MATH 273 - Problem Set 3

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Submission: Submit your answers on Gradescope (`https://www.gradescope.com/courses/925433`) by the deadline. You can either handwrite and scan your answers or type them (e.g., using LaTeX) if your handwriting is unclear or typing is more efficient. When submitting, be sure to **select the correct pages** corresponding to each problem, as this will make grading more efficient. You can assign one page to multiple problems or multiple pages to a single problem.

[Hirsch et al.] is our main textbook and [Teschl] is the supplementary textbook. Check the syllabus for the PDF files.

(20 pts) [Hirsch et al.] Exercise 7.2. Let A be an $n \times n$ matrix. Show that the Picard method for solving $X' = AX,\ X(0) = X_0$ gives the solution $\exp(tA)\, X_0$.

Solution

To see why $\exp(tA)\, X_0$ solves

$$X'(t) \;=\; A\, X(t), \quad X(0) \;=\; X_0,$$

we use the Picard iteration approach in the following steps as follows:

1. We first rewrite the ODE as an integral equation:

$$X(t) \;=\; X_0 \;+\; \int_0^t A\, X(s)\, ds.$$

2. We define the Picard iteration sequence as follows:

$$X_0(t) \;=\; X_0, \qquad X_{k+1}(t) \;=\; X_0 \;+\; \int_0^t A\, X_k(s)\, ds.$$

3. It follows by induction that

$$X_k(t) \;=\; \sum_{m=0}^{k} \frac{(At)^m}{m!}\, X_0$$

So, for $k = 0$, $X_0(t) = X_0$. We assume the form holds for $X_k(t)$, and we integrate term-by-term to see that it holds for $X_{k+1}(t)$ as well.

4. As $k \to \infty$, the series

$$\sum_{m=0}^{\infty} \frac{(At)^m}{m!}$$

converges to the matrix exponential $\exp(tA)$. We thereby get as follows:

$$X(t) \;=\; \lim_{k\to\infty} X_k(t) \;=\; \lim_{k\to\infty} \sum_{m=0}^{k} \frac{(At)^m}{m!}\, X_0 \;=\; \exp(tA)\, X_0.$$

5. A quick verification shows this limit satisfies the ODE and initial condition as follows:

$$\frac{d}{dt}\big(\exp(tA)\, X_0\big) = A\, \exp(tA)\, X_0, \quad \text{and} \quad \exp(0 \cdot A)\, X_0 = I\, X_0 = X_0.$$

Thus, the unique solution provided by Picard iteration is $\exp(tA)\, X_0$.

(20 pts) [Teschl] Problem 2.3. Show that the space $C(I, \mathbb{R}^n)$ together with the sup norm (2.3)

$$\|x\| \ = \ \sup_{t \in I} \big|x(t)\big|$$

is a Banach space if I is a compact interval. Show that the same is true for $I = [0, \infty)$ and $I = \mathbb{R}$ if one considers the vector space of bounded continuous functions $C_b(I, \mathbb{R}^n)$.

Solution
(1) Case: I is a compact interval.
Consider the normed vector space $\big(C(I, \mathbb{R}^n), \|\cdot\|\big)$ where

$$\|x\| \ = \ \sup_{t \in I} |x(t)|.$$

We want to show that every Cauchy sequence in $C(I, \mathbb{R}^n)$ converges (in the sup norm) to a limit that is also in $C(I, \mathbb{R}^n)$. Let $\{x_k\}$ be a Cauchy sequence in $C(I, \mathbb{R}^n)$. By definition, for every $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $\forall m, n \geq N$,

$$\|x_m - x_n\| \ = \ \sup_{t \in I} \big|x_m(t) - x_n(t)\big| \ < \ \varepsilon.$$

This implies $\{x_k\}$ is uniformly Cauchy on I. B/c $\mathbb{R}^n$ is complete, for each fixed $t \in I$, the sequence $\{x_k(t)\}$ converges in $\mathbb{R}^n$. We define the following:

$$x(t) \ = \ \lim_{k \to \infty} x_k(t), \quad t \in I.$$

We claim that $x \in C(I, \mathbb{R}^n)$. So, b/c $\{x_k\}$ is uniformly Cauchy, for any $\varepsilon > 0$, $\exists N$ such that $\forall m, n \geq N$,

$$\sup_{t \in I} |x_m(t) - x_n(t)| \ < \ \varepsilon.$$

By letting $n \to \infty$, we deduce the following:

$$\sup_{t \in I} |x_m(t) - x(t)| \ \leq \ \varepsilon,$$

which shows $x_k \to x$ uniformly. Uniform convergence of continuous functions on a compact set implies the limit function x is continuous. So, the convergence is in the sup norm, which implies

$$\|x_k - x\| \ = \ \sup_{t \in I} |x_k(t) - x(t)| \ \to \ 0,$$

so $x_k \to x$ in $\big(C(I, \mathbb{R}^n), \|\cdot\|\big)$. We thereby get that $C(I, \mathbb{R}^n)$ is complete (i.e., a Banach space) when I is a compact interval.

