# MATH 407 Notes

## Contents

# 1 Linear Optimization

## 1.1 Linear Optimization Modeling

A linear optimization modeling problem consists of:

1. Decision variables
2. Objective function
3. Constraints

*Example:*

Your manufacturing plant wants to optimize its production mix in order to maximize profit. The profit on a case of beer mugs is \$25 while the profit on a case of champagne glasses is \$20. Each case of beer mugs requires 20 pounds of plastic resin to produce, while champagne glasses require 12 pounds per case. The daily supply of plastic resin is limited to at most 1800 pounds. About 15 cases of total product can be produced per hour, and the workday is eight hours long. How many of each item should you produce?

Let $x_1$ be the number of beer mugs and $x_2$ be the number of champagne glasses $\leftarrow$ decision variables

$$\begin{array}{ll} \text{maximize} & 25x_1 + 20x_2 \quad \leftarrow \text{objective function} \\ \text{subject to} & \left.\begin{array}{l} 20x_1 + 12x_2 \leq 1800 \\ x_1 + x_2 \leq 120 \\ x_1, x_2 \geq 0 \end{array}\right\} \text{constraints} \end{array}$$

## 1.2 Standard Form

The standard form of a linear optimization modeling problem requires:

- Objective function is maximized
- Variable constraints are non-negative
- Inequality constraints are $\leq$ constant

*Example:*

Non-standard form:

$$\begin{array}{ll} \text{minimize} & -6x_1 - x_2 \\ \text{subject to} & 4x_1 + 5x_2 \leq 20 \\ & 2x_1 + x_2 \leq 7 \\ & x_1 \leq 0 \\ & x_2 \geq 1 \end{array}$$

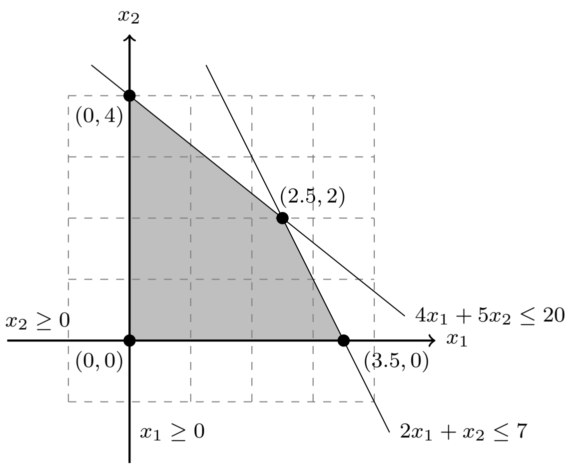
Standard form:

$$\begin{array}{ll} \text{maximize} & -6\bar{x_1} + \bar{x_2} + 1 \\ \text{subject to} & -4\bar{x_1} + 5\bar{x_2} \leq 15 \\ & -2\bar{x_1} + \bar{x_2} \leq 6 \\ & \bar{x_1}, \bar{x_2} \geq 0 \geq 1 \end{array} \qquad \text{where } \bar{x_1} = -x_1 \text{ and } \bar{x_2} = x_2 - 1$$

## 1.3 Graphical Method

*Example:*

$$\begin{array}{ll} \text{maximize} & 6x_1 + x_2 \\ \text{subject to} & 4x_1 + 5x_2 \leq 20 \\ & 2x_1 + x_2 \leq 7 \\ & x_1, x_2 \geq 0 \end{array}$$




| $(x_1, x_2)$ | $(0,0)$ | $(0,4)$ | $(3.5,0)$ | $(2.5,2)$ |
|---|---|---|---|---|
| $6x_1 + x_2$ | 0 | 4 | 21 | 17 |

$6x_1 + x_2$ is maximized at $(3.5, 0)$ with value $21$

## 1.4 Simplex Algorithm

*Example:*

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 \leq 8 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

We need to convert these inequality constraints into equations by adding slack variables

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 + w_1 = 8 \\ & x_1 - 2x_2 + w_2 = 2 \\ & x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

Then we need to make the slack variables the subject of these equations

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & w_1 = 8 - x_1 - x_2 \\ & w_2 = 2 - x_1 + 2x_2 \\ & x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

Now we can apply the simplex algorithm to solve the problem

Initial setup:

$$\begin{array}{ll} \text{maximize} & P = 2x_1 + x_2 \\ \text{subject to} & w_1 = 8 - x_1 - x_2 \\ & w_2 = 2 - x_1 + 2x_2 \\ & x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

Initial feasible solution: $(x_1, x_2, w_1, w_2) = (0, 0, 8, 2)$
Initial $P$ value: $0$

We want to increase $x_1$. The equation with $w_1$ implies $x_1 \leq 8$ and the equation with $w_2$ implies $x_1 \leq 2$. We can take $x_1 = 2$; keep $x_2$ the same. Now we find a new feasible solution:

$$\begin{aligned} w_1 &= 8 - x_1 - x_2 &&= 8 - 2 - 0 &&= 6 \\ w_2 &= 2 - x_1 + 2x_2 &&= 2 - 2 + 2(0) &&= 0 \end{aligned}$$

The equation with $w_2$ gave us the bound $x_1 \leq 2$, so solve for $x_1$:

$$\begin{aligned} w_1 &= 8 - (2 - w_2 + 2x_2) - x_2 &&= 8 - 2 + w_2 - 3x_2 &&= 6 + w_2 - 3x_2 \\ x_1 &= 2 - w_2 + 2x_2 \end{aligned}$$

