

Presha Sheet 3

1. Bell measurements

Alice has 2 qubits in $(\alpha|0\rangle + \beta|1\rangle)$ such that the overall HS is

$$|\Psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$$

She measures some non-degenerate operator - we will call B - whose eigenstates are the Bell states:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Let's first expand the tensor product.

$$|\Psi\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

Then, to find each possible outcome, we project onto the Bell states.

$$\underline{\text{Outcome 1}} \rightarrow \langle b_1 | = \frac{\langle 00 | + \langle 11 |}{\sqrt{2}}$$

$$\langle b_1 | \Psi \rangle = \frac{1}{\sqrt{2}} (\alpha^2 + \beta^2)$$

$$\text{Hence } p(b_1) = \frac{1}{2} |\alpha^2 + \beta^2|^2$$

state after measurement is $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

$$\underline{\text{Outcome 2}} \rightarrow \langle b_2 | = \frac{\langle 00 | - \langle 11 |}{\sqrt{2}}$$

$$\langle b_2 | \Psi \rangle = \frac{1}{\sqrt{2}} (\alpha^2 - \beta^2)$$

$$\therefore P(b_2) = \frac{1}{2} |\alpha^2 - \beta^2|^2$$

state after measurement is $\frac{|100\rangle - |111\rangle}{\sqrt{2}}$

Outcome 3 $\rightarrow \langle b_3 | = \frac{\langle 011 + \langle 101 |}{\sqrt{2}}$

$$\langle b_3 | \psi \rangle = \frac{1}{\sqrt{2}} (\alpha\beta + \alpha\beta) = \sqrt{2} \alpha\beta$$

$$P(b_3) = 2 |\alpha\beta|^2$$

state after measurement is $\frac{|01\rangle + |10\rangle}{\sqrt{2}}$

Outcome 4 $\rightarrow \langle b_4 | = \frac{\langle 011 - \langle 101 |}{\sqrt{2}}$

$$\langle b_4 | \psi \rangle = \frac{1}{\sqrt{2}} (\alpha\beta - \alpha\beta) = 0$$

$$\therefore P(b_4) = 0$$

state cannot occur - no state after measurement

2. Local operations /

$$\rightarrow \text{Show } (A \otimes B)^+ = A^+ \otimes B^+$$

$$\langle v_1 v_2 | (A \otimes B)^+ | w_1 w_2 \rangle = \langle w_1 w_2 | (A \otimes B) | v_1 v_2 \rangle^*$$

$$= \langle w_1 w_2 | (A |v_1\rangle^* \otimes B |v_2\rangle^*)$$

$$= \langle w_1 | A | v_1 \rangle^* + \langle w_2 | B | v_2 \rangle^*$$

$$= \langle v_1 | A^+ | w_1 \rangle + \langle v_2 | B^+ | w_2 \rangle$$

$$= \langle v_1 v_2 | A^+ \otimes B^+ | w_1 w_2 \rangle$$

$$\therefore (A \otimes B)^T = A^T \otimes B^T \text{ as required}$$

→ Show tensor product of two Hermitian operators is itself Hermitian.

Let A, B be Hermitian operators satisfying

$$A = A^T, B = B^T$$

Their tensor product is $A \otimes B$. Then

$$A \otimes B = A^T \otimes B^T = (A \otimes B)^T \leftarrow \text{from above.}$$

Hence $A \otimes B$ is Hermitian.

→ Show that the tensor product of two unitary operators is also unitary.

Let U and V be two unitary operators satisfying

$$UU^T = U^T U = VV^T = V^T V = I$$

Then $U \otimes V$ is their tensor product. Is this unitary?

$$(U \otimes V)(U \otimes V)^T = (U \otimes V)(U^T \otimes V^T) \text{ from above}$$

Using $(A \otimes B)(C \otimes D) = AC \otimes BD$, we find

$$(U \otimes V)(U \otimes V)^T = UU^T \otimes VV^T = I \otimes I$$

As well,

$$(U \otimes V)^T(U \otimes V) = U^T U \otimes V^T V = I \otimes I$$

which is unitary over the whole HS, so $U \otimes V$

is unitary.

