TEST FUNCTIONS IN CONSTRAINED INTERPOLATION

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ABSTRACT. We give a set of test functions for H_1^∞, an algebra of holomorphic functions with constraints on derivatives, whose interpolation problem was studied by Davidson, Paulsen, Raghupathi and Singh. We show that this set of test functions is minimal, by relating these ideas to realisation and interpolation problems.

1. BACKGROUND AND INTRODUCTION

There has been increased interest in constrained interpolation problems recently. These problems have many of the unusual characteristics of harder interpolation problems (they generally require collections of kernels, and have interesting behavior in the case of matrix valued interpolation, much like interpolation on multiply-connected domains), but are simple enough that we can do calculations explicitly (the kernels are typically rational functions), and we can re-use much of the theory of interpolation on the unit disc.

The basic problem, as discussed in [DPRS07], is the following: Under what circumstances can we find a bounded, holomorphic function f on the disc, with zero derivative at 0 (that is,

$$f \in H_1^\infty := \{g \in H^\infty : g'(0) = 0\}$$

for our function f), which takes prescribed values $w_1, \ldots, w_n$ at prescribed points $z_1, \ldots, z_n$? The solution, as given in [DPRS07], is analogous to the Nevanlinna-Pick theorem; such a function exists if and only if

$$\left((1 - w_i \overline{w_j}) k^s(z_j, z_i)\right)_{i,j=1}^n \geq 0 \quad \forall s \in S^2,$$

where S^2 denotes the real 2-sphere, and the kernels k^s are a particular class of kernels, parameterised by points s on the sphere[1].

In this paper, we look at interpolation with test functions in H_1^∞. The test function approach to interpolation originates with Agler (see [AM02]

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[1]They actually give two, different, Nevanlinna-Pick type theorems, but the second is not relevant here.

for a treatment of Agler's approach to this subject, as well as for general background on the field), although we'll be taking results and notation from Dritschel and McCullough's paper ([DM07]) on the subject.

We give a set of test functions for H_1^∞, broadly following the approach of [AHR08], via a Herglotz representation for H_1^∞. These test functions turn out to be rational functions, and the set of test functions is parameterised by the sphere. We show, using techniques similar to those in [DM07], that our set of test functions is minimal. We give some indication of how these techniques could yield test functions for other types of constrained interpolation problem, although the theory appears to be less elegant in these situations.

We'll also introduce the idea of *differentiating kernels*. These are a simple analogue of reproducing kernels, and whilst they're not particularly interesting in and of themselves, they have proved to be a useful tool when working with problems of this sort.

2. Differentiating Kernels

In this paper, it's convenient to introduce *differentiating kernels*. These are along much the same lines as reproducing kernels: We know by Cauchy's integral formula that differentiation is a bounded linear functional on H^2, so for each $x \in \mathbb{D}$, and for each $i = 0, 1, 2, \ldots$, there exists some function $k\left(x^{(i)}, \cdot\right) \in H^2$ such that

$$f^{(i)}(x) = \left\langle f(\cdot), k\left(x^{(i)}, \cdot\right) \right\rangle .$$

The above argument also holds for $H^2(R)$, for any finitely connected planar domain R. In the case of H^2 (that is, $H^2(\mathbb{D})$), we can use Cauchy's integral formula to calculate $k(x^{(i)}, y)$ explicitly. We note that complex contour integration is with respect to $dz = 2\pi i \cdot z\, ds$, where s is normalised arc

length measure (the measure on H^2). So, if we let $f \in H^\infty$, then

$$
\begin{aligned}
f^{(n)}(x) &= \frac{n!}{2\pi i} \oint_{\mathbb{T}} \frac{f(z)}{(z-x)^{n+1}} dz \\
&= \frac{n!}{2\pi i} \int_{\mathbb{T}} \frac{2\pi i \, z \, f(z)}{(z-x)^{n+1}} ds \\
&= \int_{\mathbb{T}} f(z) \overline{\left(n! \frac{\bar{z}}{(\bar{z}-\bar{x})^{n+1}} \right)} ds \\
&= \int_{\mathbb{T}} f(z) \overline{\left(n! \frac{z^{-1}}{(z^{-1}-\bar{x})^{n+1}} \right)} ds \\
&= \int_{\mathbb{T}} f(z) \overline{\left(n! \frac{z^n}{(1-\bar{x}z)^{n+1}} \right)} ds \\
&= \left\langle f(z), \, n! \frac{z^n}{(1-\bar{x}z)^{n+1}} \right\rangle_z .
\end{aligned}
$$

Since H^∞ is dense in H^2, this also holds for $f \in H^2$. We can now see that

$$
k(x^{(n)}, y) = n! \frac{y^n}{(1-\bar{x}y)^{n+1}} = \frac{\partial^n}{\partial \bar{x}^n} k(x, y)
$$

where $k(x, y) = (1 - \bar{x}y)^{-1}$ is the ordinary Szegő kernel. For brevity, we write $k_{x^{(i)}}$ for the function $k(x^{(i)}, \cdot)$.

If M_f is the multiplication operator of $f \in H^\infty$ on H^2, these differentiating kernels satisfy

$$
M_f^* \frac{k_{x^{(n)}}}{n!} = \sum_{i=0}^{n} \frac{\overline{f^{(i)}(x)}}{i!} \frac{k_{x^{(n-i)}}}{(n-i)!} ,
$$

as

$$
\begin{aligned}
\left\langle g, M_f^* \frac{k_{x^{(n)}}}{n!} \right\rangle &= \left\langle M_f g, \frac{k_{x^{(n)}}}{n!} \right\rangle \\
&= \left\langle f g, \frac{k_{x^{(n)}}}{n!} \right\rangle \\
&= \frac{1}{n!} (fg)^{(n)}(x) \\
&= \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(x) g^{(n-i)}(x) \\
&= \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!} \frac{g^{(n-i)}(x)}{(n-i)!} \\
&= \left\langle g, \sum_{i=0}^{n} \frac{\overline{f^{(i)}(x)}}{i!} \frac{k_{x^{(n-i)}}}{(n-i)!} \right\rangle .
\end{aligned}
$$

3. Test Functions

3.1. **Definitions.** We refer the reader to [DM07] for more in depth discussion of test functions[2]. For our purposes, we will need some basic definitions. Note that despite the similarity in name, these test functions are unrelated to their namesake in distribution theory

Definition 1. A set Ψ of complex valued functions on a set X is a set of test functions if:

(1) For each $x \in X$,
$$\sup\left\{\left|\psi(x)\right| : \psi \in \Psi\right\} < 1.$$

(2) If F is a finite set with n elements, then the unital algebra generated by $\Psi|_F$ (the restriction of Ψ to F) is n-dimensional (that is, Ψ separates points).

