Quantum Mechanics
Study Questions for the Fall 2020 Department Exam
June 16, 2020

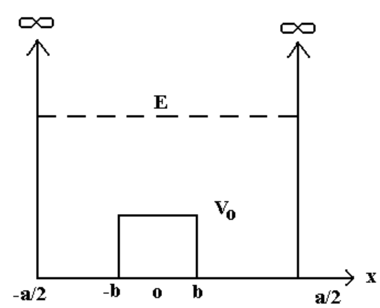
1. A particle is in the ground state of an infinite square well. That is

$$V(x) = \begin{cases} 0 & 0 \le x \le L \\ \infty & \text{otherwise} \end{cases}$$

At t=0 the wall at x=L is suddenly moved to x=2L--this happens very fast, approximately instantaneously.

a. Calculate the probability that long after t=0 the system is in the ground state of the new potential.
b. How fast must the change take place for this "instantaneous" assumption to be good?

2. a. Consider a particle of mass, *m*, and energy E, in a one-dimensional potential as shown. Derive expressions from which the eigenvalue E can be determined for the even-parity and the odd-parity solutions.
b. Describe briefly how one would obtain the numerical value of E for the even-parity solutions.



3. a. Consider a helium-like ion. The two electrons are in the 2p and 3p states. The energy levels will have definite total angular momentum $(\vec{J} = \vec{L} + \vec{s})$. What J-values can occur and how many energy levels of each J can there be?

b. How do your answers change if the electrons are both in 3p states?

4. In precise treatments of hydrogen-like atoms, the finite size of the nucleus has to be included in the calculation of the electronic energy levels. Assume the nucleus to be a sphere with a uniform charge distribution.

a. Write down the correction to the potential experienced by the electron.
b. Calculate the first order correction to the energy.

5. Two indistinguishable spin-$\frac{1}{2}$ particles of mass m move in a 1D simple harmonic oscillator with frequency ω_0. Using center-of-mass and relative coordinates, answer the following:

   a. What is the spectrum of singlet states? of triplet states? What is the ground state of the system?

   b. Now let's assume that the particles have charge q --- but do not interact with each other --- and are exposed to the time-dependent perturbation

$$H' = -q\boldsymbol{E}_0 x_1 e^{-\left(\frac{t}{\tau}\right)^2} \cos \omega t - q\boldsymbol{E}_0 x_2 e^{-\left(\frac{t}{\tau}\right)^2} \cos \omega t.$$

   Take q, $\boldsymbol{E}_0$ and τ to be real, positive constants. If the system is initially in the ground state from (a), $|\, i> = |n_1^{(i)}, n_2^{(i)}>$, calculate the probability (in first order perturbation theory)

$$P_{if}(t) = \frac{1}{\hbar^2} \,|\int_{-\infty}^{\infty} dt < f|\; H'(t)\,|i> e^{\frac{i(\varepsilon_f - \varepsilon_t)t}{\hbar}}\,|^2$$

   for finding the system in a state $|\,f> = |n_1^{(f)}, n_2^{(f)}>$ at time t.

   Potentially useful information:

$$\text{i)}\ \int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^2} = (-1)^n \frac{d^n}{d\beta^n} \int_{-\infty}^{\infty} dx\, e^{-\beta x^2} \Big|_{\beta=1} = (-1)^n \frac{d^n}{d\beta^n} \sqrt{\frac{\pi}{\beta}}\, \Big|_{\beta=1}.$$

$$\text{ii)} < k|x|n > \; = \sqrt{\frac{\hbar}{m\omega}} \left[\sqrt{\frac{n}{2}}\, \delta_{k,n-1} + \sqrt{\frac{n+1}{2}}\, \delta_{k,n+1}\right]$$

$$< n|x|0 > \; = \sqrt{\frac{\hbar}{2m\omega}}\, \delta_{n,1}$$

6. Consider an electron bound in a three dimensional simple harmonic oscillator potential in the n=1 state. Recall that the e^- has spin 1/2 and that the n=1 level of the oscillator has $\ell = 1$. Thus there are six states $\{|n=1, \ell = 1, m_\ell, m_s>\}$ with m_ℓ =+1, 0, -1 and $m_s = \pm$ ½.

   a. Using these states as a basis find the six states with definite j and m_j, where
$$\vec{J} = \vec{L} + \vec{s}.$$
   b. What are the energy levels in the presence of a spin-orbit interaction $A\vec{L}\cdot\vec{s}$ where A is the strength of the interaction. What are their degeneracies?
   c. If now a weak magnetic field B_o is applied along the z-axis how will it affect the degeneracy of the J=1/2 level?

