Problem Sheet 3

Great work, and some insightful comments along the way. I think you can manage the last part if you have time

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1. Bell measurements

Alice has 2 qubits in $(\alpha|0\rangle + \beta|1\rangle)$ such that the overall HS is

$$|\psi\rangle = (\alpha|0\rangle + \beta|1\rangle) \otimes (\alpha|0\rangle + \beta|1\rangle)$$

She measures some non-degenerate operator - we will call B - whose eigenstates are the Bell states:

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}, \quad \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \quad \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \quad \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Lets first expand the tensor product.

$$|\psi\rangle = \alpha^2|00\rangle + \alpha\beta|01\rangle + \alpha\beta|10\rangle + \beta^2|11\rangle$$

Then, to find each possible outcome, we project onto the Bell states.

Outcome 1 $\rightarrow$ $\langle b_1| = \frac{\langle 00| + \langle 11|}{\sqrt{2}}$

$$\langle b_1|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha^2 + \beta^2)$$

Hence $p(b_1) = \frac{1}{2}|\alpha^2 + \beta^2|^2$

state after measurement is $\frac{|00\rangle + |11\rangle}{\sqrt{2}}$

Outcome 2 $\rightarrow$ $\langle b_2| = \frac{\langle 00| - \langle 11|}{\sqrt{2}}$

$$\langle b_2|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha^2 - \beta^2)$$

$$\therefore\ P(b_2) = \frac{1}{2}|\alpha^2 - \beta^2|^2$$

state after measurement is $\dfrac{|00\rangle - |11\rangle}{\sqrt{2}}$

Outcome 3 $\rightarrow$ $\langle b_3| = \dfrac{\langle 01| + \langle 10|}{\sqrt{2}}$

$$\langle b_3|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha\beta + \alpha\beta) = \sqrt{2}\,\alpha\beta$$

$$p(b_3) = 2|\alpha\beta|^2$$

state after measurement is $\dfrac{|01\rangle + |10\rangle}{\sqrt{2}}$

Outcome 4 $\longrightarrow$ $\langle b_4| = \dfrac{\langle 01| - \langle 10|}{\sqrt{2}}$

$$\langle b_4|\psi\rangle = \frac{1}{\sqrt{2}}(\alpha\beta - \alpha\beta) = 0$$

$$\therefore\ P(b_4) = 0$$

state cannot occur - no state after measurement

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2. Local operations I

$\rightarrow$ Show $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

$$\langle v_1 v_2|(A \otimes B)^\dagger|w_1 w_2\rangle = \langle w_1 w_2|(A \otimes B)|v_1 v_2\rangle^*$$

$$= \langle w_1 w_2|\left(A|v_1\rangle^* \otimes B|v_2\rangle^*\right)$$

$$= \langle w_1|A|v_1\rangle^* + \langle w_2|B|v_2\rangle^*$$

$$= \langle v_1|A^\dagger|w_1\rangle + \langle v_2|B^\dagger|w_2\rangle$$

$$= \langle v_1 v_2|A^\dagger \otimes B^\dagger|w_1 w_2\rangle$$

$\therefore \; (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$ as required

It's true, but why can we deduce this? You've only shown it for product states $|v_1 v_2\rangle, |w_1 w_2\rangle$

→ Show tensor product of two Hermitian operators is itself Hermitian.

Let A, B be Hermitian operators satisfying

$$A = A^\dagger, \quad B = B^\dagger$$

Their tensor product is $A \otimes B$. Then

$$A \otimes B = A^\dagger \otimes B^\dagger = (A \otimes B)^\dagger \quad \leftarrow \text{from above.}$$

Hence $A \otimes B$ is Hermitian. ✓

→ Show that the tensor product of two unitary operators is also unitary.

