MATH 273 - Problem Set 1

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Submission: Submit your answers on Gradescope (`https://www.gradescope.com/courses/925433`) by the deadline. You can either handwrite and scan your answers or type them (e.g., using LaTeX) if your handwriting is unclear or typing is more efficient. When submitting, be sure to **select the correct pages** corresponding to each problem, as this will make grading more efficient. You can assign one page to multiple problems or multiple pages to a single problem.

(15 pts) Exercise 1.10. [Hint: Compute $(xe^{-t})'$.] Consider the differential equation $x' = x + \cos(t)$.

Part (a)
Find the general solution of this equation.

Solution
The given differential equation is as follows:

$$x' = x + \cos(t).$$

$$x' - x = \cos(t).$$

We apply the hint to compute $(xe^{-t})'$ to rewrite LHS as follows:

$$\frac{d}{dt}\left(xe^{-t}\right) = x'e^{-t} - xe^{-t}.$$

Then, we multiply the entire equation $x' - x = \cos(t)$ by e^{-t}:

$$e^{-t}x' - e^{-t}x = e^{-t}\cos(t).$$

We see LHS simplifies, and we can integrate both sides w.r.t. t to get the following:

$$\frac{d}{dt}\left(xe^{-t}\right) = e^{-t}\cos(t)$$

$$\int \frac{d}{dt}\left(xe^{-t}\right)\, dt = \int e^{-t}\cos(t)\, dt$$

$$= xe^{-t}$$

The integral of right-hand side (i.e., $\int e^{-t}\cos(t)\, dt$) can be simplified as follows:

$$\int e^{-t}\cos(t)\, dt = \frac{e^{-t}(\sin(t) - \cos(t))}{2}$$

$$xe^{-t} = \frac{e^{-t}(\sin(t) - \cos(t))}{2} + C.$$

We multiply through by e^t to isolate $x(t)$ to get the general solution as follows:

$$x(t) = \frac{\sin(t) - \cos(t)}{2} + Ce^t.$$

Part (b)
Prove that there is a unique periodic solution for this equation.

Solution
To prove the existence and uniqueness of a periodic solution, we examine the following:

1. **General solution:** From Part (a), the general solution is as follows:
$$x(t) = \frac{\sin(t) - \cos(t)}{2} + Ce^t.$$
2. **Behavior of the solution:**
   - The term $\frac{\sin(t)-\cos(t)}{2}$ is periodic w/ period 2π, as both $\sin(t)$ and $\cos(t)$ are periodic functions with period 2π.
   - The term Ce^t grows (if $C > 0$) or decays (if $C < 0$) exponentially as $t \to \infty$, so our function is not periodic unless $C = 0$.
3. **Eliminate the non-periodic term:** So, we must have $C = 0$ to be periodic, which removes the exponential term and simplifies the solution to the following:
$$x(t) = \frac{\sin(t) - \cos(t)}{2}.$$

Part (c)
Compute the Poincaré map $p : \{t = 0\} \to \{t = 2\pi\}$ for this equation and use this to verify again that there is a unique periodic solution.

Solution
To compute the Poincaré map and verify the uniqueness of the periodic solution, we proceed as follows:

1. **Define the Poincaré map:** The Poincaré map p maps the value of the solution at $t = 0$ to the value at $t = 2\pi$. Let $x(0) = x_0$. Then, at $t = 2\pi$, the solution is as follows:
$$x(2\pi) = \frac{\sin(2\pi) - \cos(2\pi)}{2} + Ce^{2\pi} = \frac{0-1}{2} + Ce^{2\pi} = -\frac{1}{2} + Ce^{2\pi}.$$
2. **Periodic solution condition:** For $x(t)$ to be periodic w/ period 2π, the Poincaré map must satisfy:
$$p(x_0) = x_0.$$
   We sub $x_0 = x(0)$ to get the following:
$$x(0) = \frac{\sin(0) - \cos(0)}{2} + C = -\frac{1}{2} + C.$$
   At $t = 2\pi$, periodicity requires:
$$x(2\pi) = x(0).$$
   So, by substitution, we get as follows:
$$-\frac{1}{2} + Ce^{2\pi} = -\frac{1}{2} + C.$$
$$Ce^{2\pi} = C.$$
   B/c $e^{2\pi} > 1$, the only solution to this equation is $C = 0$.
3. **Unique periodic solution:** When $C = 0$, the unique periodic solution becomes as follows:
$$x(t) = \frac{\sin(t) - \cos(t)}{2}.$$
   This is periodic w/ period 2π, as $\sin(t)$ and $\cos(t)$ are periodic functions w/ period 2π.

