1. **Density operators**

   A source S emits qubits at random in one of the following three states

   $$|\psi_1\rangle = |0\rangle, \qquad |\psi_2\rangle = |1\rangle, \qquad |\psi_3\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

   where $|0\rangle$ and $|1\rangle$ are the standard orthonormal basis states for $\mathbb{C}^2$. Each state is emitted with probability $\frac{1}{3}$.

   (a) Find the density operator ρ associated with this source, expressed as a matrix in the $|0\rangle, |1\rangle$ basis.

   (b) If Y is measured on the source, what are the possible outcomes, and what are their corresponding probabilities?

   (c) Compute the eigenvalues and eigenvectors of ρ.

   (d) A source S' emits qubits in the states

   $$|\phi_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |\phi_2\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}},$$

   with probabilities p and $1-p$ respectively. Using part (c), or otherwise, find a value of p for which the output of source S' is indistinguishable from the output of source S.

   (e) Note that the eigenvalues of ρ are non-negative numbers which sum to one. Show that this will always be the case for the eigenvalues of a density operator, i.e. for any Hermitian operator σ satisfying $\mathrm{tr}(\sigma) = 1$, and $\langle u|\sigma|u\rangle \geq 0$ for all $|u\rangle$.

   (f) (*Tricky*) Suppose that two sources are characterised by different density operators σ_1 and σ_2, with $\sigma_1 \neq \sigma_2$. Show that the Hermitian operator $A = \sigma_1 - \sigma_2$ ($\neq 0$) has a different expectation value when measured on the output states of the two sources (and can hence distinguish them). *Hint: You may find it useful to use 2 (b) to show this.*

2. **The trace**

   The trace of an operator is the linear operation given by

   $$\mathrm{tr}(|u\rangle\langle v|) = \langle v|u\rangle,$$

   where the states $|u\rangle$ and $|v\rangle$ are any states from the Hilbert space.

   (a) Show that the trace is equivalently given by

   $$\mathrm{tr}(A) = \sum_i \langle i|A|i\rangle,$$

   where A is an arbitrary operator and $|i\rangle$ forms an arbitrary orthonormal basis.

   (b) Show that

   $$\mathrm{tr}(A^\dagger A) = \sum_{i,j} |\langle i|A|j\rangle|^2,$$

   where A is an arbitrary operator and $|i\rangle$ forms an arbitrary orthonormal basis.

   (c) Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$, for any operators A and B.

   (d) Show that $\mathrm{tr}(ABC) = \mathrm{tr}(CAB) = \mathrm{tr}(BCA)$, for any operators A and B an C.

   (e) After undergoing unitary time evolution, a density operator ρ will be transformed into $\rho' = U\rho U^\dagger$, where U is unitary. Show that $\mathrm{tr}(U\rho U^\dagger) = \mathrm{tr}(\rho)$. What is the physical significance of this?

3. **The partial trace**

The partial trace over the first space is the linear operation given by

$$\mathrm{tr}_1\left(|u\rangle\langle v| \otimes |w\rangle\langle x|\right) = \langle v|u\rangle |w\rangle\langle x|$$

where the states $|u\rangle$ and $|v\rangle$ are any states from the Hilbert space of Alice and $|w\rangle$ and $|x\rangle$ are any states from the Hilbert space of Bob.

(a) Show that

$$\sum_i \left(\langle i| \otimes I\right)\left(|u\rangle\langle v| \otimes |w\rangle\langle x|\right)\left(|i\rangle \otimes I\right) = \langle v|u\rangle |w\rangle\langle x|.$$

(b) Explain why this implies more generally that an alternative expression for the partial trace over the first subsystem is

$$\mathrm{tr}_1(E) = \sum_i \left(\langle i| \otimes I\right) E \left(|i\rangle \otimes I\right),$$

where E is any operator on the joint Hilbert space.

(c) For the case of a two qubits (each Hilbert space is $\mathbb{C}^2$), write out the matrix in the standard basis of the operators $\langle 0| \otimes I$ and $|1\rangle \otimes I$.

(d) What are the analogous results to parts (a) and (b) for the partial trace over the second space?

4. ***(optional)*** **The Bloch sphere**

Consider the Bloch sphere representation for a qubit, given by

$$\rho = \frac{1}{2}(I + n_x X + n_y Y + n_z Z)$$

where $\mathbf{n} = (n_x, n_y, n_z)$ is a real vector of length $|\mathbf{n}| = \sqrt{n_x^2 + n_y^2 + n_z^2} \leq 1$, and X, Y and Z are the Pauli operators.

(a) By writing ρ as a matrix in the $|0\rangle, |1\rangle$ basis, or otherwise, show that the eigenvalues of ρ are given by

$$\lambda = \frac{1}{2}\left(1 \pm |\mathbf{n}|\right).$$

(b) Thus explain why points on the surface of the Bloch sphere represent pure states, whilst points in the interior of the sphere represent mixed states.

(c) Express the projectors $|+\rangle\langle+|$ and $|-\rangle\langle-|$ in terms of their Bloch vectors. Hence, show that if we perform a measurement of X on a density operator with Bloch vector $\vec{n}$, then the probability of the outcomes are given by

$$\mathrm{Prob}(+1) = \frac{1}{2}(1 + n_x), \qquad \mathrm{Prob}(-1) = \frac{1}{2}(1 - n_x). \tag{6.1}$$

(d) Consider applying the unitary transformation

$$U(\theta) = \cos\left(\frac{\theta}{2}\right) I + i \sin\left(\frac{\theta}{2}\right) Z$$

to the state ρ such that it is transformed into $\rho' = U(\theta)\rho U(\theta)^\dagger$. If the initial Bloch vector is $\mathbf{n} = (1, 0, 0)$, show that the final Bloch vector is $\mathbf{n}' = (\cos(\theta), -\sin(\theta), 0)$, i.e. that the Bloch vector is rotated by an angle $-\theta$ about the z-axis.