Definition 2. For any collection of test functions Ψ, we define a set of positive kernels
$$\mathcal{K}_\Psi := \left\{k : x \times x \to \mathbb{C} \,|\, (1 - \psi(x)\overline{\psi(y)})k(x, y) \geq 0 \, \forall \psi \in \Psi\right\}.$$

From this, we define a normed algebra $H^\infty(\mathcal{K}_\Psi)$. We say a function $f : X \to \mathbb{C}$ is in $H^\infty(\mathcal{K}_\Psi)$ with $\left\|f\right\|_{H(\mathcal{K}_\Psi)} \leq 1$ if
$$(1 - f(x)\overline{f(y)})k(x, y) \geq 0 \, \forall k \in \mathcal{K}_\Psi.$$

Since we have defined the unit ball of $H^\infty(\mathcal{K}_\Psi)$, by extension we have defined the whole of $H^\infty(\mathcal{K}_\Psi)$, and given its norm. Multiplication is defined pointwise.

3.2. **Zero Mean Probability Measures.** We start out by finding a set of test functions for H_1^∞. Since test functions have norm 1 or less, the Möbius transformation
$$m : z \to \frac{1+z}{1-z}$$
(which takes the unit disc to the right half plane), takes test functions to functions with positive real part. Since $H_1^\infty \subseteq H^\infty$, our test functions must have a Herglotz representation (see Theorem 1.1.19 of [AHR08]), so if ψ is a test function, $f(z) = m(\psi(z))$, and $f(0) > 0$,[3] then
$$f(z) = \int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w)$$
for some positive measure μ. For our test function to be in H_1^∞, we also need that $\psi'(0) = 0$. We can see that since
$$f'(z) = \psi'(z)\, m'\left(\psi(z)\right)$$

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[2]Many of these ideas are also covered in [AM02], although our notation is largely taken from [DM07].

[3]This condition isn't as problematic as it looks.

and $m' \neq 0$, we have that $\psi'(0) = 0$ if and only if $f'(0) = 0$.

Now,

$$f'(z) = \int_{\mathbb{T}} \left(\frac{d}{dz} \frac{w+z}{w-z} \right) d\mu(w) = \int_{\mathbb{T}} \frac{2w}{(w-z)^2} d\mu(w)$$

so

$$f'(0) = \int_{\mathbb{T}} \frac{2}{w} d\mu(w) = 2 \int_{\mathbb{T}} \overline{w} d\mu(w) = 2 \overline{\int_{\mathbb{T}} w d\mu(w)} \,.$$

If μ is a probability measure (which it will later be convenient to assume it is), then the condition that $\psi'(0) = 0$ is equivalent to the condition that $\mathbb{E}(\mu) = 0$, so μ has *zero-mean*. It should be noted here that requiring μ to be a probability measure is equivalent to requiring that $f(0) = 1$, or equivalently still, that $\psi(0) = 0$.

We have proved the following:

Theorem 3. *The analytic function ψ has $\|\psi\|_\infty \leq 1$, $\psi(0) = 0$, and $\psi'(0) = 0$ if and only if the corresponding measure is a zero-mean probability measure.*

3.3. **Extreme Directions.** We'll be using a lot of techniques and definitions from [AHR08]. In that paper, they used the convention that if X was a "real function space" in some sense, then X^h is the set of all real functions in X corresponding to the real parts of holomorphic functions. Here, we'll use the convention that X^1 is the set of all real functions in X corresponding to the real parts of holomorphic functions with zero derivative at 0, in ways that should be fairly clear.

If $L_\mathbb{R}^2(\mathbb{T})$ is defined in the usual way, then $L_\mathbb{R}^{2,1}(\mathbb{T})$ is the set of all functions in $L_\mathbb{R}^2(\mathbb{T})$ which are the real part of an analytic function with $f'(0) = 0$. It is easy to see that $\left(L_\mathbb{R}^{2,1}(\mathbb{T}) \right)^\perp = \text{span}\{\text{Im} z, \text{Re} z\}$, so $L_\mathbb{R}^{2,1}(\mathbb{T})$ has (real) co-dimension 2 in $L_\mathbb{R}^2(\mathbb{T})$. We let $M_\mathbb{R}(\mathbb{T})$ be the space of finite regular real Borel measures[4] on $\mathbb{T}$, and $C_\mathbb{R}(\mathbb{T})$ be the space of real continuous functions on $\mathbb{T}$, so that $M_\mathbb{R}(\mathbb{T})$ is the dual of $C_\mathbb{R}(\mathbb{T})$, under the weak-$*$ and uniform topologies, respectively. By the convention mentioned above, we define $M_\mathbb{R}^1(\mathbb{T})$ and $C_\mathbb{R}^1(\mathbb{T})$ as the subspaces of $M_\mathbb{R}(\mathbb{T})$ and $C_\mathbb{R}(\mathbb{T})$ (respectively), corresponding to holomorphic functions with zero derivative at 0. Then, as in [AHR08], we can see that

$$M_\mathbb{R}^1(\mathbb{T}) = \{\text{Im} z, \text{Re} z\}^\perp$$

and

$$C_\mathbb{R}^1(\mathbb{T})^\perp = \text{span}\{\text{Im} z \, ds, \text{Re} z \, ds\} \,,$$

where $\perp$ denotes the annihilator.

We also need to make use of extreme directions. We say a vector x in a cone C is an *extreme direction* in C if, to have $x = x_1 + x_2$ for some $x_1, x_2 \in C$ we need $x_1 = tx$ and $x_2 = sx$ for some $s, t \geq 0$.

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[4] Remember that we can associate a harmonic function to a measure $\mu \in M_\mathbb{R}(\mathbb{T})$ via the Poisson kernel.