(2) Case: $I = [0, \infty)$ or $I = \mathbb{R}$.
Here, we work with the subspace $C_b(I, \mathbb{R}^n)$ of all bounded continuous functions on I w/ the same sup norm:

$$\|x\| \ = \ \sup_{t \in I} |x(t)|.$$

Let $\{x_k\}$ be a Cauchy sequence in $C_b(I, \mathbb{R}^n)$. By the same argument as above, $\{x_k\}$ is uniformly Cauchy, so pointwise it converges to some function $x(t)$. Being uniformly Cauchy thereby ensures the following:

$$\lim_{k \to \infty} \sup_{t \in I} |x_k(t) - x(t)| \ = \ 0,$$

which implies $x_k \to x$ uniformly. B/c each x_k is continuous on I and the limit is attained uniformly, x is also continuous. So, each x_k is bounded, and uniform convergence guarantees x is bounded as well. This implies $x \in C_b(I, \mathbb{R}^n)$, and thereby $\big(C_b(I, \mathbb{R}^n), \|\cdot\|\big)$ is complete.

Conclusion: Both $\big(C(I, \mathbb{R}^n), \|\cdot\|\big)$ (for compact I) and $\big(C_b(I, \mathbb{R}^n), \|\cdot\|\big)$ (for $I = [0, \infty)$ or $I = \mathbb{R}$) are Banach spaces under the sup norm.

(20 pts) [Teschl] Problem 2.5. Show that $f \in C^1(\mathbb{R}^m, \mathbb{R}^n)$ is locally Lipschitz continuous. In fact, show that

$$|f(y) - f(x)| \leq \sup_{\varepsilon \in [0,1]} \left\| \frac{\partial f\big(x + \varepsilon(y-x)\big)}{\partial x} \right\| |x - y|,$$

where $\frac{\partial f(x_0)}{\partial x}$ denotes the Jacobian matrix at x_0 and $\|\cdot\|$ denotes the matrix norm (cf. (3.8)).

$$\|A\| = \sup_{x:\|x\|=1} \|Ax\|$$

Conclude that a function $f \in C^1(U, \mathbb{R}^n)$, $U \subseteq \mathbb{R}^{n+1}$, is locally Lipschitz continuous in the second argument, uniformly with respect to the first, and thus satisfies the hypothesis of Theorem 2.2 (Picard–Lindelöf). *(Hint: Start with the case $m = n = 1$.)*

Solution

Step 1: Prove the local Lipschitz property for $f \in C^1(\mathbb{R}^m, \mathbb{R}^n)$.

(a) The case $m = n = 1$.

When $m = n = 1$, $f'(x)$ is just the single-variable derivative of f. Let $x, y \in \mathbb{R}$. By the Mean Value Theorem (MVT), $\exists$ some point ξ on the line segment between x and y such that we get as follows:

$$f(y) - f(x) = f'(\xi)\,(y - x).$$

By taking absolute values, we get the following:

$$|f(y) - f(x)| = |f'(\xi)|\,|y - x| \leq \sup_{z \in [x,y]} |f'(z)|\,|y - x|,$$

and b/c f' is continuous (since we know f is C^1), f' is bounded in any closed (and thereby compact) interval. So, on every compact set, f is Lipschitz.

(b) The general case $m \geq 1, n \geq 1$.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, the mean value inequality in several variables states the following:

$$f(\mathbf{y}) - f(\mathbf{x}) = \int_0^1 \frac{\partial f\big(\mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})\big)}{\partial x} \big(\mathbf{y} - \mathbf{x}\big)\, d\varepsilon.$$

We take the norm on both sides and apply the definition of matrix norm $\|A\|$ to get as follows:

$$\big\|f(\mathbf{y}) - f(\mathbf{x})\big\| \leq \int_0^1 \left\| \frac{\partial f\big(\mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})\big)}{\partial x} \right\| \|\mathbf{y} - \mathbf{x}\|\, d\varepsilon = \|\mathbf{y} - \mathbf{x}\| \int_0^1 \left\| \frac{\partial f\big(\mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})\big)}{\partial x} \right\| d\varepsilon$$

$$\leq \sup_{\varepsilon \in [0,1]} \left\| \frac{\partial f\big(\mathbf{x} + \varepsilon(\mathbf{y} - \mathbf{x})\big)}{\partial x} \right\| \|\mathbf{y} - \mathbf{x}\|.$$

B/c f is C^1, $\frac{\partial f}{\partial x}$ is continuous, and so on a small enough (compact) neighborhood, this Jacobian matrix is bounded. This also consequently shows that f is locally Lipschitz.

Step 2: The case $f \in C^1(U, \mathbb{R}^n)$ with $U \subseteq \mathbb{R}^{n+1}$.