3. Local operations 2

a. $X \otimes X$. Note $X = |0\rangle\langle 1| + |1\rangle\langle 0| \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{aligned} X \otimes X &= (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|) \\ &= |00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00| \end{aligned}$$

In matrix form, this will be

$$X \otimes X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

b. $X \otimes I = (|0\rangle\langle 1| + |1\rangle\langle 0|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$
 $= |00\rangle\langle 10| + |01\rangle\langle 11| + |10\rangle\langle 00| + |11\rangle\langle 01|$

In matrix form

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

c. $I \otimes X = (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|)$
 $= |00\rangle\langle 01| + |01\rangle\langle 00| + |10\rangle\langle 11| + |11\rangle\langle 10|$

In matrix form

$$I \otimes x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{aligned} d. \quad I \otimes I &= (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) \\ &= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| \end{aligned}$$

In matrix form

$$I \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

→ Consider $(x \otimes x)^2$ using $(A \otimes B)(C \otimes D) = AC \otimes BD$

$$x = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$(x \otimes x)^2 = (x \otimes x)(x \otimes x) = xx \otimes xx$$

$$\begin{aligned} xx &= (|0\rangle\langle 1| + |1\rangle\langle 0|)(|0\rangle\langle 1| + |1\rangle\langle 0|) \\ &= |1\rangle\langle 1| + |0\rangle\langle 0| = I \end{aligned}$$

$$\therefore (x \otimes x)^2 = I \otimes I$$

Using matrix representation.

$$(x \otimes x)^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = I \otimes I \quad (\text{part d})$$

4. Local operations 4

Let operator A have matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (\text{CNOT})$$

We want to show this is non-local, i.e. A cannot be written as a tensor product of two other operators — $A \neq U \otimes V$.

Consider $|+\rangle|0\rangle$ where $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ — call this state $|4\rangle$ with

$$|4\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

A is a CNOT operator, hence

$$A|4\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

here "on" is set to 1.

i.e. qubit 2 only swaps if qubit 1 is on.

For now, let $A = U \otimes V$. Now consider the effect of applying $U \otimes V$ to $|4\rangle$.

$$U \otimes V |4\rangle = \frac{U|0\rangle \otimes V|0\rangle + U|1\rangle \otimes V|1\rangle}{\sqrt{2}}$$

To provide the same operation as A, we require

$$U|0\rangle = |0\rangle, U|1\rangle = |1\rangle$$

$$V|0\rangle = |0\rangle, V|1\rangle = |1\rangle$$

But here we see a contradiction! V must do some

definite operation, but here it must do two different actions depending on the value of qubit 1.

This is because the states are entangled - the operation of γ depends on the value of the first qubit.

Hence A cannot be written in the form $U \otimes V$, therefore it is non-local, and by extension, the states are entangled.

5. Measurements on entangled particles

Alice and Bob $|Alice\rangle |Bob\rangle$ have particles in

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}} (|y_+\rangle|0\rangle + |y_-\rangle|1\rangle)$$

where $|y_{\pm}\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm i|1\rangle)$. $|y_{\pm}\rangle$ are eigenstates of γ operator.

Alice measures γ and finds outcome +1.

