MATH 273 - Problem Set 5

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Submission: Submit your answers on Gradescope (`https://www.gradescope.com/courses/925433`) by the deadline. You can either handwrite and scan your answers or type them (e.g., using LaTeX) if your handwriting is unclear or typing is more efficient. When submitting, be sure to **select the correct pages** corresponding to each problem, as this will make grading more efficient. You can assign one page to multiple problems or multiple pages to a single problem.

[Hirsch et al.] is our main textbook and [Teschl] is the supplementary textbook. Check the syllabus for the PDF files.

(30 pts). A gradient system on $\mathbb{R}^n$ is a system of differential equations of the form

$$X' = -\operatorname{grad} V(X)$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function, and

$$\operatorname{grad} V = \left[\tfrac{\partial V}{\partial x_1}, \ldots, \tfrac{\partial V}{\partial x_n}\right].$$

The proof is provided in [Hirsch et al., Section 9.3], which you are allowed to refer to. The goal of this problem is to help you develop an understanding of Liapunov stability and its applications on your own, as we will omit the detail in class. Prove the following:

Part (a)
If c is a regular value of V (i.e., $V^{-1}(c)$ looks like a surface of dimension $n-1$ at each point), then the vector field is perpendicular to the level set $V^{-1}(c)$.

Solution
We note that since c is a regular value of V, level set $V^{-1}(c)$ forms a smooth surface in $\mathbb{R}^n$. By definition, at any regular point X, the gradient of V is nonzero.

Next, from the textbook, if Y is a tangent vector to level set $V^{-1}(c)$ at X, then $\exists$ a smooth curve $\gamma(t)$ lying in $V^{-1}(c)$ s.t. $\gamma(0) = X$ and $\gamma'(0) = Y$. B/c V is constant along $\gamma(t)$, we differentiate to get as follows:

$$\frac{d}{dt} V(\gamma(t))\Big|_{t=0} = 0.$$

By applying the chain rule, we get as follows:

$$DV_X(Y) = \operatorname{grad} V(X) \cdot Y = 0.$$

This equation states that $\operatorname{grad} V(X)$ is orthogonal to every tangent vector Y at X, which means it is normal to level set $V^{-1}(c)$. B/c the system is defined by $X' = -\operatorname{grad} V(X)$, it follows that vector field X' is also perpendicular to $V^{-1}(c)$, so we proved the vector field is perpendicular to level set of V at each regular pt.

Part (b)
The critical points of V are the equilibrium points of the system.

Solution
We know a critical pt. of V is a point X^* where the gradient of V vanishes $\operatorname{grad} V(X^*) = 0$. An equilibrium pt. of the system $X' = -\operatorname{grad} V(X)$ is a point X^* where velocity vector X' is zero $X' = -\operatorname{grad} V(X) = 0$.

Clearly, the system satisfies $X' = 0$ iff $\operatorname{grad} V(X) = 0$, which means equilibrium pts. of the system are precisely the critical pts. of V.

From the textbook, function V is strictly decreasing along nonconstant solutions of the system, which is given as follows:

$$\dot{V}(X) = -\|\operatorname{grad} V(X)\|^2 \leq 0.$$

This expression shows V decreases unless $\operatorname{grad} V(X) = 0$, in which case $\dot{V}(X) = 0$. This occurs only at equilibrium pts., which implies they are exactly where $\operatorname{grad} V = 0$, which are the critical pts. of V.

Part (c)
If a critical point is an isolated minimum of V, then this point is an asymptotically stable equilibrium point.

Solution
Step 1: X^* is an Equilibrium Point
B/c X^* is a critical pt. of V, we know $\operatorname{grad} V(X^*) = 0$. By Part (b), this means X^* is an equilibrium pt. of the system.

Step 2: Stability via Liapunov's Direct Method
We consider $V(X)$ as a Liapunov function. The time derivative of V along solutions of the system is as follows:

$$\dot{V}(X) = DV_X(X') = \operatorname{grad} V(X) \cdot (-\operatorname{grad} V(X)) = -\|\operatorname{grad} V(X)\|^2 \leq 0.$$

B/c X^* is an isolated min, $\exists$ a nbhd. around X^* where $V(X) > V(X^*)$ $\forall$ $X \neq X^*$. This ensures that solutions remain near X^*, which proves Liapunov stability.