(15 pts) Exercise 1.11. First-order differential equations need not have solutions that are defined for all time.

Part (a)
Find the general solution of the equation $x' = x^2$.

Solution
We find the general solution as follows:

1. **Rewriting the equation:** We rewrite the equation using separation of variables:
$$\frac{1}{x^2}dx = dt.$$
2. **Integrate both sides:**
$$\int \frac{1}{x^2}dx = \int dt \implies -\frac{1}{x} = t + C,$$
3. **Solve for x:** We rearrange the equation to isolate x as follows:
$$-\frac{1}{x} = t + C \implies x = -\frac{1}{t+C}.$$
4. **Including the constant solution:** We note that the differential equation $x' = x^2$ also allows for the constant solution $x = 0$, b/c subbing in $x = 0$ satisfies $x' = 0$.
5. **General solution:** By combining the non-constant and constant solutions, the general solution is as follows:
$$x(t) = \begin{cases} -\frac{1}{t+C}, & \text{for } C \neq -t, \\ 0, & \text{for the constant solution.} \end{cases}$$

Part (b)
Discuss the domains over which each solution is defined.

Solution
For non-constant solutions, $x(t) = -\frac{1}{t+C}$, there is a vertical asymptote at $t = -C$ where the solution becomes undefined. So, the domain of each solution depends on the value of constant C as follows:
$$t \in (-\infty, -C) \cup (-C, \infty).$$
The constant solution $x(t) = 0$ is defined $\forall t \in \mathbb{R}$, b/c it does not depend on C. We conclude the following:

- For $C \neq 0$, the solution $x(t) = -\frac{1}{t+C}$ is defined on intervals that exclude $t = -C$, such as $(-C, \infty)$ or $(-\infty, -C)$, depending on the initial condition.
- For $C = 0$, the constant solution $x(t) = 0$ is defined $\forall t \in \mathbb{R}$.

Part (c)
Give an example of a differential equation for which the solution satisfying $x(0) = 0$ is defined only for $-1 < t < 1$.

Solution
The choice of our equation is based on two key properties:

$$x' = \frac{x^2}{t^2 - 1}.$$

1. The denominator $t^2 - 1$ introduces singularities at $t = \pm 1$. These singularities ensure the solution is undefined at $t = \pm 1$, which restricts the domain to $-1 < t < 1$.

2. The numerator x^2 ensures the differential equation allows for both non-trivial solutions and constant solution $x(t) = 0$.

Step 1: Rewrite the equation. By using separation of variables, we rewrite the equation as follows:

$$\frac{1}{x^2} dx = \frac{1}{t^2 - 1} dt.$$

Step 2: Integrate both sides. LHS becomes:

$$\int \frac{1}{x^2} dx = -\frac{1}{x}.$$

RHS can be integrated using partial fractions:

$$\frac{1}{t^2 - 1} = \frac{1}{2(t-1)} - \frac{1}{2(t+1)}.$$

$$\int \frac{1}{t^2 - 1} dt = \frac{1}{2} \ln |t - 1| - \frac{1}{2} \ln |t + 1| + C = \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C.$$

Step 3: Combine results. We equate the integrals and rearrange to solve for x:

$$-\frac{1}{x} = \frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C.$$

$$x = -\frac{1}{\frac{1}{2} \ln \left| \frac{t-1}{t+1} \right| + C}.$$

Step 4: Domain of solution. The term $\ln \left| \frac{t-1}{t+1} \right|$ becomes undefined at $t = \pm 1$, where the denominator $t + 1 = 0$ or numerator $t - 1 = 0$. The solution is thereby valid only for $t \in (-1, 1)$.