This allows us to formulate the following:

Theorem 4. *Let* $E = \left\{ \mu \in M^1_{\mathbb{R}}(\mathbb{T}) : \mu \geq 0 \right\}$. *If μ is an extreme direction in E, then μ is supported at three or fewer points on $\mathbb{T}$.*

Proof. Suppose μ is supported on four or more points in $\mathbb{T}$, and divide the support of $\mathbb{T}$ into four non-empty parts, Δ_1 to Δ_4. Let $\mu_i = \chi_{\Delta_i}\mu$, where χ_Δ is the indicator function on Δ. Let $\mathcal{M} = \mathrm{span}\{\mu_1, \ldots, \mu_4\}$. The dimension of $\mathcal{M}$ is 4, and since $M^1_{\mathbb{R}}(\mathbb{T})$ has co-dimension 2 in $M_{\mathbb{R}}(\mathbb{T})$, we must have that

$$\dim\left(\mathcal{M} \cap M^1_{\mathbb{R}}(\mathbb{T})\right) \geq 2.$$

Therefore, there must exist a $\nu \in \mathcal{M} \cap M^1_{\mathbb{R}}(\mathbb{T})$ which is linearly independent of μ. Since a measure $\alpha_1\mu_1 + \cdots + \alpha_4\mu_4 \in \mathcal{M}$ is positive whenever $\alpha_1, \ldots, \alpha_4$ are all positive, we can choose an $\epsilon > 0$ small enough that $\mu \pm \epsilon\nu \geq 0$. Therefore, $\frac{1}{2}(\mu \pm \epsilon\nu) \in E$, but

$$\mu = \frac{1}{2}(\mu + \epsilon\nu) + \frac{1}{2}(\mu - \epsilon\nu)$$

so μ is not an extreme direction in E. $\square$

We can combine this with Theorem 3 on the preceding page, which says that $\mathbb{E}(\mu) = 0$ for $\mu \in E$:

- If μ is supported at one point of $\mathbb{T}$, then it is clearly impossible to have $\mathbb{E}(\mu) = 0$.
- If μ is supported at two points, then 0 must be in the convex hull of these two points (the line between them), and so the two points must lie opposite each other on the circle, and both have equal weight (if μ is to be a probability measure, this weight must be $\frac{1}{2}$).
- If μ is supported at three points, then these points must be such that 0 is in the *interior* of their convex hull (that is, the interior of the triangle they form; if 0 lies on one of the lines of the triangle, then μ will only be supported on the two points at either end of the line, so this is the degenerate case we had before). We can also see that if μ is a probability measure, then its weights are uniquely determined by its support – the weights are precisely the barycentric co-ordinates of 0, with respect to the three vertices of the triangle.

We can now note that, if a measure μ in E is supported on three or fewer points, it is uniquely determined by those points, up to multiplication by a scalar. In particular, if $\mu = t_1\mu_1 + t_2\mu_2$, for some $\mu_1, \mu_2 \in E$, then μ_1 and μ_2 must be supported on a subset of the support of μ, so must be scalar multiples of μ, so we have characterised the extreme directions in E.

Note that we can rescale any non-zero measure in E to a probability measure.
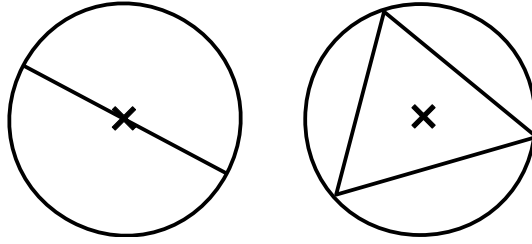
FIGURE 3.1. Types of Element in $\widehat{\Theta}$
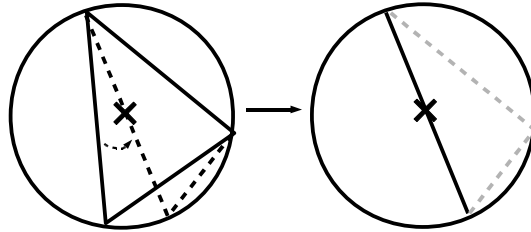
FIGURE 3.2. Convergence in $\widehat{\Theta}$

3.4. **Some Topology.** This is a convenient time to talk about the "space" of test functions. We define a set $\widehat{\Theta}$, containing two types of element, as shown in Figure 3.1:

- diameters of the circle;
- triangles, with vertices on the circumference of the circle, and the centre of the circle in their interior.

To topologise this set, we say that a sequence of triangles converges to a triangle if its vertices converge, and if we have a sequence of triangles where one of the lines converges to a diameter, then we say the sequence converges to the diameter, as in Figure 3.2. Essentially, the third point on the triangle disappears, which makes sense, as in the case of the zero-mean probability measures above, the weight of the third point would tend to zero. In fact, if we identify points in $\widehat{\Theta}$ with zero-mean probability measures on the circle, as in the discussion following Theorem 4, we can see that this topology corresponds exactly to the weak-* topology on $M_{\mathbb{R}}(\mathbb{T})$.

As we'll see later, if μ is a zero-mean probability measure supported on n points, then μ induces a test function ψ_μ, which will be an n-to-one Blaschke product, with $\psi'_\mu(0) = 0$, $\psi_\mu(0) = 0$, and $\psi_\mu(w) = 1$ precisely when w is in the support of μ. Conversely, if we have such a function ψ, then there is a corresponding zero-mean probability measure μ_ψ, with support $\psi^{-1}(\{1\})$. If we have a test function ψ_μ, corresponding to a $\mu \in \widehat{\Theta}$, then $\widetilde{\psi} := \overline{\psi_\mu(-1)}\psi_\mu$ is a two-or-three-to-one inner function of the type required, and since $\widetilde{\psi}(-1) = 1$, the corresponding measure $\widetilde{\mu}$ is supported at -1. This
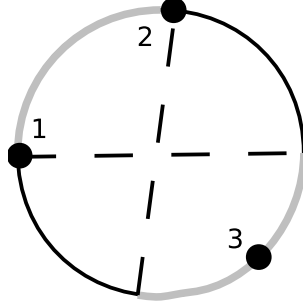
FIGURE 3.3. The setup for Θ

is useful, as we can safely identify two test functions if one is a constant, unimodular multiple of the other.

If we define an equivalence relation $\sim$ on $\widehat{\Theta}$ by

$$\mu_1 \sim \mu_2 \iff \psi_{\mu_1} = \lambda\psi_{\mu_2} \text{ for some } \|\lambda\| = 1$$

then we can define $\Theta := \widehat{\Theta}/\sim$. By the above reasoning, Θ can be represented as the set of all triangles or diameters in $\widehat{\Theta}$ with a vertex at -1 (i.e, on the leftmost point of the circle). When viewed in this sense, there is only one diameter in Θ.

It's interesting to note that Θ is homeomorphic to S^2. We show (using a certain amount of hand-waving) that Θ is homeomorphic to $\mathbb{C} \cup \{\infty\}$. First, we set the diameter (in Θ) as ∞. This leaves the set of all triangles in Θ. We know that any triangle in Θ can be represented by a triangle with a point at -1 (the left hand side of the circle), so we put point one on the left of the circle, allow point two to vary over the whole top of the circle (corresponding to the real axis), and allow point three to vary over the range of points opposite the arc between point one and point two (corresponding to the imaginary axis) as in Figure 3.3. As either of these points tend towards the edges of their ranges, the triangle tends towards the diameter – which corresponds to ∞.

