Problem Sheet 1

1// $YZ|0\rangle = Y(|0\rangle) = i|1\rangle$

$YZ|1\rangle = -Y|1\rangle = i|0\rangle$

$\therefore \quad YZ|\psi\rangle = YZ(\alpha|0\rangle + \beta|1\rangle) = \alpha YZ|0\rangle + \beta YZ|1\rangle$

$$= i\alpha|1\rangle + i\beta|0\rangle$$

$iX|\psi\rangle = i\alpha X|0\rangle + i\beta X|1\rangle = i\alpha|1\rangle + i\beta|0\rangle = YZ|\psi\rangle$ //

This implies that $iX = YZ$, as the action of both operators on <u>any</u> state is identical.

2// a. $Y|0\rangle = i|1\rangle \qquad Y|1\rangle = -i|0\rangle$

$Y|0\rangle = \underbrace{M_{00}}_{=0}|0\rangle + \underbrace{M_{01}}_{=i}|1\rangle$ where $\underset{\sim}{M}$ is matrix of Y

$Y|1\rangle = \underbrace{M_{10}}_{=-i}|0\rangle + \underbrace{M_{11}}_{=0}|1\rangle$

$$\therefore \quad \underset{\sim}{M} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

b. $Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$

c. $|+\rangle$ and $|-\rangle$ are eigenstates of X corresponding to $\lambda_\pm = \pm 1$.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow X|\pm\rangle = \pm|\pm\rangle$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \pm\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{where } |\pm\rangle = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} y \\ x \end{pmatrix} = \pm\begin{pmatrix} x \\ y \end{pmatrix}$$

set $x = 1 \rightarrow \begin{pmatrix} y \\ 1 \end{pmatrix} = \pm\begin{pmatrix} 1 \\ y \end{pmatrix} \rightarrow$ $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$$\therefore\ Y|+\rangle = \left(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\right)\cdot\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \underline{\frac{i}{\sqrt{2}}\left(-|0\rangle + |1\rangle\right)}$$

$$Y|-\rangle = \left(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\right)\cdot\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \underline{\frac{i}{\sqrt{2}}\left(+|0\rangle + |1\rangle\right)}$$

<u>or!</u>

$$Y|+\rangle = Y\frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = \frac{1}{\sqrt{2}}\left(Y|0\rangle + Y|1\rangle\right) = \frac{1}{\sqrt{2}}\left(i|1\rangle - i|0\rangle\right) \ //$$ same result!

$$Y|-\rangle = Y\frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = \frac{1}{\sqrt{2}}\left(Y|0\rangle - Y|1\rangle\right) = \frac{1}{\sqrt{2}}\left(i|1\rangle + i|0\rangle\right) //$$ same result!

3. $A|0\rangle = a|0\rangle + b|1\rangle,\quad A|1\rangle = c|0\rangle + d|1\rangle$

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$$

a. $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ w.r.t. standard basis $|0\rangle, |1\rangle$ //

b. $A = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$ //

c. $M\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha a + \beta b \\ \alpha c + \beta d \end{pmatrix}$ //

d. $A|\phi\rangle = \left(a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|\right)\left(\alpha|0\rangle + \beta|1\rangle\right)$

$$= \left(\alpha a + \beta b\right)|0\rangle + \left(\alpha c + \beta d\right)|1\rangle \ //$$ as in (c)

4. $|\psi_0\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + |1\rangle + |2\rangle\right)$

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle\right)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}\left(|0\rangle + \omega^2|1\rangle + \omega|2\rangle\right)$$

$$\omega = e^{\frac{2\pi i}{3}}$$

To be orthonormal, we must ensure:

$$\langle\psi_0|\psi_0\rangle = \langle\psi_1|\psi_1\rangle = \langle\psi_2|\psi_2\rangle = 1$$

$$\langle\psi_0|\psi_1\rangle = \langle\psi_0|\psi_2\rangle = \langle\psi_1|\psi_2\rangle = 0$$

$$\langle\psi_0|\psi_0\rangle = \frac{1}{3}\left(\langle 0| + \langle 1| + \langle 2|\right)\left(|0\rangle + |1\rangle + |2\rangle\right)$$

