g^c + g^d = T u_y - \kappa \frac{\partial T}{\partial y}$$

Discretization

- Flux at face *e*



convective flux (central approx.)

$$f_e^c = T_e (u_x)_e \approx \frac{1}{2}(T_P + T_E)(u_x)_e$$

diffusive flux (central approx.)

$$f_e^d \approx -\kappa \frac{T_E - T_P}{\Delta x}$$

$$f_e \approx \frac{1}{2}(T_P + T_E)(u_x)_e - \kappa \frac{T_E - T_P}{\Delta x}$$

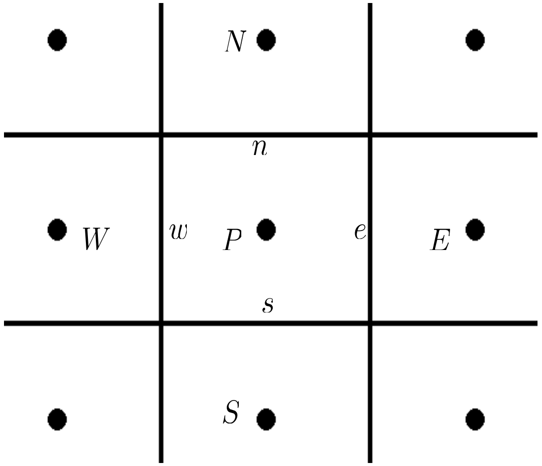
Discretization

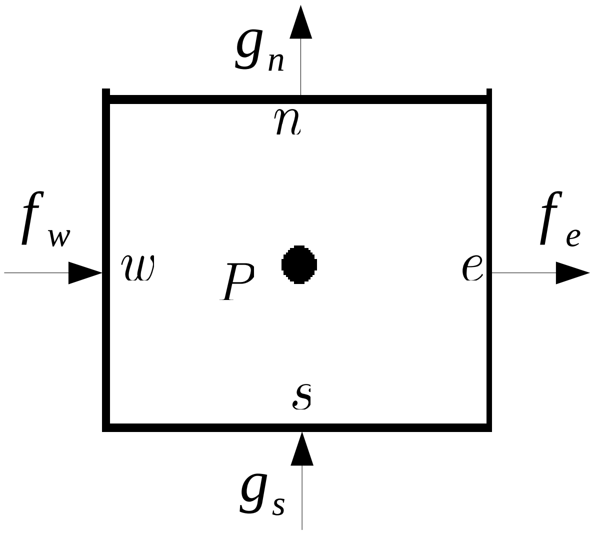
- All fluxes in an internal cell

$$(f_e - f_w)\Delta y + (g_n - g_s)\Delta x = 0$$

$$\left(\frac{1}{2}(T_P + T_E)(u_x)_e - \kappa\frac{T_E - T_P}{\Delta x}\right)\Delta y$$
$$-\left(\frac{1}{2}(T_P + T_W)(u_x)_w - \kappa\frac{T_P - T_W}{\Delta x}\right)\Delta y$$
$$+\left(\frac{1}{2}(T_P + T_N)(u_y)_n - \kappa\frac{T_N - T_P}{\Delta y}\right)\Delta x$$
$$-\left(\frac{1}{2}(T_P + T_S)(u_y)_s - \kappa\frac{T_P - T_S}{\Delta y}\right)\Delta x \quad = 0$$

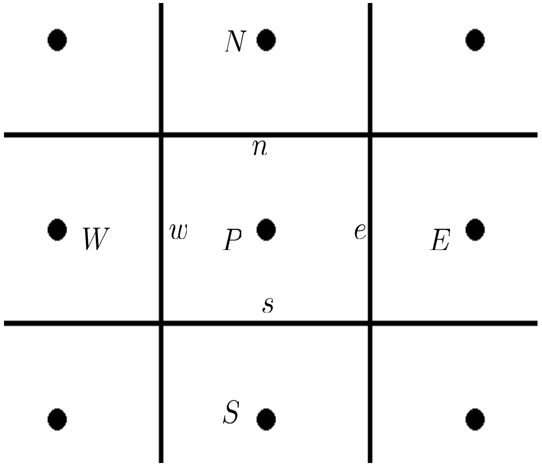
$$A_P T_P + A_E T_E + A_W T_W + A_N T_N + A_S T_S = 0$$





