MATH 273 - Problem Set 4

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Submission: Submit your answers on Gradescope (`https://www.gradescope.com/courses/925433`) by the deadline. You can either handwrite and scan your answers or type them (e.g., using LaTeX) if your handwriting is unclear or typing is more efficient. When submitting, be sure to **select the correct pages** corresponding to each problem, as this will make grading more efficient. You can assign one page to multiple problems or multiple pages to a single problem.

[Hirsch et al.] is our main textbook and [Teschl] is the supplementary textbook. Check the syllabus for the PDF files.

(20 pts) While we proved the existence and uniqueness theorem via Picard iterations, the integral is often not easy to compute. A more practical approach is Euler's method. Read [Teschl, Section 2.7] and describe how we can implement Euler's method to simulate any ODE system on a computer.

Solution

We first note that Euler's method is a numerical technique for approximating solutions to ODEs of the initial value problem (IVP) as follows:

$$\dot{x} = f(t, x), \quad x(t_0) = x_0.$$

It proceeds by breaking down time into small, discrete steps and updating the solution iteratively using the formula that follows:

$$x_{n+1} = x_n + hf(t_n, x_n),$$

where

- x_n is the approx. solution at time t_n,
- h is the step size, and
- $f(t_n, x_n)$ is the derivative at the current step.

To implement Euler's method on a computer, we could stay by doing the following basic steps:

1. Choose an initial condition (t_0, x_0).
2. Define the function $f(t, x)$.
3. Select a sufficiently small step size h and a final time T.
4. Iterate using the update formula $x_{n+1} = x_n + hf(t_n, x_n)$.
5. Visualize the results.

From the arbitrary example w/ initial condition given below, we can see that Euler's method provides an easy-to-implement approximate solution to ODEs w/ accuracy that depends on the step size h. For improved accuracy, higher-order methods such as Runge-Kutta or Heun's method can also be used.

