Problem Sheet 4

1. Tri-partite states

(a) Overall system in state

$$|\tau\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B|0\rangle_C + |1\rangle_A|1\rangle_B|1\rangle_C\right)$$

Alice measures $X = |0\rangle\langle 1| + |1\rangle\langle 0|$ so on whole Hilbert space $X_A \otimes I_B \otimes I_C$ measured.

Recall $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.
Hence we can rewrite X in terms of its eigenstates as

$$X = |+\rangle\langle +| - |-\rangle\langle -|$$

Expand Alice's qubit in the $\{|+\rangle, |-\rangle\}$ basis

$$|\tau\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|+\rangle_A + |-\rangle_A)|00\rangle_{BC} + \frac{1}{\sqrt{2}}(|+\rangle_A - |-\rangle_A)|11\rangle_{BC}\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle_A \underbrace{\left[\frac{1}{\sqrt{2}}(|00\rangle_{BC} + |11\rangle_{BC})\right]}_{|\psi_+\rangle_{BC}} + |-\rangle_A \underbrace{\left[\frac{1}{\sqrt{2}}(|00\rangle_{BC} - |11\rangle_{BC})\right]}_{|\psi_-\rangle_{BC}}\right)$$

Alice measures $X_A \otimes I_B \otimes I_C$ to find

$$X|\tau\rangle = \frac{1}{\sqrt{2}}|+\rangle_A|\psi_+\rangle_{BC} - \frac{1}{\sqrt{2}}|-\rangle_A|\psi_-\rangle_{BC}$$

If Alice's outcome is $+1$, then Bob and Charlie are left in the state

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_B|0\rangle_C + |1\rangle_B|1\rangle_C\right)$$

If Alice's outcome is -1, then Bob and Charlie are left in

$$|\psi_-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_B|0\rangle_C - |1\rangle_B|1\rangle_C\right)$$

Alice's one bit message should say whether or not she found +1. If she didn't then Bob can apply Z to the state as

$$Z_B \otimes I_C |\psi_-\rangle = \left(|0\rangle_B\langle 0|_B - |1\rangle_B\langle 1|_B \otimes I_C\right)\frac{1}{\sqrt{2}}\left(|0\rangle_B|0\rangle_C - |1\rangle_B|1\rangle_C\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle_B|0\rangle_C + |1\rangle_B|1\rangle_C\right) = |\psi_+\rangle.$$

And hence no matter what Alice measures, Bob and charlie can always end up with the state

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)$$

given Alice can communicate 1 bit.

(b) Now $|\tau\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_{A1}|0\rangle_{A2}|0\rangle_B + |1\rangle_{A1}|1\rangle_{A1}|1\rangle_B\right)$

The Bell states are

$$|\Phi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |\Phi_-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Psi_+\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad |\Psi_-\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

So our nondegenerate operator (we will call B) is

$$B = d_1\underbrace{|\Phi_+\rangle\langle\Phi_+|}_{P_1} + d_2\underbrace{|\Phi_-\rangle\langle\Phi_-|}_{P_2} + d_3\underbrace{|\Psi_+\rangle\langle\Psi_-|}_{P_3} + d_4\underbrace{|\Psi_-\rangle\langle\Psi_-|}_{P_4}$$

Alice is measuring on her two qubits. Lets put them in terms of the Bell states.

$$|\Phi_+\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad |\Phi_-\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$|\Phi_+\rangle + |\Phi_-\rangle = \sqrt{2}|00\rangle \Leftrightarrow |00\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_+\rangle + |\Phi_-\rangle\right)$$

$$|\Phi_+\rangle - |\Phi_-\rangle = \sqrt{2}\,|11\rangle \iff |11\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_+\rangle - |\Phi_-\rangle\right)$$

Hence we write $|\tau\rangle$ as

$$|\tau\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right) = \frac{1}{\sqrt{2}}\left(|00\rangle|0\rangle + |11\rangle|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left[|\Phi_+\rangle + |\Phi_-\rangle\right]|0\rangle + \frac{1}{\sqrt{2}}\left[|\Phi_+\rangle - |\Phi_-\rangle\right]|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\Phi_+\rangle\left(\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]\right) + |\Phi_-\rangle\left(\frac{1}{\sqrt{2}}\left[|0\rangle - |1\rangle\right]\right)\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\Phi_+\rangle|+\rangle + |\Phi_-\rangle|-\rangle\right)$$

$P(d_1) = \frac{1}{2}$, outcome is $|\Phi_+\rangle|+\rangle$

$P(d_2) = \frac{1}{2}$, outcome is $|\Phi_-\rangle|-\rangle$

$P(d_3) = P(d_4) = 0$ – impossible to measure states.

