Joint Congestion Control, Routing, and MAC for Stability and Fairness in Wireless Networks

Atilla Eryilmaz, *Member, IEEE*, and R. Srikant, *Fellow, IEEE*

***Abstract*—In this paper, we describe and analyze a joint scheduling, routing and congestion control mechanism for wireless networks, that asymptotically guarantees stability of the buffers and fair allocation of the network resources. The queue-lengths serve as common information to different layers of the network protocol stack. Our main contribution is to prove the asymptotic optimality of a *primal-dual congestion controller*, which is known to model different versions of transmission control protocol well.**

***Index Terms*—Congestion control, fair resource allocation, Lyapunov stability theory, nonlinear optimization, primal-dual algorithm, throughput-optimal scheduling, wireless networks.**

I. Introduction

CONSIDER A SET of flows that share the resources of a fixed wireless network. Each flow is described by its source-destination node pair, with no *a priori* established routes. The limited power resources and interference amongst concurrent transmissions necessitate multihop transmission. The nodes that constitute the network must cooperate by forwarding each others' packets toward their destinations. Thus, each node may need to maintain buffers to hold packets of those flows other than its own. For such a system, we design a joint routing, medium access control (MAC) and congestion control algorithm that stabilizes the buffers, and drives the mean flow rates to a system-wide fair allocation point.

The question of designing stable scheduling algorithms for wireless networks was first addressed by Tassiulas and Ephremides [28] under the assumption that the incoming flows are *inelastic*, i.e., the flow rates are fixed as for voice or video traffic. They showed that scheduling transmissions as a function of the buffer occupancies (queue-lengths) naturally leads to the stability of the buffers. Tassiulas [26] extended this technique to derive a joint routing and scheduling algorithm that ensures the stability of the queues. These results showed that the queue-length-based resource allocation guarantees stability of the buffers as long as the arrival rates lie within the capacity (stability) region of the network. Subsequently, there has been a large body of work that extended the same idea to different scenarios and more general settings [2], [10], [11], [21], [23], [26], [27], [29], [36]. However, these works do not consider the case of traffic whose rate can be adjusted online.

In the context of wireline networks, the idea of a distributed flow control based on a system-wide optimization problem was developed in [12], and followed by others in [1], [14], [18], [30], and [34]; see [24] for a survey. In these works, the main contribution was the design of a distributed congestion control mechanism to drive the rates of *elastic* flows towards the system-wide optimum. In [6] and [35], the authors use this idea to develop congestion control algorithms for wireless environments by reducing the available capacity region and converting the network into essentially a wireline network. The essential characteristics of wireless networks are not fully addressed there.

More recently, the problem of serving elastic traffic over wireless networks has been investigated in [5], [7], [9], [15], [16], [20], [22], and [25]. Here, the queues and the wireless characteristics of the network are included in the system model. The main idea in these works has been to combine the results on scheduling inelastic traffic in wireless networks and distributed congestion control in wireline networks to design *joint scheduling-congestion control mechanisms* that guarantee optimal routes, stability, and optimal rate allocation. These papers prove that a decentralized congestion controller at the transport layer working in conjunction with a queue-length-based scheduler at the MAC layer will asymptotically achieve buffer stability, optimal routing, and fair rate allocation. Moreover, these layers are coupled through common queue-length information.

In [9], [15], [20], and [25], the authors propose and study rate control algorithms that adapt the flow rates instantaneously as a function of the entry queue-lengths. The rate control mechanism studied in all of these works can be categorized as the *Dual Congestion Controller* since it can be interpreted as a gradient algorithm for the dual of an optimization problem. The intrinsic assumption of the dual congestion control mechanism is that the flow rates can be changed instantaneously in response to congestion feedback in the network. However, it is well known that adaptive window flow control mechanisms such as transmission control protocol (TCP) respond to congestion feedback not instantaneously, but gradually. Such a response is desired by practitioners because the rate fluctuations are small. Thus, the study of another algorithm that modifies the flow rates gradually is important. To this end, we propose and study the so called *Primal-Dual Congestion Controller* in this work. Primal-dual algorithms are well known in the optimization literature and have been studied extensively in different contexts [1], [17], [24], [30]. Since the response of the primal-dual controller is more gradual compared to the dual controller, it is not immediately clear as to whether the buffer stability and rate convergence

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A. Eryilmaz is with the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge, MA 02139-4307 USA (e-mail: eryilmaz@mit.edu; atilla.eryilmaz@gmail.com).

R. Srikant is with the Department of Electrical and Computer Engineering and Coordinated Science Laboratory, University of Illinois at Urbana–Champaign, Urbana, IL 61801 USA (e-mail: rsrikant@uiuc.edu).

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properties will be maintained. We note that the algorithm considered in [25] updates its rates somewhat differently than the algorithm in [5], [9], [15], [16], and [20]. In [25], the users' data rates are still determined instantaneously as a function of the buffer occupancies and channel conditions, but an *average flow rate* is maintained for each user and used in the algorithm. On the other hand, in our work, we update the data rates to mimic the characteristics of widely used versions of TCP [24]. Further, the proof technique used in [25] is quite different from ours, and our algorithm can be directly interpreted as a gradient algorithm for a primal optimization problem that is implemented at the sources and a gradient algorithm for a dual optimization problem that is implemented at the nodes.

Here, it must be stressed that even though the congestion control is distributed, the scheduling is still assumed to be centralized in this work. In [3], [4], [16], [31], and [32], the impact of decentralized implementations of the scheduler is studied. We note that the results of this work can be extended to distributed and asynchronous implementations for a special class of interference models using the approach in [3]. Finally, we note that a related, but different, problem has been considered in [33], where a distributed algorithm has been designed to route inelastic flows to minimize delay costs in a wireless network.

The rest of this paper is organized as follows. Section II describes the system model. In Section III, we state the objective of the resource allocation as an optimization problem and state the characteristics of the optimum point. Section IV introduces the queue-length-based resource allocation algorithm that is implemented at the MAC and network (routing) layers. We propose and provide an extensive study of the primal-dual congestion controller (transport layer) in Section V. Various modifications and extensions to the system are described in Section VI. Finally, we give concluding remarks in Section VII.

II. System Model

We assume that the network is represented by a graph, $\mathcal{G} = (\mathcal{N}, \mathcal{L})$, where $\mathcal{N}$ is a set of nodes and $\mathcal{L}$ is a set of directed links. If a link (n, m) is in $\mathcal{L}$, then it is possible to send packets from node n to node m subject to the interference constraints to be described shortly. We let $\boldsymbol{\mu} = \{\mu_l\}_{l \in \mathcal{L}}$ denote the rate vector at which data can be transferred over each link $l \in \mathcal{L}$. We assume that there is an upper bound, $\hat{\eta} < \infty$, on each μ_l, which is a reasonable assumption for any practical system. For ease of presentation, we assume that there is no fading in the environment. We will discuss the extension of the model to include time-variations in Section VI.

We let $\hat{\Gamma}$ denote a bounded region in the $|\mathcal{L}|$ dimensional real space, representing the set of $\boldsymbol{\mu}$ that can be achieved in a given time slot, i.e., it represents the interference constraint. In general, the set need not be convex. In fact, a typical case would be a discrete set of rates that can be achieved, and hence be nonconvex. We let $\Gamma := \mathcal{CH}\{\hat{\Gamma}\}$ denote the convex hull of the set Γ. It is well known that by timesharing between different rate vectors in $\hat{\Gamma}$, any point in Γ can be attained.

We use $\mathcal{F}$ to denote the set of flows that share the network resources. The routes of these flows are not specified *a priori*, but established by the back-pressure scheduling algorithm to be described in Section II. We use $b(f)$ to denote the beginning node, and $e(f)$ to denote the end node of flow f. Fig. 1 illustrates an example network with three flows passing through it.

