$\boxed{1}$ (a) $\quad X = \underset{P_+}{|+\rangle\langle+|} - \underset{P_-}{|-\rangle\langle-|}$

$$P_+ \otimes I \otimes I \, |\tau\rangle = \frac{1}{2}\left(|+\rangle|0\rangle|0\rangle + |+\rangle|1\rangle|1\rangle \right)$$

$$= \frac{1}{\sqrt{2}} |+\rangle \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right)$$

So if Alice get $\lambda = +1$, BC have the required state.

$$P_- \otimes I \otimes I \, |\tau\rangle = \frac{1}{\sqrt{2}} |-\rangle \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}} \right).$$

So if Alice gets $\lambda = +1$, she communicates "0" which means Bob should do I. If Alice gets $\lambda = -1$, she sends "1" which Bob should interpret as "do Z".

(b) $|\psi_1\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$, $|\psi_2\rangle = \dfrac{|00\rangle - |11\rangle}{\sqrt{2}}$

$|\psi_3\rangle = \dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$, $|\psi_4\rangle = \dfrac{|01\rangle - |10\rangle}{\sqrt{2}}$

So $|\psi_1\rangle\langle\psi_1| \otimes I \; |\tau\rangle$

$= \left(|\psi_1\rangle\langle\psi_1| \otimes I\right) \dfrac{|00\rangle|0\rangle + |11\rangle|1\rangle}{\sqrt{2}}$

$= \dfrac{1}{\sqrt{2}} \left(|\psi_1\rangle \langle\psi_1|00\rangle |0\rangle + |\psi_1\rangle \langle\psi_1|11\rangle |1\rangle \right)$

$= \dfrac{1}{\sqrt{2}} |\psi_1\rangle \left(\dfrac{|0\rangle + |1\rangle}{\sqrt{2}} \right) = \dfrac{1}{\sqrt{2}} |\psi_1\rangle |+\rangle$

Probability $= \left(\dfrac{1}{\sqrt{2}}\right)^2 = \dfrac{1}{2}$

and state after measurement $|\psi_1\rangle|+\rangle$.

Similarly $|\psi_2\rangle\langle\psi_2| \otimes I \; |\tau\rangle = \dfrac{1}{\sqrt{2}} |\psi_2\rangle|-\rangle$;

probability $= \dfrac{1}{2}$, state becomes $|\psi_2\rangle|-\rangle$.

And $|\psi_3\rangle\langle\psi_3| \otimes I \; |\tau\rangle = |\psi_4\rangle\langle\psi_4| \otimes I \; |\tau\rangle = 0$

$\boxed{2}$ $|\phi_2\rangle = Z \otimes I \; |\phi_1\rangle$

$|\phi_3\rangle = I \otimes X \; |\phi_1\rangle$

$|\phi_4\rangle = Z \otimes X \quad |\phi_1\rangle$.

And so using arguments in Assignment 3, Qn 6, all states are locally equivalent.

orthogonality easy to check (need to do 6 calculations: $\langle \phi_i | \phi_j \rangle$ for $i < j$) and so is normalisation. Thus we have 4 orthogonal states in $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$

$\Rightarrow$ they form a basis.

(b) $\left(\alpha|0\rangle + \beta|1\rangle\right) \frac{1}{\sqrt{2}}\left(|01\rangle + i|1\rangle|0\rangle\right)$

$$= \frac{1}{2}\left(\frac{|00\rangle + |11\rangle}{\sqrt{2}}\right)\left(\alpha|1\rangle + i\beta|0\rangle\right)$$

$$+ \frac{1}{2}\left(\frac{|00\rangle - |11\rangle}{\sqrt{2}}\right)\left(\alpha|1\rangle - i\beta|0\rangle\right)$$

$$+ \frac{1}{2}\left(\frac{|01\rangle + |10\rangle}{\sqrt{2}}\right)\left(i\alpha|0\rangle + \beta|1\rangle\right)$$

$$+ \frac{1}{2}\left(\frac{|01\rangle - |10\rangle}{\sqrt{2}}\right)\left(i\alpha|0\rangle - \beta|1\rangle\right)$$

So after Alice's measurement, Bob's state becomes

$\lambda_1 \longrightarrow \alpha|1\rangle + i\beta|0\rangle$

$\lambda_2 \longrightarrow \alpha|1\rangle - i\beta|0\rangle$

$\lambda_3 \longrightarrow i\alpha|0\rangle + \beta|1\rangle$

$\lambda_4 \longrightarrow i\alpha|0\rangle - \beta|1\rangle$

So Bob applies I, Z, X, Y respectively,

and state becomes

I : $\alpha|1\rangle + i\beta|0\rangle$

Z : $-\alpha|1\rangle - i\beta|0\rangle \equiv \alpha|1\rangle + i\beta|0\rangle$ up to overall phase

X : $i\alpha|1\rangle + \beta|0\rangle \equiv \alpha|1\rangle - i\beta|0\rangle$ "

Y : $-\alpha|1\rangle + i\beta|0\rangle \equiv \alpha|1\rangle - i\beta|0\rangle$ "

(c) Alice wants to send 01, she does X.