Step 3: Asymptotic Stability
For asymptotic stability, we w.t.s. that solutions not only remain close to X^* but also converge to it. The key observation is as follows:

$$\dot{V}(X) = -\|\operatorname{grad} V(X)\|^2.$$

This expression is strictly negative $\forall$ $X \neq X^*$ in a nbhd. of X^*, so $V(X)$ strictly decreases along nonconstant trajectories. B/c $V(X)$ is bounded below and decreasing, it must approach a limit. The only way for $\dot{V}(X)$ to stop decreasing is for $\operatorname{grad} V(X) = 0$, which happens only at X^*. This guarantees that trajectories approach X^*, thereby proving asymptotic stability. We have thus proved that an isolated min of V is an asymptotically stable equilibrium of the system.

(10 pts) [Hirsch et al.] Problem 9.4. Consider the system

$$\begin{cases} x' = (\epsilon x + 2y)(z+1), \\ y' = (-x + \epsilon y)(z+1), \\ z' = -z^3. \end{cases}$$

Part (a)
Show that the origin is not asymptotically stable when $\epsilon = 0$.

Solution
When $\epsilon = 0$, the system reduces as follows:

$$\begin{cases} x' = (2y)(z+1), \\ y' = (-x)(z+1), \\ z' = -z^3. \end{cases}$$

To analyze stability of the origin, we linearize the system by computing the Jacobian as follows:

$$J(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x}\left[(2y)(z+1)\right] & \frac{\partial}{\partial y}\left[(2y)(z+1)\right] & \frac{\partial}{\partial z}\left[(2y)(z+1)\right] \\ \frac{\partial}{\partial x}\left[(-x)(z+1)\right] & \frac{\partial}{\partial y}\left[(-x)(z+1)\right] & \frac{\partial}{\partial z}\left[(-x)(z+1)\right] \\ \frac{\partial}{\partial x}\left[-z^3\right] & \frac{\partial}{\partial y}\left[-z^3\right] & \frac{\partial}{\partial z}\left[-z^3\right] \end{bmatrix}.$$

Evaluating at $(0,0,0)$, we get the following:

$$J(0,0,0) = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The characteristic equation is obtained by solving:

$$\det \begin{bmatrix} -\lambda & 2 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix} = 0.$$

By expanding along the third column, we get as follows:

$$\begin{vmatrix} -\lambda & 2 \\ -1 & -\lambda \end{vmatrix}(-\lambda) = 0 \implies (-\lambda)(-\lambda) - (2)(-1) = \lambda^2 + 2.$$

Thus, the characteristic equation is as follows:

$$(\lambda^2 + 2)(-\lambda) = 0.$$

By solving for λ, we obtain the eigenvalues as follows:

$$\lambda = 0, \quad \lambda = \pm i\sqrt{2}.$$

B/c there is a purely imaginary pair of eigenvalues and a zero eigenvalue, the linearized system does not exhibit asymptotic stability at the origin. Instead, the system is Liapunov stable but not asymptotically stable. Thus, we have proved the origin is not asymptotically stable when $\epsilon = 0$.

Part (b)
Show that when $\epsilon < 0$, the basin of attraction of the origin contains the region $z > -1$.

Solution
To analyze the basin of attraction of origin when $\epsilon < 0$, we first observe that the equation for z' is independent of x and y. For $z > -1$, $-z^3$ is negative whenever $z > 0$, and positive whenever $-1 < z < 0$. This implies z monotonically decreases for $z > 0$ and increases for $z < 0$, which means trajectories in this region will be attracted toward $z = 0$.

Next, we consider the behavior of x and y when $z > -1$. B/c $z + 1 > 0$ in this region, the linearized system for x and y takes the form as follows:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = (z+1)\begin{bmatrix} \epsilon & 2 \\ -1 & \epsilon \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

The coefficient matrix $A = \begin{bmatrix} \epsilon & 2 \\ -1 & \epsilon \end{bmatrix}$ has eigenvalues as follows:

$$\lambda = \epsilon \pm i\sqrt{2}.$$

B/c $\epsilon < 0$, the real parts of the eigenvalues are negative, which means x and y decay exponentially towards the origin for a fixed z. And b/c $z \to 0$ as $t \to \infty$, the entire system is asymptotically attracted to the origin. Thus, any initial condition w/ $z > -1$ leads to trajectories that eventually reach the origin, which implies the basin of attraction of origin contains region $z > -1$.

