Review of basic notions of linear algebra

- We will denote the N-dimensional space of complex vectors $\mathbf{v} = (v_1, v_2, \ldots, v_N)$ by $\mathbb{C}^N$.

- The sum of two vectors $\mathbf{v}$ and $\mathbf{w}$ can be defined in a natural way by adding element by element:
$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_N + w_N)\,. \tag{1}$$

- Multiplication of a vector by a complex number α (scalar) is defined by
$$\alpha\mathbf{v} = (\alpha v_1, \alpha v_2, \ldots, \alpha v_N)\,. \tag{2}$$

- m vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m$ are *linearly independent* if
$$\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_m\mathbf{e}_m = 0 \tag{3}$$
implies $\alpha_1 = \alpha_2 = \ldots = \alpha_m = 0$.

- An N-dimensional vector space has N independent vectors. A set $\{\mathbf{e}_j\}_{j=1,\ldots,N}$ of N independent vectors is called *basis*. A canonical basis is given by the N-tuple
$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \quad \mathbf{e}_N = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}. \tag{4}$$

- Given a basis $\{\mathbf{e}_j\}_{j=1,\ldots,N}$ a vector can be written in a unique way as linear combination of the basis vectors:
$$\mathbf{v} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_N\mathbf{e}_N, \quad \alpha_j \in \mathbb{C}. \tag{5}$$
$\alpha_1, \alpha_2, \ldots, \alpha_N$ are called *components* of $\mathbf{v}$ in the basis $\{\mathbf{e}_j\}_{j=1,\ldots,N}$.

- If we write $\mathbf{v} = (v_1, v_2, \ldots, v_N)$, the v_j are the components of $\mathbf{v}$ in the canonical basis.

Elementary notions to be remembered:

- If $M = (m_{ij})$ is a $N \times N$ matrix (with real or complex elements), then $\mathbf{x} \neq \mathbf{0}$ is said to be an eigenvector of M corresponding to the eigenvalue $\lambda \in \mathbb{C}$ if $M\mathbf{x} = \lambda\mathbf{x}$.

- The eigenvalues of a matrix are computed by finding the roots of the characteristic polynomial
$$Z(\lambda) = \det(M - \lambda I) = \sum_{k=0}^{N} a_k \lambda^k \tag{6}$$

- Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of M, then
$$\det(M) = \prod_{j=1}^{N} \lambda_j. \tag{7}$$

- Many $N \times N$ matrices (not all) can be diagonalized, that is they admit N independent eigenvectors.
- Let Y be an arbitrary $N \times N$ matrix and write

$$M' = Y^{-1} M Y. \tag{8}$$

  We say that $M'$is conjugated to M by Y.
- Let λ_j be the j-th eigenvalue of M and $\mathbf{x}_j$ its corresponding eigenvector. If X is the matrix whose j-th column is $\mathbf{x}_j$, then

$$X^{-1} M X = \Lambda, \tag{9}$$

  where Λ is the diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix} \tag{10}$$

- We say that X *diagonalizes* M. (M is conjugated by X to a diagonal matrix.)
- The determinant of the matrix has the following properties:

$$\det(AB) = \det(A)\det(B), \tag{11}$$

$$\det(Y^{-1} M Y) = \det(Y^{-1})\det(M)\det(Y) = \det(M) \tag{12}$$

  which follows from the identity

$$\det(Y^{-1}) = 1/\det(Y). \tag{13}$$

- The trace $\operatorname{Tr} M$ is the sum of all the diagonal elements of M,

$$\operatorname{Tr}(M) = \sum_{j=1}^{N} m_{jj}. \tag{14}$$

  It has the cyclic property

$$\operatorname{Tr}(ABC) = \operatorname{Tr}(BCA) = \operatorname{Tr}(CAB). \tag{15}$$

- The cyclic property implies that

$$\operatorname{Tr}(X^{-1} M X) = \operatorname{Tr}(M). \tag{16}$$

  Functions of matrices that are invariant under conjugation by an arbitrary matrix are called *class functions.*
- If X diagonalizes M then

$$\operatorname{Tr}(M) = \operatorname{Tr}(X \Lambda X^{-1}) = \operatorname{Tr}(\Lambda) = \sum_{j=1}^{N} \lambda_j. \tag{17}$$

The trace of a matrix that can be diagonalized is the sum of its eigenvalues.