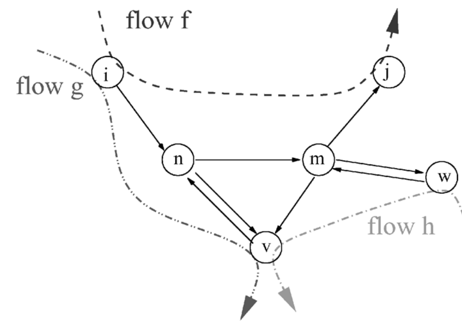

Fig. 1. An example network model with $b(f) = i$, $e(f) = j$, $b(g) = i$, $e(g) = v$, and $b(h) = w$, $e(h) = v$.

Associated with each flow f is a utility function $U_f(x_f)$, which is a function of the flow rate x_f. The utility function, denoted by $U_f(\cdot)$ for flow f, is assumed to satisfy the following conditions.

- $U_f(\cdot)$ is a twice differentiable, strictly concave, nondecreasing function of the mean flow rate x_f.
- For every m and M satisfying $0 < m < M < \infty$, there exist constants $\tilde{c}$ and $\tilde{C}$ satisfying $0 < \tilde{c} < \tilde{C} < \infty$ such that

$$\tilde{c} \le -\frac{1}{U_f''(x)} \le \tilde{C} \quad \forall x \in [m, M]. \tag{1}$$

We note that these conditions are not restrictive and hold for the following class of utility functions:

$$U_f(x) = \beta_f \frac{x^{1-\alpha_f}}{(1-\alpha_f)} \quad \forall \alpha_f > 0. \tag{2}$$

This class of utility functions is known to characterize a large class of fairness concepts including weighted-proportional and max–min fairness [19].

Next, we describe the capacity region of the network as in [15] and [21].

Definition 1 (Capacity Region): The *capacity region*, Λ, of the network contains the set of flow rates $\mathbf{x} \ge \mathbf{0}$ for which there exists a set $\{\mu_l^{(d)}\}_{l \in \mathcal{L}}^{d \in \mathcal{N}}$ that satisfies the following.

(i) $[\sum_d \mu_l^{(d)}]_l \in \Gamma$, where $\mu_l^{(d)} \ge 0$ for all $l \in \mathcal{L}, d \in \mathcal{N}$.

(ii) For each $n \in \mathcal{N}$, and $d \ne n$

$$\mu_{\text{into}(n)}^{(d)} + \sum_f x_f \mathcal{I}_{\{b(f)=n, e(f)=d\}} \le \mu_{\text{out}(n)}^{(d)}$$

where

$$\mu_{\text{into}(n)}^{(d)} := \sum_{(k,n) \in \mathcal{L}} \mu_{(k,n)}^{(d)}$$
$$\mu_{\text{out}(n)}^{(d)} := \sum_{(n,m) \in \mathcal{L}} \mu_{(n,m)}^{(d)}.$$

Observe that $\mu_{\text{into}(n)}^{(d)}$ (or $\mu_{\text{out}(n)}^{(d)}$) denotes the potential number of packets that are destined for node d, incoming to (or outgoing

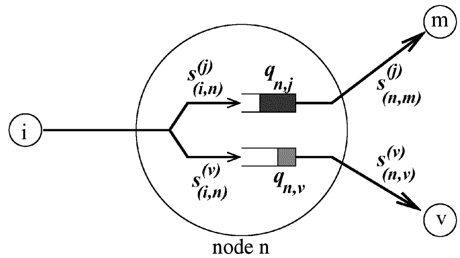

Fig. 2. Each node contains a queue for each destination node. This figure zooms into node n of Fig. 1.

from) node n. The condition (i) captures the interference constraints, while condition (ii) is the flow-conservation constraint that must hold at each node. $\diamond$

It is assumed that each node maintains a separate queue for those flows that have the same destination. We use $q_{n,d}[t]$ to denote the number of packets that are destined for node d, waiting for service at node n at time t. Fig. 2 illustrates one such node. We set $q_{nn}[t] = 0$ for all $n \in \mathcal{N}$ and for all t. For each $n \in \mathcal{N}$, and $d \in \mathcal{N} \setminus \{n\}$, the evolution of $q_{n,d}$ is given by

$$q_{n,d}[t+1] = q_{n,d}[t] + \sum_f x_f[t]\mathcal{I}_{\{b(f)=n,e(f)=d\}} + s^{(d)}_{\text{into}(n)}[t] - s^{(d)}_{\text{out}(n)}[t] \quad (3)$$

where we define $s^{(d)}_{\text{into}(n)}[t] := \sum_{(k,n)\in\mathcal{L}} s^{(d)}_{(k,n)}[t]$ and $s^{(d)}_{\text{out}(n)}[t] := \sum_{(n,m)\in\mathcal{L}} s^{(d)}_{(n,m)}[t]$, and $s^{(d)}_{(n,m)}[t]$ denotes the rate provided to d-destined packets over link (n,m) at slot t. Notice that, this is the actual amount of packets served over the link, not the potential amount denoted by $\mu^{(d)}_{(n,m)}[t]$. Clearly, we have $s_{(n,m)}[t] = \sum_d s^{(d)}_{(n,m)}[t]$. Also, $s^{(d)}_{(n,m)}[t] = \min(\mu^{(d)}_{(n,m)}[t], q_{n,d}[t])$ for all $(n,m) \in \mathcal{L}, d \neq n$.

III. Problem Statement and Characterization of the Optimal Point

Our goal is to design a congestion control, scheduling mechanism such that the flow rate vector $\mathbf{x}$ solves the following optimization problem:

$$\max_{\mathbf{x}\in\Lambda} \sum_{f\in\mathcal{F}} U_f(x_f). \quad (4)$$

We refer to (4) as the *primal problem*. Due to the strict concavity assumption of $U_f(\cdot)$ and the convexity of the capacity region Λ, there exists a unique optimizer of the primal problem, which we refer to as $\mathbf{x}^\star$. We call this the *fair rate allocation*.

One can use duality theory by defining $\lambda_{n,d}$ to be the Lagrange multiplier associated with the constraint

$$\mu^{(d)}_{\text{into}(n)} + \sum_f x_f \mathcal{I}_{\{b(f)=n,e(f)=d\}} \le \mu^{(d)}_{\text{out}(n)}$$

to get the following dual function after algebraic manipulations:

$$D(\lambda) = \sum_{f\in\mathcal{F}} \max_{x_f\ge 0} \{U_f(x_f) - x_f\lambda_{b(f),e(f)}\} + \max_{\boldsymbol{\mu}\in\Gamma} \sum_{(n,m)\in\mathcal{L}} \sum_{d\in\mathcal{N}\setminus\{n\}} \mu^{(d)}_{(n,m)}(\lambda_{n,d} - \lambda_{m,d}).$$

Here, $\lambda_{n,d}$ can be interpreted as the price of transferring a unit amount of data from node n to node d. Thus, $\lambda_{b(f),e(f)}$ is nothing but the price of transferring a unit amount of data from the source of flow f to its destination. Such an approach was taken in [15] where it was shown that for this problem, the duality gap vanishes. Hence, there exists a nonempty set $\Psi^\star$ of optimal Lagrange multipliers that satisfy

$$\sum_{f\in\mathcal{F}} U_f\left(x_f^\star\right) = D(\boldsymbol{\lambda}^\star), \text{ for all } \boldsymbol{\lambda}^\star \in \Psi^\star$$

and there is an associated rate vector $\overset{\star}{\mu} \in \Gamma$ for each $\boldsymbol{\lambda}^\star$, which satisfies the following.

(i) $\overset{\star}{\mu}{}^{(d)}_{\text{into}(n)} + \sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}} \le \overset{\star}{\mu}{}^{(d)}_{\text{out}(n)}$, for all $n \in \mathcal{N}$ and $d \in \mathcal{N} \setminus \{n\}$.

(ii) $\overset{\star}{\mu} \in \arg\max_{\boldsymbol{\mu}\in\Gamma} \sum_{(n,m)\in\mathcal{L}} \mu_{(n,m)} \max_d(\lambda^\star_{n,d} - \lambda^\star_{m,d})$.

(iii) $\lambda^\star_{n,d}(\overset{\star}{\mu}{}^{(d)}_{\text{into}(n)} + \sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}} - \overset{\star}{\mu}{}^{(d)}_{\text{out}(n)}) = 0$ for all $n \in \mathcal{N}$ and $d \in \mathcal{N} \setminus \{n\}$.