As $|y_{\pm}\rangle$ eigenstates of γ , then γ can be written as

$$\gamma = \sum_{n=+,-}^1 d_n |y_n\rangle \langle y_n| = |y_+\rangle \langle y_+| - |y_-\rangle \langle y_-|$$

Now measure. Formally we measure $\gamma \otimes I$:

$$\gamma \otimes I = |y_+\rangle \langle y_+| \otimes I - |y_-\rangle \langle y_-| \otimes I = P_+ - P_-$$

Project $|\Psi_2\rangle$ onto P_+ ($d = +1$):

$$P_+ |\Psi_2\rangle = \left(|y_+\rangle \langle y_+| \otimes I \right) \cdot \frac{1}{\sqrt{2}} (|y_+\rangle|0\rangle + |y_-\rangle|1\rangle)$$

$$= \frac{1}{\sqrt{2}} (|y_+\rangle|0\rangle)$$

Then renormalised, the state of the particles becomes simply

$$|\eta_+\rangle = |y_+\rangle |\phi\rangle$$

Now what if Bob measures the state. What is the probability he measures -1 ?

Bob measures so we use $I \otimes \gamma$:

$$\begin{aligned} I \otimes \gamma &= I \otimes (|y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|) \\ &= I \otimes |y_+\rangle\langle y_+| - I \otimes |y_-\rangle\langle y_-| = Q_+ - Q_- \end{aligned}$$

Project $|\eta_+\rangle$ onto Q_- :

$$\begin{aligned} Q_- |\eta_+\rangle &= (I \otimes |y_-\rangle\langle y_-|)(|y_+\rangle |\phi\rangle) \\ &= \frac{1}{\sqrt{2}} (I \otimes |y_-\rangle (\langle 0| + i\langle 1|))(|y_+\rangle |\phi\rangle) \\ &= \frac{1}{\sqrt{2}} |y_-\rangle |y_+\rangle \end{aligned}$$

Therefore probability is $\frac{1}{2}$.

The act of Alice's measurement "destroyed" the entangled state.

6. Local equivalence.

$|\Psi_1\rangle$ and $|\Psi_2\rangle$ "equivalent under local unitary transforms" if

$$|\Psi_1\rangle = U_1 \otimes U_2 |\Psi\rangle$$

for some unitary $U_1 \neq U_2$.

$$a. \quad |b_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \quad |b_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|b_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \quad |b_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

$$\underline{|b_1\rangle \rightarrow |b_2\rangle}$$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = U_1 \otimes U_2 \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}} \right)$$

$$\text{Set } U_1 \otimes U_2 = \mathcal{Z} \otimes I$$

$$= (|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$\text{Then } U_1 \otimes U_2 \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}} \right) = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = |b_1\rangle$$

Hence we can say that $|b_1\rangle$ and $|b_2\rangle$ are E.U.L.
u.T. since $\mathcal{Z} \otimes I$ are unitary.

$$\underline{|b_2\rangle \rightarrow |b_3\rangle}$$

$$\frac{|00\rangle - |11\rangle}{\sqrt{2}} = U_1 \otimes U_2 \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

$$\text{Here, } U_1 \otimes U_2 = I \otimes \underbrace{(|0\rangle\langle 1| - |1\rangle\langle 0|)}_{U_2}$$

Is U_2 unitary?

$$U_2^\dagger = |1\rangle\langle 0| - |0\rangle\langle 1|$$

$$-U_2 U_2 = (|0\rangle\langle 1| - |1\rangle\langle 0|)(|1\rangle\langle 0| - |0\rangle\langle 1|)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = I \therefore U_2 \text{ unitary!}$$

Therefore $|b_2\rangle$ and $|b_3\rangle$ E.U.L.U.T..

$$\underline{|b_3\rangle \rightarrow |b_4\rangle}$$

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = U_1 \otimes U_2 \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}} \right)$$

$$\begin{aligned} \text{Then } U_1 \otimes U_2 &= I \otimes (|1\rangle\langle 1| - |0\rangle\langle 0|) \\ &= -I \otimes \tau \quad \therefore \text{unitary} \end{aligned}$$

Therefore $|b_3\rangle$ and $|b_4\rangle$ E.U.L.U.T..