Discretization

- The coefficients are then



$$A_P T_P + A_E T_E + A_W T_W + A_N T_N + A_S T_S = 0$$

$$A_E = \frac{1}{2}(u_x)_e \Delta y - \kappa \frac{\Delta y}{\Delta x}$$

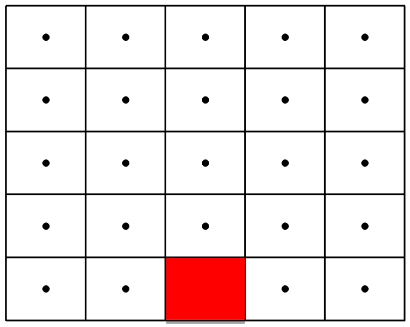
$$A_W = -\frac{1}{2}(u_x)_w \Delta y - \kappa \frac{\Delta y}{\Delta x}$$

$$A_N = \frac{1}{2}(u_y)_n \Delta x - \kappa \frac{\Delta x}{\Delta y}$$

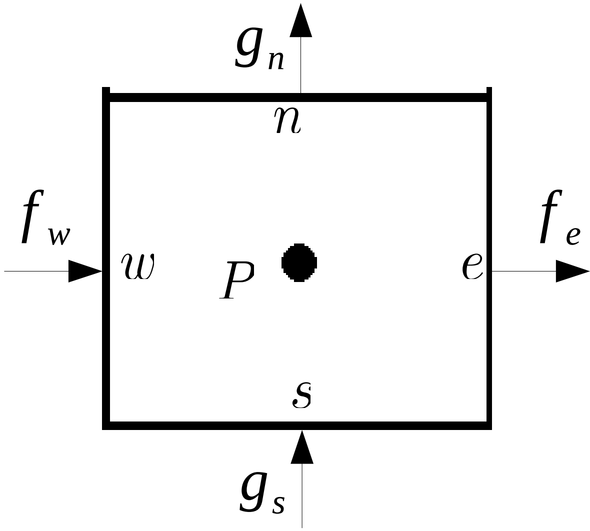
$$A_S = -\frac{1}{2}(u_y)_s \Delta x - \kappa \frac{\Delta x}{\Delta y}$$

$$A_p = \frac{1}{2} \underbrace{\left[((u_x)_e - (u_x)_w)\Delta y + ((u_y)_n - (u_y)_s)\Delta x\right]}_{=0} + 2\kappa \left[\frac{\Delta y}{\Delta x} + \frac{\Delta x}{\Delta y}\right]$$

South boundary



- $j = 2, \ldots, N_x - 1\,; \qquad k = 1$
- East, west and north fluxes unmodified
- The b.c. is $\partial_y T = 0$, so as a 1st approx. we can assume $T_P \approx T_S$
- The flux is then $g_s = T_P\,(u_y)_s$
- For this flow
  $(u_y)_s = 0 \text{ at } y = 0 \rightarrow g_s = 0$



South boundary

- $g_s = 0$
- The equation is then

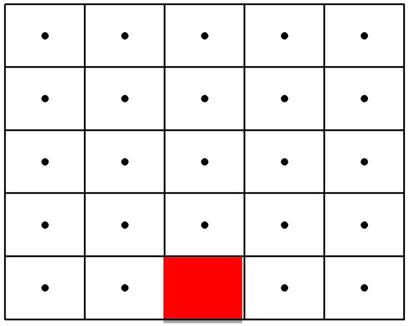
$$(f_e - f_w)\Delta y + g_n \Delta x = 0$$

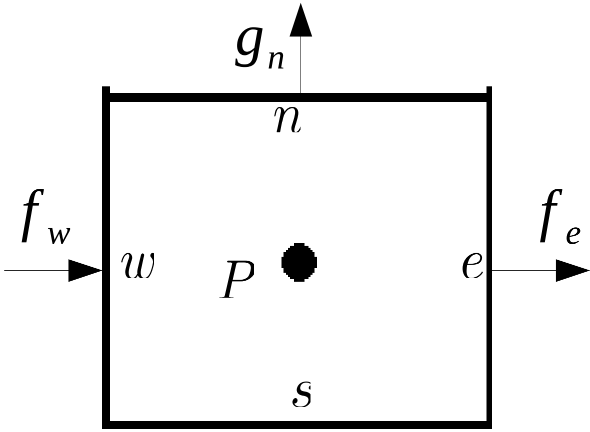
$$\left(\frac{1}{2}(T_P + T_E)(u_x)_e - \kappa\frac{T_E - T_P}{\Delta x}\right)\Delta y$$

$$-\left(\frac{1}{2}(T_P + T_W)(u_x)_w - \kappa\frac{T_P - T_W}{\Delta x}\right)\Delta y$$