Using property (iii), and summing over all $n \in \mathcal{N}$ and $d \in \mathcal{N} \setminus \{n\}$, we get

$$\begin{aligned}\sum_f x_f^\star \lambda^\star_{b(f),e(f)} &= \sum_{n\in\mathcal{N}} \sum_{d\in\mathcal{N}\setminus\{n\}} \lambda^\star_{n,d}\left(\overset{\star}{\mu}{}^{(d)}_{\text{out}(n)} - \overset{\star}{\mu}{}^{(d)}_{\text{into}(n)}\right)\\ &= \sum_{n\in\mathcal{N}} \sum_{d\in\mathcal{N}\setminus\{n\}} \overset{\star}{\mu}{}^{(d)}_{(n,m)}\left(\lambda^\star_{n,d} - \lambda^\star_{m,d}\right)\\ &= \sum_{(n,m)\in\mathcal{L}} \overset{\star}{\mu}_{(n,m)} \max_d\left(\lambda^\star_{n,d} - \lambda^\star_{m,d}\right)\\ &\ge \sum_{(n,m)\in\mathcal{L}} \mu_{(n,m)} \max_d\left(\lambda^\star_{n,d} - \lambda^\star_{m,d}\right),\\ &\quad \forall \boldsymbol{\mu} \in \Gamma \end{aligned} \quad (5)$$

where the last inequality follows from the property (ii). This inequality will later be utilized in the proof of stability of the system.

IV. Scheduling Algorithm

In this paper, we use a queue-length-based scheduler known as the *back-pressure scheduler* introduced by Tassiulas [26], which uses the differential backlog at the two end nodes of a link to determine the rate of that link. This scheduler assigns a weight to each link that equals to the maximum differential backlog between the transmitting and receiving nodes, and then provides higher service to links with larger weights. Hence, the services are allocated to equalize the queue-lengths of neighboring nodes at the fastest rate. The details of the scheduler is provided in the following definition.

Definition 2 (Back-Pressure Scheduler): At slot t, for each $(n,m) \in \mathcal{L}$, we define the differential backlog for destination node d as

$$w_{(n,m),d}[t] := (q_{n,d}[t] - q_{m,d}[t]).$$

Also, we let

$$\begin{aligned} w_{(n,m)}[t] &= \max_d \left\{w_{(n,m),d}[t]\right\} \\ d_{(n,m)}[t] &= \arg\max_d \left\{w_{(n,m),d}[t]\right\}. \end{aligned} \tag{6}$$

Choose the rate vector $\boldsymbol{\mu}[t] \in \hat{\Gamma}$ that satisfies

$$\boldsymbol{\mu}[t] \in \arg\max_{\{\boldsymbol{\eta}\in\hat{\Gamma}\}} \sum_{\{(n,m)\in\mathcal{L}\}} \eta_{(n,m)} w_{(n,m)}[t] \tag{7}$$

and then serve the queue holding packets destined for node $d_{(n,m)}[t]$ over link (n,m) at rate $\mu_{(n,m)}[t]$. That is, we set

$$\mu_{(n,m)}^{(d_{(n,m)}[t])}[t] = \mu_{(n,m)}[t].$$

The rest of the queues at node n are not served at slot t. ◇

Such a scheduling rule has been shown to achieve *throughput-optimality* [28], that is, any arrival rate that can be supported under the stability of the network using any other scheduling-routing policy can be supported by this policy. Intuitively, this fact is due to the dynamic nature of this policy that serves queues that are highly congested relative to their neighbors. Thus, no buffer is allowed to be highly congested, which leads to the throughput-optimality property. Next, we list two other facts related to the back-pressure policy that will be used later.

Fact 1: The maximization in (7) can be performed over Γ instead of $\hat{\Gamma}$, because the optimal rate vector must always contain at least one element from $\hat{\Gamma}$. This follows from the linearity of the objective function and the fact that $\Gamma = \mathcal{CH}\{\hat{\Gamma}\}$.

Fact 2: Those flows that have $w_{l,d}[t] < 0$ will get $\mu_l^{(d)}[t] = 0$, because the objective of the optimization in (7) can only decrease by choosing $\mu_l^{(d)}[t] > 0$, if $w_{l,d}[t] < 0$.

V. Primal-Dual Congestion Controller

The function of the congestion control mechanism is to observe the congestion level of the network and respond to it by increasing/decreasing the data rate of the flows so that they evolve towards the fair allocation as described in Section III. In this paper, we propose a primal-dual congestion control mechanism that can be implemented in a decentralized fashion for each flow. In particular, the source node of each flow uses its local queue-length information, as well as the utility function associated with that flow to update the flow rate in an iterative manner. This is a realistic model for window based flow control mechanisms implemented in many versions of TCP, because for such mechanisms the flow rates are gradually increased or decreased depending on the congestion feedback from the network.

Definition 3 (Primal-Dual Congestion Controller): At the beginning of time slot t, each flow, say f, has access to the queue-length of its first node, i.e., $q_{b(f),e(f)}[t]$. The data rate $x_f[t]$ of flow f satisfies

$$x_f[t+1] = \left\{x_f[t] + \alpha\left(KU_f'\left(x_f[t]\right) - q_{b(f),e(f)}[t]\right)\right\}_m^M$$

where the notation $\{y\}_a^b$ projects the value of y to the closest point in the interval $[a,b]$. We assume that m is a fixed positive valued quantity that can be arbitrarily small, $M > 2\hat{\eta}$, and $K > 0$. ◇

The constant K will be used to guarantee convergence of the achieved rates to the fair allocation. In particular, we are interested in the performance of the system for large K.

In the following sections, we prove that this congestion control mechanism, when operated in parallel with the back-pressure scheduler, achieves flow rates arbitrarily close to the fair allocation. To that end, we first study a heuristic fluid model, and then move on to the original discrete-time system.

A. Convergence of the Primal-Dual Controller

In this section, we first introduce a heuristic fluid model of the joint scheduler-congestion control mechanism, and prove its stability and convergence properties, and then show the convergence properties of the discrete-time primal-dual algorithm using the results of the fluid model.

1) Analysis of a Continuous-Time Fluid Model: Before we provide the model and analysis, we give the LaSalle's invariance principle from nonlinear systems theory, that will be useful in our subsequent analysis. Consider the differential equation: $\dot{\mathbf{y}}(t) = f(\mathbf{y}(t))$, where f is a locally Lipschitz function. Then, the following theorem can be used to determine its asymptotic behavior [13].

Theorem 1 (LaSalle's Invariance Principle): Let $Y : D \to \mathcal{R}$ be a radially unbounded,[1] continuously differentiable, positive definite[2] function such that $\dot{Y}(\mathbf{z}) \leq 0$ for all $\mathbf{z} \in D$. Let $\mathcal{E}$ be the set of points in D where $\dot{Y}(\mathbf{z}) = 0$. Let $\mathcal{M}$ be the largest invariant set[3] in $\mathcal{E}$. Then, every solution starting in D approaches $\mathcal{M}$ as $t \to \infty$.

Now, we present the heuristic fluid model of the system. We assume that the time is continuous and the evolution of each queue is governed by the differential equation: for each $n \in \mathcal{N}$, and $d \in \mathcal{N} \setminus \{n\}$

$$\dot{q}_{n,d}(t) = \left(\sum_f x_f(t)\mathcal{I}_{\{b(f)=n,e(f)=d\}} + \mu_{\text{into}(n)}^{(d)}(t) - \mu_{\text{out}(n)}^{(d)}(t)\right)_{q_{n,d}(t)\geq 0} \tag{8}$$

where $(y)_{z\geq a}$ is equal to y when $z > a$ and is equal to $\max(y,0)$ when $z = a$. Here, (t) is used instead of $[t]$ to signify that we are working in continuous-time. The back-pressure algorithm computes the link schedules and rates at every instant of time as described in Section II. Finally, the congestion controller is

[1]Function $F(\mathbf{z})$ is called *radially unbounded* if $\lim_{||\mathbf{z}||\to\infty} F(\mathbf{z}) = \infty$.