State becomes $X \otimes \mathbb{1} \, |\psi_3\rangle = \dfrac{|11\rangle + i|00\rangle}{\sqrt{2}} =: |\Phi\rangle$

Bob now measures non-degenerate operator

with eigenstates $|\psi_1\rangle, |\psi_2\rangle, |\psi_3\rangle, |\psi_4\rangle$

$$|\psi_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad |\psi_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad |\psi_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

So prob(Bob thinks 00)

$$= \| (|\psi_1\rangle\langle\psi_1|) |\Phi\rangle \|^2$$

$$= | \langle\psi_1|\Phi\rangle |^2$$

$$= \left| \left(\frac{\langle 00| + \langle 11|}{\sqrt{2}} \right) \left(\frac{|11\rangle + i|00\rangle}{\sqrt{2}} \right) \right|^2$$

$$= \left| \frac{1+i}{2} \right|^2 = \frac{1}{4}(1+1) = \frac{1}{2}$$

Similarly Prob (Bob thinks 01) = 0

Prob (Bob thinks 10) = 0

Prob (Bob thinks 11) = $\frac{1}{2}$.

$\boxed{3}$

- measure first qubit. If get 0

state becomes

$$|\psi_1\rangle = \frac{|0\rangle\langle 0| \otimes I \otimes I\, |\psi\rangle}{\| \,|0\rangle\langle 0| \otimes I \otimes I\, |\psi\rangle \|}$$

probability is

$$p_1 = \| \,|0\rangle\langle 0| \otimes I \otimes I\, |\psi\rangle \|^2$$

- Given state is $|\psi_1\rangle$, measure 2nd qubit. State becomes

$$|\psi_2\rangle = \frac{(I \otimes |0\rangle\langle 0| \otimes I)\, |\psi_1\rangle}{\| \, I \otimes |0\rangle\langle 0| \otimes I\, |\psi_1\rangle \|}$$

$$= \frac{(|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I)\, |\psi\rangle}{\| \,|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I\, |\psi\rangle \|}$$

probability of getting 0

$$\| I \otimes |0\rangle\langle 0| \otimes I \, |\psi_1\rangle \|^2$$

$$= \frac{\| (|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I) |\psi\rangle \|^2}{\| (|0\rangle\langle 0| \otimes I \otimes I) |\psi\rangle \|^2} =: p_2$$

- given state is $|\psi_2\rangle$, measure 3rd qubit.

prob of getting 0 is

$$p_3 = \| (I \otimes I \otimes |0\rangle\langle 0|) |\psi_2\rangle \|^2$$

$$= \frac{\| (|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|) |\psi\rangle \|^2}{\| (|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes I) |\psi\rangle \|^2}$$

So total probability of getting 0 ~~for~~ each of the three measurements is

$$p_1 p_2 p_3$$

$$= \| (|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0|) |\psi\rangle \|^2$$

$$= |a_{000}|^2$$

$\boxed{4}$ Note $\left(\langle \underline{x}| \langle \underline{y}|\right)\left(|\underline{z}\rangle|\underline{w}\rangle\right)$

$= \delta_{\underline{x}\underline{z}}\ \delta_{\underline{y}\underline{w}}$

where $\delta_{\underline{x}\underline{z}}$ means $\delta_{x_1 z_1}\delta_{x_2 z_2}$

where $\underline{x} = x_1 x_2$, $\underline{z} = z_1 z_2$ etc.

so $\left(\langle \underline{x}| \langle \underline{y} \oplus f(\underline{x})|\right)\left(|\underline{z}\rangle|\underline{w} \oplus f(\underline{z})\rangle\right)$

$= \delta_{\underline{x}\underline{z}}\ \langle \underline{y} \oplus f(\underline{x}) | \underline{w} \oplus f(\underline{z})\rangle$

$= \delta_{\underline{x}\underline{z}}\ \langle \underline{y} \oplus f(\underline{x}) | \underline{w} \oplus f(\underline{x})\rangle$

$= \delta_{\underline{x}\underline{z}}\ \langle \underline{y} | \underline{w}\rangle$

$= \delta_{\underline{x}\underline{z}}\ \delta_{\underline{y}\underline{w}}$

ie U_f takes an ON basis to an ON basis

(b) $|f\rangle = \frac{1}{2}\Big(|00\rangle|f(00)\rangle + |01\rangle|f(01)\rangle$

$+ |10\rangle|f(10)\rangle + |11\rangle|f(11)\rangle\Big)$

So

$\langle f_1|f_2\rangle = \frac{1}{4}\Big(\langle 00|\langle 01| + \langle 01|\langle 10| + \langle 10|\langle 11| + \langle 11|\langle 00|\Big)$

$\Big(|00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|b_1 b_2\rangle\Big)$

$$= \frac{1}{4} \langle 00 | b_1 b_2 \rangle$$

$$= \begin{cases} 1/4 & b_1 b_2 = 00 \\ 0 & \text{otherwise} \end{cases}$$