Step 5: Initial condition. To satisfy the initial condition $x(0) = 0$, we adjust the constant C to ensure the solution passes through the origin at $t = 0$. Our differential equation that satisfies $x(0) = 0$ defined only for $-1 < t < 1$ is thus as follows:

$$x' = \frac{x^2}{t^2 - 1},$$

(15 pts) Exercise 1.12. First-order differential equations need not have unique solutions satisfying a given initial condition.

Part (a)
Prove that there are infinitely many different solutions of the differential equation $x' = x^{1/3}$ satisfying $x(0) = 0$.

Solution
We show that $x' = x^{1/3}$ has infinitely many different solutions satisfying $x(0) = 0$.

Step 1: Rewrite the equation. By using separation of variables, we rewrite the equation as follows:

$$x^{-1/3}dx = dt.$$

Step 2: Integrate both sides. LHS integrates as follows:

$$\int x^{-1/3}dx = \frac{3}{2}x^{2/3}.$$

RHS integrates as follows:

$$\int dt = t + C.$$

We equate the two integrals and solve for x as follows:

$$\frac{3}{2}x^{2/3} = t + C.$$

$$x = \left(\frac{2}{3}(t + C)\right)^{3/2}.$$

Step 3: Apply the initial condition. The initial condition $x(0) = 0$ implies the following:

$$0 = \left(\frac{2}{3}(0 + C)\right)^{3/2}.$$

which happens if $C = 0$, so we get the following particular solution:

$$x(t) = \left(\frac{2}{3}t\right)^{3/2}.$$

But, this does not rule out other solutions. We observe that function $x(t)$ can be defined piecewise to satisfy the differential equation and initial condition.

Step 5: Construct infinitely many solutions. We define a family of solutions piecewise as follows:

$$x(t) = \begin{cases} 0, & \text{for } t \leq t_0, \\ \left(\frac{2}{3}(t - t_0)\right)^{3/2}, & \text{for } t > t_0, \end{cases}$$

where t_0 is any real number.

- For $t \leq t_0$, $x(t) = 0$ satisfies both $x' = x^{1/3}$ (b/c $x = 0 \implies x' = 0$) and the initial condition.
- For $t > t_0$, the non-trivial solution $\left(\frac{2}{3}(t - t_0)\right)^{3/2}$ satisfies the differential equation.

B/c t_0 can be chosen arbitrarily, there are infinitely many such piecewise-defined solutions.

Conclusion: The differential equation $x' = x^{1/3}$ admits infinitely many solutions satisfying $x(0) = 0$, which can be constructed by choosing different t_0 values that leads to non-uniqueness.

Part (b)
Discuss the corresponding situation that occurs for $x' = \frac{x}{t}$, $x(0) = x_0$.

Solution
As noted in Ed, the equation is ill-posed, b/c the term $\frac{x}{t}$ is undefined at $t = 0$. So, the equation cannot be directly solved at $t = 0$ w/o additional assumptions or interpretations. A reasonable approach is to solve the equation for $t \neq 0$ and then extend the solution continuously to $t = 0$ so the initial condition is satisfied.

Step 1: Solve for $t \neq 0$. We rewrite the equation as follows:

$$\frac{dx}{dt} = \frac{x}{t}.$$

$$\frac{1}{x}dx = \frac{1}{t}dt.$$

$$\int \frac{1}{x}dx = \int \frac{1}{t}dt.$$

$$\ln|x| = \ln|t| + C,$$

$$|x| = e^C|t|.$$

We let $k = e^C$ and get the following:

$$x = kt,$$

where k is a constant determined by the initial condition.