Let U and V be two unitary operators satisfying

$$UU^\dagger = U^\dagger U = VV^\dagger = V^\dagger V = I$$

Then $U \otimes V$ is their tensor product. Is this unitary?

$$(U \otimes V)(U \otimes V)^\dagger = (U \otimes V)(U^\dagger \otimes V^\dagger) \quad \text{from above}$$

Using $(A \otimes B)(C \otimes D) = AC \otimes BD$, we find

$$(U \otimes V)(U \otimes V)^\dagger = UU^\dagger \otimes VV^\dagger = I \otimes I$$ ✓

As well,

$$(U \otimes V)^\dagger (U \otimes V) = U^\dagger U \otimes V^\dagger V = I \otimes I$$

No need to check both

which is unitary over the whole HS, so $U \otimes V$

is unitary.

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3. Local operations 2

a. $X \otimes X$. Note $X = |0\rangle\langle 1| + |1\rangle\langle 0| \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$X \otimes X = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) \otimes \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$
$$= |00\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 00|$$

In matrix form, this will be

$$X \otimes X = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

b. $$X \otimes I = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) \otimes \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$
$$= |00\rangle\langle 10| + |01\rangle\langle 11| + |10\rangle\langle 00| + |11\rangle\langle 01|$$

In matrix form

$$X \otimes I = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

c. $$I \otimes X = \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \otimes \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right)$$
$$= |00\rangle\langle 01| + |01\rangle\langle 00| + |10\rangle\langle 11| + |11\rangle\langle 10|$$

In matrix form

$$I \otimes X = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ ✓

d. $I \otimes I = (|0\rangle\langle 0| + |1\rangle\langle 1|) \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|)$

$= |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|$

In matrix form

$$I \otimes I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ ✓

Good to see you're comfortable with Dirac notation to use it here!

→ Consider $(X \otimes X)^2$ using $(A \otimes B)(C \otimes D) = AC \otimes BD$

$X = |0\rangle\langle 1| + |1\rangle\langle 0|$

$(X \otimes X)^2 = (X \otimes X)(X \otimes X) = XX \otimes XX$

$XX = (|0\rangle\langle 1| + |1\rangle\langle 0|)(|0\rangle\langle 1| + |1\rangle\langle 0|)$

$= |1\rangle\langle 1| + |0\rangle\langle 0| = I$ ✓

$\therefore (X \otimes X)^2 = I \otimes I$ ✓

Using matrix representation.

$$(X \otimes X)^2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I \otimes I$$

(part d) ✓

4. Local operations 4

Let operator A have matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad (\text{CNOT})$$ Indeed!

We want to show this is non-local, i.e. A cannot be written as a tensor product of two other operators — $A \neq U \otimes V$.

Consider $|+\rangle|0\rangle$ where $|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ — call this state $|\psi\rangle$ with

$$|\psi\rangle = \frac{|00\rangle + |10\rangle}{\sqrt{2}}$$

A is a CNOT operator, hence

$$A|\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

here "on" is set to 1.

i.e. qubit 2 only swaps if qubit 1 is on.

For now, let $A = U \otimes V$. Now consider the effect of applying $U \otimes V$ to $|\psi\rangle$.

$$U \otimes V|\psi\rangle = \frac{U|0\rangle \otimes V|0\rangle + U|1\rangle \otimes V|0\rangle}{\sqrt{2}}$$

To provide the same operation as A, we require

$$U|0\rangle = |0\rangle, \quad U|1\rangle = |1\rangle$$

$$V|0\rangle = |0\rangle, \quad V|0\rangle = |1\rangle$$

But here we see a contradiction! V must do some definite operation, but here it must do two different actions depending on the value of qubit 1.

This is because the states are entangled - the operation of V depends on the value of the first qubit.

Hence A cannot be written in the form $U \otimes V$, therefore it is non-local, and by extension, the states are entangled. I think this argument only obviously works when U is the identity. Otherwise, I don't see how you would compare coefficients. Instead:

$(U \otimes V)|+\rangle|0\rangle = U|+\rangle \otimes V|0\rangle$
is clearly a product state

5. Measurements on entangled particles

Alice and Bob $|Alice\rangle|Bob\rangle$ have particles in

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\left(|y_+\rangle|0\rangle + |y_-\rangle|1\rangle\right)$$

where $|y_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$. $|y_\pm\rangle$ are eigenstates of γ operator.

Alice measures γ and finds outcome $+1$.

