$\boxed{1}$ $|\psi_1\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$, $|\psi_2\rangle = \dfrac{|00\rangle - |11\rangle}{\sqrt{2}}$

$|\psi_3\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$, $|\psi_4\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

$$p(\lambda_1) = \left\| \, |\psi_1\rangle\langle\psi_1| \, (\alpha|0\rangle + \beta|1\rangle)(\alpha|0\rangle + \beta|1\rangle) \right\|^2$$

$$= \left\| \, |\psi_1\rangle \left(\frac{\alpha^2 + \beta^2}{\sqrt{2}} \right) \right\|^2 = \frac{1}{2} \left| \alpha^2 + \beta^2 \right|^2$$

$$p(\lambda_2) = \frac{1}{2} \left| \alpha^2 - \beta^2 \right|^2$$

$$p(\lambda_3) = \frac{1}{2} \left| 2\alpha\beta \right|^2 = 2 |\alpha|^2 |\beta|^2$$

$$p(\lambda_4) = 0$$

State after measurement is $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle$ respectively. Outcome 4 never occurs.

$\boxed{2}$ • Consider $A \otimes B$ acting on product states :

$$\langle c| \langle d| \, (A \otimes B)^\dagger \, |a\rangle |b\rangle$$

$$= \langle a| \langle b| \, A \otimes B \, |c\rangle |d\rangle^*$$

$$= \langle a|A|c\rangle^* \, \langle b|B|d\rangle^*$$

$$= \langle c|A^\dagger|a\rangle \, \langle d|B^\dagger|b\rangle$$

$$= \langle c| \langle d| \, A^\dagger \otimes B^\dagger \, |a\rangle |b\rangle .$$

Since $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ on product states, it is true on all states by linearity.

• Let U and V be unitary. Then

$$(U \otimes V)^\dagger = U^\dagger \otimes V^\dagger$$

so $(U \otimes V)(U \otimes V)^\dagger = (U \otimes V)(U^\dagger \otimes V^\dagger)$

$$= UU^\dagger \otimes VV^\dagger$$

$$= I \otimes I = I$$

$\boxed{3}$ If $A \leftrightarrow \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}$ $B \leftrightarrow \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}$

$$A \otimes B \leftrightarrow \begin{pmatrix} a_{00}(B) & a_{01}(B) \\ a_{10}(B) & a_{11}(B) \end{pmatrix}$$

(B) means matrix associated to B.

$$\text{so } A \otimes B \leftrightarrow \begin{pmatrix} a_{00}b_{00} & a_{00}b_{01} & a_{01}b_{00} & a_{01}b_{01} \\ a_{00}b_{10} & a_{00}b_{11} & a_{01}b_{10} & a_{01}b_{11} \\ a_{10}b_{00} & a_{10}b_{01} & a_{11}b_{00} & a_{11}b_{01} \\ a_{10}b_{10} & a_{10}b_{11} & a_{11}b_{10} & a_{11}b_{11} \end{pmatrix}$$

$$\text{Thus } X \otimes X \leftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$X \otimes I \leftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$I \otimes X \leftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$I \otimes I \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$(X \otimes X)(X \otimes X) = X^2 \otimes X^2 = I \otimes I = I$$

$\boxed{4}$ Let us call the operator C.

Then $C|00\rangle = |00\rangle$

$C|10\rangle = |11\rangle$

so $C\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$.

RHS is entangled, but $|+\rangle|0\rangle$ is not,

so C cannot be of the form $A \otimes B$.

$\boxed{5}$ Alice measures Y

$$Y \otimes I = \underbrace{(|y+\rangle\langle y+| \otimes I)}_{P_+} - \underbrace{(|y-\rangle\langle y-| \otimes I)}_{P_-}$$

Note $P_+ |\Phi_2\rangle = \dfrac{|y+\rangle|0\rangle}{\sqrt{2}}$

so $|\eta+\rangle = |y+\rangle|0\rangle$ (up to overall phase).