Step 2: Apply the Initial Condition. For $t = 0$, the initial condition $x(0) = x_0 \implies x = 0$ as follows:

$$x(t) = \begin{cases} kt, & t \neq 0, \\ x_0, & t = 0. \end{cases}$$

Step 3: Extending Continuously. To satisfy the initial condition $x(0) = x_0$ and ensure continuity of the solution, we set $k = 0$ as follows:

$$x(t) = x_0, \quad \text{which is a constant function for all } t.$$

Conclusion:

1. The solution $x(t) = x_0$ satisfies the equation "almost everywhere," excluding point $t = 0$, where the equation is undefined.

2. The interpretation of the solution at $t = 0$ involves extending it continuously from $t \neq 0$ so it aligns w/ the given initial condition.

Part (c)
Discuss the situation that occurs for $x' = \frac{x}{t^2}$, $x(0) = 0$.

Solution
Step 1: Solve for $t \neq 0$. We rewrite the equation as follows:

$$\frac{dx}{dt} = \frac{x}{t^2}.$$

$$\frac{1}{x}dx = \frac{1}{t^2}dt.$$

LHS:

$$\int \frac{1}{x}dx = \ln|x|.$$

RHS:

$$\int \frac{1}{t^2}dt = -\frac{1}{t}.$$

We combine the results:
$$\ln|x| = -\frac{1}{t} + C.$$

Step 2: Solve for x. We exponentiate both sides to solve for x as follows:
$$x = e^C e^{-1/t}.$$

We let $k = e^C$, so the general solution is as follows:
$$x(t) = ke^{-1/t}, \quad \text{where } t \neq 0.$$

Step 3: Apply the Initial Condition. The initial condition $x(0) = 0$ cannot be directly applied, b/c solution $x(t)$ is undefined at $t = 0$. But, as $t \to 0^+$ or $t \to 0^-$, the term $e^{-1/t}$ diverges (depending on the sign of t).

To ensure continuity and satisfy the initial condition $x(0) = 0$, we can define the solution piecewise as follows:
$$x(t) = \begin{cases} ke^{-1/t}, & t > 0, \\ 0, & t = 0, \\ ke^{-1/t}, & t < 0. \end{cases}$$

Conclusion:

1. The solution is valid for $t \neq 0$, but the singularity at $t = 0$ requires us to extend the solution to $t = 0$ by setting $x(0) = 0$.
2. The piecewise definition ensures the solution satisfies the initial condition $x(0) = 0$, though the behavior of $e^{-1/t}$ near $t = 0$ diverges.

(20 pts) Exercises 2.2 and 2.3.

Exercise 2.2. Find the general solution of each of the following linear systems:

Part (a)
$$X' = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} X$$

Solution
We use the eigenvalue-eigenvector approach to solve this.

Step 1: Find the eigenvalues. We define matrix A as follows:
$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

The eigenvalues are the roots of the characteristic equation as follows:
$$\det(A - \lambda I) = 0.$$
$$A - \lambda I = \begin{pmatrix} 1-\lambda & 2 \\ 0 & 3-\lambda \end{pmatrix}.$$
$$\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 0 = (1-\lambda)(3-\lambda).$$

We get the following roots:

$$\lambda = 1, \quad \lambda = 3.$$

Step 2: Find the eigenvectors. For $\lambda = 1$:

$$A - I = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}.$$

By solving $(A - I)V = 0$, we find the following:

$$V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $\lambda = 3$:

$$A - 3I = \begin{pmatrix} -2 & 2 \\ 0 & 0 \end{pmatrix}.$$

By solving $(A - 3I)V = 0$, we find the following:

$$V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Step 3: Write the general solution. The general solution is a linear combination of the eigenvectors scaled by their respective exponential terms as follows:

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2.$$

We sub in the eigenvalues and eigenvectors as follows:

$$X(t) = c_1 e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

$$X(t) = \begin{pmatrix} c_1 e^t + c_2 e^{3t} \\ c_2 e^{3t} \end{pmatrix}.$$

Part (b)

$$X' = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} X$$

Solution

Step 1: Find the eigenvalues. We define matrix A as follows:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