[2]Y is positive definite if $Y(\mathbf{z}^\star) = \mathbf{0}$ for some $\mathbf{z}^\star$, and $Y(\mathbf{z}) > 0$ for all $\mathbf{z} \neq \mathbf{z}^\star$.

[3]A set $\mathcal{M}$ is said to be an *invariant set* if $\mathbf{z}(0) \in \mathcal{M}$ implies that $\mathbf{z}(t) \in \mathcal{M}$ for all $t \in \mathcal{R}$.

assumed to determine the instantaneous flow rates such that, for all $f \in \mathcal{F}$

$$\dot{x}_f(t) = \alpha \left(K U_f' \left(x_f(t) \right) - q_{b(f),e(f)}(t) \right)_{x_f(t) \geq m} \tag{9}$$

where we take $m > 0$ such that it satisfies: $m < x_f^\star$ for all $f \in \mathcal{F}$. Then, the following global asymptotic stability result holds.

We note that the local Lipschitz property of the function governing the evolution holds only if $m > 0$. For $m = 0$, the right-hand side of (9) will not be locally Lipschitz.

Theorem 2: Starting from any $\mathbf{x}(0)$ and $\mathbf{q}(0)$, the rate vector $\mathbf{x}(t)$ converges to $\mathbf{x}^\star$ as t goes to infinity. Moreover, the queue-length vector $\mathbf{q}(t)$ approaches the bounded set $\tilde{\mathcal{S}}$ described by

$$\tilde{\mathcal{S}} = \mathcal{S} \cap \left\{ \mathbf{q} \geq \mathbf{0} : \mathbf{q_{b(f),e(f)}} = \lambda^\star_{\mathbf{b(f),e(f)}} \text{ for all } \mathbf{f} \in \mathcal{F} \right\}$$

where

$$\mathcal{S} := \left\{ \mathbf{q} \geq \mathbf{0} : \sum_{\mathbf{n,d}} \left[\left(\mathbf{q_{n,d}(t)} - \lambda^\star_{\mathbf{n,d}} \right) \left(\mu^{(\mathbf{d})}_{\mathbf{out(n)}}(\mathbf{t}) - \mu^{(d)}_{\text{into}(n)}(t) - \sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}} \right) \right] = 0 \right\}.$$

Proof: The proof is based on LaSalle's invariance principle. For any given $\boldsymbol{\lambda}^\star \in \Psi^\star$, we study the following Lyapunov function

$$Y(\mathbf{x}, \mathbf{q}; \boldsymbol{\lambda}^\star) := \sum_{f \in \mathcal{F}} \frac{\left(x_f - x_f^\star \right)^2}{2\alpha} + \sum_{n \in \mathcal{N}} \sum_{d \in \mathcal{N}} \frac{\left(q_{n,d} - \lambda^\star_{n,d} \right)^2}{2} \tag{10}$$

which is introduced in [24]. It is easy to see that this is a radially unbounded function. Next, we study time-derivative of this function

$$\begin{aligned}
\dot{Y}&\left(\mathbf{x}(t), \mathbf{q}(t); \boldsymbol{\lambda}^\star \right) \\
&= \sum_{f \in \mathcal{F}} \left(x_f(t) - x_f^\star \right) \times \left(U_f' \left(x_f(t) \right) - q_{b(f),e(f)}(t) \right)_{x_f(t) \geq m} && (11) \\
&\quad + \sum_{n,d} \left(q_{n,d}(t) - \lambda^\star_{n,d} \right) \times \left(\sum_f x_f(t) \mathcal{I}_{\{b(f)=n,e(f)=d\}} + \mu^{(d)}_{\text{into}(n)}(t) - \mu^{(d)}_{\text{out}(n)}(t) \right)_{q_{n,d}(t) \geq 0} && (12) \\
&\leq \sum_{f \in \mathcal{F}} \left(x_f(t) - x_f^\star \right) \left(U_f' \left(x_f(t) \right) - q_{b(f),e(f)}(t) \right) && (13) \\
&\quad + \sum_{n,d} \left(q_{n,d}(t) - \lambda^\star_{n,d} \right) \times \left(\sum_f x_f(t) \mathcal{I}_{\{b(f)=n,e(f)=d\}} + \mu^{(d)}_{\text{into}(n)}(t) - \mu^{(d)}_{\text{out}(n)}(t) \right) && (14)
\end{aligned}$$

where (13) follows from (11) due to the assumption that $m < x_f^\star, \forall f$. Similarly, (14) follows from (12) when we note that removing the lower bound on $q_{b(f),e(f)}(t)$ can only increase the sum since $\lambda^\star_{n,d} \geq 0$ by definition.

Let us add and subtract $U_f'(x_f^\star) = \lambda^\star_{b(f),e(f)}$, and $\sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}}$ to (13) and (14), respectively, and rearrange the terms. This operation yields

$$\begin{aligned}
\dot{Y}&\left(\mathbf{x}(t), \mathbf{q}(t); \boldsymbol{\lambda}^\star \right) \\
&\leq \sum_f \left(x_f(t) - x_f^\star \right) \left(U_f' \left(x_f(t) \right) - U_f' \left(x_f^\star \right) \right) && (15) \\
&\quad + \sum_f \left(x_f(t) - x_f^\star \right) \left(\lambda^\star_{b(f),e(f)} - q_{b(f),e(f)}(t) \right) && (16) \\
&\quad + \sum_f \left(q_{b(f),e(f)}(t) - \lambda^\star_{b(f),e(f)} \right) \left(x_f(t) - x_f^\star \right) && (17) \\
&\quad + \sum_{n,d} \lambda^\star_{n,d} \left(\mu^{(d)}_{\text{out}(n)}(t) - \mu^{(d)}_{\text{into}(n)}(t) - \sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}} \right) && (18) \\
&\quad + \sum_{n,d} q_{n,d}(t) \left(\sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}} + \mu^{(d)}_{\text{into}(n)}(t) - \mu^{(d)}_{\text{out}(n)}(t) \right). && (19)
\end{aligned}$$

Notice that terms (16) and (17) cancel each other. The strict concavity of $U_f(\cdot)$ implies that (15) ≤ 0 for all $\mathbf{x}(t)$ with strict inequality whenever $\mathbf{x}(t) \neq \mathbf{x}^\star$. Next, we study (18) and (19) separately to argue that they are both upper-bounded by zero.

We start with (18). Notice that we can write

$$\begin{aligned}
\sum_{n,d} \lambda^\star_{n,d} \left(\mu^{(d)}_{\text{out}(n)}(t) - \mu^{(d)}_{\text{into}(n)}(t) \right)
&= \sum_{(n,m) \in \mathcal{L}} \sum_{d \neq n} \mu^{(d)}_{(n,m)}(t) \left(\lambda^\star_{n,d} - \lambda^\star_{m,d} \right) \\
&\leq \sum_f x_f^\star \lambda^\star_{b(f),e(f)}
\end{aligned}$$

where the equality follows from a change in the order of summation, and the inequality is due to (5). Therefore, we have (18) ≤ 0.

Next, we consider the expression (19). We start by noting that the optimum flow rate vector, $\mathbf{x}^\star$, can be translated into a link rate vector, which we denote by $\mathbf{y}(\mathbf{x}^\star)$. Since $\mathbf{x}^\star \in \Lambda$, we must have $\mathbf{y}(\mathbf{x}^\star) \in \Gamma$ by definition. Notice that we can write

$$\begin{aligned}
\sum_f x_f^\star q_{b(f),e(f)}(t) &= \sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} y_{(n,m)}^{(d)}(\mathbf{x}^\star)\left(q_{n,d}(t)-q_{m,d}(t)\right)\\
&\leq \sum_{(n,m)\in\mathcal{L}} y_{(n,m)} \max_d \left(q_{n,d}(t)-q_{m,d}(t)\right)\\
&= \sum_{(n,m)\in\mathcal{L}} y_{(n,m)} w_{(n,m)}(t)\\
&\overset{(a)}{\leq} \sum_{(n,m)\in\mathcal{L}} \mu_{(n,m)}(t) \max_d \left(q_{n,d}(t)-q_{m,d}(t)\right)\\
&= \sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} \mu_{(n,m)}^{(d)}(t)\left(q_{n,d}(t)-q_{m,d}(t)\right)
\end{aligned}$$

where the inequality (a) holds due to (7). Thus, (19) ≤ 0.