7. Consider the vector space of angular momentum eigenstates for two spin – ½ particles. One possible basis is given by the four product states

$$|\pm, \pm\rangle = |\pm\rangle_{(1)} \otimes |\pm\rangle_{(2)}$$

where $|\pm\rangle_{(1)}$ and $|\pm\rangle_{(2)}$ are eigenstates of

$\vec{S}^2_{(1)}$, $S_{z\ (1)}$ and $\vec{S}^2_{(2)}$, $S_{z\ (2)}$, respectively, such that (in atomic units, $\hbar = 1$):

$$\vec{S}^2_{(i)} \quad |\pm\rangle_{(i)} = \frac{3}{4} \quad |\pm\rangle_{(i)}$$

$$S_{z\ (i)} \quad |\pm\rangle_{(i)} = \pm \frac{1}{2} \quad |\pm\rangle_{(i)} \quad , \quad i = 1\ ,\ 2.$$

a. In this vector space, find simultaneous eigenstates of the total spin operators,

$$\vec{S}^2 = (\vec{S}_{(1)} + \vec{S}_{(2)})^2$$

$$S_z = S_{z\ (1)} + S_{z\ (2)} \quad ,$$

and the corresponding eigenvalues.

b. Assume that $|x{:}\pm\rangle_{(i)}$ are simultaneous eigenstates of $\vec{S}^2_{(i)}$ and $S_{x\ (i)}$, $i = 1,\ 2$.

Expand $|x{:}\pm\rangle_{(i)}$ in terms of the basis states $|\pm\rangle_{(i)}$.

c. In the $|x{:}\pm\rangle_{(i)}$ basis, calculate the eigenstate to the total spin operator $\vec{S}^2$ with eigenvalue (= total spin) 0.

8. The result of a measurement shows that the electron spin is along the $+x$ direction at t=0. For $t > 0$, the electron enters in a uniform magnetic field that is parallel to the $+z$ direction. Calculate the quantum mechanical probability as a function of time for finding the electron in each of the following states ($\hbar = 1$):

a. $S_x = 1/2$
b. $S_x = -1/2$
c. $S_z = 1/2$
d. $S_z = -1/2$.

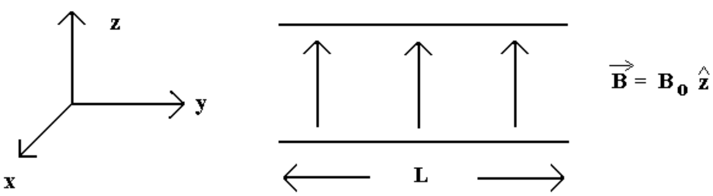
The Pauli matrices are given by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

9. In an experiment you make repeated measurements of the energy of a system. You find that for one-fourth of the measurements you obtain the value E_1; for one-third of the measurements you obtain the energy value E_2; and for the remaining measurements you obtain the value E_3.

   a. Using *only this information* write as complete a time dependent wave function for this state as you can. For parts of the wave function which you cannot specify use general functions and indicate why you cannot specify it (them).
   b. Is this wave function uniquely determined by the experimental evidence? If so, how? If not, are there further experiments which you could do to make a unique determination?

10. A polarized proton travels at velocity v toward a magnetic field as shown.



   a. If the polarization before entering the field is $\langle S_x \rangle = \hbar/2, \quad \langle S_y \rangle = \langle S_z \rangle = 0$, find the polarization when the proton leaves the field.
   b. If one wants to use this device to change the polarization to $\langle S_x \rangle = -\hbar/2, \quad \langle S_y \rangle = \langle S_z \rangle = 0$, how long should the B-field region be?

11. Consider a physical system that has two stationary states, $|1\rangle$ and $|2\rangle$, with energies $E_1 < E_2$, and Hamiltonian H_o:

$$H_o |i\rangle = E_i |i\rangle, \quad i = 1, 2$$

This system is modified by a time-independent, real-valued perturbation V, and its eigenenergies and stationary states are given by

$$(H_o + V) |\psi_\pm\rangle = E_\pm |\psi_\pm\rangle.$$

   a. Determine the energy eigenvalues E_+ and E_- of the perturbed system in terms of E_1, E_2, and $V_{12} := \langle 1 | V | 2 \rangle$. Assume $\langle 1 | V | 1 \rangle = \langle 2 | V | 2 \rangle = 0$.

   b. See what happens to E_+ and E_- as $V \to 0$; does the result make sense? Explain!

   c. Find the normalized eigenfunctions $|\psi_\pm\rangle$ in terms of E_1, E_2, and V_{12}.

   d. The perturbation was switched on at time $t = 0$ when the system was in state $|1\rangle$. Calculate the probability for finding the system in state $|2\rangle$ at time $t > 0$. Assume $|V_{12}| << |E_1 - E_2|$.

