QIT Problem Sheet 2

1/ (a) $\sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \rightarrow Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$

Eigenvectors will exist such that $Y|v\rangle = \lambda|v\rangle$ with eigenvalue λ.

Let $|v\rangle = a|0\rangle + b|1\rangle \longleftrightarrow \underline{v} = \begin{bmatrix} a \\ b \end{bmatrix}$

$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix}$ in matrix form — eigenvalues from

$$\det\left(\sigma_i - \lambda I\right) = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + i^2 = 0 \Rightarrow \underline{\underline{\lambda = \pm 1}}$$

Now back to Dirac form to find eigenvectors.

$$Y|v\rangle = \pm|v\rangle$$

$$\Rightarrow \left(-i|0\rangle\langle 1| + i|1\rangle\langle 0|\right)\left(a|0\rangle + b|1\rangle\right) = \pm\left(a|0\rangle + b|1\rangle\right)$$

$$\Rightarrow \quad ai|1\rangle - bi|0\rangle = \pm\left(a|0\rangle + b|1\rangle\right)$$

$$\Rightarrow \begin{cases} -ib = \pm a \longrightarrow b = \pm ai \\ ai = \pm b \end{cases}$$

Let $b=1$, and hence $b = \pm i$ such that we find two eigenstates

$$|v_+\rangle = |0\rangle + i|1\rangle \quad \text{with } \lambda = 1$$
$$|v_-\rangle = |0\rangle - i|1\rangle \quad \text{with } \lambda = -1$$

The normalised states are then

$$|e_+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right), \quad |e_-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

We can then write Y in diagonal form.

$$Y = \sum_i^{1} \lambda_i |e_i\rangle\langle e_i|$$

for $|e_i\rangle$ eigenvectors with corresponding eigenvalue λ_i.

$$Y = |e_+\rangle\langle e_+| - |e_-\rangle\langle e_-|$$

$$= \frac{1}{2}\Big(|0\rangle + i|1\rangle\Big)\Big(\langle 0| - i\langle 1|\Big) - \frac{1}{2}\Big(|0\rangle - i|1\rangle\Big)\Big(\langle 0| + i\langle 1|\Big)$$

$$= \frac{1}{2}\Big[\cancel{|0\rangle\langle 0|} - i|0\rangle\langle 1| + i|1\rangle\langle 0| + \cancel{|1\rangle\langle 1|}$$

$$- \cancel{|0\rangle\langle 0|} - i|0\rangle\langle 1| + i|1\rangle\langle 0| - \cancel{|1\rangle\langle 1|}\Big]$$

$$= \frac{1}{2}\Big[-2i|0\rangle\langle 1| + 2i|1\rangle\langle 0|\Big]$$

$$= \underline{-i|0\rangle\langle 1| + i|1\rangle\langle 0| \quad \text{as expected}}$$

(b) The identity operator can be defined on $\mathbb{C}^2$ as

$$I|0\rangle = |0\rangle, \quad I|1\rangle = |1\rangle$$

Check both results:

$$\Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big)|0\rangle = |0\rangle \quad \checkmark$$

$$\Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big)|1\rangle = |1\rangle \quad \checkmark$$

$\checkmark$

Consider $|y_+\rangle\langle y_+| + |y_-\rangle\langle y_-|$ - expand in $\{|0\rangle, |1\rangle\}$ using result in (a) - we find it is equal to the above $|0\rangle\langle 0| + |1\rangle\langle 1|$ - hence it is also the identity operator.

(c) Let $U = |y_+\rangle\langle 0| + |y_-\rangle\langle 1|$

$$= \frac{1}{\sqrt{2}}\Big(|0\rangle + i|1\rangle\Big)\langle 0| + \frac{1}{\sqrt{2}}\Big(|0\rangle - i|1\rangle\Big)\langle 1|$$

$$= \frac{1}{\sqrt{2}}\Big[|0\rangle\langle 0| + i|1\rangle\langle 0| + |0\rangle\langle 1| - i|1\rangle\langle 1|\Big]$$

$$U|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + i|1\rangle\right) = |y_+\rangle$$

$$U|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right) = |y_-\rangle$$

$$UU^\dagger = \left(|y_+\rangle\langle 0| + |y_-\rangle\langle 1|\right)\left(|0\rangle\langle y_+| + |1\rangle\langle y_-|\right)$$

$$= |y_+\rangle\langle y_+| + |y_-\rangle\langle y_-|$$

$$= \underline{\underline{I}} \text{ from (b)}$$

$$U^\dagger U = \left(|0\rangle\langle y_+| + |1\rangle\langle y_+|\right)\left(|y_+\rangle\langle 0| + |y_-\rangle\langle 1|\right)$$

$$= |0\rangle\langle 0| + |1\rangle\langle 1|$$

$$= \underline{\underline{I}} = UU^\dagger \text{ from (b) and (c)}$$

Hence U is <u>unitary</u>.