Combining these results, we see that

$$\dot{Y}\left(\mathbf{x}(t),\mathbf{q}(t);\boldsymbol{\lambda}^\star\right) \leq 0, \quad \text{for all } \mathbf{x}(t) \geq m\mathbf{1^T}, \mathbf{q(t)} \geq \mathbf{0}$$

with strict inequality when $\mathbf{x} \neq \mathbf{x}^\star$ or $\mathbf{q} \notin \mathcal{S}$. Let us define

$$\mathcal{E} := \left\{(\mathbf{x},\mathbf{q}) : \dot{Y}(\mathbf{x},\mathbf{q};\boldsymbol{\lambda}^\star) = 0\right\}.$$

We claim that

$$\mathcal{E} \subset \{(\mathbf{x},\mathbf{q}) : \mathbf{x} = \mathbf{x}^\star, \mathbf{q} \in \mathcal{S}\}.$$

To see that this is true, observe that for any element $(\tilde{\mathbf{x}},\tilde{\mathbf{q}}) \notin \{(\mathbf{x},\mathbf{q}) : \mathbf{x} = \mathbf{x}^\star, \mathbf{q} \in \mathcal{S}\}$, we have

$$\dot{Y}\left(\mathbf{x}(t),\mathbf{q}(t);\boldsymbol{\lambda}^\star\right)\Big|_{\mathbf{x}(t)=\tilde{\mathbf{x}}}^{\mathbf{q}(t)=\tilde{\mathbf{q}}} < 0$$

from the above discussion. Since we always have $\dot{Y}(\mathbf{x}(t),\mathbf{q}(t);\boldsymbol{\lambda}^\star) \leq 0$, any element of $\mathcal{E}$ must also be an element of $\{(\mathbf{x},\mathbf{q}) : \mathbf{x} = \mathbf{x}^\star, \mathbf{q} \in \mathcal{S}\}$.

Now, defining $\mathcal{M}$ to be the largest invariant set of the primal-dual algorithm, we claim that

$$\mathcal{M} \subset \left\{(\mathbf{x},\mathbf{q}) : \mathbf{x} = \mathbf{x}^\star, \mathbf{q} \in \tilde{\mathcal{S}}\right\}.$$

This is true due to the fact that $q_{b(f),e(f)}(t) \neq \lambda_{b(f),e(f)}^\star$ for some $f \in \mathcal{F}$ would result in $\dot{x}_f(t) \neq 0$, and hence would shift $\mathbf{x}(t)$ from $\mathbf{x}^\star$.

Hence, LaSalle's invariance principle applies, i.e., $(\mathbf{x}(t),\mathbf{q}(t))$ approaches $\mathcal{M}$. This result, in turn, implies that $\mathbf{x}(t) \to \mathbf{x}^\star$ and $\mathbf{q}(t) \to \tilde{\mathcal{S}}$ as t tends to infinity. ■

2) Analysis of the Discrete-Time Model: The evolution of each flow rate is governed by the primal-dual iteration as described in Definition 3. Also, recall that the queue-lengths evolve according to (3). We define μ_{sym} to be the maximum flow rate that can be provided to all the flows, i.e.,

$$\mu_{\text{sym}} := \max\{\eta \geq 0 : (\eta,\cdots,\eta) \in \Lambda\}.$$

We assume that $\mu_{\text{sym}} > m > 0$, and that $x_f^\star > m, \forall f$. These are reasonable assumptions given that we are free to choose m as small as necessary to satisfy them.

The following lemma provides a relationship between potential service rate and the actual service rate, that will be used in the proof of the subsequent theorem.

Lemma 1: For our system, the following relationship holds for any $\mathbf{q}[t]$ and some $B < \infty$:

$$\begin{aligned}
&\sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} s_{(n,m)}^{(d)}[t]\left(q_{n,d}[t]-q_{m,d}[t]\right)\\
&\qquad\geq \sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} \mu_{(n,m)}^{(d)}[t]\left(q_{n,d}[t]-q_{m,d}[t]\right) - B.
\end{aligned}$$

Proof: We prove this lemma by covering all the possible cases.

Case 1) $q_{n,d}[t] < q_{m,d}[t]$: then, due to Fact 2, we have $\mu_{(n,m)}^{(d)}[t] = 0$, and subsequently, we must have $s_{(n,m)}^{(d)}[t] = 0$. Thus, the claim holds with equality.

Case 2) $q_{n,d}[t] \geq q_{m,d}[t]$ and $q_{n,d}[t] \geq \hat{\eta}$: then there can be no unused service since $\mu_{(n,m)}^{(d)}[t] < \hat{\eta}$ by assumption. Thus, we have $s_{(n,m)}^{(d)}[t] = \mu_{(n,m)}^{(d)}[t]$ and the claim holds with equality.

Case 3) $\hat{\eta} > q_{n,d}[t] \geq q_{m,d}[t]$: then we have $s_{(n,m)}^{(d)}[t] < \hat{\eta}$ and $\mu_{(n,m)}^{(d)}[t](q_{n,d}[t]-q_{m,d}[t]) \leq \hat{\eta}^2$. Thus, we can write

$$\sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} \mu_{(n,m)}^{(d)}[t]\left(q_{n,d}[t]-q_{m,d}[t]\right) \leq |\mathcal{L}||\mathcal{N}|\hat{\eta}^2 =: B.$$

Noting that

$$\sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} s_{(n,m)}^{(d)}[t]\left(q_{n,d}[t]-q_{m,d}[t]\right) \geq 0$$

since $s_{(n,m)}^{(d)}[t] = 0$ whenever $q_{n,d}[t] < q_{m,d}[t]$, we finish the proof of this lemma. ■

Next proposition establishes the asymptotic boundedness of the queue-lengths, and hence the stability of the system.

Proposition 1: For $\alpha = (1/K^2)$, and for some finite constant c, we have

$$\limsup_{t\to\infty} \sum_{f\in\mathcal{F}} q_{b(f),e(f)}^2[t] \leq cK^2.$$

Proof: Let us consider the Lyapunov function

$$L(\mathbf{q}) = \sum_{n\in\mathcal{N}} \sum_{d\in\mathcal{N}} q_{n,d}^2$$

and study its drift

$$\Delta L_t(\mathbf{q}) := L(\mathbf{q}[t+1]) - L(\mathbf{q}[t]) \\ \leq B_1 + \sum_{n,d} q_{n,d}[t] \left(\sum_f x_f[t] \mathcal{I}_{\{b(f)=n, e(f)=d\}} \right. \\ \left. + s^{(d)}_{\text{into}(n)}[t] - s^{(d)}_{\text{out}(n)}[t] \right)$$

for some $B_1 < \infty$ that is a function of M and the maximum link rate $\hat{\eta}$. If we add and subtract μ_{sym} and rearrange the terms, we can state for some $\epsilon > 0$ that satisfies $\epsilon < \mu_{\text{sym}} - m$ that

$$\Delta L_t(\mathbf{q}) \leq \sum_f q_{b(f),e(f)}[t](\mu_{\text{sym}} - \epsilon) \\ + \sum_f q_{b(f),e(f)}[t](x_f[t] - \mu_{\text{sym}} + \epsilon) \\ + B_1 - \sum_{n,d} q_{n,d}[t] \left(s^{(d)}_{\text{out}(n)}[t] - s^{(d)}_{\text{into}(n)}[t] \right). \quad (20)$$