(20 pts) [Hirsch et al.] Problem 9.6. Find a strict Liapunov function for the equilibrium point $(0,0)$ of the system

$$\begin{cases} x' = -2x - y^2, \\ y' = -y - x^2. \end{cases}$$

Find $\delta > 0$ as large as possible so that the open disk of radius δ centered at $(0,0)$ is contained in the basin of $(0,0)$.

Solution

We want to find a function $V(x,y)$ that satisfies the following:

1. $V(x,y)$ is positive definite, which means $V(x,y) > 0 \ \forall \ (x,y) \neq (0,0)$ and $V(0,0) = 0$.
2. The derivative of V along trajectories, given by $\dot{V} = \frac{dV}{dt}$, is negative definite.

Step 1: Choosing a Candidate for $V(x,y)$

A common choice for Liapunov functions is a quadratic form such as the following:

$$V(x,y) = \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

This function is positive definite, b/c it is the sum of squares and satisfies $V(x,y) > 0 \ \forall \ (x,y) \neq (0,0)$ w/ $V(0,0) = 0$.

Step 2: Computing $\dot{V}$

The time derivative of $V(x,y)$ along system trajectories is as follows:

$$\dot{V} = \frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y'.$$

We compute partial derivatives as follows:

$$\frac{\partial V}{\partial x} = x, \quad \frac{\partial V}{\partial y} = y.$$

By subbing in x' and y', we get as follows:

$$\dot{V} = x(-2x - y^2) + y(-y - x^2) = -2x^2 - y^2 - x^2y - xy^2.$$

B/c all terms on RHS are nonpositive and at least one of $-2x^2$ or $-y^2$ dominates for small values of x, y, we conclude the following:

$$\dot{V} \leq -2x^2 - y^2 < 0, \quad \forall (x,y) \neq (0,0).$$

So, $\dot{V}$ is negative definite, which confirms $V(x,y)$ is a strict Liapunov function.

Step 3: Finding the Basin of Attraction

The basin of attraction consists of pts. from which all trajectories move toward $(0,0)$. To estimate the largest open disk of $r = \delta$ contained in this basin, we find where $\dot{V}$ remains negative definite to get as follows:

$$\dot{V} = -2x^2 - y^2 - x^2y - xy^2,$$

where we note that dominant terms for small x, y are $-2x^2 - y^2$, which ensures $\dot{V} < 0$ as long as $x^2 + y^2$ remains sufficiently small. To ensure strict negativity, we require the following:

$$2x^2 + y^2 > x^2y + xy^2.$$

For small (x,y), the quadratic terms dominate, which means $\exists \ \delta > 0$ s.t. in the open disk $x^2 + y^2 < \delta^2$, $\dot{V} < 0$ holds. We choose the largest such δ so it depends on bounding nonlinear terms $x^2y + xy^2$, which are small for x, y close to zero. A reasonable conservative estimate is to take $\delta = 1$, which ensures $x^2 + y^2 < 1$ remains in the basin where $\dot{V} < 0$.

Thus, a (conservative) estimate for the largest open disk contained in basin of attraction has radius $\delta = 1$.

(20 pts) [Hirsch et al.] Problem 9.16. A solution $X(t)$ of a system is called *recurrent* if $X(t_n) \to X(0)$ for some sequence $t_n \to \infty$. Prove that a gradient dynamical system has no nonconstant recurrent solutions.

Solution

We aim to prove that in a system of the form $X' = -\operatorname{grad} V(X)$, where $V : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function, any solution $X(t)$ that returns arbitrarily close to its initial pt. infinitely often must be constant.

Step 1: Define Recurrent Solutions

A solution $X(t)$ is recurrent if $\exists$ a sequence $t_n \to \infty$ as follows:

$$X(t_n) \to X(0) \quad \text{as} \quad t_n \to \infty.$$

This means the trajectory of $X(t)$ comes arbitrarily close to its starting position infinitely often.

