# MATH 273 - Problem Set 2

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**Submission:** Submit your answers on Gradescope (`https://www.gradescope.com/courses/925433`) by the deadline. You can either handwrite and scan your answers or type them (e.g., using LaTeX) if your handwriting is unclear or typing is more efficient. When submitting, be sure to **select the correct pages** corresponding to each problem, as this will make grading more efficient. You can assign one page to multiple problems or multiple pages to a single problem.

**(30 pts) Exercise 3.2.** For each of the following systems of the form $\mathbf{X}' = A\mathbf{X}$:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

**Part (a)**
Find the eigenvalues and eigenvectors of $A$.

**Solution**
To first find the eigenvalues, we solve the characteristic equation as follows:

$$\det(A - \lambda I) = 0.$$

**Case (i):** $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \det = (-\lambda)(-\lambda) - (1)(1) = \lambda^2 - 1.$$

So, the eigenvalues are $\lambda = \pm 1$.

To find the eigenvectors, we sub in $\lambda = 1$ and $\lambda = -1$ into $(A - \lambda I)\mathbf{v} = 0$ as follows:

$$\lambda = 1 \implies \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies v_1 = v_2.$$

$$\lambda = -1 \implies \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies v_1 = -v_2.$$

Eigenvectors : $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

**Case (ii):** $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - (1)(1) = -\lambda + \lambda^2 - 1.$$

$$\lambda^2 - \lambda - 1 = 0.$$

The eigenvalues are $\lambda = \frac{1 \pm \sqrt{5}}{2}$.

To find the eigenvectors, we sub in each eigenvalue into $(A - \lambda I)\mathbf{v} = 0$ as follows:

$$\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

We solve this to get eigenvector $\mathbf{v}_1 = \begin{bmatrix} \sqrt{5}+1 \\ 2 \end{bmatrix}$, and for $\lambda = \frac{1-\sqrt{5}}{2}$, we get $\mathbf{v}_2 = \begin{bmatrix} \sqrt{5}-1 \\ 2 \end{bmatrix}$.

Eigenvectors : $\mathbf{v}_1 = \begin{bmatrix} \sqrt{5}+1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \sqrt{5}-1 \\ 2 \end{bmatrix}$.

**Case (iii):** $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = (1-\lambda)(-\lambda) - (1)(-1) = -\lambda + \lambda^2 + 1.$$

$$\lambda^2 - \lambda + 1 = 0.$$

The eigenvalues are $\lambda = \frac{1 \pm i\sqrt{3}}{2}$.

To find the eigenvectors, we sub in each eigenvalue into $(A - \lambda I)\mathbf{v} = 0$ as follows:

$$\begin{bmatrix} 1 - \frac{1+i\sqrt{3}}{2} & 1 \\ -1 & -\frac{1+i\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

We solve to get eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \end{bmatrix}$, and for $\lambda = \frac{1-i\sqrt{3}}{2}$, we get $\mathbf{v}_2 = \begin{bmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \end{bmatrix}$.

Eigenvectors : $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \end{bmatrix}$.

**Case (iv):** $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - (1)(-1) = \lambda^2 - 4\lambda + 4.$$

The eigenvalues are $\lambda = 2$ (repeated eigenvalue).

For $\lambda = 2$:

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \implies v_1 = v_2.$$

Eigenvector : $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

**Case (v):** $A = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 1 \\ -1 & -3-\lambda \end{bmatrix} = \lambda^2 + 2\lambda - 4.$$

The eigenvalues are $\lambda = -1 \pm \sqrt{5}$.

To find the eigenvectors, we sub in each eigenvalue into $(A - \lambda I)\mathbf{v} = 0$ as follows:

$$\begin{bmatrix} 1 - (-1+\sqrt{5}) & 1 \\ -1 & -3 - (-1+\sqrt{5}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -\sqrt{5} & 1 \\ -1 & -2-\sqrt{5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

We solve to get eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{5} \end{bmatrix}$, and for $\lambda = -1 - \sqrt{5}$, we get $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{5} \end{bmatrix}$.

Eigenvectors : $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{5} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{5} \end{bmatrix}$.

**Case (vi):** $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \det \begin{bmatrix} 1-\lambda & 1 \\ 1 & -1-\lambda \end{bmatrix} = \lambda^2 - 0\lambda - 2.$$

The eigenvalues are $\lambda = \pm\sqrt{2}$.

To find the eigenvectors, we sub in each eigenvalue into $(A - \lambda I)\mathbf{v} = 0$ as follows:

$$\begin{bmatrix} 1-\sqrt{2} & 1 \\ 1 & -1-\sqrt{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

We solve to get eigenvector $\mathbf{v}_1 = \begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix}$, and for $\lambda = -\sqrt{2}$, we get $\mathbf{v}_2 = \begin{bmatrix} -\sqrt{2}+1 \\ 1 \end{bmatrix}$.

Eigenvectors : $\mathbf{v}_1 = \begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\sqrt{2}+1 \\ 1 \end{bmatrix}$.