3.5. **The Test Functions.** If we take the set of probability measures in $M_{\mathbb{R}}^1(\mathbb{T})$, this is a subset of E, and in fact corresponds to E_ρ, in the sense of Lemma 3.4 of [AHR08], where $\rho(\mu) = \mu(\mathbb{T})$. We know, therefore, that E_ρ is convex, and we showed before that its extreme points are given by $\widehat{\Theta}$. E_ρ is compact by the Banach-Alaoglu theorem, so we can apply the Choquet-Bishop-de Leeuw theorem, and we see that for any $\mu \subseteq E_\rho$, there

exists a probability measure ν_μ on $\widehat{\Theta}$ such that[5]

$$\mu = \int_{\widehat{\Theta}} \vartheta d\nu_\mu(\vartheta)$$

Now, probability measures in $M^1_{\mathbb{R}}(\mathbb{T})$ correspond to analytic functions f on $\mathbb{D}$ with positive real part, $f'(0) = 0$, and $f(0) = 1$, using the Herglotz representation theorem. We define

$$h_\vartheta(z) := \int_{\mathbb{T}} \frac{w+z}{w-z} d\vartheta(w)$$

and

$$\psi_\vartheta = \frac{h_\vartheta - 1}{h_\vartheta + 1}$$

for all $\vartheta \in \widehat{\Theta}$. Using the reasoning above, we can write

$$\begin{aligned}
f(z) &= \int_{\mathbb{T}} \frac{w+z}{w-z} d\mu(w) \\
&= \int_{\mathbb{T}} \frac{w+z}{w-z} \left[\int_{\widehat{\Theta}} (d\vartheta(w)) \, d\nu_\mu(\vartheta) \right] \\
&= \int_{\widehat{\Theta}} \left[\int_{\mathbb{T}} \frac{w+z}{w-z} d\vartheta(w) \right] d\nu_\mu(\vartheta) \\
&= \int_{\widehat{\Theta}} h_\vartheta(z) d\nu_\mu(\vartheta) \, .
\end{aligned}$$

This gives us a new "Herglotz representation", which I'll call an *Agler-Herglotz representation*:

Theorem 5. *If f is an analytic function on $\mathbb{D}$ with positive real part, $f'(0) = 0$, and $f(0) > 0$, then there exists some positive real measure ν on $\widehat{\Theta}$ such that*

$$f(z) = \int_{\widehat{\Theta}} h_\vartheta(z) d\nu(\vartheta) \, .$$

We now prove the main result, which uses terminology from [DM07]. Here, Ψ refers to the set of test functions associated with Θ, so $\Psi := \{\psi_\vartheta : \vartheta \in \Theta\}$):

Theorem 6. *The two spaces $H^\infty(\mathcal{K}_\Psi)$ and $H^\infty_1(\mathbb{D})$ are isometrically isomorphic, that is, $H^\infty(\mathcal{K}_\Psi) = H^\infty_1(\mathbb{D})$ and $\| \cdot \|_{\mathcal{K}_\Psi} = \| \cdot \|_{H^\infty_1(\mathbb{D})}$*

Proof. One way is simple. Since $\Psi \subseteq H^\infty_1(\mathbb{D})$, we know that $\mathcal{K}_\Psi$ contains the set $\mathcal{K}^\infty_1$ of reproducing kernels given in [DPRS07], so if $\zeta \in H^\infty(\mathcal{K}_\Psi)$, and $\|\zeta\|_{\mathcal{K}_\Psi} \leq 1$, then

$$\left(\left[1 - \zeta(z)\overline{\zeta(w)} \right] k(x, w) \right) \geq 0$$

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[5]We are, confusingly but unavoidably, talking about integrating a measure-valued function (that's ϑ), with respect to a measure (that's ν_μ). Also, the points in the space that ν_μ integrates over (that's $\widehat{\Theta}$) are, themselves, measures (they're measures on $\mathbb{T}$)

for all $k \in \mathcal{K}_1^\infty$. Therefore, ζ must be in $H_1^\infty(\mathbb{D})$, with $\|\zeta\|_{H_1^\infty(\mathbb{D})} \leq 1$, so $H^\infty(\mathcal{K}_\Psi) \subseteq H_1^\infty(\mathbb{D})$.

Now, suppose that $\zeta \in H_1^\infty(\mathbb{D})$ and $\|\zeta\|_{H_1^\infty(\mathbb{D})} \leq 1$. For now, we also suppose that $\zeta(0) = 0$. We let

$$f := \frac{1+\zeta}{1-\zeta},$$

so

$$\zeta = \frac{f-1}{f+1}.$$

Hence

$$(3.1) \qquad\qquad 1 - \zeta(z)\overline{\zeta(w)} = 2\frac{f(z) + \overline{f(w)}}{(f(z)+1)(\overline{f(w)}+1)}.$$

We know that f has positive real part, $f(0) = 1$ and $f'(0) = 0$, so we can use our Agler-Herglotz representation, Theorem 5 on the preceding page, and find that there is a measure ν on $\widehat{\Theta}$ such that

$$f = \int_{\widehat{\Theta}} h_\vartheta d\nu(\vartheta).$$

We can then show, using the definition of ψ_ϑ and (3.1), that

$$1 - \zeta(z)\overline{\zeta(w)} = \int_{\widehat{\Theta}} \frac{1 - \psi_\vartheta(z)\overline{\psi_\vartheta(w)}}{(f(z)+1)(1-\psi_\vartheta(z))\left(1 - \overline{\psi_\vartheta(w)}\right)\left(\overline{f(w)}+1\right)} d\nu(\vartheta).$$

We know that if $\vartheta \in \widehat{\Theta}$, then $\overline{\psi_\vartheta(-1)}\psi_\vartheta \in \Psi$. If we define a positive kernel $\Gamma : \mathbb{D} \times \mathbb{D} \to C_b(\Psi)^*$ by

$$\Gamma(z,w)\alpha = \int_{\widehat{\Theta}} \frac{\alpha\left(\overline{\psi_\vartheta(-1)}\psi_\vartheta\right)}{(f(z)+1)(1-\psi_\vartheta(z))\left(1 - \overline{\psi_\vartheta(w)}\right)\left(\overline{f(w)}+1\right)} d\nu(\vartheta).$$

We can then see that

$$1 - \zeta(z)\overline{\zeta(w)} = \Gamma(z,w)(1 - E(z)E(w)^*),$$

so $\zeta \in H^\infty(\mathcal{K}_\Psi)$ and $\|\zeta\|_{\mathcal{K}_\Theta} \leq 1$.