2. (a) A matrix A is unitary if $AA^\dagger = A^\dagger A = I$, and self-adjoint if $A = A^\dagger$

Write in Dirac form:

$$V = |1\rangle\langle 0| + i|2\rangle\langle 1| - i|0\rangle\langle 1|$$
$$W = |0\rangle\langle 0| + i|1\rangle\langle 1| + \omega|2\rangle\langle 2|$$

Then

$$\left.\begin{aligned} V^\dagger &= |0\rangle\langle 1| - i|1\rangle\langle 2| + i|1\rangle\langle 0| \neq V \\ W^\dagger &= |0\rangle\langle 0| - i|1\rangle\langle 1| + w^*|2\rangle\langle 2| \neq W \end{aligned}\right\} \underline{\underline{\text{not self-adjoint}}}$$

Check if unitary. First V

$$\left.\begin{aligned} VV^\dagger &= |1\rangle\langle 1| + |2\rangle\langle 2| + |0\rangle\langle 0| = I \\ V^\dagger V &= |0\rangle\langle 0| + |2\rangle\langle 2| + |1\rangle\langle 1| = I \end{aligned}\right\} \underline{\underline{V \text{ unitary}}}$$

Now W:

$$WW^\dagger = |0\rangle\langle 0| + |1\rangle\langle 1| + \underbrace{\omega\omega^*}_{=|\omega|^2 = 1} |2\rangle\langle 2| = I$$
$$W^\dagger W = |0\rangle\langle 0| + |1\rangle\langle 1| + \omega^*\omega |2\rangle\langle 2| = I$$

$\Big\}$ W unitary

(b) Use $W|0\rangle = |0\rangle$, $W|1\rangle = i|1\rangle$, $W|2\rangle = \omega|2\rangle$

Now consider W^n.

$$W^n = \underbrace{WWW\ldots W}_{\text{apply } W \ n \text{ times}}$$

Consider effect on basis

$W|0\rangle = |0\rangle \rightarrow$ apply W as many times as we want - always acts as identity

$W|1\rangle = i|1\rangle \rightarrow W$ only acts as identity for $\frac{n}{4} \in \mathbb{Z}^+$

$\ldots \rightarrow \underline{1} \rightarrow i \rightarrow -1 \rightarrow -i \rightarrow \underline{1} \rightarrow \ldots$

$W|2\rangle = e^{\frac{2\pi i}{3}} \rightarrow W$ only acts as identity for $\frac{n}{3} \in \mathbb{Z}^+$

$\ldots \rightarrow \underline{1} \rightarrow e^{\frac{2\pi i}{3}} \rightarrow e^{\frac{4\pi i}{3}} \rightarrow \underline{1} \rightarrow \ldots$

Therefore $n = \mathrm{LCM}(3,4) = \underline{12}$ i.e. $\underline{W^{12} = I_3}$

3. (a) $|v_1\rangle = |0\rangle$, $|v_2\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$

These two states must be orthogonal for Bob to be able to determine with certainty.

$$\langle v_1 | v_2\rangle = 0$$

$$\Rightarrow \langle 0| \left(\cos\theta|0\rangle + \sin\theta|1\rangle \right) = 0$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \underline{\theta = \frac{\pi}{2}, \frac{3\pi}{2}}$$

For these values of θ, we have

$$|v_1\rangle = |0\rangle, \quad |v_2\rangle = \pm|1\rangle$$

We can measure $M = d_1|v_1\rangle\langle v_1| + d_2|v_2\rangle\langle v_2|$

$$= d_1|0\rangle\langle 0| + d_2|1\rangle\langle 1| \quad \text{for any } d_1 \neq d_2$$

(b) $\langle v_1^\perp|v_1\rangle = \langle v_2^\perp|v_2\rangle = 0$

$\langle v_1^\perp|v_2^\perp\rangle = \langle v_1|v_2\rangle$

$\langle v_1|v_2\rangle = \cos\theta$ from (a)

Then we have orthogonality conditions. Consider $\langle v_1^\perp|v_1\rangle = 0$. A natural choice here is to simply make $|v_1^\perp\rangle = |1\rangle$.