**Part (b)**
Find the matrix $T$ that puts $A$ in canonical form.

**Solution**

**Case (i):** $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The eigenvalues of $A$ are $\lambda = \pm 1$, and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We form the matrix $T$ using these eigenvectors:

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Then, the canonical form of $A$ is as follows:

$$T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

**Case (ii):** $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

The eigenvalues of $A$ are $\lambda = \frac{1\pm\sqrt{5}}{2}$, and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} \sqrt{5}+1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \sqrt{5}-1 \\ 2 \end{bmatrix}$. We form the matrix $T$ as follows:

$$T = \begin{bmatrix} \sqrt{5}+1 & \sqrt{5}-1 \\ 2 & 2 \end{bmatrix}.$$

The canonical form of $A$ is as follows:

$$T^{-1}AT = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

**Case (iii):** $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

The eigenvalues of $A$ are $\lambda = \frac{1\pm i\sqrt{3}}{2}$, and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \end{bmatrix}$. We form the matrix $T$ as follows:

$$T = \begin{bmatrix} 1 & 1 \\ \frac{1+i\sqrt{3}}{2} & \frac{1-i\sqrt{3}}{2} \end{bmatrix}.$$

The canonical form of $A$ is as follows:

$$T^{-1}AT = \begin{bmatrix} \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{1-i\sqrt{3}}{2} \end{bmatrix}.$$

**Case (iv):** $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

The eigenvalue of $A$ is $\lambda = 2$ (repeated). The eigenvector is $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To find the generalized eigenvector, we solve the following:

$$(A - 2I)\mathbf{w} = \mathbf{v}.$$

We let $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$. Then, we get the following:

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We solve to get $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and we then form the matrix $T$ as follows:

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

The canonical form of $A$ is as follows:

$$T^{-1}AT = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

**Case (v):** $A = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

The eigenvalues of $A$ are $\lambda = -1 \pm \sqrt{5}$, and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{5} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{5} \end{bmatrix}$. We form the matrix $T$ as follows:

$$T = \begin{bmatrix} 1 & 1 \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}.$$

The canonical form of $A$ is as follows:

$$T^{-1}AT = \begin{bmatrix} -1+\sqrt{5} & 0 \\ 0 & -1-\sqrt{5} \end{bmatrix}.$$

**Case (vi):** $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

The eigenvalues of $A$ are $\lambda = \pm\sqrt{2}$, and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\sqrt{2}+1 \\ 1 \end{bmatrix}$. We form the matrix $T$ as follows:

$$T = \begin{bmatrix} \sqrt{2}+1 & -\sqrt{2}+1 \\ 1 & 1 \end{bmatrix}.$$

The canonical form of $A$ is as follows:

$$T^{-1}AT = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}.$$

**Part (c)**
Find the general solution of both $\mathbf{X}' = A\mathbf{X}$ and $\mathbf{Y}' = (T^{-1}AT)\mathbf{Y}$.

**Solution**
**Case (i):** $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

The eigenvalues of $A$ are $\lambda = \pm 1$, the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and the general solution of $\mathbf{X}' = A\mathbf{X}$ is thereby as follows:

$$\mathbf{X}(t) = c_1 e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

For $\mathbf{Y}' = (T^{-1}AT)\mathbf{Y}$, where $T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, the solution is as follows:

$$\mathbf{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^t \\ c_2 e^{-t} \end{bmatrix}.$$

**Case (ii):** $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

The eigenvalues are $\lambda = \frac{1 \pm \sqrt{5}}{2}$, the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} \sqrt{5}+1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} \sqrt{5}-1 \\ 2 \end{bmatrix}$, and the general solution of $\mathbf{X}' = A\mathbf{X}$ is thereby as follows:

$$\mathbf{X}(t) = c_1 e^{\frac{1+\sqrt{5}}{2}t} \begin{bmatrix} \sqrt{5}+1 \\ 2 \end{bmatrix} + c_2 e^{\frac{1-\sqrt{5}}{2}t} \begin{bmatrix} \sqrt{5}-1 \\ 2 \end{bmatrix}.$$

For $\mathbf{Y}' = (T^{-1}AT)\mathbf{Y}$, where $T$ is formed from the eigenvectors and $T^{-1}AT = \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix}$, the solution is as follows:

$$\mathbf{Y}(t) = \begin{bmatrix} c_1 e^{\frac{1+\sqrt{5}}{2}t} \\ c_2 e^{\frac{1-\sqrt{5}}{2}t} \end{bmatrix}.$$

**Case (iii):** $A = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$

The eigenvalues are $\lambda = \frac{1 \pm i\sqrt{3}}{2}$, the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \end{bmatrix}$, and the general solution of $\mathbf{X}' = A\mathbf{X}$ is as follows:

$$\mathbf{X}(t) = c_1 e^{\frac{1+i\sqrt{3}}{2}t} \begin{bmatrix} 1 \\ \frac{1+i\sqrt{3}}{2} \end{bmatrix} + c_2 e^{\frac{1-i\sqrt{3}}{2}t} \begin{bmatrix} 1 \\ \frac{1-i\sqrt{3}}{2} \end{bmatrix}.$$

For $\mathbf{Y}' = (T^{-1}AT)\mathbf{Y}$, where $T = \begin{bmatrix} 1 & 1 \\ \frac{1+i\sqrt{3}}{2} & \frac{1-i\sqrt{3}}{2} \end{bmatrix}$ and $T^{-1}AT = \begin{bmatrix} \frac{1+i\sqrt{3}}{2} & 0 \\ 0 & \frac{1-i\sqrt{3}}{2} \end{bmatrix}$, solution is as follows:

$$\mathbf{Y}(t) = \begin{bmatrix} c_1 e^{\frac{1+i\sqrt{3}}{2}t} \\ c_2 e^{\frac{1-i\sqrt{3}}{2}t} \end{bmatrix}.$$

**Case (iv):** $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$

The eigenvalue is $\lambda = 2$ (repeated) w/ eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the generalized eigenvector is $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and the general solution of $\mathbf{X}' = A\mathbf{X}$ is as follows:

$$\mathbf{X}(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 t e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For $\mathbf{Y}' = (T^{-1}AT)\mathbf{Y}$, where $T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $T^{-1}AT = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, the solution is as follows:

$$\mathbf{Y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1e^{2t} + c_2te^{2t} \\ c_2e^{2t} \end{bmatrix}.$$