To see that this holds when $\zeta(0) \neq 0$, simply note that

$$1 - \left(\frac{\zeta(z)-a}{1-\bar{a}\zeta(z)}\right)\overline{\left(\frac{\zeta(w)-a}{1-\bar{a}\zeta(w)}\right)} = \frac{(1-a\bar{a})\left(1 - \zeta(z)\overline{\zeta(w)}\right)}{(1-\bar{a}\zeta(z))\left(1 - a\overline{\zeta(w)}\right)}.$$

Therefore, $\zeta \in H^\infty(\mathcal{K}_\Psi)$, and $\|\zeta\|_{\mathcal{K}_\Psi} \leq 1$, so $H^\infty(\mathcal{K}_\Psi) = H_1^\infty(\mathbb{D})$ and $\|\cdot\|_{\mathcal{K}_\Psi} = \|\cdot\|_{H_1^\infty(\mathbb{D})}$, as required. $\qquad\square$

3.6. **Minimality.** As in [DM07], we show that this set of test functions is minimal, in the sense that there is no closed subset C of Ψ so that C is a set of test functions for H_1^∞. First, we need a lemma.

Lemma 7. *If, for some measure μ on a space C, and some separable Hilbert space H, $M \in B(H)$, $f \in B(H) \otimes L^1(\mu)$, $f(x) \geq 0$ μ-almost everywhere, and $M \geq \int_C f(x) \, d\mu(x)$, then for all scalars $\delta > 0$ there exists some $C' \subseteq C$ and some scalar $N_\delta > 0$, such that $\mu(C - C') < \delta$ and $M \geq N_\delta f(x)$ for all $x \in C'$.*

Proof. Define, for $N > 0$, and $\varphi \in H$,

$$C_N^\varphi = \{x \in C : \langle (M - Nf(x)) \varphi, \varphi \rangle \geq 0\},$$

$$C_N = \{x \in C : M \geq Nf(x)\},$$

$$C_0 = \bigcup_{N>0} C_N, \qquad C_0^\varphi = \bigcup_{N>0} C_N^\varphi.$$

We can see that, for any given $\varphi \in H$, $C - C_0^\varphi$ is a μ-null set. To see this, note that for x to be in $C - C_0^\varphi$, we'd need to have $\langle M\varphi, \varphi \rangle = 0$ but $\langle f(x)\varphi, \varphi \rangle > 0$. However,

$$\langle M\varphi, \varphi \rangle \geq \left\langle \int_C f(x) \, d\mu(x) \varphi, \varphi \right\rangle \geq \int_{C - C_0^\varphi} \langle f(x) \varphi, \varphi \rangle \, d\mu(x) > 0,$$

which is a contradiction.

If Φ is a countable dense subset of the unit ball in H, we can see that

$$C_N = \bigcap_{\varphi \in \Phi} C_N^\varphi$$

and so

$$C - C_0 = \bigcap_{N>0} (C - C_N) = \bigcap_{N>0} \left(\bigcup_{\varphi \in \Phi} \left[C - C_N^\varphi \right] \right)$$

$$= \bigcup_{\varphi \in \Phi} \left(\bigcap_{N>0} \left[C - C_N^\varphi \right] \right) = \bigcup_{\varphi \in \Phi} \left(C - C_0^\varphi \right),$$

which is a countable union of null-sets, and so is a null set.

We can now see that as $N \to 0$, $\mu(C - C_N) \to 0$, and $M \geq Nf(x)$ for all $x \in C_N$, so our result is proved. $\square$

Theorem 8. *No proper closed subset C of Ψ is a set of test functions for H_1^∞.*

Proof. Suppose, towards an eventual contradiction, that C is a proper closed subset of Ψ and $\psi_0 = \psi_{\vartheta_0} \notin C$. Since C is closed, its complement is open, so we can safely assume that ϑ_0 is not a diameter, and not an equilateral triangle[6].

We notice that the differentiating kernels we defined in Section 2 on page 2 are rational functions, so we can extend them to the entire Riemann

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[6]If we do the calculations, we discover that these two possibilities correspond to the test functions z^2 and $-z^3$, which are inconvenient corner cases

sphere[7], $\mathbb{C} \cup \{\infty\}$. If we do this, then we see that if $x \neq 0$, $k_{x^{(n)}}$ has $n + 1$ poles, all at $\overline{x^{-1}}$, and $k_{0^{(n)}}$ has n poles, all at ∞.

The kernels
$$\Delta_\vartheta(z, w) := \left(1 - \psi_\vartheta(z)\overline{\psi_\vartheta(w)}\right) k(w, z)$$

are positive and have rank at most three (k is just the Szegő kernel). To see this, first note that ψ_ϑ has at most three zeroes[8], and that at least two of them must be at zero, as $\psi_\vartheta(0) = 0$ and $\psi'_\vartheta(0) = 0$. Also note that M_ϑ, the operator of multiplication by ψ_ϑ, is an isometry on H^2, so $1 - M_\vartheta M_\vartheta^*$ is the projection onto
$$\mathfrak{M}_\vartheta := \ker M_\vartheta^* = \operatorname{Span}\left\{k_0, k_{0'}, k_{a_\vartheta}\right\},$$

where a_ϑ is the third zero of ϑ (if ϑ has three zeroes at zero, then $a_\vartheta = 0''$; if ϑ is a diameter, then the span doesn't include k_{a_ϑ}).

Now,
$$\Delta_\vartheta(z, w) = \left\langle \left(1 - M_\vartheta M_\vartheta^*\right) k_w, k_z \right\rangle := \langle P_\vartheta k_w, k_z \rangle = \langle P_\vartheta k_w, P_\vartheta k_z \rangle,$$

and we can view this as a holomorphic function in z, and an antiholomorphic function in w. If we think of the antiholomorphic function as being in the dual of H^2, then
$$\Delta_\vartheta \in H^2 \otimes \left(H^2\right)^* \cong B\left(H^2\right).$$

More explicitly, Δ_ϑ defines an operator on H^2 as
$$\Delta_\vartheta f(z) := \int_\mathbb{T} \Delta_\vartheta(z, w) f(w) \, ds(w),$$

so
$$\langle \Delta_\vartheta f, g \rangle = \int_\mathbb{T} \int_\mathbb{T} \overline{g(z)} \, \Delta_\vartheta(z, w) f(w) \, ds(w) \, ds(z)$$
$$= \int_\mathbb{T} \int_\mathbb{T} \overline{g(z)} \, \langle P_\vartheta k_w, P_\vartheta k_z \rangle \, f(w) \, ds(w) \, ds(z)$$
$$= \int_\mathbb{T} \int_\mathbb{T} \langle f(w) P_\vartheta k_w, g(z) P_\vartheta k_z \rangle \, ds(w) \, ds(z)$$
$$= \left\langle \int_\mathbb{T} f(w) P_\vartheta k_w \, ds(w), \int_\mathbb{T} g(z) P_\vartheta k_z \, ds(z) \right\rangle$$
$$= \left\langle \int_\mathbb{T} f(w) P_\vartheta k_w \, ds(w), \int_\mathbb{T} g(z) P_\vartheta k_z \, ds(z) \right\rangle_{\mathfrak{M}_\vartheta}.$$

Hence, we have factorised Δ_ϑ as $A_\vartheta^* A_\vartheta$, where $A_\vartheta : H^2 \to \mathfrak{M}_\vartheta$ is given by
$$A_\vartheta f := \int_\mathbb{T} f(w) P_\vartheta k_w \, ds(w).$$

---

[7]The fact that this is homeomorphic to Θ is not relevant here, and appears to be a coincidence.
[8]If ϑ is a diameter, then it has two zeroes, otherwise it has three.