As $|y_\pm\rangle$ eigenstates of γ, then γ can be written as

$$\gamma = \sum_{n=+,-}' d_n |y_n\rangle\langle y_n| = |y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|$$

Now measure. Formally we measure $\gamma \otimes I$:

$$\gamma \otimes I = |y_+\rangle\langle y_+| \otimes I - |y_-\rangle\langle y_-| \otimes I = P_+ - P_-$$

Project $|\Psi_2\rangle$ onto P_+ $(d = +1)$:

$$P_+|\Psi_2\rangle = \left(|y_+\rangle\langle y_+| \otimes I\right) \cdot \frac{1}{\sqrt{2}}\left(|y_+\rangle|0\rangle + |y_-\rangle|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|y_+\rangle|0\rangle\right)$$

Then renormalised, the state of the particles becomes simply

$$|\eta_+\rangle = |y_+\rangle|0\rangle$$

Now what if Bob measures the state. What is the probability he measures -1?

Bob measures so we use $I \otimes Y$:

$$I \otimes Y = I \otimes \left(|y_+\rangle\langle y_+| - |y_-\rangle\langle y_-|\right)$$

$$= I \otimes |y_+\rangle\langle y_+| - I \otimes |y_-\rangle\langle y_-| = Q_+ - Q_-$$

Project $|\eta_+\rangle$ onto Q_-:

$$Q_-|\eta_+\rangle = \left(I \otimes |y_-\rangle\langle y_-|\right)\left(|y_+\rangle|0\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(I \oplus |y_-\rangle\left(\langle 0| + i\langle 1|\right)\right)\left(|y_+\rangle|0\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|y_-\rangle|y_+\rangle$$

Therefore probability is $1/2$.

The act of Alice's measurement "destroyed" the entangled state. Indeed, but if you notice, it doesn't change Bob's measurement probabilities. This is important later in the course (it's called no-signalling)

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6. Local equivalence.

$|\psi_1\rangle$ and $|\psi_2\rangle$ "equivalent under local unitary transforms" if

$$|\psi_1\rangle = U_1 \otimes U_2 |\psi\rangle$$

for some unitary U_1 & U_2.

a. $|b_1\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$ $\quad |b_2\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$

$|b_3\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}$ $\quad |b_4\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$

$\underline{|b_1\rangle \to |b_2\rangle}$

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = U_1 \otimes U_2 \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}} \right)$$

Set $U_1 \otimes U_2 = Z \otimes I$

$$= \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) \otimes \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

Then $U_1 \otimes U_2 \left(\frac{|00\rangle - |11\rangle}{\sqrt{2}} \right) = \left(\frac{|00\rangle + |11\rangle}{\sqrt{2}} \right) = |b_1\rangle$

Hence we can say that $|b_1\rangle$ and $|b_2\rangle$ are E.u.L. u.T. since Z & I are unitary.

$\underline{|b_2\rangle \to |b_3\rangle}$

$$\frac{|00\rangle - |11\rangle}{\sqrt{2}} = U_1 \otimes U_2 \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

Here, $U_1 \otimes U_2 = I \otimes \underbrace{\left(|0\rangle\langle 1| - |1\rangle\langle 0| \right)}_{U_2}$

Is U_2 unitary?

$$U_2^\dagger = |1\rangle\langle 0| - |0\rangle\langle 1|$$

$$-U_2 U_2 = \left(|0\rangle\langle 1| - |1\rangle\langle 0|\right)\left(|1\rangle\langle 0| - |0\rangle\langle 1|\right)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1| = I \quad \therefore U_2 \text{ unitary!}$$

In fact $U_2 = -iY$

Therefore $|b_2\rangle$ and $|b_3\rangle$ E.U.L.U.T..

$|b_3\rangle \to |b_4\rangle$

$$\frac{|01\rangle - |10\rangle}{\sqrt{2}} = U_1 \otimes U_2 \left(\frac{|01\rangle + |10\rangle}{\sqrt{2}} \right)$$

Then $U_1 \otimes U_2 = I \otimes (|1\rangle\langle 1| - |0\rangle\langle 0|)$

$= -I \otimes Z \quad \therefore$ unitary

Therefore $|b_3\rangle$ and $|b_4\rangle$ E.U.L.U.T..