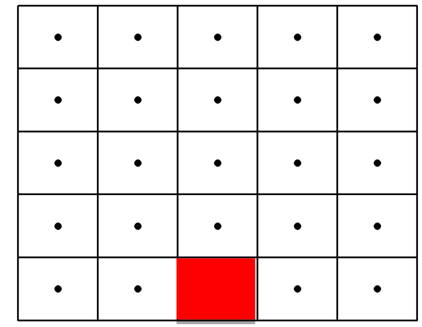
$$+\left(\frac{1}{2}(T_P + T_N)(u_y)_n - \kappa\frac{T_N - T_P}{\Delta y}\right)\Delta x = 0$$

$$A_P T_P + A_E T_E + A_W T_W + A_N T_N = 0$$





South boundary



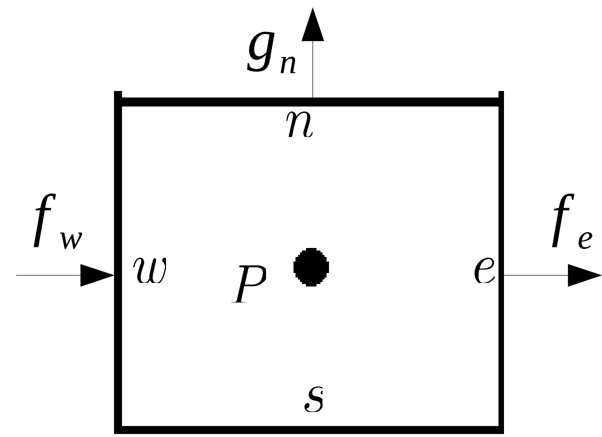
- $g_s = 0$
- The equation is then

$$(f_e - f_w)\Delta y + \; g_n \Delta x = 0$$

$$\left(\frac{1}{2}(T_P + T_E)(u_x)_e - \kappa \frac{T_E - T_P}{\Delta x}\right) \; \Delta y$$

$$-\left(\frac{1}{2}(T_P + T_W)(u_x)_w - \kappa \frac{T_P - T_W}{\Delta x}\right) \; \Delta y$$

$$+\left(\frac{1}{2}(T_P + T_N)(u_y)_n - \kappa \frac{T_N - T_P}{\Delta y}\right) \; \Delta x \quad = 0$$



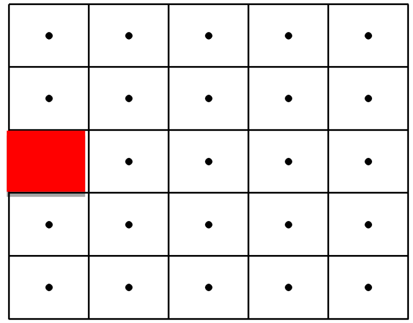
$$A_P T_P + A_E T_E + A_W T_W + A_N T_N = 0$$

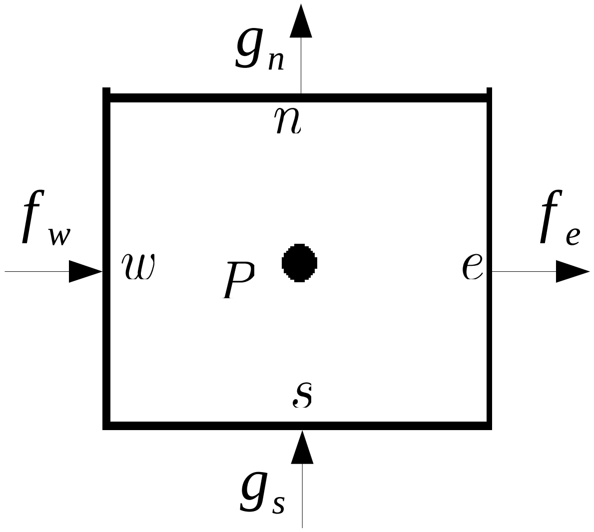
- Changed coef.: $A_P = 2\kappa \frac{\Delta y}{\Delta x} + \kappa \frac{\Delta x}{\Delta y} + \frac{1}{2} \underbrace{\left[(u_x)_e \Delta y - (u_x)_w \Delta y + (u_y)^n \Delta x \right]}_{=0}$