2 Teleportation/Superdense coding

(a) For two vectors to be locally equivalent

$$|\psi_2\rangle = U_1 \otimes U_2 |\psi_1\rangle$$

Quick proof – if $|\psi_2\rangle$ is locally equivalent with $|\psi_1\rangle$ and $|\psi_3\rangle$ is locally equivalent with $|\psi_2\rangle$, then is $|\psi_3\rangle$ locally equivalent with $|\psi_1\rangle$?

$$|\psi_2\rangle = U_1 \otimes U_2 |\psi_1\rangle, \quad |\psi_2\rangle = V_1 \otimes V_2 |\psi_3\rangle$$

then $|\psi_3\rangle = (V_1 \otimes V_2)^{-1}(U_1 \otimes U_2)|\psi_1\rangle$

$$= \underbrace{(W_1 \otimes W_2)}_{=(V_1 \otimes V_2)^{-1},\ \text{also unitary}}(U_1 \otimes U_2)|\psi_1\rangle$$

$$= U_a \otimes U_b |\psi_1\rangle$$

as the product of unitary matrices are also unitary.
Hence statement true.

States are

$$|\phi_1\rangle = \frac{|00\rangle + i|11\rangle}{\sqrt{2}} \qquad |\phi_2\rangle = \frac{|00\rangle - i|11\rangle}{\sqrt{2}}$$

$$|\phi_3\rangle = \frac{|01\rangle + i|10\rangle}{\sqrt{2}} \qquad |\phi_4\rangle = \frac{|01\rangle - i|10\rangle}{\sqrt{2}}$$

$|\phi_1\rangle \to |\phi_2\rangle$: $Z \otimes I |\phi_1\rangle = |\phi_2\rangle$

$\therefore$ locally equivalent

$|\phi_2\rangle \to |\phi_3\rangle$: $Z \otimes X |\phi_2\rangle = \frac{|01\rangle + i|10\rangle}{\sqrt{2}} = |\phi_3\rangle$

$\therefore$ locally equivalent

$|\phi_3\rangle \to |\phi_4\rangle$: $Z \otimes I |\phi_3\rangle = |\phi_4\rangle$

$\therefore$ locally equivalent

Therefore all four states are locally equivalent.

To form an orthonormal basis

$$\langle\phi_i|\phi_j\rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

i	j	$\langle \phi_i \mid \phi_j \rangle$	Valid?
1	1	$\frac{1}{2}(\langle 00\| - i\langle 11\|)(\|00\rangle + i\|11\rangle) = \frac{1}{2}(1+1) = 1$	✓
1	2	$\frac{1}{2}(\langle 00\| - i\langle 11\|)(\|00\rangle - i\|11\rangle) = \frac{1}{2}(1-1) = 0$	✓
1	3	$\frac{1}{2}(\langle 00\| - i\langle 11\|)(\|01\rangle + i\|10\rangle) = \frac{1}{2}0 = 0$	✓
1	4	$\frac{1}{2}(\langle 00\| - i\langle 11\|)(\|01\rangle - i\|10\rangle = 0$	✓
2	2	$\frac{1}{2}(\langle 00\| + i\langle 11\|)(\|00\rangle - i\|11\rangle) = \frac{1}{2}(1+1) = 1$	✓
2	3	$\frac{1}{2}(\langle 00\| + i\langle 11\|)(\|01\rangle + i\|10\rangle) = 0$	✓
2	4	$\frac{1}{2}(\langle 00\| + i\langle 11\|)(\|01\rangle - i\|10\rangle) = 0$	✓
3	3	$\frac{1}{2}(\langle 01\| - i\langle 10\|)(\|01\rangle + i\|10\rangle) = \frac{1}{2}(1+1) = 1$	✓
3	4	$\frac{1}{2}(\langle 01\| - i\langle 10\|)(\|01\rangle - i\|10\rangle) = \frac{1}{2}(1-1) = 0$	✓
4	4	$\frac{1}{2}(\langle 01\| + i\langle 10\|)(\|01\rangle - i\|10\rangle) = \frac{1}{2}(1+1) = 1$	✓

All valid, so states form an orthonormal basis.

(b) Alice and Bob share

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|1\rangle_B + i|1\rangle_A|0\rangle_B\right)$$

Alice wants to teleport $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$

A and B think they are in $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$.

For $|\psi\rangle$, A would measure the Bell states (specifically a non-degenerate operator with eigenstates equal to the

Bell states). But, we are not actually in $|\psi_3\rangle$ - we are in $|\phi_3\rangle$.