If $|b_1\rangle$ and $|b_2\rangle$ equivalent, and $|b_2\rangle$ and $|b_3\rangle$ equivalent, are $|b_1\rangle$ and $|b_3\rangle$?

$$|b_1\rangle = U_1 \otimes U_2 |b_2\rangle = U_1 \otimes U_2 (U_3 \otimes U_4 |b_3\rangle)$$

Strictly, you also need to be able to invert operations as well. Of course, you can

Two successive unitary operations $(U_3 \otimes U_4$ then $U_1 \otimes U_2)$ is unitary, hence the above statement is true, and by extension, all Bell states are equivalent under local unitary transformation

b. $$\frac{|00\rangle + |11\rangle}{\sqrt{2}} = \underbrace{\left(|0\rangle\langle 0| - e^{\frac{-\pi i}{4}} |1\rangle\langle 1|\right)}_{A} \otimes I \; \frac{|00\rangle - e^{\frac{\pi i}{4}} |11\rangle}{\sqrt{2}}$$

A unitary? $AA^\dagger = \left(|0\rangle\langle 0| - e^{\frac{-\pi i}{4}} |1\rangle\langle 1|\right)\left(|0\rangle\langle 0| - e^{\frac{+\pi i}{4}} |1\rangle\langle 1|\right)$

$= |0\rangle\langle 0| + |1\rangle\langle 1| \quad \checkmark$ A unitary

States 1 & 2 EULUT

$$\frac{|00\rangle - e^{\frac{\pi i}{4}} |11\rangle\langle 1|}{\sqrt{2}} = \underbrace{\left(|0\rangle\langle 0| - e^{\frac{\pi i}{4}} |1\rangle\langle 1|\right)}_{\text{unitary - see above}} \otimes \underbrace{\left(|1\rangle\langle 0| + |1\rangle\langle 0|\right)}_{X\text{ - unitary}} \frac{|01\rangle + |10\rangle}{\sqrt{2}}$$

States 2 and 3 EULUT.

$$\frac{|01\rangle + |11\rangle}{\sqrt{2}} = \underbrace{I}_{\text{unitary}} \otimes \underbrace{\left(|1\rangle\langle 1| - |0\rangle\langle 0|\right)}_{-Z\text{ - unitary}} \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

States 3 and 4 EULUT

By extension then, all states are EULUT.

→ Orthonormal basis? Must have

$$\langle \text{state } i | \text{state } j \rangle = \delta_{ij}$$

We can see that ⟨state 1|state 2⟩ is

$$\frac{\langle 00| + \langle 11|}{\sqrt{2}} \; \frac{|00\rangle - e^{\frac{\pi i}{4}}|11\rangle}{\sqrt{2}} = \frac{1 - e^{\frac{\pi i}{4}}}{2} \neq 0$$

and hence these states do not form an orthonormal basis.

c. Lets denote our state as $|a_i\rangle \otimes |b_i\rangle$, such that our normalised state is

$$|\psi_i\rangle = |a_i\rangle \otimes |b_i\rangle$$

Now consider some separate $|\psi_j\rangle = |a_j\rangle \otimes |b_j\rangle$. We want to show that for any i, j,

$$|\psi_j\rangle = U_1 \otimes U_2 \left(|a_i\rangle \otimes |b_i\rangle \right)$$

$$\therefore |a_j\rangle \otimes |b_j\rangle = U_1 |a_i\rangle \otimes U_2 |b_i\rangle$$

This gives us two equations

$$|a_j\rangle = U_1 |a_i\rangle, \quad |b_j\rangle = U_2 |b_i\rangle$$

On $\mathbb{C}^2$, there are two basis vectors, $|0\rangle$ and $|1\rangle$. Consider

$$|a_i\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \alpha |0\rangle + \beta |1\rangle$$