We also note now, for use later, that $A_\vartheta^* = I_{\mathfrak{M}_\vartheta}$, the embedding map of $\mathfrak{M}_\vartheta$ into H^2 , as

$$
\begin{aligned}
\langle A_\vartheta f, g \rangle &= \left\langle \int_{\mathbb{T}} f(w)\, P_\vartheta k_w \, ds(w), g \right\rangle_{\mathfrak{M}_\vartheta} \\
&= \int_{\mathbb{T}} f(w)\, \langle P_\vartheta k_w, g \rangle_{\mathfrak{M}_\vartheta}\, ds(w) \\
&= \int_{\mathbb{T}} f(w)\, \left\langle P_\vartheta k_w, I_{\mathfrak{M}_\vartheta} g \right\rangle ds(w) \\
&= \int_{\mathbb{T}} f(w)\, \left\langle k_w, I_{\mathfrak{M}_\vartheta} g \right\rangle ds(w) \\
&= \int_{\mathbb{T}} f(w) \overline{\left(I_{\mathfrak{M}_\vartheta} g \right)(w)}\, ds \\
&= \left\langle f, I_{\mathfrak{M}_\vartheta} g \right\rangle .
\end{aligned}
$$

We choose any set of four points $F = \{z_1, z_2, z_3, z_4\} \in \mathbb{D}$, and consider the classical Nevanlinna-Pick problem, of finding a contractive function $\varphi \in H^\infty$ such that $\varphi(z_i) = \psi_0(z_i)$ for all i. Since $\Delta_0(z, w)$ has rank at most three, the 4×4 matrix

$$
\left(\left[1 - \psi_0(z_i)\overline{\psi_0(z_j)} \right] k(z_j, z_i) \right)_{i,j=1}^4
$$

must be singular, so the problem has a unique solution, $\varphi = \psi_0$.

Now, if we assume that C is a set of test functions for H_1^∞, then by Theorem 2.3 of [DM07] there must be a positive kernel $\Gamma : F \times F \to C(C)^*$ such that

$$
1 - \psi_0(z_i)\overline{\psi_0(z_j)} = \Gamma(z_i, z_j)\left(1 - E(z_i)E(z_j)^* \right) .
$$

Indeed, by Theorem 2.2 of [DM07], this kernel must extend to the whole of $\mathbb{D} \times \mathbb{D}$. We can rewrite this, in our case, by saying that there exists a measure μ on C, and functions $h_l(z, \cdot) \in L^2(\mu)$, for $l = 1, \ldots, 4$, such that

$$
(3.2) \qquad 1 - \psi_0(z)\overline{\psi_0(w)} = \int_C \sum_{l=1}^4 h_l(z, \psi)\overline{h_l(w, \psi)}\left(1 - \psi(z)\overline{\psi(w)} \right) d\mu(\psi) .
$$

Multiplying this equation by $k(z, w)$ gives

$$
\Delta_0(z, w) = \int_C \sum_{l=1}^4 h_l(z, \psi)\Delta_\psi(z, w)\overline{h_l(w, \psi)}\, d\mu(\psi) .
$$

Since Δ_ψ is a positive kernel and a positive operator, when seen as an operator on H^2, as above, we can say that for all l,

$$
\Delta_0(z, w) \geq \int_C h_l(z, \psi)\Delta_\psi(z, w)\overline{h_l(w, \psi)}\, d\mu(\psi) .
$$

We now know, by Lemma 7 on page 10, that for any $\delta > 0$, there is a set C', and a constant $c_\delta > 0$ such that $\mu(C - C') < \delta$, and

$$\Delta_0(z, w) \geq c_\delta h_l(z, \psi)\Delta_\psi(z, w)\overline{h_l(w, \psi)},$$

for all $\psi \in C'$.

If we use our factorisation of Δ_ϑ from above, we see that

$$A_0^* A_0 \geq c_\delta h_l(z, \psi)A_\psi^* A_\psi \overline{h_l(w, \psi)}$$

and so by Douglas' Lemma, the range of $h_l(\cdot, \psi)A_\psi^*$ is contained in the range of A_0^*. Therefore, there exist constants $c_1, \ldots, c_9$ so that

$$(3.3) \qquad\qquad h_l(\cdot, \psi)k_0 = c_1 k_0 + c_2 k_{0'} + c_3 k_{a_0}$$

$$(3.4) \qquad\qquad h_l(\cdot, \psi)k_{0'} = c_4 k_0 + c_5 k_{0'} + c_6 k_{a_0}$$

$$(3.5) \qquad\qquad h_l(\cdot, \psi)k_{a_\psi} = c_7 k_0 + c_8 k_{0'} + c_9 k_{a_0}.$$

By letting δ go to zero, we see that these equations must hold for μ-almost-all $\psi \in C$.

Equation (3.3) tells us that $h_l(\cdot, \psi) = c_1 k_0 + c_2 k_{0'} + c_3 k_{a_0}$, as k_0 is constant. We can also see that $h_l(\cdot, \psi)$ must extend meromorphically to the Riemann sphere, as these kernels do so. Equation (3.4) tells us that $c_2 = 0$, as otherwise the left hand side of the equation has a triple pole at ∞, but the right hand side has at most only a double pole.

We consider equation (3.5) in three cases. Firstly, if ψ has only two zeroes, there is no equation (3.5), so $h_l(\cdot, \psi) = c_1 + c_3 k_{a_0}$.

