If a basis is not specified in a question, it is assumed to be the standard one e.g. $|0\rangle, |1\rangle$ for single qubit states, and $|00\rangle, |01\rangle, |10\rangle, |11\rangle$, in that order, for two qubit states.

The Pauli operators X, Y and Z on $\mathbb{C}^2$ are defined by

$$\begin{aligned} X|0\rangle &= |1\rangle; & X|1\rangle &= |0\rangle \\ Y|0\rangle &= i|1\rangle; & Y|1\rangle &= -i|0\rangle \\ Z|0\rangle &= |0\rangle; & Z|1\rangle &= -|1\rangle. \end{aligned}$$

1. Pauli operators/Pauli matrices

(a) Calculate $YZ|0\rangle$ and $YZ|1\rangle$, and hence show for any state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ that $YZ|\psi\rangle = iX|\psi\rangle$. This shows that $YZ = iX$ (why?).

(b) Calculate the Pauli matrices σ_x, σ_y and σ_z, the matrices associated to X, Y and Z in the standard basis. Show that $\sigma_y\sigma_z = i\sigma_x$. (Notice the correspondence to part (a)).

2. Dirac Form

(a) Find the matrix of Y with respect to the standard basis and hence show that Y is self-adjoint and unitary.

(b) Write Y in Dirac form in the standard basis.

(c) Using the Dirac form for Y, calculate $Y|+\rangle$ and $Y|-\rangle$, where $|+\rangle$ and $|-\rangle$ are the $+1$ and -1 eigenstates of X. Confirm that you get the same result by calculating these states directly from the definition of Y.

3. Matrix versus operator notation

Consider the operator A and the state $|\phi\rangle$ on $\mathbb{C}^2$ defined by

$$\begin{aligned} A|0\rangle &= a|0\rangle + b|1\rangle; \qquad \text{and} \qquad & |\phi\rangle &= \alpha|0\rangle + \beta|1\rangle \\ A|1\rangle &= c|0\rangle + d|1\rangle; \end{aligned}$$

$a, b, c, d, \alpha, \beta$ are constants.

(a) Find the matrix M of A with respect to the standard basis.

(b) Find the Dirac form for A (i.e. write A in the form $\sum A_{ij}|i\rangle\langle j|$) with respect to the standard basis.

(c) Calculate the action of the matrix M on the column vector $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

(d) Calculate $A|\phi\rangle$ directly using operator notation. Confirm that your two calculations agree.

4. States and operators on $\mathbb{C}^3$

Show that the following three states form an orthonormal basis for $\mathbb{C}^3$

$$\begin{aligned} |\psi_0\rangle &= \frac{1}{\sqrt{3}}\Big(|0\rangle + |1\rangle + |2\rangle\Big); \\ |\psi_1\rangle &= \frac{1}{\sqrt{3}}\Big(|0\rangle + \omega|1\rangle + \omega^2|2\rangle\Big); \\ |\psi_2\rangle &= \frac{1}{\sqrt{3}}\Big(|0\rangle + \omega^2|1\rangle + \omega|2\rangle\Big), \end{aligned}$$

where $|0\rangle$, $|1\rangle$ and $|2\rangle$ are orthonormal basis elements for $\mathbb{C}^3$ and $\omega = \exp(2\pi i/3)$.

Write down the matrix of the operator M that satisfies $M|a\rangle = |\psi_a\rangle$ for $a = 0, 1, 2$. What can you say about M?

5. Self-adjoint and unitary operators

Let the operator W on $\mathbb{C}^2$ be defined by

$$\begin{aligned} W|0\rangle &= |0\rangle + |1\rangle; \\ W|1\rangle &= |1\rangle. \end{aligned}$$

(a) Find the matrix of W in the standard basis, find its adjoint, $W^\dagger$; you should write down $W^\dagger|0\rangle$ and $W^\dagger|1\rangle$ explicitly. Show that W is neither self-adjoint nor unitary.

(b) Let the operator T be defined by $T = WW^\dagger$. Calculate $T|0\rangle$ and $T|1\rangle$. Show that T is self-adjoint, but not unitary.