If $|b_1\rangle$ and $|b_2\rangle$ equivalent, and $|b_2\rangle$ and $|b_3\rangle$ equivalent,
are $|b_1\rangle$ and $|b_3\rangle$?

$$|b_1\rangle = U_1 \otimes U_2 |b_2\rangle = U_1 \otimes U_2 (U_3 \otimes U_4 |b_3\rangle)$$

Two successive unitary operations ($U_3 \otimes U_4$ then $U_1 \otimes U_2$) is unitary, hence the above statement is true, and by extension,
all Bell states are equivalent under local unitary transformation.

$$b. \frac{|00\rangle + |11\rangle}{\sqrt{2}} = \underbrace{\left(|0\rangle\langle 0| - e^{\frac{-\pi i}{4}} |1\rangle\langle 1| \right)}_A \otimes I \frac{|00\rangle - e^{\frac{\pi i}{4}} |11\rangle}{\sqrt{2}}$$

$$\begin{aligned} \text{A unitary? } AA^\dagger &= \left(|0\rangle\langle 0| - e^{\frac{-\pi i}{4}} |1\rangle\langle 1| \right) \left(|0\rangle\langle 0| - e^{\frac{+\pi i}{4}} |1\rangle\langle 1| \right) \\ &= |0\rangle\langle 0| + |1\rangle\langle 1| \quad \checkmark_{\text{unitary}}^A \end{aligned}$$

States 1 & 2 EULUT

$$\frac{|00\rangle - e^{\frac{\pi i}{4}} |11\rangle}{\sqrt{2}} = \underbrace{\left(|0\rangle\langle 0| - e^{\frac{\pi i}{4}} |1\rangle\langle 1| \right)}_{\text{unitary - see above}} \otimes \underbrace{\left(|1\rangle\langle 0| + |0\rangle\langle 1| \right)}_{X - \text{unitary}} \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

States 2 and 3 EULUT.

$$\frac{|01\rangle + |11\rangle}{\sqrt{2}} = \underbrace{I \otimes}_{\text{unitary}} \underbrace{\left(|1\rangle\langle 1| - |0\rangle\langle 0| \right)}_{-\tau - \text{unitary}} \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

States 3 and 4 EULUT

By extension then, all states are EULUT.

→ Orthonormal basis? Must have

$$\langle \text{state } i | \text{state } j \rangle = S_{ij}$$

We can see that $\langle \text{state } 1 | \text{state } 2 \rangle$ is

$$\frac{\langle 001 + \langle 111 | 100 \rangle - e^{\frac{\pi i}{4}} | 111 \rangle}{\sqrt{2}} = \frac{1 - e^{\frac{\pi i}{4}}}{2} \neq 0$$

and hence these states do not form an orthonormal basis.

c. Lets denote our state as $|a_i\rangle \otimes |b_i\rangle$, such that our normalised state is

$$|\Psi_i\rangle = |a_i\rangle \otimes |b_i\rangle$$

Now consider some separate $|\Psi_j\rangle = |a_j\rangle \otimes |b_j\rangle$. We want to show that for any i, j ,

$$|\Psi_j\rangle = U_1 \otimes U_2 (|a_i\rangle \otimes |b_i\rangle)$$

$$\therefore |a_j\rangle \otimes |b_j\rangle = U_1 |a_i\rangle \otimes U_2 |b_i\rangle$$

This gives us two equations

$$|a_j\rangle = U_1 |a_i\rangle, \quad |b_j\rangle = U_2 |b_i\rangle$$

On \mathbb{C}^2 , there are two basis vectors, $|0\rangle$ and $|1\rangle$. Consider

$$|a_i\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha |0\rangle + \beta |1\rangle$$

