Exercise sheet 6

This is assignment is due on March 6th at noon, to be submitted electronically via Blackboard.

1. Let X be a non-negative random variable, i.e. for some sample space Ω, $X : \Omega \to \mathbb{R}_{\geq 0}$. Take $\lambda > 0$ and define the set

$$\Delta_\lambda = \{\omega \in \Omega : \; X(\omega) \geq \lambda\}.$$

For example, if $X : \{Heads, Tails\} \to \mathbb{R}_{\geq 0}$ *is a Bernoulli random variable defined by* $X(Heads) = 1$, $X(Tails) = 0$ *then, say,* $\Delta_{1/2} = \{Heads\}$, $\Delta_{51} = \emptyset$.

   (a) Show that for any such X and λ,

$$\lambda \cdot \chi_{\Delta_\lambda}(\omega) \leq X(\omega) \qquad \text{where} \qquad \chi_A(\omega) := \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases}.$$

   (b) Hence show the inequality

$$\mathbb{P}(X \geq \lambda) \leq \frac{1}{\lambda}\mathbb{E}[X]. \tag{1}$$

   (c) Now relax the condition on the codomain of X, so let $X : \Omega \to \mathbb{R}$ be a *signed* random variable (i.e. one that can take negative values). Use (1) to derive, for any $k \in \mathbb{N}$ and $\varepsilon > 0$,

$$\mathbb{P}(|X| \geq \varepsilon) \leq \frac{1}{\varepsilon^k}\mathbb{E}\big[|X|^k\big],$$

   (d) Hence show the *Chebyshev inequality*

$$\mathbb{P}\big(|X - \mathbb{E}[X]| \geq \varepsilon\big) \leq \frac{1}{\varepsilon^2}\operatorname{Var}(X).$$

2. Let $\{\lambda_j\}_{j=1}^N$ be a collection of, possibly N-dependent, i.i.d. real-valued random variables, all with the same (possibly N dependent) law

$$F_N(\Delta) = \mathbb{P}_N(\lambda_1 \in \Delta). \tag{2}$$

Consider the diagonal $N \times N$ random matrix $M_N(\xi_{j,k})_{j,k=1}^N$, with entries

$$\xi_{j,k} := \begin{cases} \lambda_j, & j = k \\ 0, & j \neq k \end{cases}.$$

Let $\overline{\mathcal{N}}_N(\Delta)$ denote the normalized counting measure of its eigenvalues with expectation $\mathbb{E}[\overline{\mathcal{N}}_N(\Delta)]$. Recall that we call $\mathbb{P}(\mathcal{N}_N(\Delta) = \ell)$ the *occupation probabilities*, for some interval $\Delta \subset \mathbb{R}$. Show that

(a) $\mathbb{E}_N[\overline{\mathcal{N}}_N(\Delta)] = F_N(\Delta)$,

(b) $\mathrm{Var}_N\left(\overline{\mathcal{N}}_N(\Delta)\right) = \frac{1}{N}F_N(\Delta)\big(1 - F_N(\Delta)\big)$,

(c) $\mathbb{P}_N(\mathcal{N}_N(\Delta) = \ell) = \binom{N}{\ell}\big(F_N(\Delta)\big)^{\ell}\big(1 - F_N(\Delta)\big)^{N-\ell}$.

3. Prove the $\ell = 2$ case for Lemma 15.5, i.e. that for any $\Delta \subset \mathbb{R}$,

$$\mathbb{P}_N(\mathcal{N}_N(\Delta) = 2) = \frac{1}{2}\frac{d^2}{da^2}\mathbb{E}_N\left[\prod_{j=1}^{N}(1 - a\chi_\Delta(\lambda_j))\right]\Bigg|_{a=1}.$$

4. Suppose that in (2) the law F_N satisfies $F({}_N\Delta) = \int_\Delta \rho_N(x)dx$, with smooth density $\rho_N : \mathbb{R} \to \mathbb{R}_{>0}$. For $s \geq 0, x_0 \in \mathbb{R}$, consider the interval $\Delta_N \subset \mathbb{R}$ defined by

$$\Delta_N := \left(x_0, x_0 + \frac{s}{N\rho(x_0)}\right) \subset \mathbb{R}.$$

Show that the occupation probabilities in 2.(c) satisfy

$$\lim_{N\to\infty} \mathbb{P}_N(\mathcal{N}_N(\Delta_N) = \ell) = \frac{s^\ell}{\ell!}e^{-s},$$

for any $\ell \in \mathbb{Z}_{\geq 0}$ and $s \geq 0$.

5. Let $\Phi : X \to \mathbb{R}$ be a map from a vector space X to $\mathbb{R}$. We say Φ is *Gateaux differentiable* at $\varphi \in X$ in direction $\psi \in X$, if the limit

$$\lim_{\varepsilon\to 0}\frac{\Phi[\varphi + \varepsilon\psi] - \Phi[\varphi]}{\varepsilon} = \frac{d}{d\varepsilon}\Phi[\varphi + \varepsilon\psi]\Big|_{\varepsilon=0} \tag{3}$$

exists. In case the above exists for all directions $\psi \in X$, then Φ is called Gateaux differentiable at $\varphi \in X$, written

$$\left(\frac{\delta}{\delta\varphi}\Phi[\varphi]\right)[\psi] := \lim_{\epsilon\to 0}\frac{\Phi[\varphi + \epsilon\psi] - \Phi[\varphi]}{\epsilon}. \tag{4}$$

For any $y \in \mathbb{R}$, consider the map $\delta_y : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} f(x)\delta_y(x)dx = f(y)$$

for any integrable function $f : \mathbb{R} \to \mathbb{R}$ and abbreviate

$$\frac{\delta}{\delta\varphi(y)}\Phi[\varphi] := \left(\frac{\delta}{\delta\varphi}\Phi[\varphi]\right)[\delta_y].$$

Using the expression derived in lectures for the gap probability (cf. (15.8)), show that

$$\frac{\delta}{\delta\varphi(y)}\mathbb{E}_N\left[\prod_{j=1}^{N}(1-\varphi(\lambda_j))\right]\Bigg|_{\varphi=0} = -NP_{N,1}(y)$$

$$\frac{\delta}{\delta\varphi(y)}\ln\mathbb{E}_N\left[\prod_{j=1}^{N}(1-\varphi(\lambda_j))\right]\Bigg|_{\varphi=0} = -NP_{N,1}(y).$$

(Hint for the second: for x approaching 0*,* $\ln(1+x) = x + O(x^2)$*. Using this we can compare to the first equality.)*