**1. Tri-partite states**

(a) Consider that Alice, Bob and Charlie each have one qubit and the three qubits are in the following state

$$|\tau\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|1\rangle\right).$$

Show that if Alice measures $X$ on her particle, then communicates one bit to Bob, they can arrange things so that Bob and Charlie's particles are in the state

$$\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right).$$

(b) Now let Alice have two particles and Bob one. The three particles are in the state $|\tau\rangle$. Alice now makes a measurement of a non-degenerate operator with eigenstates the Bell states. For each measurement outcome, find the probability that the outcome occurs and the state after the measurement.

**2. Teleportation/Superdense Coding**

(a) By considering the action of operators such as $X \otimes I$ (where $I$ is the identity operator on $\mathbb{C}^2$), show that the following four states on $\mathbb{C}^2 \otimes \mathbb{C}^2$ are locally equivalent.

$$\begin{aligned} |\phi_1\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + i|1\rangle|1\rangle\right) \qquad |\phi_2\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - i|1\rangle|1\rangle\right) \\ |\phi_3\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + i|1\rangle|0\rangle\right) \qquad |\phi_4\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - i|1\rangle|0\rangle\right) \end{aligned}$$

Show also that these states form an orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$.

(b) Let Alice and Bob share two particles, particles 1 and 2, say, in the state $|\phi_3\rangle$.

Alice wishes to teleport an unknown state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ of a third particle (particle 3) to Bob; $\alpha$ and $\beta$ are complex constants satisfying $|\alpha|^2 + |\beta|^2 = 1$.

Alice and Bob think their particles 1 and 2 are in fact in the state

$$|\Psi\rangle = \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}.$$

Let them now do the standard teleportation protocol [i.e. what they would have done had they shared qubits in the state $|\Psi\rangle$]. Compute the state of Bob's particle for each of Alice's measurement outcomes, after the full protocol has taken place.

(c) Let Alice and Bob again share two particles, particles 1 and 2, in the state $|\phi_3\rangle$. Let Alice and Bob do all the operations that would have done in the standard super-dense coding protocol. i.e. in particular they imagine that they in fact share particles in the state $|\Psi\rangle$ and Alice does one of the standard Pauli operators or the identity etc.

If Alice had intended to send the message "01", find the probabilities that Bob interprets the message as each of the possible outcomes "00", "01", "10", "11".

**3. Measuring in the computation basis**

Consider the three general qubit state

$$|\psi\rangle = \sum_{j,k,m=0}^{1} a_{jkm}|j,k,m\rangle.$$

Show that if each qubit is measured in the computational basis, the total probability that every measurement yields the value 0 is $|a_{000}|^2$.

**4. Functions from two bits to two bits**

We now consider functions $f(\mathbf{x})$ from 2 bits to 2 bits $f : \{0,1\}^2 \mapsto \{0,1\}^2$.

For any function $f$ from 2 bits to 2 bits, we define

$$U_f|\mathbf{x}\rangle|\mathbf{y}\rangle = |\mathbf{x}\rangle|\mathbf{y} \oplus \mathbf{f}(\mathbf{x})\rangle,$$

and

$$|f\rangle = U_f \frac{1}{2}\sum_{\mathbf{x}} |\mathbf{x}\rangle|00\rangle$$

where for a 2 bit string $\mathbf{x} = x_1x_2$, where $x_1$ and $x_2$ are bits, $|\mathbf{x}\rangle = |x_1x_2\rangle$ denotes $|x_1\rangle|x_2\rangle$. Similarly for a 2 bit string $\mathbf{y} = y_1y_2$, $|\mathbf{y}\rangle = |y_1y_2\rangle$ denotes $|y_1\rangle|y_2\rangle$. $\oplus$ denotes bit-wise addition modulo 2, e.g. $01 \oplus 11 = 10$.

(a) Show that $U_f$ is unitary, e.g. by showing that it takes an ON-basis to an ON-basis.

(b) The functions $f_1$ and $f_2$ have the following lists of values :

$$f_1(00) = 01,\ f_1(01) = 10,\ f_1(10) = 11,\ f_1(11) = 00.$$

$$f_2(00) = 00,\ f_2(01) = 01,\ f_2(10) = 10,\ f_2(11) = b_1b_2.$$

Find the value of $\langle f_1|f_2\rangle$ for all the four possible two-bit strings $b_1b_2$.