We w.t.s that $f(t, x)$ is locally Lipschitz continuous in the second argument x and uniformly in the first argument t. Let $(t_0, x_0) \in U$. B/c f is continuously differentiable, its partials $\frac{\partial f}{\partial x}(t, x)$ are continuous on a small product neighborhood of (t_0, x_0). So, $\exists$ a neighborhood $N = [t_0 - \delta, t_0 + \delta] \times \overline{B_\rho(x_0)} \subset U$ within which the Jacobian $\frac{\partial f}{\partial x}$ is bounded by some constant M. This means that for $(t, x), (t, \tilde{x}) \in N$, we get as follows:

$$\big\|f(t, x) - f(t, \tilde{x})\big\| \leq M\,\|x - \tilde{x}\|.$$

This bound does not depend on t, and instead depends only on the local neighborhood around (t_0, x_0). So, f is locally Lipschitz in x and uniformly in t. This proves it satisfies the hypothesis of the Picard–Lindelöf Theorem (Theorem 2.2).

Step 3: Conclusion
A C^1 function is locally Lipschitz. Specifically, the bound

$$|f(y) - f(x)| \;\leq\; \sup_{\varepsilon\in[0,1]} \left\| \frac{\partial f\big(x + \varepsilon(y-x)\big)}{\partial x} \right\| |y - x|$$

demonstrates that the local behavior of f is controlled by the sup of its derivative, which establishes the desired local Lipschitz property.

> **(20 pts) [Teschl] Problem 2.14.** Suppose $f \in C(U, \mathbb{R}^n)$ satisfies $|f(t,x) - f(t,y)| \leq L(t)\,|x - y|$. Show that the solution $\varphi(t, x_0)$ of (2.10)
>
> $$\dot{x} \;=\; f(t,x), \quad x(t_0) \;=\; x_0$$
>
> satisfies
>
> $$\big|\varphi(t,x_0) - \varphi(t,y_0)\big| \;\leq\; |x_0 - y_0| \,\exp\Big(\int_{t_0}^{t} L(s)\,ds\Big).$$

Solution
Let $\varphi(t, x_0)$ and $\varphi(t, y_0)$ be two solutions of

$$\dot{x} \;=\; f(t,x), \quad x(t_0) \;=\; x_0,$$

w/ different initial conditions $x_0 \neq y_0$. We define

$$z(t) \;=\; \varphi(t,x_0) \;-\; \varphi(t,y_0)$$

and claim that $|z(t)|$ satisfies a Grönwall-type inequality.

Step 1. Differentiate $z(t)$.
By definition of φ, we have as follows:

$$\dot{\varphi}(t,x_0) \;=\; f\big(t,\varphi(t,x_0)\big), \quad \dot{\varphi}(t,y_0) \;=\; f\big(t,\varphi(t,y_0)\big).$$

So, we get as follows:

$$\dot{z}(t) \;=\; \dot{\varphi}(t,x_0) - \dot{\varphi}(t,y_0) \;=\; f\big(t,\varphi(t,x_0)\big) \;-\; f\big(t,\varphi(t,y_0)\big).$$

Step 2. Apply the Lipschitz condition.
Using the hypothesis that $|f(t,x) - f(t,y)| \leq L(t)\,|x-y|$, we get as follows:

$$|\dot{z}(t)| \;=\; \big|f\big(t,\varphi(t,x_0)\big) - f\big(t,\varphi(t,y_0)\big)\big| \;\leq\; L(t)\,\big|\varphi(t,x_0) - \varphi(t,y_0)\big| \;=\; L(t)\,|z(t)|.$$

Step 3. Grönwall's inequality.
We define $w(t) = |z(t)|$. B/c $w(t)$ is absolutely continuous, we may differentiate $w(t)$ almost everywhere and obtain the following:

$$\frac{d}{dt}w(t) \;=\; \frac{d}{dt}|z(t)| \;\leq\; \big|\dot{z}(t)\big| \;\leq\; L(t)\,|z(t)| \;=\; L(t)\,w(t).$$

So, $w'(t) \leq L(t)\,w(t)$, and applying Grönwall's Lemma from t_0 to t, we find the following:

$$w(t) \;\leq\; w(t_0)\,\exp\Big(\int_{t_0}^{t} L(s)\,ds\Big).$$

But, $w(t_0) = |z(t_0)| = |x_0 - y_0|$, so we get as follows:

$$|z(t)| \;=\; \big|\varphi(t,x_0) - \varphi(t,y_0)\big| \;\leq\; |x_0 - y_0|\,\exp\Big(\int_{t_0}^{t} L(s)\,ds\Big).$$

Conclusion. We thereby can conclude the following:

$$\big|\varphi(t,x_0) - \varphi(t,y_0)\big| \;\leq\; |x_0 - y_0|\,\exp\Big(\int_{t_0}^{t} L(s)\,ds\Big).$$