Note that due to Lemma 1, we can lower bound the sum in (20) as follows:

$$\sum_{n,d} q_{n,d}[t] \left(s^{(d)}_{\text{out}(n)}[t] - s^{(d)}_{\text{into}(n)}[t] \right) \\ = \sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} s^{(d)}_{(n,m)}[t] \left(q_{n,d}[t] - q_{m,d}[t] \right) \\ \geq \sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} \mu^{(d)}_{(n,m)}[t] \left(q_{n,d}[t] - q_{m,d}[t] \right) - B_2 \quad (21)$$

for some $B_2 < \infty$. Also, note that the back-pressure algorithm is designed specifically to allocate the service rate vector $\boldsymbol{\mu}[t]$ to maximize the double sum in (21). We define $\boldsymbol{\mu}^{\text{sym}} = (\mu_{\text{sym}}, \cdots, \mu_{\text{sym}})$, and note that $\boldsymbol{\mu}^{\text{sym}} \in \Lambda$. By the definition of Λ, we can find a set of link rate vectors, $\mathbf{y}(\boldsymbol{\mu}^{\text{sym}})$, that correspond to the flow rate vector $\boldsymbol{\mu}^{\text{sym}}$ such that $\mathbf{y}(\boldsymbol{\mu}^{\text{sym}}) \in \Gamma$, i.e., we have

$$\sum_f \mu_{\text{sym}} q_{b(f),e(f)}[t] \\ = \sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} y^{(d)}_{(n,m)}(\boldsymbol{\mu}^{\text{sym}}) \left(q_{n,d}[t] - q_{m,d}[t] \right) \\ \leq \sum_{(n,m)\in\mathcal{L}} \sum_{d\neq n} \mu^{(d)}_{(n,m)}[t] \left(q_{n,d}[t] - q_{m,d}[t] \right) \quad (22)$$

where the last step is due to the back-pressure scheduling algorithm. By substituting (21) and (22) into the upper bound in (20), we get

$$\Delta L_t(\mathbf{q}) \leq -\epsilon \sum_f q_{b(f),e(f)}[t] + B_1 + B_2 \\ + \sum_f q_{b(f),e(f)}[t] \left(x_f[t] - \mu_{\text{sym}} + \epsilon \right). \quad (23)$$

Next, we will prove that

$$\limsup_{t\to\infty} \sum_f q_{b(f),e(f)}[t] \left(x_f[t] - \mu_{\text{sym}} + \epsilon \right) \leq c_1 K$$

for some $c_1 < \infty$. Towards this end, we define r to be a constant that satisfies $r > M + \hat{\eta} + \max_f U'_f(m)$. Then, we study the following two cases.

Case 1) If in any time slot, say t, we have $q_{b(f),e(f)}[t] > rK$ for some flow f, then even if $x_f[t]$ were at the maximum possible rate M, $x_f[t+K]$ will be m. This is true because of the way we chose r: at every slot $q_{b(f),e(f)}$ can decrease by at most $\hat{\eta}$, and thus $q_{b(f),e(f)}[t+i] \geq (r-\hat{\eta})K$ for all $i \in \{0, \cdots, K\}$. On the other hand, for all the time slots between t and $t+K$, the rate x_f will decrease by at least

$$\frac{1}{K^2} \left(K U'_f(x_f[t+i]) - q_{b(f),e(f)}[t+i] \right) \\ \geq \frac{1}{K} \left(U'_f(m) - (r - \hat{\eta}) \right) \\ \geq \frac{M}{K}$$

unless x_f hits its lower bound m.

Now, observe that when $x_f[t+K] = m$, we have $q_{b(f),e(f)}[t+K](x_f[t+K] - \mu_{\text{sym}} + \epsilon) \leq 0$ since $m < \mu_{\text{sym}} - \epsilon$. This is clearly true for all time slots after $t+K$ for as long as $q_{b(f),e(f)}$ stays above rK.

Case 2) If we have $q_{b(f),e(f)}[t] \leq rK$, then we have $q_{b(f),e(f)}[t](x_f[t] - \mu_{\text{sym}} + \epsilon) \leq rMK$. And, even if $q_{b(f),e(f)}$ exceeds rK, we know from our discussion in Case 1 that x_f will hit m in at most K slots. Noting that $q_{b(f),e(f)}$ cannot increase by more than M in a slot, we can state that the largest value $q_{b(f),e(f)}$ can attain before $q_{b(f),e(f)}(x_f - \mu_{\text{sym}} + \epsilon) \leq 0$ is $(r+M)K$. Therefore, we have $q_{b(f),e(f)}(x_f - \mu_{\text{sym}} + \epsilon) \leq M(r+M)K$.

Combining the arguments of the two cases, we can conclude that there exists a constant c_1 for which

$$\limsup_{t\to\infty} \sum_f q_{b(f),e(f)}[t] \left(x_f[t] - \mu_{\text{sym}} + \epsilon \right) \leq c_1 K$$

is true.

After substituting this result in (23) and finding a large enough constant d, we can write

$$\Delta L_t(\mathbf{q}) \leq -\epsilon \mathcal{I}_{\left\{\sum_f q_{b(f),e(f)}[t] \geq dK\right\}} \\ + c_1 K \mathcal{I}_{\left\{\sum_f q_{b(f),e(f)}[t] < dK\right\}}$$

which simply states that $\sum_f q^2_{b(f),e(f)}$ will decrease if $\sum_f q_{b(f),e(f)}[t] \geq dK$ and possibly increase by an $O(K)$

expression if $q_{b(f),e(f)}[t] < dK$. This naturally implies that we can find a d_2 for which we have

$$\limsup_{t\to\infty} \sum_{f\in\mathcal{F}} q_{b(f),e(f)}[t] \le d_2 K. \tag{24}$$

Hence, we can find some c for which the statement of the proposition must hold. ■

Equation (24) states that the total queue-length expression grows at most linearly with K. Hence, we would expect delay to grow with K. Next, we state the main theorem which shows that the average rate obtained by each user can be made arbitrarily close to its *fair* share [as defined by the resource allocation problem (4)] by choosing K sufficiently large. Thus, there exists a tradeoff between delay and fairness, which can be controlled with the choice of K. In particular, by increasing K, the fairness characteristics can be improved with the cost of larger delay.

Theorem 3: For $\alpha = (1/K^2)$, and for some finite $B > 0$, we have: for all $f \in \mathcal{F}$

$$x_f^\star - \frac{B}{\sqrt{K}} \le \liminf_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} x_f[t] \le \limsup_{T\to\infty} \frac{1}{T}\sum_{t=0}^{T-1} x_f[t] \le x_f^\star + \frac{B}{\sqrt{K}}.$$

Proof: We study the drift of the Lyapunov function $Y(\cdot)$ that is given in (10)

$$\Delta Y_t(\mathbf{x},\mathbf{q};\boldsymbol{\lambda}^\star) := Y\left(\mathbf{x}[t+1],\mathbf{q}[t+1];\boldsymbol{\lambda}^\star\right) - Y\left(\mathbf{x}[t],\mathbf{q}[t];\boldsymbol{\lambda}^\star\right)$$

which can be upper-bounded by using the same line of reasoning we followed in the proof of Theorem 2, i.e., we handle the boundary constraints of the rates and queue-lengths, add and subtract $KU_f'(x_f^\star) = \lambda^\star_{b(f),e(f)}$, and $\sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}}$, and rearrange terms to get:

$$\begin{aligned}
\Delta Y_t(\mathbf{x},\mathbf{q};\boldsymbol{\lambda}^\star) &\le K\sum_f \left(U_f'(x_f[t]) - U_f'(x_f^\star)\right)\left(x_f[t]-x_f^\star\right) && (25)\\
&+ \sum_f \left(x_f[t]-x_f^\star\right)\left(\lambda^\star_{b(f),e(f)} - q_{b(f),e(f)}[t]\right) && (26)\\
&+ \sum_f \frac{\alpha}{2}\left(KU_f'(x_f[t]) - q_f[t]\right)^2 && (27)\\
&+ \sum_f \left(q_{b(f),e(f)}[t] - \lambda^\star_{b(f),e(f)}\right)\left(x_f[t]-x_f^\star\right) && (28)\\
&+ \sum_{n,d} \lambda^\star_{n,d}\left(s^{(d)}_{\mathrm{out}(n)}[t] - s^{(d)}_{\mathrm{into}(n)}[t] - \sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}}\right) && (29)\\
&+ \sum_{n,d} q_{n,d}[t]\left(\sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}} + s^{(d)}_{\mathrm{into}(n)}[t] - s^{(d)}_{\mathrm{out}(n)}[t]\right) && (30)\\
&+ \frac{1}{2}\sum_{n,d}\left(\sum_f x_f^\star \mathcal{I}_{\{b(f)=n,e(f)=d\}} + s^{(d)}_{\mathrm{into}(n)}[t] - s^{(d)}_{\mathrm{out}(n)}[t]\right)^2. && (31)
\end{aligned}$$

