1. Eigenvalues/eigenvectors and unitary operators

(a) Write down the Dirac form for Y in the standard basis. Show that Y has eigenvalues $+1$ and -1 and find the associated normalised eigenvectors $|y_+\rangle$ and $|y_-\rangle$. Hence write Y in diagonal form. Verify that the two expressions you have given for Y coincide.

(b) Show that $|0\rangle\langle 0| + |1\rangle\langle 1|$ and $|y_+\rangle\langle y_+| + |y_-\rangle\langle y_-|$ are both expressions for the identity operator on $\mathbb{C}^2$.

(c) Let $U = |y_+\rangle\langle 0| + |y_-\rangle\langle 1|$. Calculate the action of U on $|0\rangle$ and $|1\rangle$. Show that $UU^\dagger = U^\dagger U = I$, where I is the identity operator on $\mathbb{C}^2$.

2. Unitary operators

The operators V and W on $\mathbb{C}^3$ are defined by

$$\begin{aligned} V|0\rangle &= |1\rangle; \quad V|1\rangle = i|2\rangle; \quad V|2\rangle = -i|0\rangle \\ W|0\rangle &= |0\rangle; \quad W|1\rangle = i|1\rangle; \quad W|2\rangle = \omega|2\rangle \end{aligned}$$

where $|0\rangle$, $|1\rangle$ and $|2\rangle$ are orthonormal basis elements for $\mathbb{C}^3$, and $\omega = \exp(2\pi i/3)$.

(a) Show that V and W are unitary but not self-adjoint.

(b) What is the smallest integer $n > 0$ such that $W^n = I_3$, where I_3 is the identity operator on $\mathbb{C}^3$?

3. Distinguishing non-orthogonal states

(a) Let Alice send a qubit to Bob in one of the two states

$$|v_1\rangle = |0\rangle \quad \text{or} \quad |v_2\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle,$$

where θ is a real constant $0 \le \theta < 2\pi$. For what values of θ can Bob determine the state that Alice sent, with certainty? For these values of θ, give a measurement operator which Bob can use to distinguish the states.

(b) We now consider the states $|v_1\rangle$ and $|v_2\rangle$ as in part (a), but now for values of θ for which the states are not orthogonal. Find normalised states $|v_1^\perp\rangle$ and $|v_2^\perp\rangle$ satisfying

$$\langle v_1^\perp|v_1\rangle = \langle v_2^\perp|v_2\rangle = 0, \quad \langle v_1^\perp|v_2^\perp\rangle = \langle v_1|v_2\rangle.$$

(c) Alice now sends a particle to Bob in either state $|v_1\rangle$ or $|v_2\rangle$. Consider the operator

$$B_2 = \lambda_1|v_1\rangle\langle v_1| + \lambda_2|v_1^\perp\rangle\langle v_1^\perp|,$$

where $\lambda_1 < \lambda_2$. Bob measures B_2 on the qubit Alice sent. What can Bob say about the state of the qubit if he gets the outcome (a) λ_1, (b) λ_2?

4. Entangled and un-entangled states of two qubits

(a) Show that

$$\frac{|0\rangle|1\rangle + |1\rangle|0\rangle}{\sqrt{2}}$$

is entangled.

(b) By explicitly writing them in product form, show that

$$\frac{|0\rangle|1\rangle + |0\rangle|0\rangle}{\sqrt{2}} \qquad \text{and} \qquad \frac{|0\rangle|0\rangle + |0\rangle|1\rangle + |1\rangle|0\rangle + |1\rangle|1\rangle}{2}$$

are un-entangled.

5. Product states of two systems

Prove that a state of two qubits $|\psi\rangle = \sum_{i,j=0}^{1} \alpha_{ij}|i\rangle|j\rangle$ is of product form, (i.e. it may be written in the form $|\psi_1\rangle|\psi_2\rangle$ for some $|\psi_1\rangle$ and $|\psi_2\rangle$) if and only if $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10}$.

6. Adjoints

(a) Use the definition of the adjoint of an operator which was given in the lectures to show that $(A+B)^\dagger = A^\dagger + B^\dagger$ for two operators A and B.

(b) Show also that $(AB)^\dagger = B^\dagger A^\dagger$, and thus $(A^\dagger)^n = (A^n)^\dagger$ when n is a integer ≥ 1.

(c) Let U and V be unitary operators. Show that U^2 and U^2V^3 are both unitary.