If $a_\psi \neq 0$, then

$$h_l(y, \psi)k_{a_\psi}(y) = c_7 k_0(y) + c_8 k_{0'}(y) + c_9 k_{a_0}(y),$$

$$\left[c_1 + c_3 \frac{1}{1 - \overline{a_0}y}\right]\frac{1}{1 - \overline{a_\psi}y} = c_7 + c_8 y + c_9 \frac{1}{1 - \overline{a_0}y},$$

$$c_1 + c_3 \frac{1}{1 - \overline{a_0}y} = c_7 - c_7\overline{a_\psi}y + c_8 y - c_8\overline{a_\psi}y^2 + \frac{c_9 - c_9\overline{a_\psi}y}{1 - \overline{a_0}y},$$

$$c_1 - c_1\overline{a_0}y + c_3 = \begin{cases} c_7 - c_7\overline{a_\psi}y - c_7\overline{a_0}y + c_7\overline{a_\psi a_0}y^2 \\ +c_8 y - c_8\overline{a_\psi}y^2 - c_8\overline{a_0}y^2 + c_8\overline{a_\psi a_0}y^3 \\ +c_9 - c_9\overline{a_\psi}y \end{cases}.$$

Looking at the y^3 coefficient tells us that $c_8 = 0$, looking at the y^2 coefficient then tells us that $c_7 = 0$, and so comparing the constant and y coefficients, we see that

$$\begin{cases} c_1 + c_3 = c_9 \\ \quad c_1\overline{a_0} = c_9\overline{a_\psi} \end{cases},$$

$$\begin{cases} c_1 = c_9 \dfrac{\overline{a_\psi}}{\overline{a_0}} \\ c_3 = c_9 \left(1 - \dfrac{\overline{a_\psi}}{\overline{a_0}}\right) \end{cases},$$

and so

$$h_l(y, \psi) = c_9\left(\frac{\overline{a_\psi}}{\overline{a_0}} + \left(1 - \frac{\overline{a_\psi}}{\overline{a_0}}\right)\frac{1}{1 - \overline{a_0}y}\right) = c\frac{1 - \overline{a_\psi}y}{1 - \overline{a_0}y}.$$

Alternately, if $a_\psi = 0$, then equation (3.5) becomes

$$h_l(y, \psi)y^2 = c_7 + c_8 y + c_9\frac{1}{1 - \overline{a_0}y}$$

$$c_1 y^2 + c_3\frac{y^2}{1 - \overline{a_0}y} = c_7 + c_8 y + c_9\frac{1}{1 - \overline{a_0}y}$$

$$c_1 y^2 - c_1\overline{a_0}y^3 + c_3 y^2 = c_7 + c_7\overline{a_0}y + c_8 y + c_8\overline{a_0}y^2 + c_9.$$

Looking at the y^3 coefficient tells us that $c_1\overline{a_0} = 0$, so $c_1 = 0$. The y^2 terms then tell us that $c_3 = c_8\overline{a_0}$. We then see that

$$h_l(y, \psi) = c_3\frac{1}{1 - \overline{a_0}y} = c\frac{1 - \overline{a_\psi}y}{1 - \overline{a_0}y}$$

as before.

Combining these consequences of equations (3.3)–(3.5) with (3.2) gives a more explicit realisation than the one in (3.2), namely,

$$1 - \psi_0(z)\overline{\psi_0(w)} =$$

$$\sum_{l=1}^{4}\left(\alpha_l + \frac{\beta_l}{1 - \overline{a_0}z}\right)\left(\overline{\alpha_l} + \frac{\overline{\beta_l}}{1 - a_0\overline{w}}\right)\left(1 - \psi_\infty(z)\overline{\psi_\infty(w)}\right)$$

$$+ \int_{C\backslash\{\infty\}} c(\psi)\left(\frac{1 - \overline{a_\psi}z}{1 - \overline{a_0}z}\right)\left(\frac{1 - a_\psi\overline{w}}{1 - a_0\overline{w}}\right)\left(1 - \psi(z)\overline{\psi(w)}\right)d\mu(\psi)$$

for some positive $c \in L^1(\mu)$ and some $\alpha_1, \beta_1, \ldots, a_4, \beta_4 \in \mathbb{C}$. We know that the ψs are Blaschke products, and we know their roots, so we can write this even more explicitly as

$$1 - z^2\overline{w}^2\frac{z - a_0}{1 - \overline{a_0}z}\frac{\overline{w} - \overline{a_0}}{1 - a_0\overline{w}} =$$

$$\sum_{l=1}^{4}\left(\alpha_l + \frac{\beta_l}{1 - \overline{a_0}z}\right)\left(\overline{\alpha_l} + \frac{\overline{\beta_l}}{1 - a_0\overline{w}}\right)\left(1 - z^2\overline{w}^2\right)$$

$$+ \int_{C\backslash\{\infty\}} c(\psi)\frac{1 - \overline{a_\psi}z}{1 - \overline{a_0}z}\frac{1 - a_\psi\overline{w}}{1 - a_0\overline{w}}\left(1 - z^2\overline{w}^2\frac{z - a_\psi}{1 - \overline{a_\psi}z}\frac{\overline{w} - \overline{a_\psi}}{1 - a_\psi\overline{w}}\right)d\mu(\psi).$$

If we multiply both sides by $(1 - \overline{a_0}z)(1 - a_0\overline{w})$ we get

$$(1 - \overline{a_0}z)(1 - a_0\overline{w}) - z^2\overline{w}^2(z - a_0)(\overline{w} - \overline{a_0}) =$$

$$\sum_{l=1}^{4}\left(\alpha_l(1 - \overline{a_0}z) + \beta_l\right)\left(\overline{\alpha_l}(1 - a_0\overline{w}) + \overline{\beta_l}\right)\left(1 - z^2\overline{w}^2\right)$$

$$+ \int_{C\backslash\{\infty\}} c(\psi)\left(\left(1 - \overline{a_\psi}z\right)\left(1 - a_\psi\overline{w}\right) - z^2\overline{w}^2\left(z - a_\psi\right)\left(\overline{w} - \overline{a_\psi}\right)\right) d\mu(\psi),$$

which we can expand to get

$$1 - \overline{a_0}z - a_0\overline{w} + |a_0|^2 z\overline{w} - z^3\overline{w}^3 + a_0 z^2\overline{w}^3 + \overline{a_0}z^3\overline{w}^2 - |a_0|^2 z^2\overline{w}^2 =$$

$$\sum_{l=1}^{4}\left\{ \begin{array}{c} \left|\alpha_l + \beta_l\right|^2 - \left|\alpha_l + \beta_l\right|^2 z^2\overline{w}^2 \\ -(\alpha_l + \beta_l)\overline{\alpha_l}a_0\overline{w} + (\alpha_l + \beta_l)\overline{\alpha_l}a_0 z^2\overline{w}^3 \\ -(\overline{\alpha_l} + \overline{\beta_l})\alpha_l\overline{a_0}z + (\overline{\alpha_l} + \overline{\beta_l})\alpha_l\overline{a_0}z^3\overline{w}^2 \\ + |\alpha_l|^2 |a_0|^2 z\overline{w} - |\alpha_l|^2 |a_0|^2 z^3\overline{w}^3 \end{array} \right.$$

$$+ \int_{C\backslash\{\infty\}} c(\psi)\left[\begin{array}{c} 1 - \overline{a_\psi}z - a_\psi\overline{w} + \left|a_\psi\right|^2 z\overline{w} \\ -z^3\overline{w}^3 + \overline{a_\psi}z^3\overline{w}^2 + a_\psi z^2\overline{w}^3 - \left|a_\psi\right|^2 z^2\overline{w}^2 \end{array} \right] d\mu(\psi).$$