**Case (v):** $A = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$

The eigenvalues are $\lambda = -1 \pm \sqrt{5}$, the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 1 \\ \sqrt{5} \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -\sqrt{5} \end{bmatrix}$, and the general solution of $\mathbf{X}' = A\mathbf{X}$ is as follows:

$$\mathbf{X}(t) = c_1e^{(-1+\sqrt{5})t} \begin{bmatrix} 1 \\ \sqrt{5} \end{bmatrix} + c_2e^{(-1-\sqrt{5})t} \begin{bmatrix} 1 \\ -\sqrt{5} \end{bmatrix}.$$

For $\mathbf{Y}' = (T^{-1}AT)\mathbf{Y}$, where $T = \begin{bmatrix} 1 & 1 \\ \sqrt{5} & -\sqrt{5} \end{bmatrix}$ and $T^{-1}AT = \begin{bmatrix} -1+\sqrt{5} & 0 \\ 0 & -1-\sqrt{5} \end{bmatrix}$, solution is as follows:

$$\mathbf{Y}(t) = \begin{bmatrix} c_1e^{(-1+\sqrt{5})t} \\ c_2e^{(-1-\sqrt{5})t} \end{bmatrix}.$$

**Case (vi):** $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

The eigenvalues are $\lambda = \pm\sqrt{2}$, the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -\sqrt{2}+1 \\ 1 \end{bmatrix}$, and the general solution of $\mathbf{X}' = A\mathbf{X}$ is as follows:

$$\mathbf{X}(t) = c_1e^{\sqrt{2}t} \begin{bmatrix} \sqrt{2}+1 \\ 1 \end{bmatrix} + c_2e^{-\sqrt{2}t} \begin{bmatrix} -\sqrt{2}+1 \\ 1 \end{bmatrix}.$$

For $\mathbf{Y}' = (T^{-1}AT)\mathbf{Y}$, where $T = \begin{bmatrix} \sqrt{2}+1 & -\sqrt{2}+1 \\ 1 & 1 \end{bmatrix}$ and $T^{-1}AT = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{bmatrix}$, the solution is as follows:

$$\mathbf{Y}(t) = \begin{bmatrix} c_1e^{\sqrt{2}t} \\ c_2e^{-\sqrt{2}t} \end{bmatrix}.$$

**Part (d)**
Sketch the phase portraits of both systems.

```
# Code for sketching phase portraits of both systems for cases i-vi

# Matrices for cases i-vi
matrices_cases = {
    "Case (i)": np.array([[0, 1], [1, 0]]),
    "Case (ii)": np.array([[1, 1], [1, 0]]),
    "Case (iii)": np.array([[1, 1], [-1, 0]]),
    "Case (iv)": np.array([[1, 1], [-1, 3]]),
    "Case (v)": np.array([[1, 1], [-1, -3]]),
    "Case (vi)": np.array([[1, 1], [1, -1]]),
}

# Corresponding canonical forms (T^-1 A T)
canonical_forms = {
    "Case (i)": np.array([[1, 0], [0, -1]]),
    "Case (ii)": np.array([[1.618, 0], [0, -0.618]]),
    "Case (iii)": np.array([[0.5 + 0.866j, 0], [0, 0.5 - 0.866j]]),
    "Case (iv)": np.array([[2, 1], [0, 2]]),
    "Case (v)": np.array([[-1 + 2.236, 0], [0, -1 - 2.236]]),
    "Case (vi)": np.array([[1.414, 0], [0, -1.414]]),
}

fig, axes = plt.subplots(6, 2, figsize=(15, 25))
axes = axes.flatten()

# Plotting each case
for i, (case, A) in enumerate(matrices_cases.items()):
    plot_phase_portrait(A, f"{case} - Original System", axes[2 * i]) # Original

    plot_phase_portrait(canonical_forms[case], f"{case} - Canonical System", axes
        [2 * i + 1]) # Canonical
plt.tight_layout()
plt.show()
```