The eigenvalues are the roots of the characteristic equation as follows:

$$\det(A - \lambda I) = 0.$$

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 2 \\ 3 & 6 - \lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = (1 - \lambda)(6 - \lambda) - (2)(3) = (1 - \lambda)(6 - \lambda) - 6.$$

$$\det(A - \lambda I) = 6 - 7\lambda + \lambda^2 - 6 = \lambda^2 - 7\lambda.$$

$$\lambda(\lambda - 7) = 0.$$

We get the following roots:

$$\lambda = 0, \quad \lambda = 7.$$

Step 2: Find the eigenvectors. For $\lambda = 0$:

$$A - 0I = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix}.$$

By solving $(A - 0I)V = 0$, we find the following:

$$V_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

For $\lambda = 7$:

$$A - 7I = \begin{pmatrix} -6 & 2 \\ 3 & -1 \end{pmatrix}.$$

By solving $(A - 7I)V = 0$, we find the following:

$$V_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

Step 3: Write the general solution. The general solution is a linear combination of the eigenvectors scaled by their respective exponential terms as follows:

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2.$$

We sub in the eigenvalues and eigenvectors as follows:

$$X(t) = c_1 e^{0t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

$$X(t) = \begin{pmatrix} -2c_1 + c_2 e^{7t} \\ c_1 + 3c_2 e^{7t} \end{pmatrix}.$$

Part (c)

$$X' = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} X$$

Solution

Step 1: Find the eigenvalues. We define matrix A as follows:

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues are the roots of the characteristic equation as follows:

$$\det(A - \lambda I) = 0.$$

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 2 \\ 1 & -\lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = (1 - \lambda)(-\lambda) - (2)(1) = -\lambda + \lambda^2 - 2.$$

$$\det(A - \lambda I) = \lambda^2 - \lambda - 2.$$

$$(\lambda - 2)(\lambda + 1) = 0.$$

We get the following roots:

$$\lambda = 2, \quad \lambda = -1.$$

Step 2: Find the eigenvectors. For $\lambda = 2$:

$$A - 2I = \begin{pmatrix} -1 & 2 \\ 1 & -2 \end{pmatrix}.$$

By solving $(A - 2I)V = 0$, we find the following:

$$V_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

For $\lambda = -1$:

$$A + I = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}.$$

By solving $(A + I)V = 0$, we find the following:

$$V_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Step 3: Write the general solution. The general solution is a linear combination of the eigenvectors scaled by their respective exponential terms as follows:

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2.$$

We sub in the eigenvalues and eigenvectors as follows:

$$X(t) = c_1 e^{2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

$$X(t) = \begin{pmatrix} 2c_1 e^{2t} - c_2 e^{-t} \\ c_1 e^{2t} + c_2 e^{-t} \end{pmatrix}.$$

Part (d)

$$X' = \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix} X$$

Solution

Step 1: Find the eigenvalues. We define matrix A as follows:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -3 \end{pmatrix}.$$

The eigenvalues are the roots of the characteristic equation as follows:

$$\det(A - \lambda I) = 0.$$

$$A - \lambda I = \begin{pmatrix} 1 - \lambda & 2 \\ 3 & -3 - \lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = (1 - \lambda)(-3 - \lambda) - (2)(3) = (-3 - \lambda)(1 - \lambda) - 6.$$

$$\det(A - \lambda I) = -3 + 3\lambda - \lambda - \lambda^2 - 6 = -\lambda^2 + 2\lambda - 9.$$

$$\lambda^2 - 2\lambda + 9 = 0.$$

Step 2: Solve for eigenvalues. We use the quadratic formula to solve for the eigenvalues:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad a = 1,\ b = -2,\ c = 9.$$

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(9)}}{2(1)} = \frac{2 \pm \sqrt{4-36}}{2}.$$
$$\lambda = \frac{2 \pm \sqrt{-32}}{2} = \frac{2 \pm 4i\sqrt{2}}{2}.$$
$$\lambda = 1 \pm 2i\sqrt{2}.$$