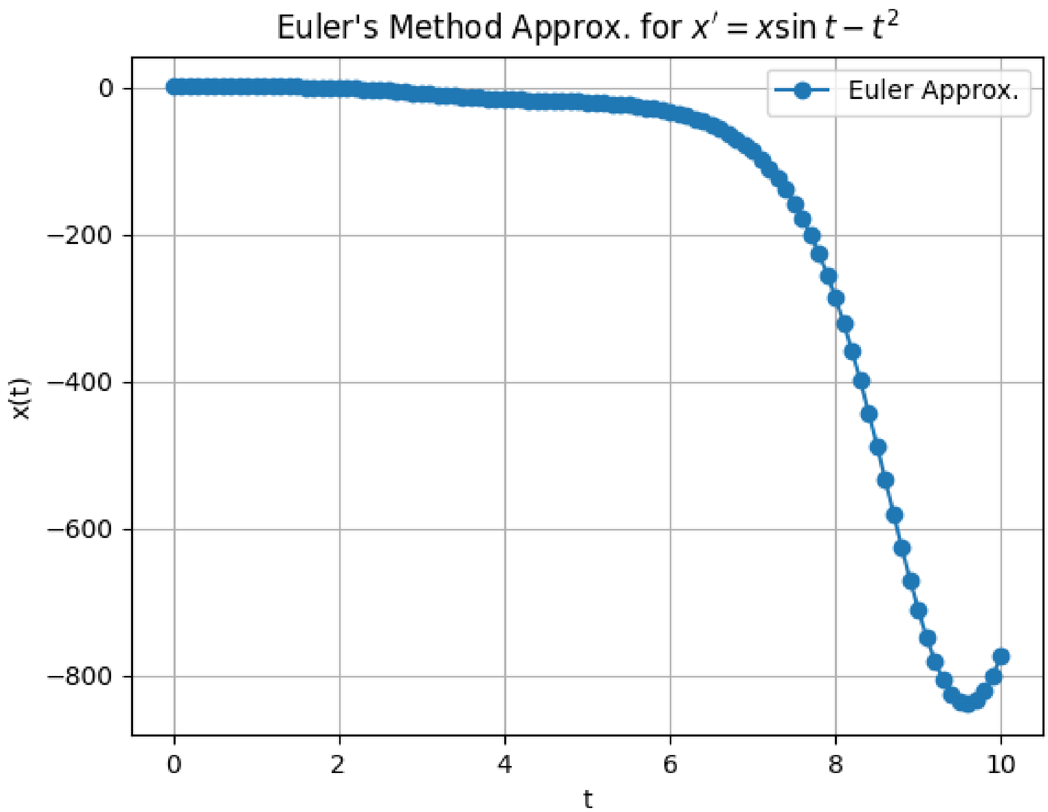


Figure 1: Euler Method Approx. for $x' = xsin(t) - t^2$ and $x(0) = 1$

Python Code Example:

```
import numpy as np
import matplotlib.pyplot as plt
# Define the ODE
def f(t, x):
    return x * np.sin(t) - t**2
# Implement Euler's Method
def euler(f, t0, x0, h, T):
    t_values = np.arange(t0, T + h, h)
    x_values = np.zeros(len(t_values))
    x_values[0] = x0
    for n in range(1, len(t_values)):
        x_values[n] = x_values[n-1] + h * f(t_values[n-1], x_values[n-1])
    return t_values, x_values
# Parameters
t0, x0 = 0, 1
h = 0.1
T = 10
# Compute numerical solution and plot results
t_vals, x_vals = euler(f, t0, x0, h, T)
plt.plot(t_vals, x_vals, label="Euler Approx.", marker="o", linestyle="-")
plt.xlabel("t")
plt.ylabel("x(t)")
plt.title("Euler's Method Approx. for $x' = x \\sin t - t^2$")
plt.legend()
plt.grid()
plt.show()
```

(20 pts) Justify why Euler's method is a good approximation of the actual solution by proving [Teschl, Theorem 2.19], which is as follows:

> Suppose f is continuous on $V = [t_0, t_0+T] \times \overline{B_\delta(x_0)} \subset U$ and denote the maximum of $|f|$ by M. Then there exists at least one solution of the initial value problem ($\dot{x} = f(t,x), \quad x(t_0) = x_0$) for $t \in [t_0, t_0 + T_0]$ which remains in $\overline{B_\delta(x_0)}$, where $T_0 = \min\{T, \frac{\delta}{M}\}$. The analogous result holds for the interval $[t_0 - T_0, t_0]$.

The textbook provides a proof, so focus on understanding it and rewriting it in your own words. Review the Arzelà–Ascoli theorem if needed.

Solution

To justify why Euler's method is a good approx. of the actual solution, we analyze the method's error and its connection to Theorem 2.19 (Peano's Thm.) and Theorem 2.18 (Arzelà–Ascoli).

We start by saying that to prove the existence of a solution to the IVP

$$\dot{x} = f(t,x), \quad x(t_0) = x_0,$$

we assume that f is cts. on set $V = [t_0, t_0 + T] \times \overline{B_\delta(x_0)} \subset U$. We let M be an upper bound for $|f|$ on V, which means $|f(t,x)| \leq M \ \forall \ (t,x) \in V$. We w.t.s. $\exists$ at least one solution on an interval of length $T_0 = \min\{T, \delta/M\}$ that remains inside $\overline{B_\delta(x_0)}$.

- Step 1: Constructing Euler Approximations.
  Consider the sequence of approx. solutions $x_h(t)$ obtained using Euler's method w/ step size h. We define these approxs. recursively as follows:

  $$x_{n+1} = x_n + hf(t_n, x_n),$$

  where $t_n = t_0 + nh$, and we extend $x_h(t)$ piecewise linearly. B/c $|f(t,x)| \leq M$, it follows that the approxs. remain bounded s.t.

  $$|x_h(t) - x_0| \leq M|t - t_0|.$$

  We choose $T_0 = \min\{T, \delta/M\}$ to ensure $x_h(t)$ stays within $\overline{B_\delta(x_0)}$ for $t \in [t_0, t_0 + T_0]$.