Initial state is then

$$|AB\rangle = (\alpha|0\rangle + \beta|1\rangle)|\phi_3\rangle$$

$$= \frac{1}{\sqrt{2}}(\alpha|0\rangle_T + \beta|1\rangle_T)(|0\rangle_A|1\rangle_B + i|1\rangle_A|0\rangle_B)$$

We want to rewrite this in terms of the Bell states.

by inspection

$$|AB\rangle = \frac{1}{2\sqrt{2}}\left[|0\rangle_T|0\rangle_A + |1\rangle_T|1\rangle_A\right]|\tau_1\rangle \qquad |\tau_1\rangle = \alpha|0\rangle + i\beta|1\rangle$$

$$+ \frac{1}{2\sqrt{2}}\left[|0\rangle_T|0\rangle_A - |1\rangle_T|1\rangle_A\right]|\tau_2\rangle \qquad |\tau_2\rangle = \alpha|0\rangle - i\beta|1\rangle$$

$$+ \frac{1}{2\sqrt{2}}\left[|0\rangle_T|1\rangle_A + |1\rangle_T|0\rangle_A\right]|\tau_3\rangle \qquad |\tau_3\rangle = i\alpha|0\rangle + \beta|1\rangle$$

$$+ \frac{1}{2\sqrt{2}}\left[|0\rangle_T|1\rangle_A - |1\rangle_T|0\rangle_A\right]|\tau_4\rangle \qquad |\tau_3\rangle = i\alpha|0\rangle - \beta|1\rangle$$

$$= \frac{1}{2\sqrt{2}}\Big(|\Phi_+\rangle(\alpha|0\rangle + i\beta|1\rangle) + |\Phi_-\rangle(\alpha|0\rangle - i\beta|1\rangle)$$
$$+ |\psi_+\rangle(i\alpha|0\rangle + \beta|1\rangle) + |\psi_-\rangle(i\alpha|0\rangle - \beta|1\rangle)\Big)$$

If Alice measures

$$A = \lambda_1|\Phi_+\rangle\langle\Phi_+| + \lambda_2|\Phi_-\rangle\langle\Phi_-| + \lambda_3|\psi_+\rangle\langle\psi_+| + \lambda_4|\psi_-\rangle\langle\psi_-|$$

then the outcomes based on measurement are

Observed eigenvalue	Bob's state
λ_1	$\alpha\|0\rangle + i\beta\|1\rangle$
λ_2	$\alpha\|0\rangle - i\beta\|1\rangle$
λ_3	$i\alpha\|0\rangle + \beta\|1\rangle$

d_4	$i\alpha|0\rangle - \beta|1\rangle$

(c) A and B share

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle + i|10\rangle)$$

A wants to send 01. So she applies $X \otimes I$.

$$X \otimes I|\phi_3\rangle = \frac{1}{\sqrt{2}}\left[|11\rangle + i|00\rangle\right] = |\text{shared}\rangle$$

She sends this to Bob, but he measures in the Bell Basis as he thinks that was their state. Lets rewrite |shared> in this Basis

$$|11\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle) \quad |00\rangle = \frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle).$$

$$\therefore \; |\text{shared}\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|\Phi_+\rangle - |\Phi_-\rangle) + i\frac{1}{\sqrt{2}}(|\Phi_+\rangle + |\Phi_-\rangle)\right]$$

$$= \frac{1}{\sqrt{2}}\left[|\Phi_+\rangle\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) + |\Phi_-\rangle\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{\frac{i\pi}{4}}|\Phi_+\rangle + e^{\frac{3\pi}{4}}|\Phi_-\rangle\right]$$

Hence as Bob measures

$$B = d_{00}|\Phi_+\rangle\langle\Phi_+| + d_{01}|\Phi_-\rangle\langle\Phi_-| + d_{10}|\Psi_+\rangle\langle\Psi_+| + d_{11}|\Psi_-\rangle\langle\Psi_-|$$

$$P(d_{00}) = P(d_{01}) = \frac{1}{2}, \quad P(d_{10}) = P(d_{11}) = 0$$

Therefore he has a 50% chance to mistakenly measure 00, and a 50% chance to correctly measure 01.

3. Measuring in the Computation basis

$$|\psi\rangle = \sum_{j,k,m=0}^{1} a_{jkm} |jkm\rangle$$
$$= a_{000}|000\rangle + \ldots + a_{111}|111\rangle$$

If we measure a qubit in this basis, and find that the outcome is zero, we measure $|0\rangle\langle 0|$. Therefore, to find the outcome is zero for all three qubits is

$$|0\rangle\langle 0| \otimes |0\rangle\langle 0| \otimes |0\rangle\langle 0| \sum_{j,k,m=0}^{1} a_{jkm} |jkm\rangle$$
$$= a_{000}|000\rangle$$

Therefore, the probability of this happening is

$$|a_{000}|^2$$

as required.