$$= \frac{1}{3}(1+1+1) = 1 \checkmark$$

$$\langle\psi_1|\psi_1\rangle = \frac{1}{3}\left(\langle 0| + \omega^*\langle 1| + (\omega^2)^*\langle 2|\right)\left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle\right)$$

$$= \frac{1}{3}\left(1 + \omega^*\omega + (\omega^2)^*\omega^2\right)$$

$$= \frac{1}{3}(1+1+1) = 1 \checkmark$$

$$\langle\psi_2|\psi_2\rangle = \frac{1}{3}\left(\langle 0| + e^{\frac{-4\pi i}{3}}\langle 1| + e^{-\frac{2\pi i}{3}}\langle 2|\right)\left(|0\rangle + e^{\frac{4\pi i}{3}}|1\rangle + e^{\frac{2\pi i}{3}}|2\rangle\right)$$

$$= \frac{1}{3}(1+1+1) = 1 \checkmark$$

$$\langle\psi_0|\psi_1\rangle = \frac{1}{3}\left(\langle 0| + \langle 1| + \langle 2|\right)\left(|0\rangle + \omega|1\rangle + \omega^2|2\rangle\right)$$

$$= \frac{1}{3}\left(1 + \omega + \omega^2\right) = \frac{1}{3}\Big(1 + \underbrace{e^{\frac{2\pi i}{3}}}_{-0.5+\frac{\sqrt{3}}{2}i} + \underbrace{e^{\frac{4\pi i}{3}}}_{-0.5-\frac{\sqrt{3}}{2}i}\Big) = 0 \checkmark$$

$$\langle\psi_0|\psi_2\rangle = \frac{1}{3}\left(\langle 0| + \langle 1| + \langle 2|\right)\left(|0\rangle + \omega^2|1\rangle + \omega|2\rangle\right)$$

$$= \frac{1}{3}\left(1 + \omega^2 + \omega\right) = 0 \checkmark \quad \text{(as before)}$$

$$\langle\psi_1|\psi_2\rangle = \frac{1}{3}\left(\langle 0| + e^{-\frac{2\pi i}{3}}\langle 1| + e^{-\frac{4\pi i}{3}}\langle 2|\right)\left(|0\rangle + e^{\frac{4\pi i}{3}}|1\rangle + e^{\frac{2\pi i}{3}}|2\rangle\right)$$

$$= \frac{1}{3}\left(1 + e^{\frac{2\pi i}{3}} + e^{-\frac{2\pi i}{3}}\right) = \frac{1}{3}\left(1 + \omega + \omega^2\right) = 0 \checkmark$$

Hence these vectors form an orthonormal basis in $\mathbb{C}^3$.
We needn't check $\langle\psi_1|\psi_0\rangle$ etc since $\langle\psi_0|\psi_1\rangle^* = \langle\psi_1|\psi_0\rangle = 0$ etc.

$M|a\rangle = |\psi_a\rangle$

$$\therefore \quad M = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \quad \text{by inspection}$$

M is symmetric. $(\omega^2)^* = \left(e^{\frac{4\pi i}{3}}\right)^* = e^{-\frac{4\pi i}{3}} = e^{\frac{2\pi i}{3}} = \omega \qquad M^\dagger = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^* & \omega \\ 1 & \omega & \omega^* \end{pmatrix}$ not Hermitian

5. $W|0\rangle = |0\rangle + |1\rangle, \quad W|1\rangle = |1\rangle$

a. $W = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ w.r.t. standard basis

$W^\dagger = (W^T)^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$W^\dagger|0\rangle = |0\rangle, \quad W^\dagger|1\rangle = |0\rangle + |1\rangle$

As $W \neq W^\dagger$ $\therefore$ W not self-adjoint //

$WW^\dagger = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \overset{2\times2}{=} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq I \quad \therefore$ W not unitary //

b. $T = WW^\dagger = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$T|0\rangle = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \overset{2\times1}{=} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2|0\rangle + |1\rangle$

$T|1\rangle = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \overset{2\times1}{=} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |0\rangle + |1\rangle$

$T^\dagger = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = T \quad \therefore$ T self adjoint //

$TT^\dagger = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \overset{2\times2}{=} \begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix} \neq I \quad \therefore$ T not unitary //