- Step 2: Establishing Equicontinuity.
  The family $x_h(t)$ is equicontinuous b/c, for any s, t within the given interval, we have

  $$|x_h(t) - x_h(s)| \leq M|t - s|.$$

  This uniform Lipschitz condition implies that the sequence of functions does not oscillate too wildly and satisfies the conditions of the Arzelà–Ascoli theorem.

- Step 3: Extracting a Convergent Subsequence.
  B/c the approximations $x_h(t)$ are uniformly bounded and equicontinuous, the Arzelà–Ascoli thm. guarantees existence of a uniformly convergent subsequence $\phi_m(t) \to \phi(t)$, where $\phi(t)$ is a continuous function.

- Step 4: Showing that the Limit is a Solution.
  By taking the limit as $h \to 0$, we pass to the integral formulation of the IVP

  $$x_h(t) = x_0 + \int_{t_0}^{t} f(s, x_h(s))\, ds.$$

  B/c f is cts, we can use uniform convergence to show the limit function $\phi(t)$ satisfies

  $$\phi(t) = x_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds.$$

  By differentiating both sides, we see $\phi(t)$ is a solution to the original ODE.

(20 pts) [Hirsch et al.] Exercise 8.2. Find a global change of coordinates that linearizes the system

$$\begin{cases} x' = x + y^2, \\ y' = -y, \\ z' = -z + y^2. \end{cases}$$

Solution
We aim to transform this system into a linear one by introducing new variables.

- Step 1: Identify Nonlinear Terms.
  The nonlinear terms in the system are y^2 in both the x' and z' equations.

- Step 2: Define a Change of Variables.
  B/c y^2 appears in both x' and z', we introduce three new variables

  $$v = x - cy^2, \quad w = z - dy^2, \quad u = y.$$

  where some constants c and d to be determined. The idea is to absorb the nonlinear term y^2 into these new variables, so the derivatives v' and w' do not contain any nonlinearities and to track y via u.

- Step 3: Computing the Derivatives.
  We differentiate v using the chain rule as follows:

  $$v' = x' - c\frac{d}{dt}(y^2).$$

  B/c $\frac{d}{dt}(y^2) = 2yy' = -2y^2$ (i.e., b/c $y' = -y$), we obtain the following:

  $$v' = (x + y^2) - c(-2y^2) = x + y^2 + 2cy^2.$$

  Similarly, differentiating w yields us the following:

  $$w' = z' - d\frac{d}{dt}(y^2) = (-z + y^2) - d(-2y^2) = -z + y^2 + 2dy^2.$$

  Lastly, differentiating u is simply done by the following:

  $$u' = y' = -y.$$

- Step 4: Choosing c and d to Eliminate y^2.
  To remove the nonlinear term y^2, we require that

  $$c + 1 = 0, \quad d + 1 = 0.$$

  By solving for c and d, we see

  $$c = -\frac{1}{2}, \quad d = -\frac{1}{2}.$$

  So, the final change of variables is as follows:

  $$v = x - \frac{y^2}{2}, \quad w = z - \frac{y^2}{2}, \quad u = y.$$

- Step 5: Verifying the Transformation.
  Using these new variables, we now compute v' and w' as follows:

  $$v' = (x + y^2) - \frac{d}{dt}\left(\frac{y^2}{2}\right).$$

B/c $\frac{d}{dt}\left(\frac{y^2}{2}\right) = yy' = -y^2$, we obtain the following:

$$v' = (x + y^2) - (-y^2) = x.$$

Similarly, we get the following:

$$w' = (-z + y^2) - \frac{d}{dt}\left(\frac{y^2}{2}\right) = (-z + y^2) - (-y^2) = -z.$$

Finally, the equation for u remains as the following:

$$u' = -u.$$

Thus, the system transforms into the following:

$$\begin{cases} v' = v, \\ u' = -u, \\ w' = -w. \end{cases}$$

which is now a fully linear system having used a global change of coordinates.