To get our contradiction, we look at the $\overline{w}$, $z^3\overline{w}^3$ and $z^2\overline{w}^2$ coefficients of this equation. These yield

$$(3.6) \qquad a_0 = \sum_{l=1}^{4}(\alpha_l + \beta_l)\overline{\alpha_l}a_0 + \int_{C\backslash\{\infty\}} c(\psi)a_\psi d\mu(\psi),$$

$$(3.7) \qquad 1 = \sum_{l=1}^{4}|\alpha_l|^2 |a_0|^2 + \int_{C\backslash\{\infty\}} c(\psi)d\mu(\psi),$$

$$(3.8) \qquad |a_0|^2 = \sum_{l=1}^{4}\left|\alpha_l + \beta_l\right|^2 + \int_{C\backslash\{\infty\}} c(\psi)\left|a_\psi\right|^2.$$

We can easily see that (3.6) implies

$$|a_0|^2 = \left|\sum_{l=1}^{4}(\alpha_l + \beta_l)\overline{\alpha_l}a_0 + \int_{C\backslash\{\infty\}} c(\psi)a_\psi d\mu(\psi)\right|^2,$$

and if we define a Hilbert space $H = \mathbb{C}^4 \oplus L^2(\mu)$, then we can rewrite this as

$$(3.9) \qquad |a_0|^2 = \left|\left\langle \begin{pmatrix} (a_0\overline{\alpha_l})_l \\ c(\psi)^{1/2} \end{pmatrix}, \begin{pmatrix} (\overline{\alpha_l} + \overline{\beta_l})_l \\ c(\psi)^{1/2}\overline{a_\psi} \end{pmatrix} \right\rangle\right|^2.$$

Then by the Cauchy-Schwarz inequality,

$$|a_0|^2 \leq$$

$$\underbrace{\left(\sum_{l=1}^{4} |\alpha_l|^2 |a_0|^2 + \int_{C\backslash\{\infty\}} c(\psi)d\mu(\psi) \right)}_{=1 \text{ by } (3.7)} \underbrace{\left(\sum_{l=1}^{4} |\alpha_l + \beta_l|^2 + \int_{C\backslash\{\infty\}} c(\psi) |a_\psi|^2 \right)}_{=|a_0|^2 \text{ by } (3.8)}$$

$$= |a_0|^2 \ .$$

Further, the two vectors in (3.9) are linearly independent, as $\overline{a_\psi}$ is (the complex conjugate of) the third root of ψ, which is different for each ψ (meaning $c(\psi)^{1/2}$ and $c(\psi)^{1/2}\overline{a_\psi}$ are linearly independent), so the inequality is strict, which is a contradiction[9]. □

Remark 9. Clearly, this proof is not very good. It would seem more natural to use the fact that the Agler-Herglotz representation from Theorem 5 was parameterised by extreme measures, although there is no obvious way to do this.

The fact that the minimum set of test functions is parameterised by the sphere is interesting, as the set of kernels given in [DPRS07] is also parameterised by the sphere, and conjectured to be minimal (that paper contains some partial results, towards this aim).

Similarly, Abrahamse gave (in [Abr79]) a set of kernels corresponding to interpolation on a multiply connected (*n*-holed) domain, parameterised by the *n*-torus, and conjectured that this set of kernels was minimal (there are some partial results in this direction in [BC96]); in [DM07] and [Pic08], the authors give sets of test functions for *n*-holed domains, which are also parameterised by the *n*-torus, and minimal[10]. The test functions in [DM07] and [Pic08] are based on a construction given in [DM05].

Perhaps there is some sort of duality between minimal sets of test functions and minimal sets of kernels? A possible counterexample to such a duality may occur with the bidisc; it's well known (see for example [AM02]) that only two test functions are needed for the bidisc, whereas in [MP02], the authors conjecture that infinitely many kernels are required.

3.7. **Generalisation.** It's worth briefly discussing how these ideas could be generalised to other spaces. In [Rag08], the author looks at spaces of the

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[9]This is a slight oversimplification. If $c(\psi)$ is non-zero at exactly one point ψ (and that ψ is singular with respect to μ), then this linear independence argument doesn't hold. However, a simple calculation by equating coefficients (which is omitted) shows that if the vectors are linearly dependent, so $c(\psi)$ is non-zero at exactly one point, then that point must be $\psi_0 \notin C$, which is also a contradiction.

[10]The main aim of [Pic08] was not to investigate test functions, although Note 2.10 of [Pic08] gives the set of test functions, and the results of Section 5.2 of [Pic08] are broadly analogous to the results used in the proof of Theorem 8 in this paper, and in Proposition 5.3 of [DM07].

form[11] $\mathbb{C} + BH^\infty$. These spaces are a natural generalisation of the space H_1^∞, and the author provides a generalisation of the Nevanlinna-Pick theorem from [DPRS07][12].

We used the Herglotz representation trick to turn a linear equation ($f'(0) = 0$) into a constraint on probability measures (that they have zero mean), found the extreme points of the set of constrained probability measures, and then used those extreme probability measures to generate our test functions.

Suppose we want to apply these techniques to $\mathbb{C} + BH^\infty$. It's fairly clear that we can come up with a set of linear equations that all functions in $\mathbb{C} + BH^\infty$ must satisfy (functions must be constant at zeroes of B, and have a prescribed number of zero derivatives at repeated zeroes of B). It should also be fairly easy to turn this set of linear equations into a set of constraints on probability measures.

The difficulty comes when we calculate extreme points. Theorem 4 should work well enough, so if we have n independent equations, we can say that extreme measures are supported on at most $2n+1$ points. However, the reasoning that followed Theorem 4 (which showed precisely which such measures would give functions in H_1^∞) relied on a geometric interpretation of the probability constraint, which doesn't obviously generalise to other types of constraint.

If we can calculate the extreme measures, these should generate test functions in precisely the same way as above. However, the proof that the test functions we've found are minimal is heavily dependent – perhaps overly dependent – on explicit calculations using the test functions.

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[11]Here, B is a Blaschke product.

[12]The author only generalises the first form here; a generalisation of the second form is given in [BBT08].

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