West boundary

- $j = 1 \qquad\qquad k = 2, \ldots, N_y - 1$
- East, south and north fluxes unmodified
- The b.c. is $T_w = 1 - y$
- We need to change the approx. for the spatial derivative, e.g. $\left.\dfrac{\partial T}{\partial x}\right|_w \approx \dfrac{T_P - T_w}{\Delta x/2}$
- Also at this boundary $(u_x)_w = 0$
- So that the flux is

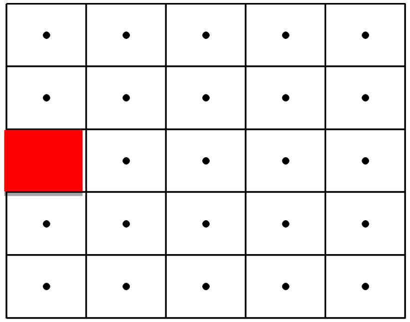
$$f_w = -2\,\kappa \frac{T_P - T_w}{\Delta x}$$

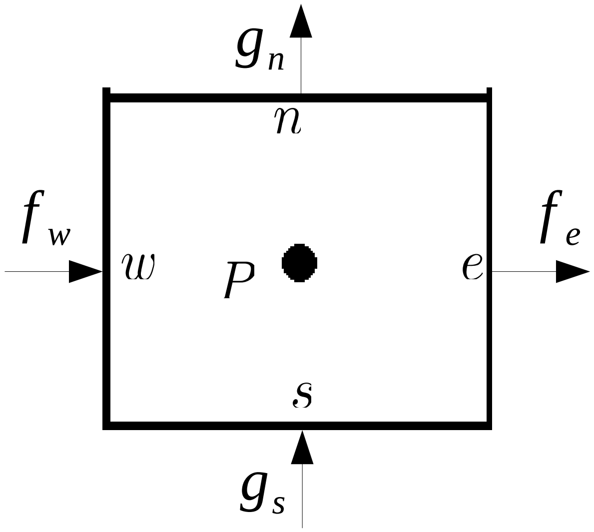




West boundary

- $f_w = -2\,\kappa \dfrac{T_P - T_w}{\Delta x}$
- $(f_e - f_w)\Delta y + (g_n - g_s)\Delta x = 0$





West boundary

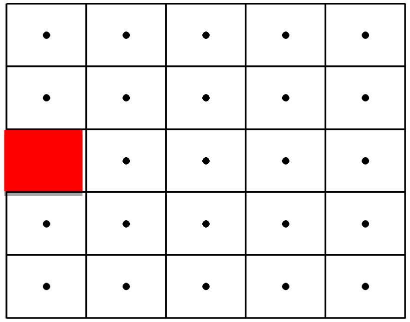
- $f_w = -2\,\kappa \dfrac{T_P - T_w}{\Delta x}$
- $(f_e - f_w)\Delta y + (g_n - g_s)\Delta x = 0$

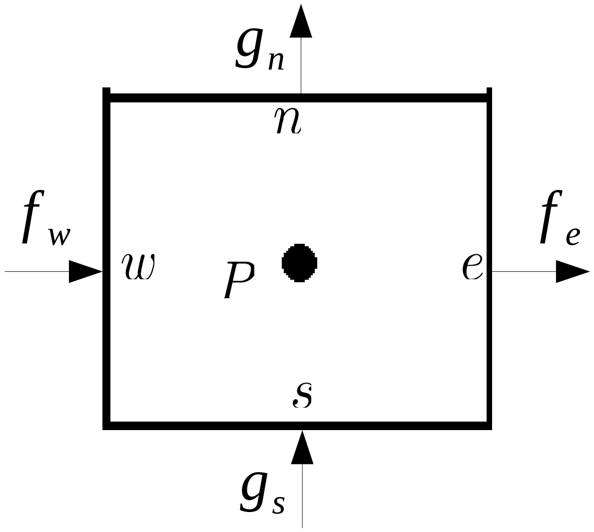
$$\left(\frac{1}{2}(T_P + T_E)(u_x)_e - \kappa \frac{T_E - T_P}{\Delta x}\right) \Delta y$$

$$-2\,\kappa \frac{T_P - T_w}{\Delta x} \quad \Delta y$$

$$+\left(\frac{1}{2}(T_P + T_N)(u_y)_n - \kappa \frac{T_N - T_P}{\Delta y}\right) \Delta x$$

$$-\left(\frac{1}{2}(T_P + T_S)(u_y)_s - \kappa \frac{T_P - T_S}{\Delta y}\right) \Delta x \quad = 0$$