(20 pts) [Hirsch et al.] Exercise 8.5. Consider the system

$$\begin{cases} x' = x^2 + y, \\ y' = x - y + a, \end{cases}$$

where a is a parameter.

Part (a)
Find all equilibrium points and compute the linearized equation at each.

Solution
We solve this problem step-by-step as follows.
Step 1: Solve for Equilibrium Points

- From the second equation, we express y in terms of x as follows:

$$y = x + a.$$

- By subbing this into the first equation, we get

$$x^2 + (x + a) = 0 \implies x^2 + x + a = 0.$$

- We solve for x using the quadratic formula as follows:

$$x = \frac{-1 \pm \sqrt{1 - 4a}}{2}.$$

- By subbing these values of x into $y = x + a$, we obtain the equilibrium points in the following:

$$\left(\frac{-1 + \sqrt{1 - 4a}}{2}, \frac{-1 + \sqrt{1 - 4a}}{2} + a\right),$$

$$\left(\frac{-1 - \sqrt{1 - 4a}}{2}, \frac{-1 - \sqrt{1 - 4a}}{2} + a\right).$$

Step 2: Compute the Linearized System.
We note that the Jacobian matrix of the system is given by the following:

$$J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix}.$$

Evaluating at each equilibrium point yields the following:

- At
$$(x_1, y_1) = \left(\frac{-1+\sqrt{1-4a}}{2}, \frac{-1+\sqrt{1-4a}}{2} + a \right),$$
the Jacobian matrix is
$$J(x_1,y_1) = \begin{bmatrix} 2x_1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1+\sqrt{1-4a} & 1 \\ 1 & -1 \end{bmatrix}.$$

- Similarly, at
$$(x_2, y_2) = \left(\frac{-1-\sqrt{1-4a}}{2}, \frac{-1-\sqrt{1-4a}}{2} + a \right),$$
the Jacobian matrix is
$$J(x_2,y_2) = \begin{bmatrix} -1-\sqrt{1-4a} & 1 \\ 1 & -1 \end{bmatrix}.$$

Thus, we have found the equilibrium points and the corresponding linearized systems.

Part (b)
Describe the behavior of the linearized system at each equilibrium point.

Solution
To analyze the behavior of the linearized system at each equilibrium point, we first examine the eigenvalues of the Jacobian matrix.

Step 1: Compute the Characteristic Equation.
For each equilibrium point, the Jacobian matrix is as follows:

$$J = \begin{bmatrix} 2x & 1 \\ 1 & -1 \end{bmatrix}.$$

The characteristic equation $\det(J - \lambda I) = 0$ allows us to expand the determinant to get as follows:

$$\begin{vmatrix} 2x-\lambda & 1 \\ 1 & -1-\lambda \end{vmatrix} = (2x-\lambda)(-1-\lambda) - 1(1) = -2x - 2x\lambda + \lambda + \lambda^2 - 1 = \lambda^2 - (2x+1)\lambda - (2x+1).$$

By solving for the eigenvalues, we get

$$\lambda = \frac{(2x+1) \pm \sqrt{(2x+1)^2 + 4(2x+1)}}{2} = \frac{(2x+1) \pm \sqrt{(2x+1)(2x+1+4)}}{2}.$$

Step 2: Classify the Behavior at Each Equilibrium.

- If both eigenvalues are real and the same sign, the equilibrium is a node (stable if negative, unstable if positive).
- If the eigenvalues are real and opposite signs, the equilibrium is a saddle point (unstable).
- If the eigenvalues are complex w/ nonzero real part, the equilibrium is a spiral (stable if real part is negative, unstable if positive).