Then $|a_j\rangle = \begin{bmatrix} \gamma \\ \delta \end{bmatrix} = \gamma |0\rangle + \delta |1\rangle$. Can we find a U_1 such that it is unitary and

$$U_1 |a_i\rangle = |a_j\rangle$$

$$U_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$$

$$U_1 = \begin{bmatrix} \frac{\gamma}{2\alpha} & \frac{\gamma}{2\beta} \\ \frac{\delta}{2\alpha} & \frac{\delta}{2\beta} \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\gamma}{2\alpha} & \frac{\gamma}{2\beta} \\ \frac{\delta}{2\alpha} & \frac{\delta}{2\beta} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \gamma + \gamma \\ \delta + \delta \end{bmatrix} = \begin{bmatrix} \gamma \\ \delta \end{bmatrix}$$

Is U_1 unitary?

$$U_1^\dagger = \begin{bmatrix} \frac{\gamma}{2\alpha} & \left(\frac{\delta}{2\alpha}\right)^* \\ \left(\frac{\gamma}{2\beta}\right)^* & \frac{\delta}{2\beta} \end{bmatrix}$$

$$\begin{aligned} \therefore U_1 U_1^\dagger &= \begin{bmatrix} \frac{\gamma}{2\alpha} & \frac{\gamma}{2\beta} \\ \frac{\delta}{2\alpha} & \frac{\delta}{2\beta} \end{bmatrix} \begin{bmatrix} \frac{\gamma}{2\alpha} & \frac{\delta}{2\alpha}^* \\ \frac{\gamma}{2\beta}^* & \frac{\delta}{2\beta} \end{bmatrix} \\ &= \begin{bmatrix} \left(\frac{\gamma}{2\alpha}\right)^2 + \left(\frac{\gamma}{2\beta}\right)\left(\frac{\gamma}{2\beta}\right)^* & \frac{\gamma}{2\alpha}\left(\frac{\gamma}{2\alpha}\right)^* + \frac{\gamma}{2\beta}\frac{\delta}{2\beta} \\ \frac{\delta}{2\alpha}\frac{\gamma}{2\alpha} + \frac{\delta}{2\beta}\left(\frac{\gamma}{2\beta}\right)^* & \frac{\delta}{2\alpha}\left(\frac{\delta}{2\alpha}\right)^* + \left(\frac{\delta}{2\beta}\right)^2 \end{bmatrix} \\ &= \begin{bmatrix} \left|\frac{\gamma}{2\alpha}\right|^2 + \left|\frac{\gamma}{2\beta}\right|^2 & \left(\frac{\gamma}{2\alpha}\right)^* + \frac{\gamma\delta}{(2\alpha)(2\beta)} \\ \frac{\delta\gamma}{(2\alpha)^2} + \frac{\delta\gamma^*}{(2\beta)^2} & \left|\frac{\delta}{2\alpha}\right|^2 + \left|\frac{\delta}{2\beta}\right|^2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} (0,0) \rightarrow &= |\gamma|^2 \left(\frac{1}{4\alpha^2} + \frac{1}{4\beta^2} \right) = \frac{|\gamma|^2}{4} \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} \right) \xrightarrow{\text{for normalisation}} = 1 \\ &= \frac{|\gamma|^2}{4} \left(\frac{\beta^2 + \alpha^2}{\alpha^2 \beta^2} \right) \\ &= \frac{|\gamma|^2}{4\alpha^2 \beta^2} \end{aligned}$$

Rewrite U_1 as

$$u_1 = \frac{1}{(2\alpha\beta)^2} \begin{bmatrix} \gamma^2 + |\gamma|^2 & \gamma\gamma^* + \gamma\delta \\ \gamma\delta + \gamma\gamma^* & \delta^2 + |\delta|^2 \end{bmatrix}$$

$$= \frac{1}{(2\alpha\beta)^2} \begin{bmatrix} \gamma^2 + |\gamma|^2 & 2\delta \operatorname{Re}\gamma \\ 2\delta \operatorname{Re}\gamma & \delta^2 + |\delta|^2 \end{bmatrix}$$

Where we have the conditions $\gamma^2 + \delta^2 = 1$ and $\alpha^2 + \beta^2 = 1$

Think I must have done this wrong.
Not sure where to go here.