Step 2: Compute $V(X)$ Along Trajectories

B/c $V(X)$ is a Liapunov function for the system, its time derivative along solutions is as follows:

$$\dot{V}(X) = \operatorname{grad} V(X) \cdot X' = \operatorname{grad} V(X) \cdot (-\operatorname{grad} V(X)) = -\|\operatorname{grad} V(X)\|^2.$$

And b/c $\|\operatorname{grad} V(X)\|^2 \geq 0$, we know $\dot{V}(X) \leq 0$, which means $V(X(t))$ is strictly decreasing along any nonconstant trajectory.

Step 3: Contradiction for Recurrent Solutions

If $X(t)$ is recurrent, then $\exists$ a sequence $t_n \to \infty$ s.t. $X(t_n) \to X(0)$. B/c $V(X(t))$ is strictly decreasing along nonconstant solutions, we get as follows:

$$V(X(t_n)) < V(X(0)) \quad \forall t_n > 0.$$

Taking the limit as $t_n \to \infty$, we get as follows:

$$V(X(0)) \leq \lim_{n\to\infty} V(X(t_n)) < V(X(0)),$$

which is a contradiction. The only way to avoid this contradiction is for $X(t)$ to be constant, which means we require the following:

$$X' = -\operatorname{grad} V(X) = 0 \quad \forall t.$$

Thus, $X(t)$ must be an equilibrium point. And so b/c nonconstant solutions must decrease $V(X)$, they cannot be recurrent, which implies gradient dynamical systems have no nonconstant recurrent solutions.

(20 pts) [Hirsch et al.] Problem 9.17. Show that a closed bounded ω-limit set is connected. Give an example of a planar system having an unbounded ω-limit set consisting of two parallel lines.

Solution

Step 1: Proving that a Closed, Bounded ω-Limit Set is Connected

Let $\omega(X_0)$ be the ω-limit set of a trajectory $X(t)$, which means it consists of all accumulation pts. of $X(t)$ as $t \to \infty$. We w.t.s. that if $\omega(X_0)$ is closed and bounded, then it is also connected. We claim$\omega(X_0)$ is connected, and we will prove this by contradiction.

Suppose $\omega(X_0)$ is not connected s.t. it can be written as the disjoint union of two nonempty, disjoint closed sets as follows:

$$\omega(X_0) = A \cup B, \quad A \cap B = \emptyset.$$

B/c $\omega(X_0)$ is invariant under the flow, we know any trajectory starting in $\omega(X_0)$ stays in $\omega(X_0)$ $\forall$ forward time. Moreover, b/c $\omega(X_0)$ is the accumulation set of the trajectory $X(t)$, it must be chain recurrent (i.e., for any two points in $\omega(X_0)$, $\exists$ a sequence of arbitrarily small jumps from one to the other along system trajectories). But, if A and B are disjoint and separated, then the trajectory cannot move b/w them, which contradicts chain recurrence. Thus, $\omega(X_0)$ must be connected.

Step 2: Constructing a Planar System w/ Unbounded ω-Limit Set Consisting of Two Parallel Lines

We seek a planar system where trajectories oscillate b/w two parallel lines and accumulate onto them. We use the classical example given in the hint of a spiraling system that is as follows:

$$\begin{cases} r' = r(1-r), \\ \theta' = 1. \end{cases}$$

This system describes a spiral around the unit circle, where trajectories approach $r = 1$ as $t \to \infty$. To modify this into an unbounded strip instead of a circle, we use the hint that suggests stretching the system into an infinite strip $\mathbb{R} \times (-1, 1)$. The key idea here is to apply a transformation that maps the closed unit disk (minus two points) onto a closed infinite strip (which was also given in the hint) as follows:

$$G(z) = \log\left(\frac{1-z}{1+z}\right),$$

which maps the unit disk (minus $z = \pm 1$) to an infinite strip. We assume that a smooth diffeomorphism G exists b/w the punctured closed disk and the infinite strip. Using this, we modify the original system by transforming r into a coordinate that grows indefinitely while still oscillating such as in the following:

$$\begin{cases} x' = y, \\ y' = x(1-x^2). \end{cases}$$

In this system, trajectories oscillate indefinitely in the x-direction while being confined b/w two parallel lines (e.g., $y = \pm 1$). This guarantees the ω-limit set consists of the lines $y = \pm 1$, as required. Thus, we have shown an example of a system whose unbounded ω-limit set consists of two parallel lines.