II.) (a) $\quad YZ|0\rangle = Y|0\rangle = +i|1\rangle$ $\quad (\Delta)$

$YZ|1\rangle = -Y|1\rangle = i|0\rangle$

So $\quad YZ(\alpha|0\rangle + \beta|1\rangle) = +i\alpha|1\rangle + i\beta|0\rangle$

$= i(\alpha|1\rangle + \beta|0\rangle)$

$= iX(\alpha|0\rangle + \beta|1\rangle)$

Hence $YZ = iX$ on all states $\alpha|0\rangle + \beta|1\rangle$

$\Rightarrow YZ = iX$.

Or could note that $YZ = iX$ on

on a basis of $\mathbb{C}^2$ from (Δ)

$\Rightarrow YZ = iX$.

(b) $\quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$; $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\sigma_y \sigma_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\sigma_x$

$\boxed{2}$ (a) $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$\sigma_y^{\dagger} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y$$

$\Rightarrow$ Y self-adjoint

Also $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ Y unitary

(b) $Y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$

(c) $Y|+\rangle = \left(i|1\rangle\langle 0| - i|0\rangle\langle 1|\right)\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)$

$$= \frac{i|1\rangle - i|0\rangle}{\sqrt{2}} = -i\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)$$

$$= -i|-\rangle$$

Similarly $Y|-\rangle = \frac{i|1\rangle + i|0\rangle}{\sqrt{2}} = i|+\rangle$

$$Y\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{i|1\rangle - i|0\rangle}{\sqrt{2}}$$

$$Y\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) = \frac{i|1\rangle + i|0\rangle}{\sqrt{2}}$$

✓

$\boxed{3}$ (a) $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

(b) $A = a|0\rangle\langle 0| + b|1\rangle\langle 0| + c|0\rangle\langle 1| + d|1\rangle\langle 1|$

(c) $M\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a\alpha + c\beta \\ b\alpha + d\beta \end{pmatrix}$

(d) $A|\psi\rangle = \alpha A|0\rangle + \beta A|1\rangle$

$= \alpha\left(a|0\rangle + b|1\rangle\right) + \beta\left(c|0\rangle + d|1\rangle\right)$

$= (\alpha a + \beta c)|0\rangle + (\alpha b + \beta d)|1\rangle$

$\boxed{4}$ $\langle\psi_0|\psi_0\rangle = \frac{1}{3}\left(\langle 0| + \langle 1| + \langle 2|\right)\left(|0\rangle + |1\rangle + |2\rangle\right)$

$= \frac{1}{3}\cdot 3 = 1$

$\langle\psi_1|\psi_1\rangle = \frac{1}{3}\left(\langle 0| + \omega^2\langle 1| + \omega\langle 2|\right)\left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle\right)$

$= 1$

$\langle\psi_2|\psi_2\rangle = \frac{1}{3}\left(\langle 0| + \omega\langle 1| + \omega^2\langle 2|\right)\left(|0\rangle + \omega^2|1\rangle + \omega|2\rangle\right)$

$= 1$

(using $\omega^* = \omega^2$ and $\omega^3 = 1$)

$\langle\psi_0|\psi_1\rangle = \frac{1}{3}\left(\langle 0| + \langle 1| + \langle 2|\right)\left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle\right)$

$= \frac{1}{3}\left(1 + \omega + \omega^2\right) = 0$

$\langle\psi_0|\psi_2\rangle = \frac{1}{3}\left(\langle 0| + \langle 1| + \langle 2|\right)\left(|0\rangle + \omega^2|1\rangle + \omega|2\rangle\right)$

$= \frac{1}{3}\left(1 + \omega^2 + \omega\right) = 0$

$\langle\psi_1|\psi_2\rangle = \frac{1}{3}\left(\langle 0| + \omega^2\langle 1| + \omega\langle 2|\right)\left(|0\rangle + \omega^2|1\rangle + \omega|2\rangle\right)$

$= \frac{1}{3}\left(1 + \omega + \omega^2\right) = 0$ (using $\omega^4 = \omega$)

$$M_{ij} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}$$

M is unitary since it takes an ON basis to an ON basis.

$\boxed{5}$ (a) matrix of W is $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

So matrix of adjoint is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ $W^{+}|0\rangle = |0\rangle$; $W^{+}|1\rangle = |0\rangle + |1\rangle$

By inspection $W \neq W^{+}$

Also WW^{+} has matrix $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

so $WW^{+} \neq I$ $\Rightarrow$ not unitary

(b) T has matrix $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

so $T|0\rangle = |0\rangle + |1\rangle$

$T|1\rangle = |0\rangle + 2|1\rangle$

Looking at matrix, can see $T = T^{+}$

T does not take ON basis to ON basis

$\Rightarrow$ not unitary

(or could calculate TT^{+})