On this problem sheet, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

1. Consider the set of $N \times N$ complex matrices $M = (m_{jk})$ with no constraints imposed on their elements. Denote the generic entry by $m_{jk} = x_{jk} + iy_{jk}$, where x_{jk} and y_{jk} are the real and imaginary parts, respectively. The Ginibre ensemble is defined by requiring that the x_{jk}'s and y_{jk}'s are independently identically distributed with probability density functions

$$p(x_{jk}) = \frac{1}{\sqrt{\pi}} e^{-x_{jk}^2} \qquad \text{and} \qquad p(y_{jk}) = \frac{1}{\sqrt{\pi}} e^{-y_{jk}^2}, \qquad j,k = 1,\ldots,N. \tag{1}$$

The eigenvalues of M are complex numbers, $z_1,\ldots,z_N$. You are given that their joint probability density function (*j.p.d.f.*) is

$$P(z_1,\ldots,z_N) = \frac{1}{K} \exp\left(-\sum_{j=1}^{N} |z_j|^2\right) \prod_{1\le j<k\le N} |z_k - z_j|^2, \quad z_j \in \mathbb{C}, \quad j = 1,\ldots,N, \tag{2}$$

where $K = \pi^N \prod_{j=1}^N j!$.

(a) Denote by $p_j(z)$ a polynomial of degree j. The scalar product between two polynomials in the complex plane is defined by the two-dimensional integral

$$(p_j, p_k) = \int_{\mathbb{C}} w(z) p_j(z) \overline{p_k(z)} d^2 z, \tag{3}$$

where if $z = x + iy$ then $d^2 z = dxdy$ is the two-dimensional differential. The notation $\overline{p(z)}$ indicates the complex conjugate of $p(z)$. Here $w(z) \ge 0$ is a non-negative function from the complex numbers to the real line. Now set $w(z) = \exp(-|z|^2)$.

i. **(5 marks)**
Show that with this choice of weighting function the monomials z^j, $j = 1,2,\ldots$, form a system of orthogonal polynomials with respect to scalar product (3).

ii. **(10 marks)**
Show that we can write the *j.p.d.f.* (2) as

$$P(z_1,\ldots,z_N) = \frac{1}{N!} \det_{N\times N} \left(K_N(z_j, z_k)\right), \tag{4}$$

where

$$K_N(z,w) = \frac{e^{-\frac{|z|^2}{2} - \frac{|w|^2}{2}}}{\pi} \sum_{l=0}^{N-1} \frac{z^l \overline{w}_k^l}{l!}. \tag{5}$$

iii. **(10 marks)**
Check that the kernel (5) satisfies the hypotheses of Gaudin's lemma.

(b) **(10 marks)**
Recall that the n-point correlation function is defined by

$$R_n(z_1, \ldots, z_n) = \frac{N!}{(N-n)!} \int_{\mathbb{C}} \cdots \int_{\mathbb{C}} P(z_1, \ldots, z_N) d^2 z_{n+1} \cdots d^2 z_N. \tag{6}$$

Prove that

$$R_n(z_1, \ldots, z_n) = \det_{n \times n} \left(K_N(z_j, z_k) \right). \tag{7}$$

(c) **(5 marks)**
Show that the one-point correlation function (or one-level density) is

$$R_1(z) = \pi^{-1} e^{-|z|^2} \sum_{j=0}^{N-1} \frac{|z|^{2j}}{j!}. \tag{8}$$

(d) **(15 marks)**
Write $|z| = r$. You are given the following inequalities

$$\pi R_1(z) \leq e^{-r^2} \frac{r^{2N}}{N!} \frac{N}{r^2 + 1 - N} \qquad \text{for} \qquad r^2 > N, \tag{9a}$$

$$1 - \pi R_1(z) \leq e^{-r^2} \frac{r^{2N}}{N!} \frac{N+1}{N+1-r^2} \qquad \text{for} \qquad r^2 < N. \tag{9b}$$

Now set $r = N^{1/2} \pm \frac{u}{\sqrt{2}}$, where $0 \leq u \ll \sqrt{N}$, the $+$ sign applies to Eq. (9a) and the $-$ to Eq. (9b). Show that for large N the right-hand sides of Eqs. (9a) and (9b) can be approximated by

$$e^{-u^2} / (2\sqrt{\pi} u) \tag{10}$$

in the sense that what is neglected tends to 0 as $N \to \infty$.
Hint: You may find the following formulae useful:

$$N! \approx \sqrt{2\pi N} \left(\frac{N}{e} \right)^N \qquad \text{for large } N. \tag{11a}$$

$$\log(1 \pm t) = \pm t - \frac{t^2}{2} + O(t^3) \qquad \text{as } t \to 0. \tag{11b}$$

(e) (**5 marks**)
Using Eqs. (9a), (9b) and (10) describe the behaviour of $R_1(z)$.