For each equilibrium x_1 and x_2, we know the following:

$$x_1 = \frac{-1 + \sqrt{1-4a}}{2}, \quad x_2 = \frac{-1 - \sqrt{1-4a}}{2}.$$

By subbing these into the characteristic equation, we can determine the nature of each equilibrium as follows:

- If $1 - 4a > 0$, both x_1 and x_2 are real and distinct.
- If $1 - 4a < 0$, then x_1 and x_2 become complex, which leads to spiral behavior.
- The sign of $2x + 1$ determines whether the equilibria are stable or unstable.

Summary: The exact classification depends on the value of a. Generally:

- For small a, one equilibrium behaves as a saddle, and the other may be a stable or unstable node.
- For larger a, complex eigenvalues may introduce spiral behavior.

Part (c)
Describe any bifurcations that occur.

Solution
To determine bifurcations in the system, we analyze how the number and nature of equilibrium points change as parameter a varies (which we already did a bit in Part b).

Step 1: Identify Critical Points of Change.
The equilibrium points are given by the solutions to

$$x^2 + x + a = 0.$$

The discriminant of this quadratic equation is known to be

$$\Delta = 1 - 4a.$$

The nature of equilibrium points depends on the value of Δ, which we discuss in the following:

- If $\Delta > 0$ (i.e., $a < \frac{1}{4}$), there are two distinct real equilibria.
- If $\Delta = 0$ (i.e., $a = \frac{1}{4}$), there is a single equilibrium w/ a double root.
- If $\Delta < 0$ (i.e., $a > \frac{1}{4}$), there are no real equilibria.

Step 2: Identify the Bifurcation Type.
At $a = \frac{1}{4}$, the two real equilibria merge into one and then disappear for $a > \frac{1}{4}$. This suggests a saddle-node bifurcation, where two equilibrium points (one stable, one unstable) annihilate each other as parameter a increases past a critical threshold. Thus, the system undergoes a saddle-node bifurcation at $a = \frac{1}{4}$.

(20 pts) [Hirsch et al.] Exercise 8.9. Consider the system

$$\begin{cases} r' = r - r^2, \\ \theta' = \sin\theta + a. \end{cases}$$

> **Part (a)**
> For which values of a does this system undergo a bifurcation?

Solution
To determine bifurcations in the system, we analyze equilibrium points and their dependence on a.

Step 1: Identify Equilibrium Points.
The equilibrium points are found by setting $r' = 0$ and $\theta' = 0$, which we do in the following:

$$r - r^2 = 0, \quad \sin\theta + a = 0.$$

By solving for r and θ, we get the following:

$$r(1-r) = 0 \implies r = 0 \quad \text{or} \quad r = 1.$$

$$\sin\theta = -a.$$

For equilibrium values of θ to exist, we require

$$-1 \leq -a \leq 1 \quad \Rightarrow \quad -1 \leq a \leq 1.$$

So, equilibria in θ $\exists$ only for $a \in [-1, 1]$.

Step 2: Identify Bifurcations.
Bifurcations occur when number or nature of equilibrium points changes, which happens at critical values

$$a = \pm 1.$$

For $|a| < 1$, $\exists$ two equilibrium values of θ given by

$$\theta = \arcsin(-a) + 2\pi k, \quad k \in \mathbb{Z}.$$

For $a = \pm 1$, a unique equilibrium exists at

$$\theta = \frac{3\pi}{2} \quad \text{for} \quad a = 1, \quad \theta = \frac{\pi}{2} \quad \text{for} \quad a = -1.$$

For $|a| > 1$, no equilibrium exists in θ, which means trajectories in θ will not stabilize.

Step 3: Classify the Bifurcation.
At $a = \pm 1$, equilibrium solutions in θ appear or disappear, which indicates a saddle-node bifurcation. We note the following:

- When a crosses ± 1, the number of equilibrium solutions for θ changes from two to zero.
- The loss of equilibria implies a qualitative change in the system's dynamics.

> **Part (b)**
> Describe the local behavior of solutions near the bifurcation values (at, before, and after the bifurcation).

Solution
To describe the local behavior of solutions near bifurcation values $a = \pm 1$, we analyze dynamics of θ in different parameter regions.