12. Consider a hydrogen atom in its ground state. An electric field, $E(t)$, is applied in the z-direction.

$$E(t) = \begin{cases} 0 & t < 0 \\ E_o e^{-t/\tau} & t > 0. \end{cases}$$

a. Apply first-order time-dependent perturbation theory to calculate an expression for the probability that the hydrogen atom is in an excited state.
b. What is the probability if the excited state is (i) $2s$ and (ii) $2p$? Explain briefly the justification for your answer.
c. How do your answers to (b) behave in the limits $\tau = 0$ and $\tau = \infty$. Explain physically whether this behavior is sensible.

13. A spin 1/2 particle is placed in a uniform magnetic field H_0 directed along $\hat{z}$. A small rf field $H_1 \cos\omega t$ is applied along $\hat{x}$. Calculate the time dependent probability for the spin to flip from up to down along $\hat{z}$. The Pauli spin matrices are

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

14. The scattering amplitude for a proton on a target atom is $f(\theta)$. That is, if the target is at $\vec{r}_T = 0$ then far from the target the wavefunction is

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + f(\theta)\frac{e^{ikr}}{r}.$$

a. Suppose that there are $N(N >> 1)$ target atoms located at the points $\vec{r}_1 = \vec{A}$, $\vec{r}_2 = 2\vec{A}$, $\vec{r}_3 = 3\vec{A}$ and so on.

   What is $\psi(\vec{r})$ for positions $\vec{r}$ far from the line? That is, find $\psi(\vec{r})$ for $|\vec{r}| >> N|\vec{A}|$. It will have the form

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} + F(\theta)\frac{e^{ikr}}{r}.$$

   Find $F(\theta)$ and $d\sigma/d\Omega = |F(\theta)|^2$. Is $d\sigma/d\Omega \sim N$?

b. Suppose now that the $N(N >> 1)$ target atoms are located at <u>random</u> positions, but all inside a small finite ball of radius R. Calculate $F(\theta)$ and $d\sigma/d\Omega$. Is $d\sigma/d\Omega \sim N$?

15. A free particle of mass m is represented by a Gaussian wave packet, the momentum space representation of which at time $t = 0$ is given by

$$\Psi(p) = (2\pi\sigma)^{-1/4} \exp\left[-\frac{(p-p_o)^2}{4\sigma^2} - i\frac{px_o}{\hbar} \right].$$

a. Find the wave function in coordinate space, $\Psi(x,t)$, for $t \geq 0$.

b. Calculate the expectation values $<x>$ and $<x^2>$.

c. Determine the uncertainty $\Delta x = \left(< \left(x - < x > \right)^2 > \right)^{1/2}$ and comment on its time evolution.

Hint:

$$\int dx\, x^{2n} e^{-x^2} = (-1)^n \frac{d^n}{d\beta^n} \int dx\, e^{-\beta x^2} \Big|_{\beta=1} = (-1)^n \frac{d^n}{d\beta^n} \sqrt{\frac{\pi}{\beta}} \Big|_{\beta=1}$$

16. Let $|\alpha\rangle$ be a stationary state of the harmonic oscillator and that

$$\alpha_- |\alpha\rangle = \alpha |\alpha\rangle$$

where α is a complex number. Calculate $<x>$, $<p>$, $<x^2>$ and $<p^2>$ where the expectation values are defined with respect to the state $|\alpha\rangle$.

Also show that $\sigma_x \sigma_p = \hbar/2$. Note that $|\alpha\rangle$ is called a coherent state. It is a general minimum uncertainty wave packet.

Hint: Remember that α_+ is the Hermitian conjugate of α_-. Do not assume that α is real.

$$x = \sqrt{\frac{\hbar}{2m\omega}} \left(a_+ + a_- \right) \quad p = i\sqrt{\frac{\hbar m\omega}{2}} \left(a_+ - a_- \right)$$

17. Consider a hydrogen atom in an electric field for which $H' = eEz = eEr\cos\theta$. The energies of $2s$ and $2p$ are degenerate without the presence of the electric field.

a. What matrix elements of H' are nonzero between the degenerate states? Give your reasons for your answer.

b. Consider now only those states where H' removes the degeneracy. Find the energies of these states.