Now consider $\langle v_2^\perp|v_2\rangle = 0$. Let $|v_2^\perp\rangle = \alpha|0\rangle + \beta|1\rangle$ such that

$$\langle v_2^\perp|v_1\rangle = (\alpha^*\langle 0| + \beta^*\langle 1|)(\cos\theta|0\rangle + \sin\theta|1\rangle)$$

$$= \alpha^*\cos\theta + \beta^*\sin\theta = 0$$

We also have $\langle v_1^\perp|v_2^\perp\rangle = \cos\theta$ and so

$$\langle v_1^\perp|v_2^\perp\rangle = \beta = \cos\theta$$

We can then say that

$$\alpha^*\cos\theta + \cos\theta\sin\theta = 0$$

$$\Rightarrow \alpha^* = -\sin\theta \Rightarrow \alpha = -\sin\theta$$

Hence we find $|v_2^\perp\rangle = -\sin\theta|0\rangle + \cos\theta|1\rangle$

$$|v_1^\perp\rangle = |1\rangle, \quad |v_2^\perp\rangle = -\sin\theta|0\rangle + \cos\theta|1\rangle$$

(c) $B_2 = d_1 |v_1\rangle\langle v_2| + d_2 |v_1^\perp\rangle\langle v_1^\perp| \quad (d_1 < d_2)$

$$= d_1 |0\rangle\langle 0| + d_2 |1\rangle\langle 1|$$

$$B_2 |v_1\rangle = d_1 |0\rangle$$

$$B_2 |v_2\rangle = d_1 \cos\theta |0\rangle + d_2 \sin\theta |1\rangle$$

So, if Bob measures d_1, then Alice could have sent either $|v_1\rangle$ or $|v_2\rangle$.

But, if Bob measures d_2, then Alice <u>must have</u> sent $|v_2\rangle$, since this measurement cannot occur if Bob measures $|v_1\rangle$.

↳ more specifically, Bob will measure d_1 projected along $|v_1\rangle$ if $|v_1\rangle$ is sent. If $|v_2\rangle$ sent, some projection along $|v_1^\perp\rangle$ will be measured.

4. (a) To be entangled, a state must not be of product form, i.e. it cannot be written as

$$a_0 b_0 |00\rangle + a_1 b_0 |10\rangle + a_0 b_1 |01\rangle + a_1 b_1 |11\rangle$$

Here, we have the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$.

This requires that $a_1 b_0 \neq 0$, $a_0 b_1 \neq 0$ <u>and</u> $a_0 b_0 = 0$, $a_1 b_1 = 0$. This is a direct contradiction - if any of a_0, b_0, a_1 or b_1 are equal to zero (which at least two must be), then $a_1 b_0 = 0$ and $a_0 b_1 = 0$ - hence $|\psi\rangle$ is entangled.

(b) $|\psi_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |00\rangle)$

$$= a_0 b_0 |00\rangle + a_1 b_0 |10\rangle + a_0 b_1 |01\rangle + a_1 b_1 |11\rangle$$

$$a_0 b_0 = \frac{1}{\sqrt{2}}, \quad a_1 b_0 = 0, \quad a_0 b_1 = \frac{1}{\sqrt{2}}, \quad a_1 b_1 = 0$$

$$= \left[a_0 |0\rangle\right] \otimes \left[b_0 |0\rangle + b_1 |1\rangle\right]$$

Including normalisation we find

$$|\psi_1\rangle = |0\rangle \otimes \left[\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right]$$ hence unentangled

Now consider $|\psi_2\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$

$$= \left[\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right] \otimes \left[\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right]$$

hence unentangled.

S// $|\psi\rangle = \sum_{i,j=0}^{1} \alpha_{ij}|i\rangle|j\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

Let $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$ and $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$|\psi_1\rangle|\psi_2\rangle = \left[\alpha_1|0\rangle + \beta_1|1\rangle\right] \otimes \left[\alpha_2|0\rangle + \beta_2|1\rangle\right]$$

$$= \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \beta_1\alpha_2|10\rangle + \beta_1\beta_2|11\rangle$$