Step 1: Behavior of r.
The equation for r is

$$r' = r - r^2.$$

This has equilibrium points at $r = 0$ and $r = 1$:

- $r = 0$ is an unstable equilibrium, b/c $r' > 0$ for small r.
- $r = 1$ is a stable equilibrium, b/c for $r > 1$, $r' < 0$ and for $0 < r < 1$, $r' > 0$.

So, solutions for r tend to 1 as $t \to \infty$, unless they start at $r = 0$.

Step 2: Behavior of θ.
The equation for θ is:

$$\theta' = \sin\theta + a.$$

- Before bifurcation ($|a| < 1$): The equation $\sin\theta + a = 0$ has two solutions for θ given by

  $$\theta = \arcsin(-a) + 2\pi k, \quad k \in \mathbb{Z}.$$

  Linearizing near these equilibria, we compute

  $$\frac{d}{d\theta}(\sin\theta + a) = \cos\theta.$$

  B/c $\cos\theta$ can be $+$ or $-$, one equilibrium is stable (at $\cos\theta < 0$), and one is unstable (at $\cos\theta > 0$).

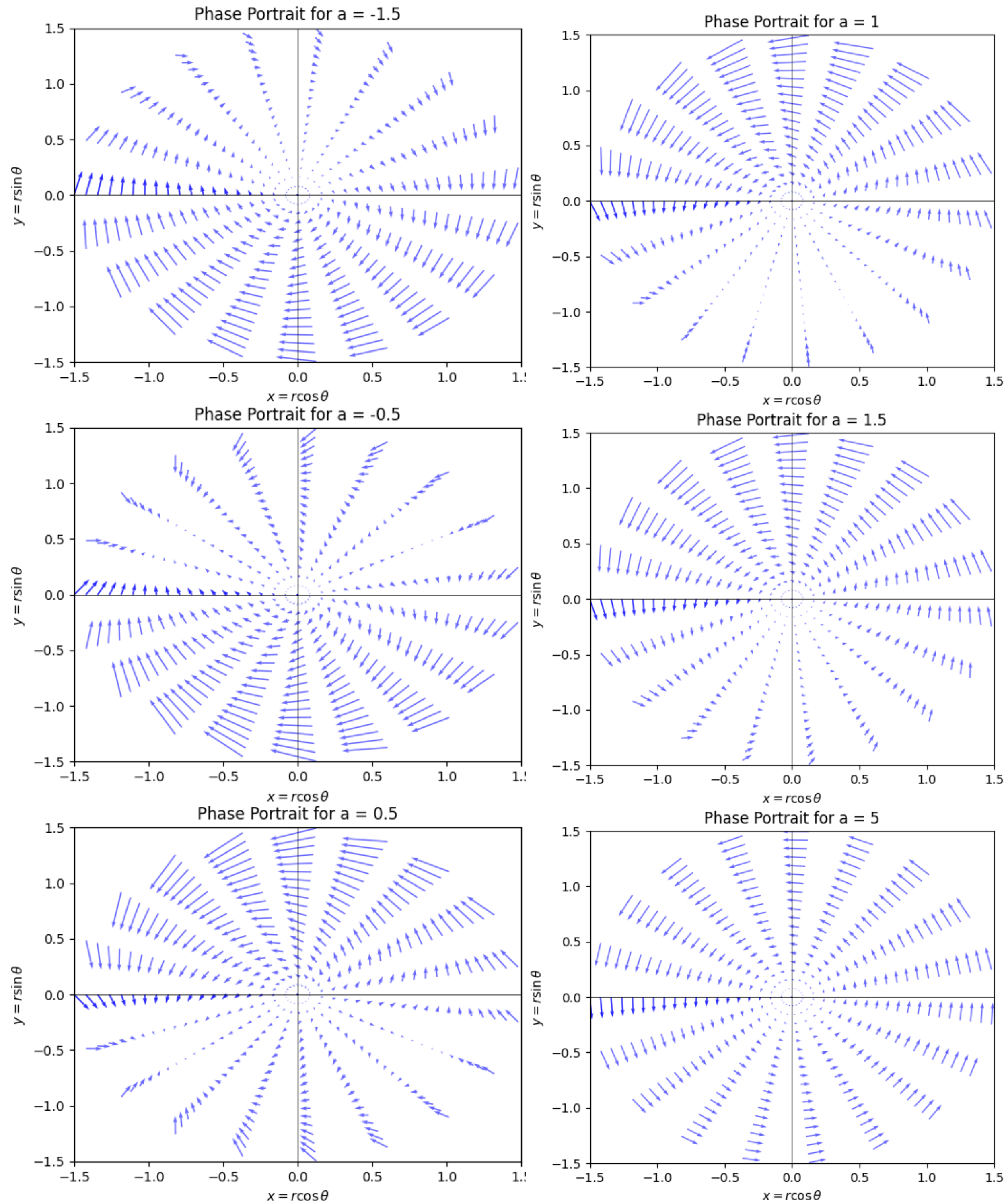
- At bifurcation ($a = \pm 1$): The equilibrium points merge into a single solution:

  $$\theta = \frac{3\pi}{2}, \quad \text{for } a = 1, \quad \theta = \frac{\pi}{2}, \quad \text{for } a = -1.$$

  At these points, $\cos\theta = 0$, so the linearized system has a zero eigenvalue, which indicates a saddle-node bifurcation.

- After bifurcation ($|a| > 1$): The equation $\sin\theta + a = 0$ has no real solutions. B/c $\sin\theta$ is bounded b/w -1 and 1, but $|a| > 1$, the function $\theta' = \sin\theta + a$ is always nonzero.
  - If $a > 1$, then $\theta' > 0 \;\forall\, \theta$, which means θ increases monotonically.
  - If $a < -1$, then $\theta' < 0 \;\forall\, \theta$, which means θ decreases monotonically.
  - In both cases, θ no longer stabilizes and instead grows unbounded in a periodic or secular fashion.

Part (c)
Sketch the phase portrait of the system for all possible different cases.



(a) Phase portraits for $a = -1.5, -0.5, 0.5$

(b) Phase portraits for $a = 1, 1.5, 5$

Figure 2: Phase portraits for different values of a.

Part (d)
Discuss any global changes that occur at the bifurcations.

Solution

1. Behavior for $|a| < 1$ (Before Bifurcation).
   - The equation $\theta' = \sin\theta + a$ has two equilibrium solutions in θ, one stable and one unstable.
   - This means trajectories are locally attracted to stable equilibrium and repelled from unstable one.
   - The existence of equilibria leads to bounded oscillatory behavior in θ w/ solutions tending to one of the equilibria.
2. Behavior at $a = \pm 1$ (Bifurcation Point).
   - The two equilibrium solutions merge into a single degenerate equilibrium, which indicates a saddle-node bifurcation.
   - At this critical value, the system undergoes a qualitative shift: instead of stable and unstable points, there is now only one equilibrium w/ neutral stability.
   - This marks the transition from bounded behavior to unbounded motion in θ.
3. Behavior for $|a| > 1$ (After Bifurcation).
   - When $|a| > 1$, the equation $\sin\theta + a = 0$ has no real solutions, which means $\exists$ no equilibrium pts in θ.
   - B/c θ' never vanishes, trajectories no longer settle into an equilibrium but instead exhibit cts rotation.
   - This results in a global change where θ grows (or decreases) indefinitely over time.
   - The system shifts from an equilibrium-based phase to one where θ evolves monotonically w/o stabilizing.
4. Global Impact of Bifurcation.
   - Before bifurcation, solutions in θ are constrained by stable equilibria, which leads to bounded behavior.
   - After bifurcation, solutions no longer settle but exhibit unbounded rotational motion.
   - Thus, the bifurcation at $a = \pm 1$ marks a global transition from a system with stable equilibrium points to one where solutions in θ rotate indefinitely w/o stabilization.