Then $|a_j\rangle = \begin{bmatrix}\gamma\\ \delta\end{bmatrix} = \gamma|0\rangle + \delta|1\rangle$. Can we find a U_1 such that it is unitary <u>and</u>

$$U_1|a_i\rangle = |a_j\rangle$$

$$U_1\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\gamma\\ \delta\end{bmatrix}$$

$$U_1 = \begin{bmatrix}\frac{\gamma}{2\alpha} & \frac{\gamma}{2\beta}\\ \frac{\delta}{2\alpha} & \frac{\delta}{2\beta}\end{bmatrix} \rightarrow \begin{bmatrix}\frac{\gamma}{2\alpha} & \frac{\gamma}{2\beta}\\ \frac{\delta}{2\alpha} & \frac{\delta}{2\beta}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\gamma+\gamma\\ \delta+\delta\end{bmatrix}$$

$$= \begin{bmatrix}\gamma\\ \delta\end{bmatrix} \checkmark$$

Is U_1 unitary?

$$U_1^\dagger = \begin{bmatrix}\frac{\gamma}{2\alpha} & \left(\frac{\delta}{2\alpha}\right)^*\\ \left(\frac{\gamma}{2\beta}\right)^* & \frac{\delta}{2\beta}\end{bmatrix}$$

$$\therefore \quad U_1U^\dagger = \begin{bmatrix}\gamma/2\alpha & \gamma/2\beta\\ \delta/2\alpha & \delta/2\beta\end{bmatrix}\begin{bmatrix}\gamma/2\alpha & \delta^*/2\alpha^*\\ \gamma^*/2\beta^* & \delta/2\beta\end{bmatrix}$$

$$= \begin{bmatrix}\left(\frac{\gamma}{2\alpha}\right)^2 + \left(\frac{\gamma}{2\beta}\right)\left(\frac{\gamma}{2\beta}\right)^* & \frac{\gamma}{2\alpha}\left(\frac{\delta}{2\alpha}\right)^* + \frac{\gamma}{2\beta}\frac{\delta}{2\beta}\\ \frac{\delta}{2\alpha}\frac{\gamma}{2\alpha} + \frac{\delta}{2\beta}\left(\frac{\gamma}{2\beta}\right)^* & \frac{\delta}{2\alpha}\left(\frac{\delta}{2\alpha}\right)^* + \left(\frac{\delta}{2\beta}\right)^2\end{bmatrix}$$

$$= \begin{bmatrix}\left|\frac{\gamma}{2\alpha}\right|^2 + \left|\frac{\gamma}{2\beta}\right|^2 & \frac{\gamma\delta^*}{(2\alpha)^2} + \frac{\gamma\delta}{(2\beta)^2}\\ \frac{\delta\gamma}{(2\alpha)^2} + \frac{\delta\gamma^*}{(2\beta)^2} & \left|\frac{\delta}{2\alpha}\right|^2 + \left|\frac{\delta}{2\beta}\right|^2\end{bmatrix}$$

$$(0,0) \rightarrow \; = |\gamma|^2\left(\frac{1}{4\alpha^2} + \frac{1}{4\beta^2}\right) = \frac{|\gamma|^2}{4}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$$ $=1$ for normalisation

$$= \frac{|\gamma|^2}{4}\left(\frac{\beta^2+\alpha^2}{\alpha^2\beta^2}\right)$$

$$= \frac{|\gamma|^2}{4\alpha^2\beta^2}$$

Rewrite U_1 as

$$u_1 = \frac{1}{(2\alpha\beta)^2}\begin{bmatrix} \gamma^2+|\gamma|^2 & \gamma\delta^* + \gamma\delta \\ \delta\gamma + \delta\gamma^* & \delta^2 + |\delta|^2 \end{bmatrix}$$

$$= \frac{1}{(2\alpha\beta)^2}\begin{bmatrix} \gamma^2+|\gamma|^2 & 2\gamma \operatorname{Re}\delta \\ 2\delta \operatorname{Re}\gamma & \delta^2 + |\delta|^2 \end{bmatrix}$$

where we have the conditions $\gamma^2 + \delta^2 = 1$ and $\alpha^2 + \beta^2 = 1$

Think I must have done this wrong. ¿:

Not sure where to go here.

It's a good attempt, you're trying the right things. It might be easier to prove that any product state is locally unitarily equivalent to $|00\rangle$, and then use a similar argument to the first two parts to deduce that any two product states are locally unitarily equivalent