1. Bell measurements

Alice has a pair of qubits that are both in the state $\alpha|0\rangle + \beta|1\rangle$. i.e. the pair of particles is in the state $(\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$. She now measures a non-degenerate operator whose eigenstates are the Bell states - they are defined in Q6a below. Calculate the probabilities of the four measurement outcomes. What are the states after the measurement in each case.

2. Local Operations 1

Show that $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$, and that the tensor product of two hermitian operators is hermitian. Show also that the tensor product of two unitary operators is unitary (you will need to use the rule $(A \otimes B)(C \otimes D) = AC \otimes BD$).

3. Local Operations 2

Calculate the matrix representatives of the operators

$$(a)\ X \otimes X; \quad (b)\ X \otimes I; \quad (c)\ I \otimes X; \quad (d)\ I \otimes I.$$

Use the rule $(A \otimes B)(C \otimes D) = AC \otimes BD$ to calculate $(X \otimes X)^2$. Check your answer by using the matrix representation of $X \otimes X$.

4. Local Operations 3

By considering its action on the state $|+\rangle|0\rangle$, show that the operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the following matrix, w.r.t. the standard basis, is non-local (i.e. it is not of the form $U \otimes V$ for any U and V):

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

5. Measurements on entangled particles

(a) Consider that Alice and Bob each have a quantum particle and that the pair of particles start in the state $|\Psi_2\rangle = \frac{1}{\sqrt{2}}(|y_+\rangle|0\rangle + |y_-\rangle|1\rangle)$, where $|y_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$ are the eigenstates of the operator Y. If Alice measures Y on her particle, find the state $|\eta_+\rangle$ of the pair of particles immediately after measurement if Alice finds the measurement outcome $+1$. If Bob now measures Y on his particle (the pair is in the state $|\eta_+\rangle$), what is the probability that he gets outcome -1?

6. Local equivalence

Two states $|\psi_1\rangle$ and $|\psi_2\rangle$ are said to be equivalent under local unitary transformations if $|\psi_1\rangle = U_1 \otimes U_2\, |\psi_2\rangle$ for some unitary operators U_1 and U_2.

(a) By finding explicit local transformations which take you from one state to another, show that the four Bell states are equivalent under local unitary transformations:

$$\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \qquad \frac{|0\rangle|0\rangle - |1\rangle|1\rangle}{\sqrt{2}} \qquad \frac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}} \qquad \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}$$

(b) By finding explicit local transformations which take you from one state to another, show that the following four states are equivalent under local unitary transformations:

$$\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \qquad \frac{|0\rangle|0\rangle - e^{i\pi/4}|1\rangle|1\rangle}{\sqrt{2}} \qquad \frac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}} \qquad \frac{|0\rangle|1\rangle - |1\rangle|0\rangle}{\sqrt{2}}$$

Do they form an orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$?

(c) [Harder] Show that all [normalised] states of product form on $\mathbb{C}^2 \otimes \mathbb{C}^2$ are equivalent under local unitary transformations.

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