This implies that for the condition $|\psi\rangle = |\psi_1\psi_2\rangle$, we require

$$\left.\begin{aligned} \alpha_{00} &= \alpha_1\alpha_2 \\ \alpha_{01} &= \alpha_1\beta_2 \\ \alpha_{10} &= \alpha_2\beta_1 \\ \alpha_{11} &= \beta_1\beta_2 \end{aligned}\right\} \quad \left.\begin{aligned} \alpha_1 &= \frac{\alpha_{00}}{\alpha_2} \\ \alpha_1 &= \frac{\alpha_{01}}{\beta_2} \end{aligned}\right\} \quad \frac{\alpha_{00}}{\alpha_2} = \frac{\alpha_{01}}{\beta_2} \Rightarrow \frac{\alpha_2}{\beta_2} = \frac{\alpha_{00}}{\alpha_{01}}$$

$$\left.\begin{aligned} \beta_1 &= \frac{\alpha_{10}}{\alpha_2} \\ \beta_1 &= \frac{\alpha_{11}}{\beta_2} \end{aligned}\right\} \quad \frac{\alpha_{10}}{\alpha_2} = \frac{\alpha_{11}}{\beta_2} \Rightarrow \frac{\alpha_2}{\beta_2} = \frac{\alpha_{10}}{\alpha_{11}}$$

$$\therefore \underline{\alpha_{00}\alpha_{11} = \alpha_{10}\alpha_{01}}$$

And so $|\psi\rangle$ can be written in product form on the condition that

$$\underline{\alpha_{00}\alpha_{11} = \alpha_{10}\alpha_{01}}$$

b// (a) Prove $(A+B)^T = A^T + B^T$

Definition of an adjoint of say x is that $\langle v|X^\dagger|w\rangle = \langle w|X|v\rangle^*$ for any $|v\rangle, |w\rangle$.

$$\langle v|(A+B)^\dagger|w\rangle = \langle w|(A+B)|v\rangle^*$$

$$= \langle w|(A|v\rangle + B|v\rangle)^*$$

$$= \langle w|A|v\rangle^* + \langle w|B|v\rangle^*$$

$$= \langle v|A^\dagger|w\rangle + \langle v|B^\dagger|w\rangle$$

$$= \langle v|(A^\dagger + B^\dagger)|w\rangle$$

Therefore $\underline{(A+B)^\dagger = A^\dagger + B^\dagger}$

(b) Show $(AB)^\dagger = B^\dagger A^\dagger$.

$$\langle v|(AB)^\dagger|w\rangle = \langle w|AB|v\rangle^*$$

$$= \langle w|A\left(B|v\rangle^*\right)$$

Let $|u\rangle = B|v\rangle^*$.

$$\langle v|AB|w\rangle = \langle w|A|u\rangle = \langle u|A^\dagger|w\rangle$$

Now recall that $\langle v|B^\dagger = \langle u|$ and hence

$$\langle v|(AB)^\dagger|w\rangle = \langle v|B^\dagger A^\dagger|w\rangle$$

$\underline{\text{Therefore } (AB)^\dagger = B^\dagger A^\dagger}$.

Now show $(A^\dagger)^n = (A^n)^\dagger$

$$(A^\dagger)^n = \underbrace{A^\dagger A^\dagger A^\dagger \dots A^\dagger}_{n \text{ times}} = \underbrace{(AA)^\dagger (AA)^\dagger (AA)^\dagger \dots (AA)^\dagger}_{\frac{n}{2} \text{ times}} = \left((AA)^{\frac{n}{2}}\right)^\dagger$$

$$= \left((A^2)^{n/2}\right)^\dagger = \underline{\underline{(A^n)^\dagger}} \quad \text{if } n \text{ even}$$

Note, if n is odd, we have an extra A as

$$(A^\dagger)^n = \underbrace{A^\dagger A^\dagger \ldots A^\dagger}_{n \text{ times}} = A^\dagger \underbrace{(AA)^\dagger \ldots (AA)^\dagger}_{n-1/2 \text{ times}} = A^\dagger (A^{n-1})^\dagger$$

$$= (A^{n-1}A)^\dagger = \underline{\underline{(A^n)^\dagger}} \quad \text{if } n \text{ odd}$$

<u>Therefore $(A^\dagger)^n = (A^n)^\dagger$ for all n integer ≥ 1.</u>

(c) U, V unitary

$U^2 \rightarrow \; U^2 = UU$.

$$U^2(U^2)^\dagger = UU(UU)^\dagger = UUU^\dagger U^\dagger = UIU^\dagger = UU^\dagger = I \quad \therefore \; \underline{U^2 \text{ unitary}}$$

$U^2V^3 \rightarrow \; U^2V^3 = UUVVV$

$$U^2V^3(U^2V^3)^\dagger = UUVVV(UUVVV)^\dagger$$

$$= UUVVVV^\dagger V^\dagger V^\dagger U^\dagger U^\dagger = I \quad \therefore \; \underline{U^2V^3 \text{ unitary}}$$

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