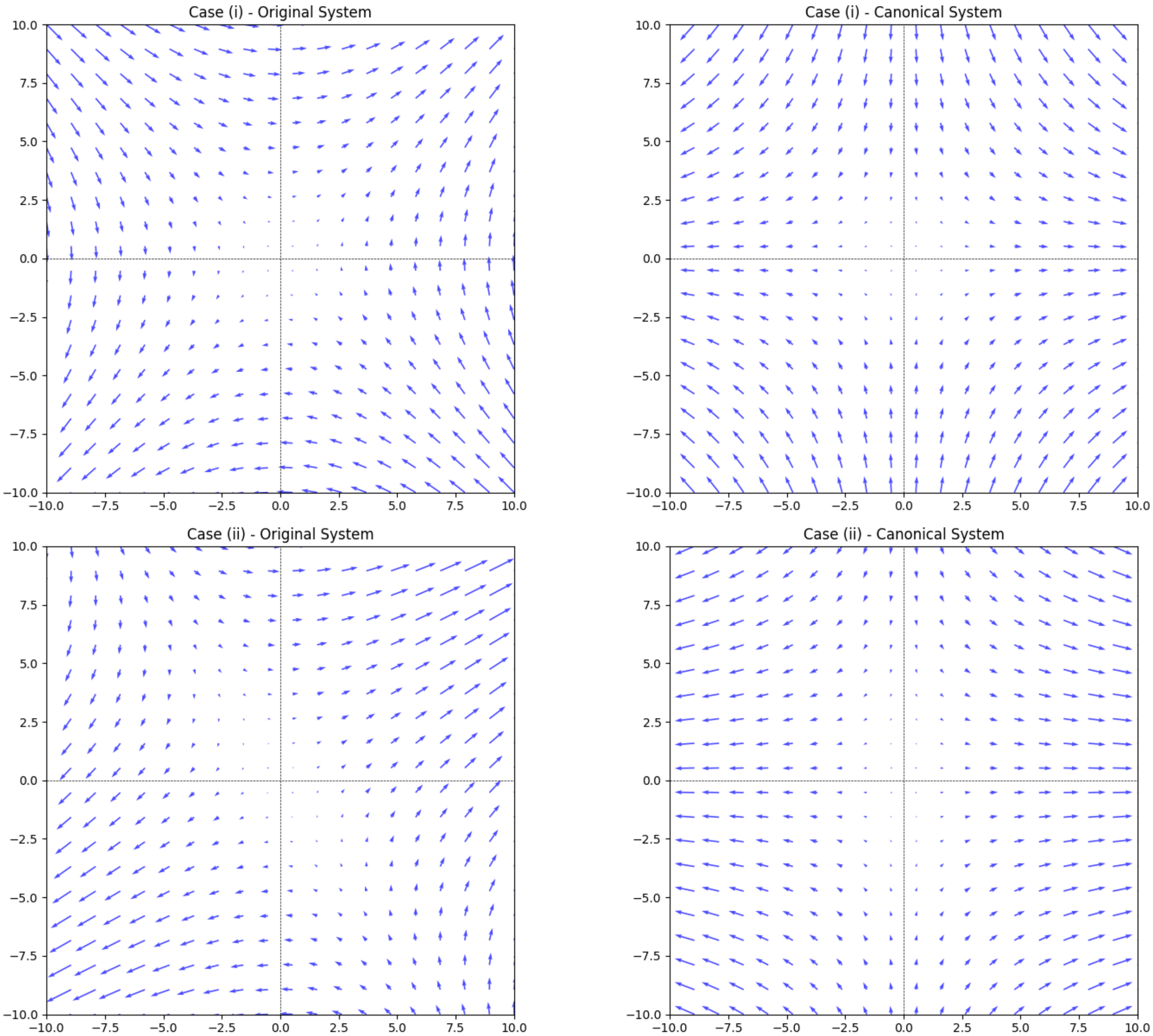


Figure 1: Plots of Original and Canonical Forms of Case i and ii

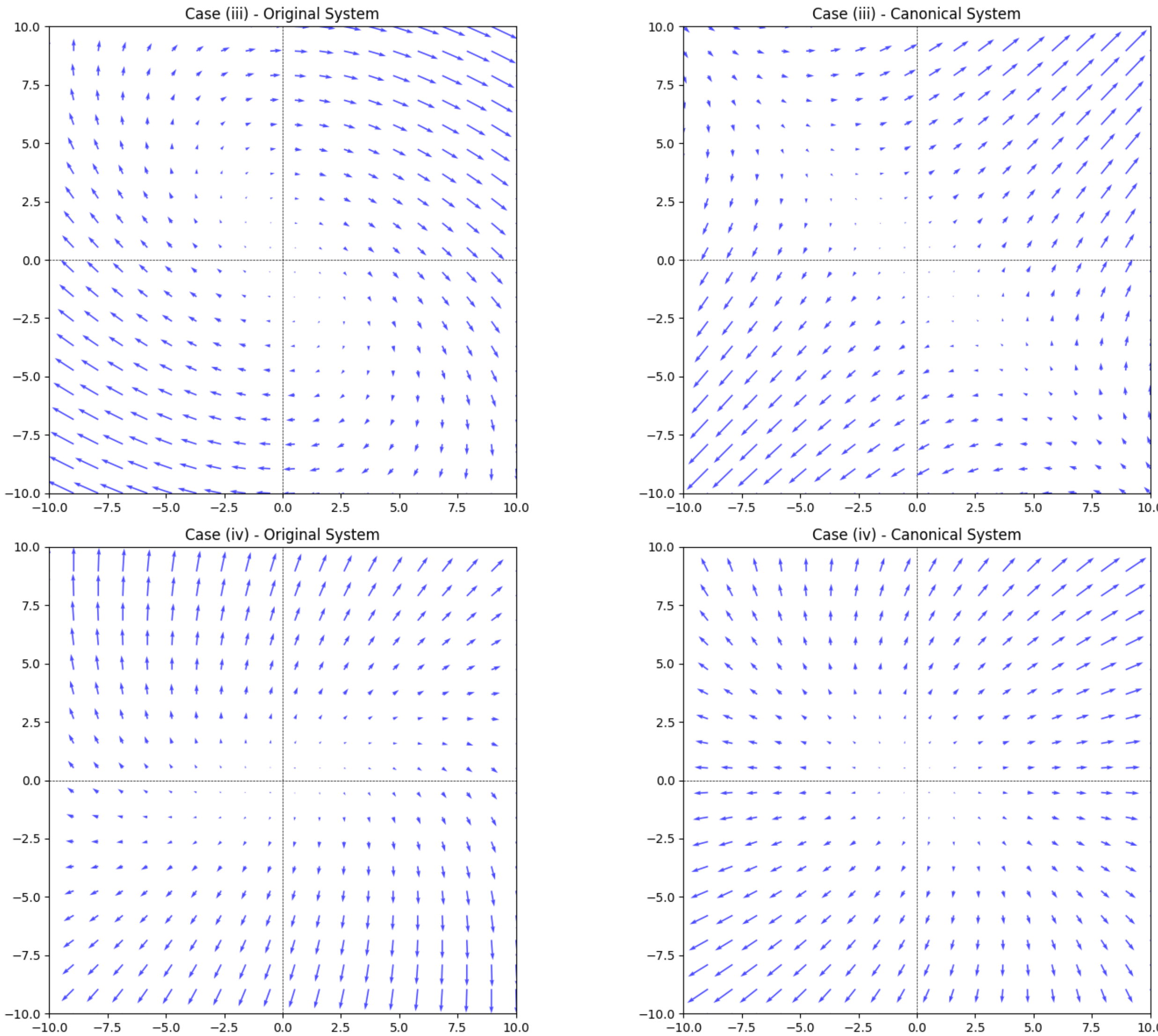


Figure 2: Plots of Original and Canonical Forms of Case iii and iv

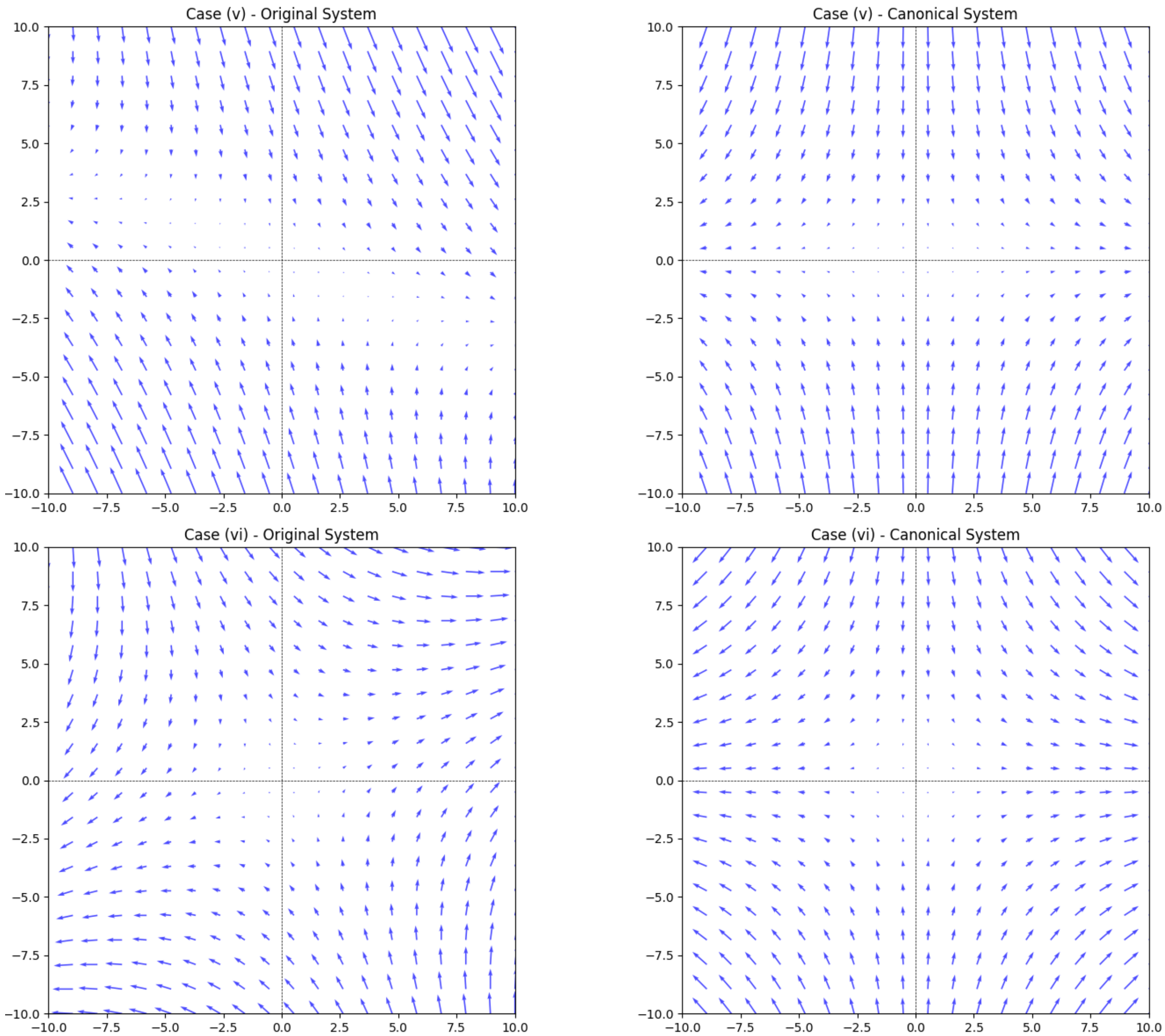


Figure 3: Plots of Original and Canonical Forms of Case v and vi

**(10 pts) Exercise 3.3.** Find the general solution of the following harmonic oscillator equations:

**Part (a)**
$x'' + x' + x = 0$

**Solution**
The characteristic equation for $x'' + x' + x = 0$ is as follows:

$$\lambda^2 + \lambda + 1 = 0.$$

We solve for $\lambda$ using the quadratic formula as follows:

$$\lambda = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}.$$

The roots are the complex conjugates as follows: $\lambda = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$ and $\lambda = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$.

The general solution is as follows:

$$x(t) = e^{-\frac{1}{2}t}\left(C_1\cos\left(\frac{\sqrt{3}}{2}t\right) + C_2\sin\left(\frac{\sqrt{3}}{2}t\right)\right),$$

**Part (b)**
$x'' + 2x' + x = 0$

**Solution**
The characteristic equation for $x'' + 2x' + x = 0$ is as follows:

$$\lambda^2 + 2\lambda + 1 = 0.$$

$$(\lambda + 1)^2 = 0.$$

The root is the repeated real root that follows:

$$\lambda = -1.$$

The general solution is as follows:

$$x(t) = (C_1 + C_2 t)e^{-t},$$

**(10 pts) Exercise 4.1.** Consider the one-parameter family of linear systems given by:

$$\mathbf{X}' = \begin{bmatrix} a & \sqrt{2} + \frac{a}{2} \\ \sqrt{2} - \frac{a}{2} & 0 \end{bmatrix}\mathbf{X}.$$