West boundary

- $f_w = -2\,\kappa \dfrac{T_P - T_w}{\Delta x}$
- $(f_e - f_w)\Delta y + (g_n - g_s)\Delta x = 0$

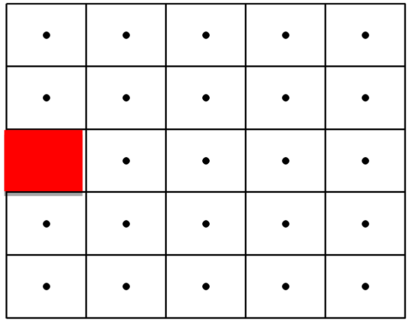
$$\left(\frac{1}{2}(T_P + T_E)(u_x)_e - \kappa\frac{T_E - T_P}{\Delta x}\right)\ \Delta y$$

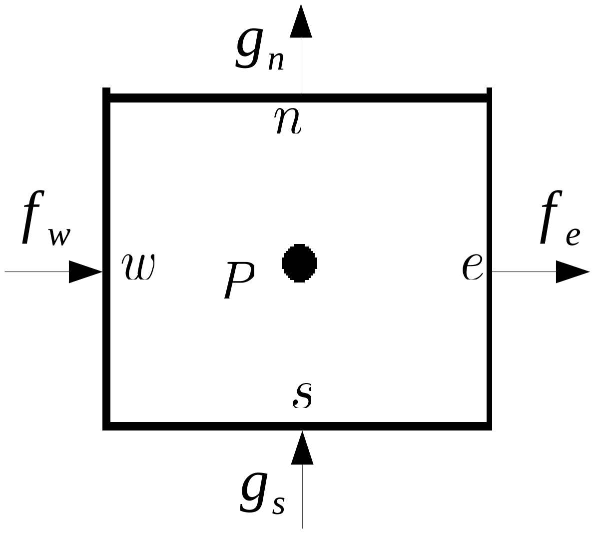
$$-2\,\kappa\frac{T_P - T_w}{\Delta x}\ \ \Delta y$$

$$+\left(\frac{1}{2}(T_P + T_N)(u_y)_n - \kappa\frac{T_N - T_P}{\Delta y}\right)\ \Delta x$$

$$-\left(\frac{1}{2}(T_P + T_S)(u_y)_s - \kappa\frac{T_P - T_S}{\Delta y}\right)\ \Delta x \quad = 0$$

$$A_P T_P + A_E T_E + A_S T_S + A_N T_N = B$$





West boundary

- $f_w = -2\kappa \dfrac{T_P - T_w}{\Delta x}$
- $(f_e - f_w)\Delta y + (g_n - g_s)\Delta x = 0$

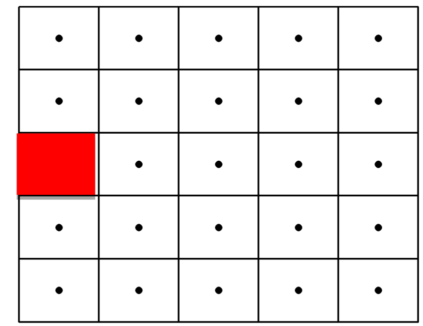
$$\left(\frac{1}{2}(T_P+T_E)(u_x)_e - \kappa\frac{T_E - T_P}{\Delta x}\right)\Delta y$$

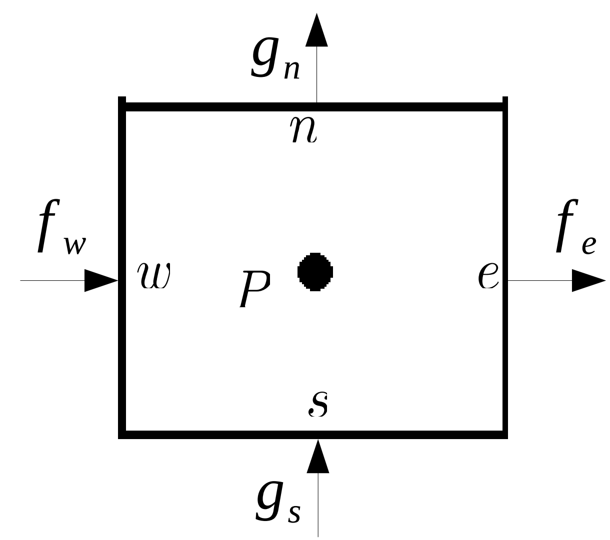
$$+\,2\kappa\frac{T_P - T_w}{\Delta x}\,\Delta y$$