We claim that $(25) \le -\tilde{C}K\|\mathbf{x}[t]-\mathbf{x}^\star\|^2$, where $\tilde{C}$ is a positive constant that is independent of K. To see this, first note that the strict concavity assumption of the utility functions allows us to write

$$K\left(U_f'(x_f[t]) - U_f'(x_f^\star)\right)\left(x_f[t]-x_f^\star\right) = -K\left|U_f'(x_f[t]) - U_f'(x_f^\star)\right|\left|x_f[t]-x_f^\star\right| \tag{32}$$

for each $f \in \mathcal{F}$. Also, due to Taylor expansion, we can find some $y_f[t]$ between $x_f[t]$ and $x_f^\star$ for which, $U_f'(x_f[t]) - U_f'(x_f^\star) = (x_f[t]-x_f^\star)U_f''(y_f[t])$. Using the assumption in (1), we can thus claim that there exists some $\tilde{C} > 0$ which yields $|U_f'(x_f[t]) - U_f'(x_f^\star)| \ge \tilde{C}|x_f[t]-x_f^\star|$, which can be substituted into (32) to prove the claim.

Observe that the terms (26) and (28) cancel each other. Also, notice that (29) and (30) are almost the same as (18) and (19), respectively, except for the fact that actual service rates appear in them instead of potential service rates. This difference can be handled by using Lemma 1, which allows us to state that (29) and (30) are both upper-bounded by some bounded quantity, say B_0. Since the link rates and the flow rates are upper-bounded at any slot, we can find some $B_1 < \infty$ such that $(31) < B_1$. Thus, we have

$$\Delta Y_t(\mathbf{x},\mathbf{q};\boldsymbol{\lambda}^\star) \le B_0 + B_1 - \tilde{C}K\|\mathbf{x}[t]-\mathbf{x}^\star\|^2 + \sum_f \frac{\alpha}{2}\left(KU_f'(x_f[t]) - q_{b(f),e(f)}[t]\right)^2.$$

If we write this drift expression for $t = 0,\cdots,T-1$ and add both sides of the inequalities, then we get

$$\begin{aligned}
Y\left(\mathbf{x}[T],\mathbf{q}[T];\boldsymbol{\lambda}^\star\right) &- Y\left(\mathbf{x}[0],\mathbf{q}[0];\boldsymbol{\lambda}^\star\right)\\
&\le T(B_0+B_1) - \tilde{C}K\sum_{t=0}^{T-1}\|\mathbf{x}[t]-\mathbf{x}^\star\|^2\\
&+ \sum_{t=0}^{T-1}\sum_f \frac{\alpha}{2}\left(KU_f'(x_f[t]) - q_{b(f),e(f)}[t]\right)^2.
\end{aligned}$$

By rearranging terms, noting that $Y(\cdot)$ is a non-negative quantity, dividing both sides by T, and taking the limit as T goes to infinity yields

$$\limsup_{T\to\infty} \frac{\tilde{C}K}{T} \sum_{t=0}^{T-1} \|\mathbf{x}[t]-\mathbf{x}^{\star}\|^2 \leq \limsup_{T\to\infty} \frac{1}{T} \times \sum_{t=0}^{T-1} \sum_{f\in\mathcal{F}} \frac{\alpha}{2} \left(KU_f'\left(x_f[t]\right) - q_{b(f),e(f)}[t]\right)^2 + B_0 + B_1. \quad (33)$$

Thus, the proof will be complete once we show that, when $\alpha = 1/K^2$, we have

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{f\in\mathcal{F}} \frac{\alpha}{2} \left(KU_f'\left(x_f[t]\right) - q_{b(f),e(f)}[t]\right)^2 \leq B_2 < \infty \quad (34)$$

for some B_2. To justify this claim, we ignore the $\alpha/2$ factor for now, and write

$$\begin{aligned}
&\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{f\in\mathcal{F}} \left(KU_f'\left(x_f[t]\right) - q_{b(f),e(f)}[t]\right)^2 \\
&= \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{f\in\mathcal{F}} \Big[\left(KU_f'\left(x_f[t]\right)\right)^2 \\
&\qquad -2KU_f'\left(x_f[t]\right) q_{b(f),e(f)}[t] \\
&\qquad +q^2_{b(f),e(f)}[t]\Big] \quad (35)\\
&\overset{(a)}{\leq} K^2 \sum_{f\in\mathcal{F}} \left(U_f'(m)\right)^2 + \limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{f\in\mathcal{F}} q^2_{b(f),e(f)}[t] \\
&\overset{(b)}{\leq} K^2 \sum_{f\in\mathcal{F}} \left(U_f'(m)\right)^2 + cK^2 \quad (36)
\end{aligned}$$

where the inequality (a) is true since $x_f[t] \in [m, M]$, for all t and $f \in \mathcal{F}$ due to the nature of the primal-dual congestion controller, and since $U_f'(y) \geq 0$, for all $y \in [m, M]$. Also, inequality (b) follows from Proposition 1. Clearly, (36) implies (34), which, when substituted into (33) results in

$$\limsup_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} \|\mathbf{x}[t]-\mathbf{x}^{\star}\|^2 \leq \frac{B^2}{K} \quad (37)$$

where we have $B^2 := (B_0 + B_1 + B_2/\tilde{C})$. Thus, for T large enough and for any $f \in \mathcal{F}$, we have

$$\left|\frac{1}{T}\sum_{t=0}^{T-1}\left(x_f[t]-x_f^{\star}\right)\right| \leq \frac{1}{T}\sum_{t=0}^{T-1}\left|x_f[t]-x_f^{\star}\right| \overset{(a)}{\leq} \sqrt{\frac{B^2}{K}} = \frac{B}{\sqrt{K}}$$

where inequality (a) follows from (37). ■

Theorem 3 directly implies that

$$\lim_{K\to\infty} \bar{x}_f(K) = x_f^{\star}, \quad \text{for all } f \in \mathcal{F}$$

where $\bar{x}_f(K)$ denotes the average rate of flow f, when the congestion controller uses the given K as its parameter i.e.,

$$\bar{x}_f(K) := \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} x_f[t].$$

Hence, the fair allocation is asymptotically attained by the joint scheduling, routing, and primal-dual congestion control mechanism.

VI. Extensions and Variations

In this section, we discuss possible extensions and variations to the joint mechanism that we studied up to this point. The goal is two-fold. First, we emphasize that the analysis can be extended to consider more realistic models. Towards this end, we discuss the inclusion of time-variations in the channel conditions into the model. Second, we consider different versions of congestion controllers that may be of interest, and show that their stability and convergence properties can be proved using the same techniques introduced earlier. In the interest of space, we only provide an outline of the arguments and do not provide complete proofs.

A. Stochastic Channel Models

Recall that our wireless network model assumes time-invariant channel conditions. However, in reality, channel conditions will fluctuate due to the change in the environment. To accommodate this situation, the network model can be extended by assuming a finite set, say $\mathcal{J}$, of states that the channel conditions can be in. Then, we let Γ_j denote the set of link rates that can be achieved when the current state is $j \in \mathcal{J}$. Let π_j^s denote the stationary probability of the channel state being j. Then, we can define the average capacity region as $\Gamma = \sum_{j\in\mathcal{J}} \pi_j^s \mathcal{CH}\{\Gamma_j\}$. Recall that $\mathcal{CH}\{A\}$ denotes the convex hull of the set A. Moreover, we define the average capacity region Λ as in Definition 1 for the average capacity region Γ. Then, the goal is to set the mean flow rates so that $\sum_f U_f(x_f)$ is maximized over all the rates in Λ.