**Part (a)**
Sketch the path traced out by this family of linear systems in the trace-determinant plane as $a$ varies.

**Solution**
The trace $\mathrm{tr}(A)$ and determinant $\det(A)$ of the matrix $A$ are as follows:

$$\mathrm{tr}(A) = a, \quad \det(A) = -\left(\sqrt{2} + \frac{a}{2}\right)\left(\sqrt{2} - \frac{a}{2}\right) = -\left(2 - \frac{a^2}{4}\right) = \frac{a^2}{4} - 2.$$

The path traced in the trace-determinant plane as $a$ varies is given parametrically by the following:

$$(\mathrm{tr}(A), \det(A)) = \left(a, \frac{a^2}{4} - 2\right).$$

This represents a parabola opening upwards with its vertex at $(0, -2)$ and symmetry about the $\mathrm{tr}(A)$-axis shown below:

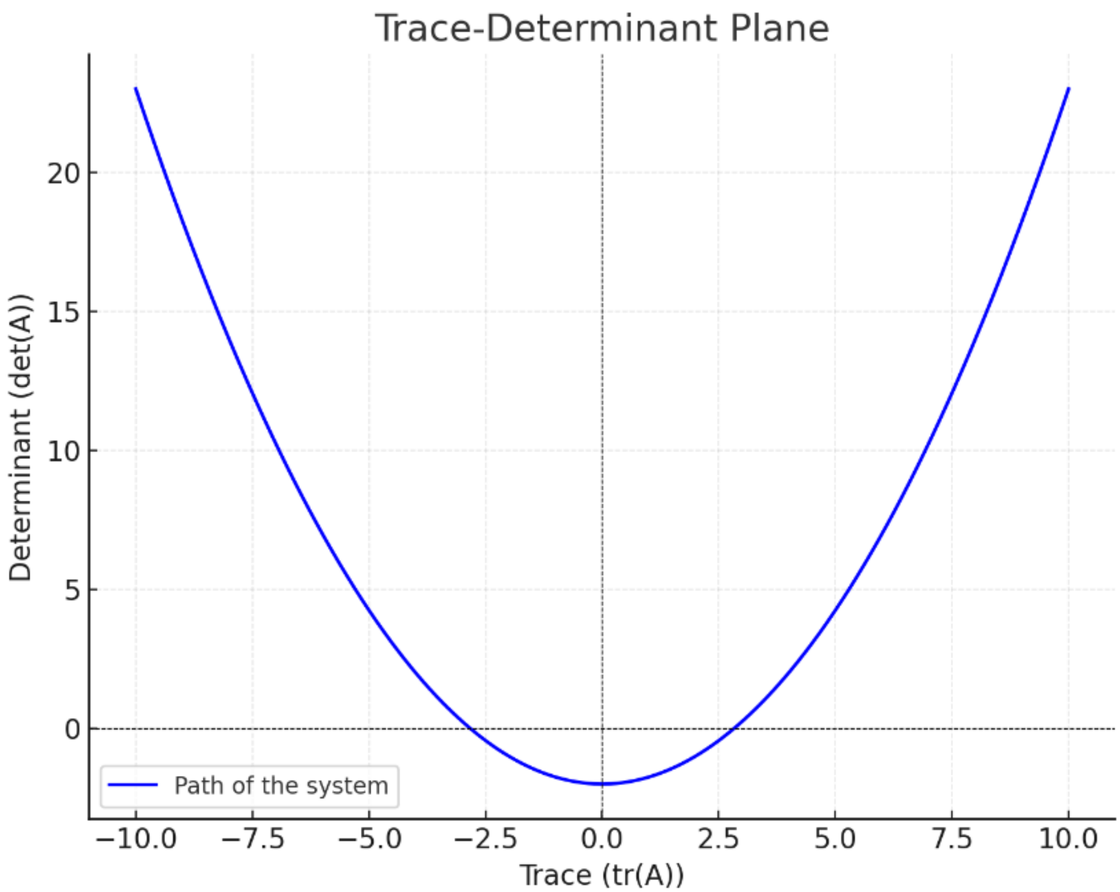


Figure 4: Trace-Determinant Plane of Linear System

**Part (b)**
Discuss any bifurcations that occur along this path and compute the corresponding values of $a$.

**Solution**
Bifurcations occur when the nature of eigenvalues of the system changes, which corresponds to transitions b/w different types of fixed points. These transitions can be analyzed using the trace-determinant plane. The eigenvalues of the matrix $A$ are given as follows:

$$\lambda = \frac{\mathrm{tr}(A)}{2} \pm \sqrt{\left(\frac{\mathrm{tr}(A)}{2}\right)^2 - \det(A)}.$$

We sub in $\mathrm{tr}(A) = a$ and $\det(A) = \frac{a^2}{4} - 2$ as follows:

$$\lambda = \frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - \left(\frac{a^2}{4} - 2\right)} = \frac{a}{2} \pm \sqrt{2}.$$

1. **Change from Real to Complex Eigenvalues:** This occurs when the discriminant of the eigenvalues changes sign as follows:
$$\left(\frac{a}{2}\right)^2 - \det(A) = \left(\frac{a}{2}\right)^2 - \left(\frac{a^2}{4} - 2\right) = 2.$$
B/c the discriminant is always positive $(2 > 0)$, there are no transitions to complex eigenvalues.

2. **Degenerate Eigenvalues:** Degeneracy occurs when both eigenvalues are equal, which happens when the square root term vanishes:
$$\sqrt{\left(\frac{a}{2}\right)^2 - \det(A)} = 0.$$
$$\left(\frac{a}{2}\right)^2 - \det(A) = 0.$$
We sub in $\det(A) = \frac{a^2}{4} - 2$ as follows:
$$\left(\frac{a}{2}\right)^2 - \left(\frac{a^2}{4} - 2\right) = 0 \implies 2 = 0.$$
So, bifurcations occur at $\boxed{a = \pm\sqrt{2}}$. The bifurcations correspond to changes in the stability or nature of the fixed point as $a$ varies. The critical values of $a$ where these occur are $\pm\sqrt{2}$.