$$+\left(\frac{1}{2}(T_P+T_N)(u_y)_n - \kappa\frac{T_N - T_P}{\Delta y}\right)\Delta x$$

$$-\left(\frac{1}{2}(T_P+T_S)(u_y)_s - \kappa\frac{T_P - T_S}{\Delta y}\right)\Delta x \quad = 0$$

$$A_P T_P + A_E T_E + A_S T_S + A_N T_N = B$$

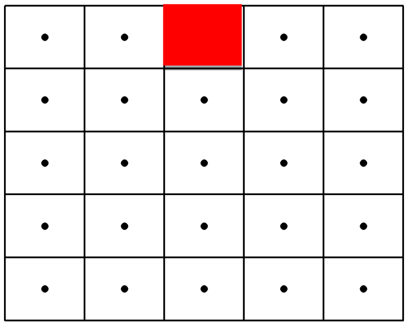




$$A_P = 3\kappa\frac{\Delta y}{\Delta x} + 2\kappa\frac{\Delta x}{\Delta y}$$

$$B = 2\kappa\frac{\Delta y}{\Delta x}\;T_w$$

North boundary

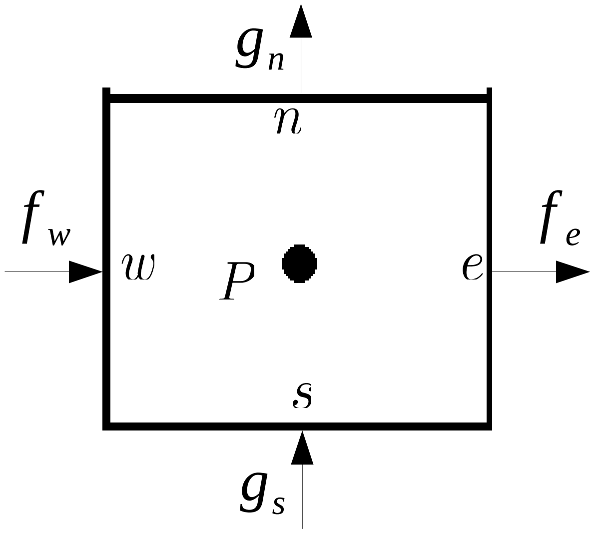


- $j = 2, \ldots, N_x - 1 \qquad k = N_y$
- West, east and south fluxes unmodified
- The b.c. is $T_n = 0$
- As before

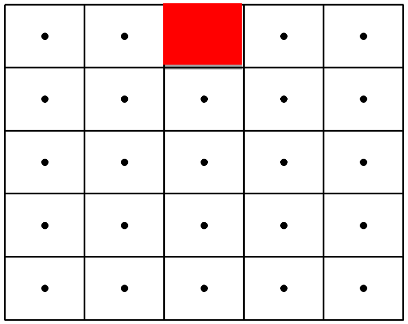
$$\left.\frac{\partial T}{\partial y}\right|_n \approx \frac{T_n - T_P}{\Delta y/2} = -\frac{T_P}{\Delta y/2}$$

