$\boxed{1}$ (a) $\quad Y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$

matrix wrt standard basis $\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Eigenvectors

$\lambda = +1 \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \longleftrightarrow \frac{|0\rangle + i|1\rangle}{\sqrt{2}} =: |y+\rangle$

$\lambda = -1 \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \longleftrightarrow \frac{|0\rangle - i|1\rangle}{\sqrt{2}} =: |y-\rangle$

$\Rightarrow \quad Y = |y+\rangle\langle y+| - |y-\rangle\langle y-|$

(b) $\quad \{|0\rangle, |1\rangle\}$ is a basis, so arbitrary state is

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

So $\quad \big(|0\rangle\langle 0| + |1\rangle\langle 1|\big)\big(\alpha|0\rangle + \beta|1\rangle\big)$

$= \alpha|0\rangle + \beta|1\rangle$

$\Rightarrow \quad |0\rangle\langle 0| + |1\rangle\langle 1|$ is identity

Also $\{|y+\rangle, |y-\rangle\}$ is a basis

so can write $|\psi\rangle = a|y+\rangle + b|y-\rangle$

and similar calculation to above shows

$|y+\rangle\langle y+| + |y-\rangle\langle y-|$ is identity.

(c) $U|0\rangle = |y+\rangle$

$U|1\rangle = |y-\rangle$

$$UU^{\dagger} = \Big(|y+\rangle\langle 0| + |y-\rangle\langle 1|\Big)\Big(|0\rangle\langle y+| + |1\rangle\langle y-|\Big)$$

$$= |y+\rangle\langle y+| + |y-\rangle\langle y-| = I$$

$\boxed{2}$ (a) Matrices of V and W wrt standard basis are

$$\begin{pmatrix} 0 & 0 & -i \\ 1 & 0 & 0 \\ 0 & -i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & \omega \end{pmatrix}$$

can easily check from the matrices

$$VV^\dagger = WW^\dagger = I$$

and $V^\dagger \neq V$ $\quad W^\dagger \neq W$

(b) Need $i^n = 1 \Rightarrow n = 4m$

$\omega^n = 1 \Rightarrow n = 3k$

(m, k are integers) $\Rightarrow$ smallest integer

$n = 12$

$\boxed{3}$

(a) Need $\langle v_1 | v_2 \rangle = 0$

$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

(b) • $\langle 0 | v_1^{\perp} \rangle = 0 \Rightarrow |v_1^{\perp}\rangle = e^{i\phi} |1\rangle$

• Let $|v_2^{\perp}\rangle = \alpha|0\rangle + \beta|1\rangle$. $\quad$ I

Then $\langle v_2 | v_2^{\perp} \rangle = 0 \Rightarrow \alpha\cos\theta + \beta\sin\theta = 0$

Also $\langle v_1 | v_2 \rangle = \cos\theta$ $\quad$ II

$\Rightarrow$ we want $\langle v_1^{\perp} | v_2^{\perp} \rangle = e^{-i\phi}\beta = \cos\theta$

So I and II $\Rightarrow \alpha = -e^{i\phi}\sin\theta$

and so could take $\phi = 0$ and

$|v_1^{\perp}\rangle = |1\rangle$

$|v_2^{\perp}\rangle = -\sin\theta\,|0\rangle + \cos\theta\,|1\rangle$

(e) If Bob gets λ_1, state could have been $|v_1\rangle$ or $|v_2\rangle$.

If Bob gets λ_2, state cannot be $|v_1\rangle$ $\Rightarrow$ state was $|v_2\rangle$.

$\boxed{4}$ (a) If it were a product state, then

$$\frac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}} = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$$

for some a, b, c, d.

Considering $|00\rangle$ term: $\quad 0 = ac$

$\Rightarrow$ $a = 0$ or $c = 0$, but neither consistent

with the equation.

(b) $$\frac{|0\rangle|1\rangle + |0\rangle|0\rangle}{\sqrt{2}} = |0\rangle \left(\frac{|1\rangle + |0\rangle}{\sqrt{2}} \right)$$

$$\frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle}{2} = \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right)$$

[5] NB two things to prove

a) $|\psi\rangle$ of product form $\Rightarrow$ $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$

b) $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$ $\Rightarrow$ $|\psi\rangle$ of product form.

a) : $|\psi\rangle = (a|0\rangle + b|1\rangle)(c|0\rangle + d|1\rangle)$

$\Rightarrow$ $\alpha_{00}\alpha_{11} = ac \cdot bd$

$\alpha_{01}\alpha_{10} = ad \cdot bc$

b) If $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$

Assume $\alpha_{11} \neq 0$ (can easily check that

$\alpha_{11} = 0$ case is product state since

then either α_{01} or $\alpha_{10} = 0$) ,

then

$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$$= \frac{\alpha_{01}\alpha_{10}}{\alpha_{11}} |00\rangle + \alpha_{01} |01\rangle + \alpha_{10} |10\rangle + \alpha_{11} |11\rangle$$

$$= \frac{1}{\alpha_{11}} \left(\alpha_{01} |0\rangle + \alpha_{11} |1\rangle \right) \left(\alpha_{10} |0\rangle + \alpha_{11} |1\rangle \right) \quad \square$$

$\boxed{6}$ (a) For any $|v\rangle$, $|w\rangle$, $(A+B)^\dagger$ is

defined by

$$\langle v|(A+B)^\dagger|w\rangle = \langle w|A+B|v\rangle^*$$

$$= \langle w|A|v\rangle^* + \langle w|B|v\rangle^*$$

$$= \langle v|A^\dagger|w\rangle + \langle v|B^\dagger|w\rangle$$

$$= \langle v|(A^\dagger + B^\dagger)|w\rangle \qquad \text{II}$$

(b) Similarly

$$\langle v|(AB)^\dagger|w\rangle = \langle w|AB|v\rangle^*$$

$$= \Big(\big(\langle w|A\big)\big(B|v\rangle\big)\Big)^*$$

$$= \langle v|B^\dagger A^\dagger|w\rangle$$

where we have used the fact that the bra vector of $B|v\rangle = \langle v|B^\dagger$.

To prove $(A^\dagger)^n = (A^n)^\dagger$ $(*)$

need to use mathematical induction:

(i) $n=1$

- obvious $A^\dagger$

(ii) If $(A^+)^k = (A^k)^+$

then $(A^+)^{k+1} = (A^+)^k A^+ = (A^k)^+ A^+$

$= (A A^k)^+$ from previous part

$= (A^{k+1})^+$

so if $(*)$ true for $n=k$, then it is true for $n=k+1$, and since it is true for $n=1$, it is true for all $n \geq 1$ $\square$.

(c) $u^2 (u^2)^+ = u^2 (u^+)^2$

$= u u u^+ u^+ = u u^+ = I$

$(u^2 v^3)^+ = (v^3)^+ (u^2)^+ = (v^+)^3 (u^+)^2$

$\Rightarrow u^2 v^3 (u^2 v^3)^+ = u^2 v^3 (v^+)^3 (u^+)^2$

$= u^2 (u^+)^2 = I$ $\square$