4. Functions from two bits to two bits

$$f(x) \quad \text{2bit} \to \text{2bit} \quad f : \{0,1\}^2 \longrightarrow \{0,1\}^2$$

Define $\quad U_f |\underline{x}\rangle|\underline{y}\rangle = |\underline{x}\rangle|\underline{y} \oplus \underline{f}(\underline{x})\rangle$

and $\quad |f\rangle = U_f \cdot \frac{1}{2} \sum_{\underline{x}}^{1} |x\rangle|00\rangle$

where $\quad \underline{x} = x_1 x_2, \quad |\underline{x}\rangle = |x_1 x_2\rangle$

$\qquad\quad \underline{y} = y_1 y_2, \quad |\underline{y}\rangle = |y_1 y_2\rangle$

(a) To take ON -> ON, compute inner product of $U_f|\underline{x}_1\rangle|\underline{y}_1\rangle$ with $U_f|\underline{x}_2\rangle|\underline{y}_2\rangle$.

$$\left(\langle \underline{x}_2| \langle \underline{y}_2 \oplus \underline{f}(\underline{x}_2)| \;\middle|\; |\underline{x}_1\rangle |\underline{y}_1 + \underline{f}(\underline{x}_1)\rangle\right)$$

$$= \delta_{\underline{x}_1, \underline{x}_2} \langle \underline{y}_2 \oplus \underline{f}(\underline{x}_2) | \underline{y}_1 + \underline{f}(\underline{x}_1)\rangle$$

$$= \delta_{\underline{x}_1, \underline{x}_2} \langle \underline{y}_2 \oplus \underline{f}(\underline{x}_1) | \underline{y}_1 + \underline{f}(\underline{x}_1)\rangle$$

$$= \delta_{\underline{x}_1, \underline{x}_2} \langle \underline{y}_2 | \underline{y}_1\rangle$$

$$= \delta_{\underline{x}_1, \underline{x}_2} \delta_{\underline{y}_1, \underline{y}_2} = \langle \underline{x}_2 | \langle \underline{y}_1 | \underline{y}_2\rangle | \underline{x}_1\rangle$$ ON basis

Hence U_f is unitary.

(b) $f_1(00) = 01$, $f_1(01) = 10$, $f_1(10) = 11$, $f_1(11) = 00$

$f_2(00) = 00$, $f_2(01) = 01$, $f_2(10) = 10$, $f_2(11) = b_1 b_2$

$$|f_1\rangle = U_f \cdot \frac{1}{2} \sum_{\underline{x}}^{1} |\underline{x}\rangle |00\rangle$$

$$= \frac{U_f}{2}\left(|00\rangle|00\rangle + |01\rangle|00\rangle + |10\rangle|00\rangle + |11\rangle|00\rangle\right)$$

$$= \frac{1}{2}\Big[|00\rangle|00 \oplus 01\rangle + |01\rangle|00 \oplus 10\rangle$$
$$+ |10\rangle|00 \oplus 11\rangle + |11\rangle|00 \oplus 00\rangle\Big]$$

$$= \frac{1}{2}\left[|00\rangle|01\rangle + |01\rangle|10\rangle + |10\rangle|11\rangle + |11\rangle|00\rangle\right]$$

$$|f_2\rangle = U_f \cdot \frac{1}{2} \sum_{\underline{x}}^{1} |\underline{x}\rangle |00\rangle$$

$$= \frac{U_f}{2}\left[|00\rangle|00\rangle + |01\rangle|00\rangle + |10\rangle|00\rangle + |11\rangle|00\rangle\right]$$

$$= \frac{1}{2}\left[|00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|00 \oplus b_1 b_2\rangle\right]$$

$$\therefore \langle f_1 | f_2\rangle = \frac{1}{4}\left[\langle 00|\langle 01| + \langle 01|\langle 10| + \langle 10|\langle 11| + \langle 11|\langle 00|\right]$$

$$\times \left[|00\rangle|00\rangle + |01\rangle|01\rangle + |10\rangle|10\rangle + |11\rangle|00 \oplus b_1 b_2\rangle \right]$$

$$= \frac{1}{4}\left(\langle 00|01\rangle + \langle 01|10\rangle + \langle 10|00 \oplus b_1 b_2\rangle + \langle 11|00\rangle \right]$$

$$= \frac{1}{4}\left(\langle 10|00 \oplus b_1 b_2\rangle \right)$$

This is zero for $b_1 b_2 = 00, 01, 11$. However, for $b_1 b_2 = 10$, this means that

$$\underline{\langle \gamma_1 | \gamma_2 \rangle = \frac{1}{4}} \quad \text{when } b_1 b_2 = 10$$