Assuming that the channel state is j at time t, the back-pressure policy performs the following optimization to determine the link rates:

$$\boldsymbol{\mu}[t] \in \arg\max_{\{\boldsymbol{\eta}\in\Gamma_j\}} \sum_{\{(n,m)\in\mathcal{L}\}} \eta_{(n,m)} w_{(n,m)}[t]$$

where $w_{(n,m)}[t]$ is defined as in (6). Then, the arguments of Section V can easily be modified to argue the stability and convergence properties of the system. Further, we can allow randomness in the arrival process to model various implementation details. For example, the flows can be assumed to satisfy

$$\begin{aligned}
&E\left[x_f[t+1] | q_{b(f),e(f)}[t]\right] \\
&\quad = \left\{x_f[t] + \alpha\left(KU_f'\left(x_f[t]\right) - q_{b(f),e(f)}[t]\right)\right\}_m^M, \text{ and} \\
&E\left[x_f^2[t] | q_{b(f),e(f)}[t]\right] \\
&\quad \leq A < \infty, \quad \forall q_{b(f),e(f)}[t]. \quad (38)
\end{aligned}$$

Under these modifications, a stochastic version of the convergence results can be proven (as in Results 1 and 2 in Section VI-B).

B. Dual Congestion Controller

A dual congestion controller aims to change the end-to-end flow rates in a direction so as to minimize the dual objective of (4). For this controller, the data rate $x_f[t]$ of flow f is an independently distributed random variable that satisfies (38) and

$$E\left[x_f[t]|q_{b(f),e(f)}[t]\right] = \min\left\{U_f'^{-1}\left(\frac{q_{b(f),e(f)}[t]}{K}\right), M\right\}.$$

The heuristic fluid model of this controller is given by

$$x_f(t) = U_f'^{-1}\left(\frac{q_{b(f),e(f)}(t)}{K}\right), \quad \text{for all } f \in \mathcal{F}.$$

For this model, the global asymptotic stability of the joint scheduling and congestion control mechanism can be proved using LaSalle's invariance principle by studying the Lyapunov function

$$V(\mathbf{q};\boldsymbol{\lambda}^\star) = \frac{1}{2}\sum_{n\in\mathcal{N}}\sum_{d\in\mathcal{N}}\left(q_{n,d} - \lambda^\star_{n,d}\right)^2$$

for any given $\boldsymbol{\lambda}^\star \in K\Psi^\star$. Hence, for the fluid model, system evolves toward the optimal rate allocation, and the queue-lengths change in such a way that the backlogs converge to the optimal Lagrange multiplier set, $K\Psi^\star$. Then, we can return to the original discrete-time, stochastic system model described before and study its performance as we did in Section V-A-2.

Here, due to the direct relationship between the queue-lengths and mean flow rates, we can also give bounds for the queue-length vector's proximity to $K\Psi^\star$ by adding a new condition the utility functions: $U_f'^{-1}(\cdot)$ is a convex function, and satisfies:[4]

$$\left(1 - \frac{U_f'^{-1}\left(\kappa + \frac{\beta}{K^{1-\sigma}}\right)}{U_f'^{-1}(\kappa)}\right) = O(K^{-\gamma})$$

for any fixed $\kappa, \beta > 0$ and for some $\gamma \in (0,1)$ that is determined as a function of $\sigma \in (0,1)$. Again, this condition is satisfied by the class of utility functions described by (2). Then, we can state the following two results (see [8]).

Result 1: There exists positive constants $\bar{c} < \infty$ and $\sigma \in (0,1)$, that depend on Λ, the utility function set $\{U_f(\cdot)\}$, and the moments of the arrival processes, such that for each $\boldsymbol{\lambda}^\star \in K\Psi^\star$

$$E\left[\|\mathbf{q}^\infty - \boldsymbol{\lambda}^\star\|\right] \le \bar{c}(K)^\sigma, \quad \text{for large } K \tag{39}$$

where $\mathbf{q}^\infty$ is a notation used to denote the state of the Markov chain in steady-state and $\|\cdot\|$ denotes the Euclidean distance.

Result 2: Let $\mathbf{q}^\infty$ be a random vector that is distributed according to the steady-state distribution of the Markov chain $\{\mathbf{q}[t]\}_t$. Then, mean of the stationary rate vector converges to $\mathbf{x}^\star$ as K increases, i.e.,

$$\lim_{K\to\infty} E\left[\mathbf{x}^\infty(K)\right] = \mathbf{x}^\star$$

where $\mathbf{x}^\infty(K)$ is a random vector that is defined by

$$x_f^\infty(K) = \min\left\{U_f'^{-1}\left(\frac{q^\infty_{b(f),e(f)}}{K}\right), M\right\}, \quad \text{for all } f \in \mathcal{F}.$$

Thus, Result 1 establishes the stability of the buffers, and Result 2 proves the asymptotic optimality of the rates achieved by the dual controller, and serves as an alternate proof of the results in [15], [20], and [25].

C. Capturing TCP Behavior

It is shown in [24] that different versions of TCP can be modeled by the following rate evolution: for all $f \in \mathcal{F}$:

$$\dot{x}_f(t) = \kappa_f\left(x_f(t)\right)\left(KU_f'\left(x_f(t)\right) - q_{b(f),e(f)}(t)\right)_{x_f(t)\ge m}$$

where $\kappa_f(\cdot)$ is any nondecreasing, continuous function with $\kappa_f(y) > 0$ for any $y > 0$. For the discrete-time version of this rate evolution, the approach used to prove the primal-dual algorithm of Section V can be applied. However, the Lyapunov function must be modified to be of the form [24]

$$Z(\mathbf{x},\mathbf{q};\boldsymbol{\lambda}^\star) := \frac{1}{2\alpha}\sum_{f\in\mathcal{F}}\int_{x_f^\star}^{x_f}\frac{\left(y - x_f^\star\right)}{\kappa_f(y)}dy + \frac{1}{2}\sum_{n\in\mathcal{N}}\sum_{d\in\mathcal{N}}\left(q_{n,d} - \lambda^\star_{n,d}\right)^2.$$

Then, convergence of the model can be proved using the same line of reasoning as in Section V.

VII. Conclusion

In this paper, we propose and study a cross-layer scheduling-routing-congestion control mechanism for wireless networks. We model many existing versions of TCP/AQM schemes using the primal-dual congestion controller. It is shown that this controller, along with suitable MAC/routing protocol achieves fairness and stability. Architecturally, we maintain the traditional protocol stack, but couple them through the use of queue-length information.

[4] $f(x) = O(g(x))$ implies that $\limsup_{x\to\infty} |(f(x)/g(x))| < \infty$.

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Atilla Eryilmaz (S'00–M'06) received the B.S. degree in electrical and electronics engineering from Boğaziçi University, Istanbul, in 1999, and the M.S. and Ph.D. degrees in electrical and computer engineering from the University of Illinois at Urbana–Champaign, Urbana, in 2001 and 2005, respectively.

He is currently working as a Postdoctoral Associate at the Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, Cambridge. His research interests include communication networks, optimal control of stochastic networks, optimization theory, stochastic processes, and network coding.

R. Srikant (S'90–M'91–SM'01–F'06) received the B.Tech. degree from the Indian Institute of Technology, Madras, in 1985, and the M.S. and Ph.D. degrees from the University of Illinois at Urbana–Champaign, Urbana, in 1988 and 1991, respectively, all in electrical engineering.

He was a Member of Technical Staff at AT&T Bell Laboratories from 1991 to 1995. He is currently with the University of Illinois at Urbana–Champaign, where he is a Professor in the Department of Electrical and Computer Engineering, and a Research Professor in the Coordinated Science Laboratory. His research interests include communication networks, stochastic processes, queueing theory, information theory, and game theory.

Dr. Srikant was an Associate Editor of *Automatica* and the IEEE TRANSACTIONS ON AUTOMATIC CONTROL, and is currently an Associate Editor of the IEEE/ACM TRANSACTIONS ON NETWORKING. He has also served on the Editorial Boards of special issues of the IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS and the IEEE TRANSACTIONS ON INFORMATION THEORY. He was the Chair of the 2002 IEEE Computer Communications Workshop, Santa Fe, NM, and will be a Program Co-Chair of IEEE INFOCOM 2007.