$$\psi_{2s} = \frac{1}{2\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \left(1 - \frac{r}{2a_0} \right) e^{-r/2a_0}$$

$$\psi_{2p0} = \frac{1}{4\sqrt{2\pi}} \frac{1}{a_0^{3/2}} \left(\frac{r}{a_0} \right) e^{-r/2a_0} \cos\theta$$

Note: $$\int_o^\infty e^{-cr} r^n\, dr = \frac{n!}{c^{n+1}}$$

18. a. If $\mathsf{H}=\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, what are the eigenvalues of this Hamiltonian?

b. Prove that [H, exp{H}]=0, that is, the two operators commute.

c. What are the eigenvalues of exp{H}?

d. If a small perturbation $\mathsf{H}'=\lambda\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, where λ is a small positive number, is applied to this system, calculate the change in the eigenenergy of the ground state, to first order in λ.

e. Carry out the change in the ground state energy to second order in λ.

Hint: $$\int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^2} = (-1)^n \frac{d^n}{d\beta^n} \int_{-\infty}^{\infty} dx\, e^{-\beta x^2} \Big|_{\beta=1} = (-1)^n \frac{d^n}{d\beta^n} \sqrt{\frac{\pi}{\beta}} \Big|_{\beta=1}.$$

19. Consider a particle in an infinitely deep three-dimensional box with the length on each side equal to *a*.

a. Write down the Hamiltonian describing such a particle.

b. Find the first 5 lowest eigenenergies.

c. Find the degeneracy of each energy eigenstate above.

d. If there are 10 noninteracting, indistinguishable fermions in such a box, what is the ground state energy of the whole system? Consider separately the cases for spin 1/2 and spin 3/2 particles.

20. A particle of mass *m* experiences the potential

$$V(x) = -V_0 \ell \left[\delta(x-a) + \delta(x+a) \right]$$

with ℓ and *a* positive, real constants with units of length and V_0 a positive, real constant with units of energy.

a. What symmetries does $V(x)$ possess?

b. Can you construct simultaneous eigenstates of energy and the symmetries you identified in (a)? Why or why not?

c. For a single δ-function potential, you know there is only a single bound state, so it should be plausible that for $V(x)$ above there are at most two bound states. Find the transcendental equations that determine the energies of these states. Sketch the corresponding wave functions and point out the symmetries from (a) and (b).

d. What are the energies of each state in the limits $a \to 0$ and $a \to \infty$? Explain why your results make sense physically.

21. The potential for the isotropic harmonic oscillator is given by

$$V(r) = (1/2)\, m\omega^2 r^2.$$

a. Show that the eigenfunctions can be expressed in the form of $R_{nl}(r)\, Y_{lm}$.

b. The problem is separable in Cartesian coordinates, show that the eigenenergies can be expressed as

$$E_n = (n + 3/2)\hbar\omega.$$

c. Find the degeneracy d(n) of the lowest four energies E_n. You obtain the degeneracy from the solutions in Cartesian coordinates. Identify the value(s) of orbital angular momentum (or momenta if more than one) for each eigenenergy.

22. A tritium atom, ^3H, is a hydrogen-like atom whose nucleus (one proton and two neutrons) is unstable. It can decay by beta emission to $^3He^+$. This decay process occurs essentially instantaneously on atomic time scales. As a result, the electron suddenly sees a nucleus with twice the charge. Assuming that the tritium atom is initially in its ground state, find the probability immediately after the decay that the $^3He^+$ can be found:

a. In the $1s$ state.
b. In the $2s$ state.
c. In the $2p_0$ state.
d. In any excited or ionized state (i.e. any state besides the ground state).

Potentially useful information:

$$\int_0^\infty dx\, x^n e^{-\alpha x} = \frac{n!}{\alpha^{n+1}}$$

$$\psi_{1s}(\mathbf{r}) = \sqrt{\frac{Z^3}{\pi a_0^3}}\, e^{-Zr/a_0}$$

$$\psi_{2s}(\mathbf{r}) = \sqrt{\frac{Z^3}{8\pi a_0^3}} \left(1 - \frac{Zr}{2a_0}\right) e^{-Zr/(2a_0)}$$

$$\psi_{2p_0}(\mathbf{r}) = \sqrt{\frac{Z^3}{8\pi a_0^3}}\, \frac{Zr}{2a_0}\, e^{-Zr/(2a_0)} \cos\theta.$$

23. a) Consider the trial wave functions $\phi(x) = A e^{-\lambda^2 x^2}$ where λ is an adjustable parameter. Determine the normalization constant A(λ).

b) Use the trial functions of part a) to obtain the best approximation to the energy of the ground state of the Hamiltonian

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + x^4$$

describing the motion of a particle of mass m in 1D.