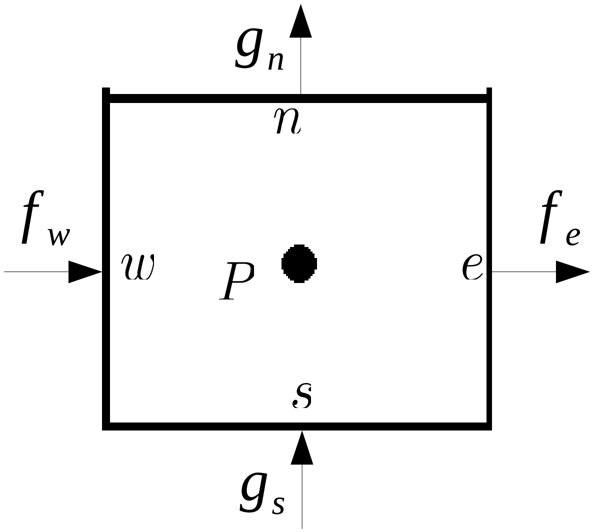
- So that the diffusive flux is

$$g_n = 2\kappa \frac{T_P}{\Delta y} \quad \underbrace{+T_n}_{=0}(u_y)_n$$



North boundary





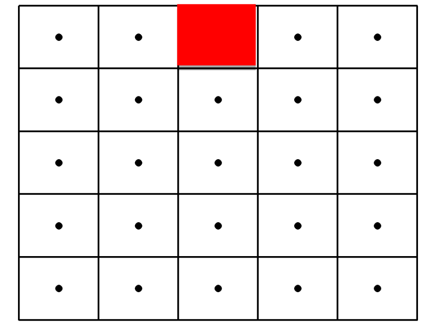
- $g_n = 2\kappa \dfrac{T_P}{\Delta y}$
- $(f_e - f_w)\Delta y + (g_n - g_s)\Delta x = 0$

$$
\begin{aligned}
&\left(\frac{1}{2}(T_P + T_E)(u_x)_e - \kappa \frac{T_E - T_P}{\Delta x}\right) \Delta y \\
-&\left(\frac{1}{2}(T_P + T_W)(u_x)_w - \kappa \frac{T_P - T_W}{\Delta x}\right) \Delta y \\
&\qquad\qquad\qquad + 2\kappa \frac{T_P}{\Delta y} \quad \Delta x \\
-&\left(\frac{1}{2}(T_P + T_S)(u_y)_s - \kappa \frac{T_P - T_S}{\Delta y}\right) \Delta x \quad = 0
\end{aligned}
$$

$$A_P T_P + A_E T_E + A_W T_W + A_S T_S = 0$$

$$A_P = -\frac{1}{2}(u_y)_n \Delta x + 2\kappa \frac{\Delta y}{\Delta x} + 3\kappa \frac{\Delta x}{\Delta y}$$

North boundary



- $g_n = 2\kappa \frac{T_P}{\Delta y}$
- $(f_e - f_w)\Delta y + (g_n - g_s)\Delta x = 0$

$$\left(\frac{1}{2}(T_P + T_E)(u_x)_e - \kappa \frac{T_E - T_P}{\Delta x}\right) \Delta y$$

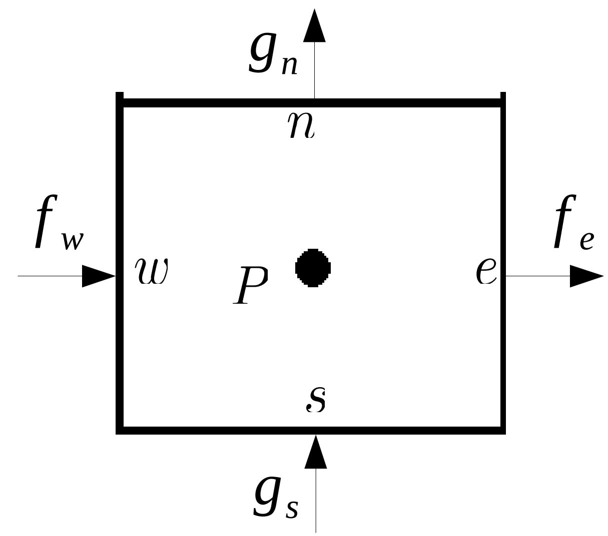
$$-\left(\frac{1}{2}(T_P + T_W)(u_x)_w - \kappa \frac{T_P - T_W}{\Delta x}\right) \Delta y$$

$$+2\kappa \frac{T_P}{\Delta y} \quad \Delta x$$

$$-\left(\frac{1}{2}(T_P + T_S)(u_y)_s - \kappa \frac{T_P - T_S}{\Delta y}\right) \Delta x \quad = 0$$

$$A_P T_P + A_E T_E + A_W T_W + A_S T_S = 0$$

$$A_P = -\frac{1}{2}(u_y)_n \Delta x + 2\kappa \frac{\Delta y}{\Delta x} + 3\kappa \frac{\Delta x}{\Delta y}$$



Mass balance:

$$\left[(u_x)_e - (u_x)_w\right]\Delta y + \left[(u_y)_n - (u_y)_s\right]\Delta x = 0$$