**(15 pts)** Let $M$ be the space of $n \times n$ complex matrices. Define
$$\|A\| = \sup_{|\mathbf{x}|=1} |A\mathbf{x}|,$$
where $|\cdot|$ is the standard Euclidean norm on $\mathbb{C}^n$.

**Part (a)**
Prove that $\|\cdot\|$ is a norm on $M$.

**Solution**
To prove $\|\cdot\|$ is a norm on $M$, we need to verify the properties of norms as follows:

1. **Positive Definiteness:** $\|A\| \geq 0$ and $\|A\| = 0$ iff $A = 0$.

   B/c $\|A\| = \sup_{|\mathbf{x}|=1} |A\mathbf{x}|$, we have $|A\mathbf{x}| \geq 0$ $\forall \mathbf{x}$, and hence $\|A\| \geq 0$. If $\|A\| = 0$, then $|A\mathbf{x}| = 0 \quad \forall\, \mathbf{x}$ w/ $|\mathbf{x}| = 1$, which implies $A\mathbf{x} = 0$. This means $A = 0$.

2. **Homogeneity:** $\|cA\| = |c| \cdot \|A\|$ for any scalar $c$ and matrix $A$.

   For $\|cA\|$:
$$\|cA\| = \sup_{|\mathbf{x}|=1} |(cA)\mathbf{x}| = \sup_{|\mathbf{x}|=1} |c| \cdot |A\mathbf{x}| = |c| \cdot \sup_{|\mathbf{x}|=1} |A\mathbf{x}| = |c| \cdot \|A\|.$$

3. **Triangle Inequality:** $\|A + B\| \leq \|A\| + \|B\| \quad \forall A, B \in M$.

   For $\|A + B\|$:
$$\|A + B\| = \sup_{|\mathbf{x}|=1} |(A + B)\mathbf{x}| \leq \sup_{|\mathbf{x}|=1} \big(|A\mathbf{x}| + |B\mathbf{x}|\big) \leq \sup_{|\mathbf{x}|=1} |A\mathbf{x}| + \sup_{|\mathbf{x}|=1} |B\mathbf{x}| = \|A\| + \|B\|.$$

**Part (b)**
Prove that:
$$\max_{j,k} |A_{jk}| \leq \|A\| \leq n \max_{j,k} |A_{jk}|.$$

**Solution**
We need to prove the double inequality $\max_{j,k} |A_{jk}| \leq \|A\| \leq n \max_{j,k} |A_{jk}|$.

1. **Lower Bound:** $\|A\| \geq \max_{j,k} |A_{jk}|$.

   We consider the standard basis vector $\mathbf{e}_k$ w/ $|\mathbf{e}_k| = 1$. For any matrix $A$, the $j$-th component of $A\mathbf{e}_k$ is $A_{jk}$, so we get as follows:
   $$|A\mathbf{e}_k| \geq |A_{jk}| \quad \text{for all } j, k.$$
   We take the supremum over all unit vectors $\mathbf{x}$ as follows:
   $$\|A\| = \sup_{|\mathbf{x}|=1} |A\mathbf{x}| \geq \max_{j,k} |A_{jk}|.$$

2. **Upper Bound:** $\|A\| \leq n \max_{j,k} |A_{jk}|$.

   We let $\mathbf{x}$ be a unit vector as follows:
   $$|A\mathbf{x}|^2 = \sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_{jk} x_k \right|^2 .$$
   We use the triangle inequality and Cauchy-Schwarz inequality as follows:
   $$\left| \sum_{k=1}^{n} A_{jk} x_k \right| \leq \sum_{k=1}^{n} |A_{jk}||x_k| \leq \max_{j,k} |A_{jk}| \sum_{k=1}^{n} |x_k|.$$
   B/c $\mathbf{x}$ is a unit vector, $\sum_{k=1}^{n} |x_k| \leq \sqrt{n}$, and we get as follows:
   $$|A\mathbf{x}| \leq \sqrt{n} \cdot \max_{j,k} |A_{jk}|.$$
   We take the supremum over all $\mathbf{x}$ w/ $|\mathbf{x}| = 1$ to get as follows:
   $$\|A\| \leq n \max_{j,k} |A_{jk}|.$$

**Part (c)**
Prove that $(M, \|\cdot\|)$ is a Banach space (i.e., a complete normed vector space).

**Solution**
To prove $(M, \|\cdot\|)$ is a Banach space, we w.t.s. that every Cauchy sequence of matrices in $M$ converges to a matrix in $M$ under norm $\|\cdot\|$.

We let $\{A_k\}_{k=1}^{\infty}$ be a Cauchy sequence in $(M, \|\cdot\|)$. By definition of Cauchy sequence, we know the following:
$$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \|A_k - A_m\| < \epsilon \quad \forall k, m \geq N.$$
B/c the entries of the matrices $A_k$ are in $\mathbb{C}$, each entry $(A_k)_{ij}$ forms a Cauchy sequence in $\mathbb{C}$, which is complete. So, $(A_k)_{ij}$ converges to some $(A)_{ij} \in \mathbb{C}$. We define $A = \lim_{k\to\infty} A_k$ entry-wise.

Next, we show that $A_k \to A$ in the norm $\|\cdot\|$. For any $\epsilon > 0$, $\exists N$ s.t. $\|A_k - A_m\| < \epsilon/2 \ \forall\ k, m \geq N$. We fix $m$ and let $k \to \infty$. Then, $A_m \to A$ entry-wise, and by continuity of the norm, we have as follows:
$$\|A_k - A\| \leq \|A_k - A_m\| + \|A_m - A\| < \epsilon/2 + \epsilon/2 = \epsilon \quad \text{for all } k \geq N.$$
So, $A_k \to A$ in the norm $\|\cdot\|$ and $A \in M$, which shows that every Cauchy sequence in $M$ converges to a limit in $M$. Thus, $(M, \|\cdot\|)$ is a Banach space.