Step 3: Find the eigenvectors. For $\lambda = 1 + 2i\sqrt{2}$, we sub λ into $A - \lambda I$ as follows:

$$A - \lambda I = \begin{pmatrix} 1-(1+2i\sqrt{2}) & 2 \\ 3 & -3-(1+2i\sqrt{2}) \end{pmatrix}.$$
$$A - \lambda I = \begin{pmatrix} -2i\sqrt{2} & 2 \\ 3 & -4-2i\sqrt{2} \end{pmatrix}.$$

We solve $(A - \lambda I)V = 0$ to find the following:

$$V_1 = \begin{pmatrix} 1 \\ \frac{2i\sqrt{2}+4}{3} \end{pmatrix}.$$

For $\lambda = 1 - 2i\sqrt{2}$:

$$V_2 = \begin{pmatrix} 1 \\ \frac{-2i\sqrt{2}+4}{3} \end{pmatrix}.$$

Step 4: Write the general solution. The general solution is a linear combination of the eigenvectors scaled by their respective exponential terms as follows:

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2.$$

We sub in the eigenvalues and eigenvectors as follows to get our final solution:

$$X(t) = c_1 e^{(1+2i\sqrt{2})t} \begin{pmatrix} 1 \\ \frac{2i\sqrt{2}+4}{3} \end{pmatrix} + c_2 e^{(1-2i\sqrt{2})t} \begin{pmatrix} 1 \\ \frac{-2i\sqrt{2}+4}{3} \end{pmatrix}.$$

Exercise 2.3. In Figure 2.2, you see four direction fields. Match each of these direction fields with one of the systems in the previous exercise.

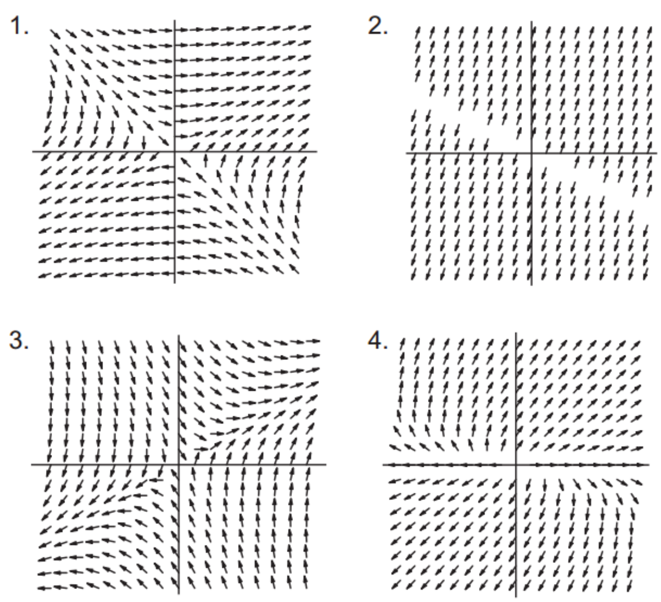

Figure 2.2 **Match these direction fields with the systems in Exercise 2.**

Solution
Direction Field 1: This clearly shows trajectories w/ saddle-point behavior (inward along one direction and outward along another). This corresponds to **Part (c)**, where $\lambda = 2$ and $\lambda = -1$.

Direction Field 2: Straight-line trajectories radiating outward in one direction (dominant eigenvalue) and being degenerate in the other is what we see here. This corresponds to **Part (b)**, where $\lambda = 0$ and $\lambda = 7$.

Direction Field 3: This looks like another saddle point, but the axis of inward and outward flow differs from Field 1. It corresponds to **Part (a)**, where $\lambda = 1$ and $\lambda = 3$, b/c while it's not strictly a saddle (both eigenvalues are positive), the dominance of one direction gives it a visual similarity to one.

Direction Field 4: We see a clear spiraling behavior outward, which is characteristic of a spiral source. This corresponds to **Part (d)**, where $\lambda = 1 \pm 2i\sqrt{2}$.