Bob now measure Y

$$I \otimes Y = \underbrace{(I \otimes |y+\rangle\langle y+|)}_{Q_+} - \underbrace{(I \otimes |y-\rangle\langle y-|)}_{Q_-}$$

so prob $\lambda = -1$ is

$$\| Q_- |\eta+\rangle \|^2 = \| Q_- |y+\rangle|0\rangle \|^2$$

$$= \| |y+\rangle|y-\rangle\langle y-|0\rangle \|^2$$

$$= \| |y+\rangle|y-\rangle \tfrac{1}{\sqrt{2}} \|^2 = \tfrac{1}{2}$$

eg

$\boxed{6}$ $|00\rangle - |11\rangle = Z\otimes I\left(|00\rangle + |11\rangle\right)$

$$|01\rangle + |10\rangle = I\otimes X\left(|00\rangle + |11\rangle\right)$$

$$|01\rangle - |10\rangle = I\otimes(-iY)\left(|00\rangle + |11\rangle\right).$$

Also can use fact that unitaries are invertible to get between any pair:

eg

$$|00\rangle - |11\rangle = (Z\otimes I)\left(|00\rangle + |11\rangle\right)$$

$$= (Z\otimes I)\,(I\otimes X)^{-1}\left(|01\rangle + |10\rangle\right)$$

$$= (Z\otimes I)\,(I\otimes X)\,\left(|01\rangle + |10\rangle\right)$$

$$= Z\otimes X\,\left(|01\rangle + |10\rangle\right) \text{ etc.}$$

Alternative: you could prove generally that if $|\psi_1\rangle$ and $|\psi_2\rangle$ are locally equivalent and $|\psi_2\rangle$ and $|\psi_3\rangle$ are locally equivalent then $|\psi_1\rangle$ and $|\psi_3\rangle$ are locally equivalent.

Then local equivalence of all 4 follows from first part.

ie. $|\psi_1\rangle \doteq |\psi_2\rangle$ means $|\psi_2\rangle = U_1 \otimes U_2 |\psi_1\rangle$

$|\psi_2\rangle \doteq |\psi_3\rangle$ means $|\psi_3\rangle = V_1 \otimes V_2 |\psi_2\rangle$

So $|\psi_3\rangle = V_1 \otimes V_2 (U_1 \otimes U_2) |\psi_1\rangle$

$= V_1 U_1 \otimes V_2 U_2 \; |\psi_1\rangle$

But $V_1 U_1$ and $V_2 U_2$ are unitary.

(b) States 1, 3 and 4, locally equivalent from (a).

Also $|00\rangle - e^{i\pi/4}|11\rangle = U \otimes I \left(|00\rangle + |11\rangle \right)$

where $U|0\rangle = |0\rangle$
$U|1\rangle = -e^{i\pi/4}|1\rangle$

and U unitary as can be easily checked (eg by finding matrix w.r.t. the standard basis).

So all 4 states locally equivalent by argument in (a).

(c) Consider two normalised qubit states $|a_1\rangle$ and $|a_2\rangle$, and let $|a_1^\perp\rangle$ and $|a_2^\perp\rangle$ be normalised states satisfying $\langle a_1|a_1^\perp\rangle = 0$ and $\langle a_2|a_2^\perp\rangle = 0$.

Then $U = |a_2\rangle\langle a_1| + |a_2^\perp\rangle\langle a_1^\perp|$ is unitary (why?) and $U|a_1\rangle = |a_2\rangle$.

So consider arbitrary product states

$$|\psi_1\rangle = |a_1\rangle|b_1\rangle$$

$$|\psi_2\rangle = |a_2\rangle|b_2\rangle .$$

Then $|\psi_2\rangle = U\otimes V\,|\psi_1\rangle$

where U is as above and

$$V = |b_2\rangle\langle b_1| + |b_2^\perp\rangle\langle b_1^\perp|$$