**(15 pts)** Let $(X, \|\cdot\|)$ be a normed vector space.

**Part (a)**
Prove that if $X$ is a Banach space, then any absolutely convergent series converges.

**Solution**
To prove this, we let $\sum_{k=1}^{\infty} x_k$ be an absolutely convergent series in $X$ as follows:

$$\sum_{k=1}^{\infty} \|x_k\| < \infty.$$

We w.t.s. that the series $\sum_{k=1}^{\infty} x_k$ converges in $X$.

1. We define the partial sums $S_n = \sum_{k=1}^{n} x_k$. B/c $X$ is a Banach space, it is complete under norm $\|\cdot\|$.
2. To prove convergence, we show that $\{S_n\}$ is a Cauchy sequence. We let $m > n$ as follows:

   $$\|S_m - S_n\| = \left\| \sum_{k=n+1}^{m} x_k \right\| \leq \sum_{k=n+1}^{m} \|x_k\|.$$

   B/c $\sum_{k=1}^{\infty} \|x_k\| < \infty$, for any $\epsilon > 0$, $\exists\ N \in \mathbb{N}$ s.t. we have as follows:

   $$\sum_{k=N+1}^{\infty} \|x_k\| < \epsilon.$$

   So $\forall\ m, n \geq N$, we have as follows:

   $$\|S_m - S_n\| \leq \sum_{k=n+1}^{m} \|x_k\| < \epsilon.$$

   This proves $\{S_n\}$ is a Cauchy sequence in $X$.
3. B/c $X$ is a Banach space (complete normed space), every Cauchy sequence converges. This means$\{S_n\}$ converges to some $S \in X$, which implies:

   $$\sum_{k=1}^{\infty} x_k = S.$$

Thus, any absolutely convergent series in a Banach space $X$ converges.

**Part (b)**
Prove or disprove: if any absolutely convergent series converges, then the space is a Banach space.

**Solution**
This statement is **true**. To prove it, we show that if every absolutely convergent series in $X$ converges, then $X$ is a Banach space.

1. Suppose $\{x_k\}$ is a Cauchy sequence in $X$. By definition of a Cauchy sequence, we know the following:

   $$\forall \epsilon > 0, \exists N \in \mathbb{N} \text{ such that } \|x_k - x_m\| < \epsilon \quad \forall k, m \geq N.$$

   We define a series $\sum_{k=1}^{\infty} y_k$, where $y_k = x_k - x_{k-1}$ and $x_0 = 0$. B/c $\{x_k\}$ is Cauchy, $\|y_k\| \to 0$ as $k \to \infty$, and $\sum \|y_k\| < \infty$, so the series is absolutely convergent.
2. By the assumption, any absolutely convergent series in $X$ converges. So, $\sum_{k=1}^{\infty} y_k$ converges to some $z \in X$.
3. The partial sums $S_n = \sum_{k=1}^{n} y_k$ satisfy the following:

   $$S_n = x_n - x_0 = x_n.$$

   This means $x_n \to z$ in $X$, which shows $\{x_k\}$ converges.

B/c every Cauchy sequence in $X$ converges, $X$ is complete, and $X$ is a Banach space.

**(10 pts)** Let $A$ be an $n \times n$ complex matrix. Prove that $\det(\exp(A)) = \exp(\operatorname{tr}(A))$, where det denotes the determinant and tr denotes the trace. [Hint: first prove it for $A$ in a Jordan normal form.]

**Solution**

To prove $\det(\exp(A)) = \exp(\operatorname{tr}(A))$, we start by assuming $A$ is in Jordan normal form. The Jordan normal form of $A$ is as follows:

$$A = PJP^{-1},$$

where $P$ is an invertible matrix and $J$ is the Jordan matrix.

**Step 1: Matrix Exponential of a Jordan Matrix:**

For a Jordan block $J_k$ of size $k$, the exponential $\exp(J_k)$ can be computed as follows:

$$\exp(J_k) = \sum_{m=0}^{\infty} \frac{J_k^m}{m!}.$$

B/c $J_k$ is upper triangular, $J_k^m$ remains upper triangular, and the diagonal entries of $\exp(J_k)$ are $e^{\lambda}$, where $\lambda$ is the eigenvalue associated with $J_k$.

So, the matrix $\exp(J)$ for the entire Jordan matrix $J$ is also upper triangular, and the diagonal entries of $\exp(J)$ are $e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $A$.

**Step 2: Determinant of $\exp(J)$:**

The determinant of an upper triangular matrix is the product of its diagonal entries, which yields the following::

$$\det(\exp(J)) = e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n} = \exp(\lambda_1 + \lambda_2 + \cdots + \lambda_n) = \exp(\operatorname{tr}(J)).$$

**Step 3: Transform Back to $A$:**

B/c $A = PJP^{-1}$, we get as follows:

$$\exp(A) = P \exp(J) P^{-1}.$$

We can then use the multiplicative property of determinants as follows:

$$\det(\exp(A)) = \det(P \exp(J) P^{-1}) = \det(P) \det(\exp(J)) \det(P^{-1}) = \det(\exp(J)) = \exp(\operatorname{tr}(J)).$$

**Step 4: Trace of $A$:**

The trace is invariant under similarity transformations, so $\operatorname{tr}(A) = \operatorname{tr}(J)$, and we get as follows:

$$\det(\exp(A)) = \exp(\operatorname{tr}(A)).$$