Useful integral:

$$\int_0^\infty dx\; x^{2n}\, e^{-x^2/a^2} = \sqrt{\pi}\,\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$$

24. A simple harmonic oscillator with charge q and frequency ω_0 is subjected to the perturbation

$$V'(x,t) = -qx\mathcal{E}_0 e^{-\left(\frac{t}{\tau}\right)^2} \cos\omega t.$$

The constants $\mathcal{E}_0$, τ, and ω are positive, real constants. Using first order perturbation theory, answer the following:

a. Assuming the oscillator is initially $(t \to -\infty)$ in its ground state, what states are populated after the perturbation and with what probabilities?
b. Specialize your probabilities from (a) to the cases: (i) $\omega = \omega_0,\ \ \tau = 20/\omega_0$, (ii) $\omega = 2\omega_0,\ \ \tau = 20/\omega_0$, and (iii) $\omega = 2\omega_0,\ \ \tau = 20/\omega_0$. Compare these probabilities and explain the relative magnitudes physically.
c. Assume now that the initial state is instead the first excited state. What states are populated after the perturbation and with what probabilities?

Potentially useful information:

$$\int_{-\infty}^{\infty} dx\, x^{2n} e^{-x^2} = (-1)^n \frac{d^n}{d\beta^n}\int_{-\infty}^{\infty} dx\; e^{-\beta x^2}\Big|_{\beta=1} = (-1)^n \frac{d^n}{d\beta^n}\sqrt{\frac{\pi}{\beta}}\;\Big|_{\beta=1}.$$

25. If the spin state of an electron is prepared such that it has spin $\hbar/2$ along the x-direction, what is the probability of finding this spin to have eigenvalue $\hbar/2$ if the spin is measured along the y-direction? Show the steps of your calculation. The Pauli spin matrices are:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

26. A pair of electrons occupies two 3d levels (they are in $3d^2$ configuration). Here we assume that LS coupling is valid.

   a. What needs to be true for LS coupling to be the correct procedure?
   b. Using LS-coupling, what terms written as ${}^{2S+1}L_J$ are possible for the two electrons, if the Pauli exclusion principle is not applied?
   c. What terms are possible after the Pauli exclusion principle is applied?

27. Let

$$|\psi\rangle = \sqrt{1/3}\ Y_{10}\ \alpha + \sqrt{2/3}\ Y_{11}\beta.$$

   a. If you measure S_z, what is the probability that you will get $\hbar/2$?
   b. If indeed you do get $\hbar/2$ in (a), what is the new state (or wavefunction) after the measurement?
   c. If you measure L_z, after (b), what is the probability that you will get $-\hbar$?
   (Note: α for spin up and β for spin down, and Y's are the spherical harmonics.)

28. Consider a simple harmonic oscillator in one dimension,

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

At time t=0 the wave function is given by

$$\psi(x,0) = \sqrt{\frac{1}{3}}\ \psi_0(x) + \sqrt{\frac{2}{3}}\ \psi_2(x)$$

where $\psi_n(x)$ denotes the exact eigenstate of the harmonic oscillator with eigen energy

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right).$$

   a. Give ψ (x,t) for $t \geq 0$ in terms of $\psi_0(x)$ and $\psi_2(x)$.
   b. What is the parity of ψ (x,t)? Does it change with time?
   d. Find the expectation value for the energy of this state. Does it change with time?
   e. Find $\langle \psi | x | \psi \rangle$ for all times.

29. Consider two spin 1/2 particles described by the Hamiltonian

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(x_1) + V(x_2),$$

where $V(x) = \infty$ for x<0 and for x>a; V(x) = 0 for 0<x<a. Assume that the electrons are in the **opposite spin state**, that is, the total S=0.

a. Write down the spin wave function(s). Use the standard notations: α for spin up, and β for spin down.
b. Find the energy and wavefunction of the ground state of this Hamiltonian.
c. Find the energy and wavefunction of the lowest state for S=1.
d. Find the energy and wavefunction of the second S=0 state. Show that the energy is the same as in (c).
e. If the two particles have a small interaction $W(x_1,x_2)= b\, x_1 x_2$ where b is small and positive, show that the degeneracy in (c) and (d) is removed. Which one has the lower energy?

30. a. A two-dimensional harmonic oscillator has the potential $V(x,y)=\frac{1}{2}m\omega^2(x^2+4y^2)$.

Calculate the energies of the first **three** lowest states, and identify the **degrees of degeneracy** for each energy.

b. If there is an additional small coupling term $W(x,y)=ax^2 y$ present, where a is a small constant. Calculate the first-order correction to each of the three levels.