(5 pts) For a 2×2 matrix M, recall that:

$$e^M = \sum_{n=0}^{\infty} \frac{M^n}{n!}.$$

Compute e^{At} where $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Solution
To compute e^{At}, we use the matrix exponential definition:

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!}.$$

Step 1: Compute powers of A. The first few powers of A are as follows:

$$A^1 = A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

$$A^2 = A \cdot A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix}.$$

$$A^3 = A \cdot A^2 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{pmatrix} = \begin{pmatrix} \lambda^3 & 3\lambda^2 \\ 0 & \lambda^3 \end{pmatrix}.$$

By induction, for any $n \geq 1$, we write:

$$A^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}.$$

Step 2: Expand e^{At} using the series. We sub A^n into the series definition of e^{At} as follows:

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

From the expression for A^n, we get the following:

$$\frac{(At)^n}{n!} = \begin{pmatrix} \frac{t^n \lambda^n}{n!} & \frac{t^n n \lambda^{n-1}}{n!} \\ 0 & \frac{t^n \lambda^n}{n!} \end{pmatrix}.$$

We sum over all n:

$$e^{At} = \sum_{n=0}^{\infty} \begin{pmatrix} \frac{t^n \lambda^n}{n!} & \frac{t^n n \lambda^{n-1}}{n!} \\ 0 & \frac{t^n \lambda^n}{n!} \end{pmatrix}.$$

Step 3: Simplify the entries. The $(1,1)$ and $(2,2)$ entries are as follows:

$$\sum_{n=0}^{\infty} \frac{t^n \lambda^n}{n!} = e^{\lambda t}.$$

The $(1,2)$ entry is as follows:

$$\sum_{n=1}^{\infty} \frac{t^n n \lambda^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{t^n \lambda^{n-1}}{(n-1)!}.$$

We let $m = n - 1$:

$$\sum_{n=1}^{\infty} \frac{t^n \lambda^{n-1}}{(n-1)!} = te^{\lambda t}.$$

Thus, the matrix exponential is the following:

$$e^{At} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

(10 pts) Consider the first-order autonomous differential equation $x' = f(x)$, where $f \in C(\mathbb{R})$. Assume that an initial value problem has a unique solution $x(t)$. Furthermore, suppose that $\lim_{t\to\infty} x(t) = x_1$ for some $x_1 \in \mathbb{R}$. Show that $\lim_{t\to\infty} x'(t) = 0$ and that $f(x_1) = 0$.

Note: The fact that $x(t) \to x_1$ does not, by itself, imply $x'(t) \to 0$. For example, consider $x(t) = \frac{\sin(t^2)}{t}$.

Solution
We let $x'(t) = f(x(t))$ represent the rate of change of $x(t)$ w.r.t. t.

Step 1: Behavior of $f(x(t))$ as $t \to \infty$. B/c $x(t) \to x_1$ as $t \to \infty$, the continuity of $f(x)$ ($f \in C(\mathbb{R})$) implies the following:

$$f(x(t)) \to f(x_1) \quad \text{as } t \to \infty.$$

So, we conclude:

$$x'(t) = f(x(t)) \to f(x_1) \quad \text{as } t \to \infty.$$

Step 2: Show $f(x_1) = 0$. If $f(x_1) \neq 0$, then for sufficiently large t, $x'(t) = f(x(t))$ would remain bounded away from zero. This implies $x(t)$ continues to change at a nonzero rate for large t, which contradicts the assumption that $x(t) \to x_1$, so we know $f(x_1) = 0$ by contradiction.

Step 3: Show $x'(t) \to 0$ as $t \to \infty$. From Step 2, we know $f(x_1) = 0$. And b/c $x(t) \to x_1$ and $f(x)$ is continuous, the value of $f(x(t)) = x'(t)$ must approach $f(x_1) = 0$. We thereby know the following:

$$\lim_{t\to\infty} x'(t) = 0.$$

Conclusion: We have shown the following:

$$\lim_{t\to\infty} x'(t) = 0 \quad \text{and} \quad f(x_1) = 0.$$