New setup:

$$\begin{array}{ll} \text{maximize} & P = 4 - 2w_2 + 5x_2 \\ \text{subject to} & w_1 = 6 + w_2 - 3x_2 \\ & x_1 = 2 - w_2 + 2x_2 \\ & x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

New feasible solution: $(x_1, x_2, w_1, w_2) = (2, 0, 6, 0)$
New $P$ value: $4$

We want to increase $x_2$. The equation with $w_1$ gives us $x_2 \leq 2$ and the equation with $x$ gives us $x_2 \geq -1$. We can take $x_2 = 2$; keep $w_2$ the same. Now we find a new feasible solution:

$$\begin{aligned} w_1 &= 6 + w_2 - 3x_2 &&= 6 + 0 - 3(2) &&= 0 \\ x_1 &= 2 - w_2 + 2x_2 &&= 2 - 0 + 2(2) &&= 6 \end{aligned}$$

The equation with $w_1$ gave us the bound $x_2 \leq 2$, so solve for $x_2$:

$$\begin{aligned} x_2 &= 2 - \tfrac{1}{3}w_1 + \tfrac{1}{3}w_2 \\ x_1 &= 2 - w_2 + 2(2 - \tfrac{1}{3}w_1 + \tfrac{1}{3}w_2) &&= 6 - \tfrac{2}{3}w_1 - \tfrac{1}{3}w_2 \end{aligned}$$

New setup:

$$\begin{array}{ll} \text{maximize} & P = 14 - \frac{5}{3}w_1 - \frac{1}{3}w_2 \\ \text{subject to} & x_2 = 2 - \frac{1}{3}w_1 + \frac{1}{3}w_2 \\ & x_1 = 6 - \frac{2}{3}w_1 - \frac{1}{3}w_2 \\ & x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

New feasible solution: $(x_1, x_2, w_1, w_2) = (6, 2, 0, 0)$
New $P$ value: $14$

We cannot increase any more variables as increasing them would cause $P$ to decrease. We stop the algorithm here

$2x_1 + x_2$ is maximized at $(6, 2)$ with value $14$

### 1.5 Simplex Algorithm Tableau Notation

*Example:*

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 \\ \text{subject to} & x_1 + x_2 \leq 8 \\ & x_1 - 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

We need to express these as equations

$$\begin{array}{ll} \text{maximize} & P - 2x_1 - x_2 + 0w_1 + 0w_2 = 0 \\ \text{subject to} & 0P + x_1 + x_2 + w_1 + 0w_2 = 8 \\ & 0P + x_1 - 2x_2 + 0w_1 + w_2 = 2 \\ & x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

Writing this in tableau notation we get

| $P$ | $x_1$ | $x_2$ | $w_1$ | $w_2$ | constant |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 0 | 8 |
| 0 | 1 | $-2$ | 0 | 1 | 2 |
| 1 | $-2$ | $-1$ | 0 | 0 | 0 |

To pivot:

1. Pick column $j$ with negative element in bottom row. This is the pivot column
2. Pick row $i$ with smallest positive ratio of $\frac{\text{constant}}{\text{pivot column entry}}$. This is the pivot row
3. Apply row operations to the pivot column such that the entry $(i, j)$ has value $1$ and all other entries in the column have value $0$

Result after one pivot:

| $P$ | $x_1$ | $x_2$ | $w_1$ | $w_2$ | constant |
|---|---|---|---|---|---|
| 0 | 0 | 3 | 1 | $-1$ | 6 |
| 0 | 1 | $-2$ | 0 | 1 | 2 |
| 1 | 0 | $-5$ | 0 | 2 | 4 |

Final result:

| $P$ | $x_1$ | $x_2$ | $w_1$ | $w_2$ | constant |
|---|---|---|---|---|---|
| 0 | 0 | 1 | $\frac{1}{3}$ | $-\frac{1}{3}$ | 2 |
| 0 | 1 | 0 | $\frac{2}{3}$ | $\frac{1}{3}$ | 6 |
| 1 | 0 | 0 | $\frac{5}{3}$ | $\frac{1}{3}$ | 14 |

$x_1$ and $x_2$ are equal to their corresponding row constants $2$ and $6$ respectively with $P$ value $14$

## 1.6 Simplex Algorithm Terminology

In the example:

$$\begin{array}{ll} \text{maximize} & P = 2x_1 + x_2 \\ \text{subject to} & w_1 = 8 - x_1 - x_2 \\ & w_2 = 2 - x_1 + 2x_2 \\ & x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

- The entire model is called the *dictionary*
- The variables on the left-hand side of an equality are called *basic variables* (i.e. $w_1, w_2$)
- The variables on the right-hand side of an equality are called *non-basic variables* (i.e. $x_1, x_2$)
- A *pivot* is a step in which a non-basic variable becomes a basic variable and vice-versa
  - An *entering variable* goes from non-basic to basic
  - A *leaving variable* goes from basic to non-basic
- A *degenerate pivot* is a step which does not change objective function value
  - The maximum value for the entering variable stays at $0$
- A *basic solution* is a feasible solution found by setting all non-basic variables to $0$
  - Simplex algorithm always generates basic feasible solutions
  - In 2D problems, the basic feasible solutions correspond to corners of the feasible region

## 1.7 A Guide to the Simplex Algorithm

*A comprehensive guide to the topics covered from 1.4 to 1.9 can be found in* ***MATH 407 Project 1***

## 1.8 Simplex Algorithm on Unbounded Problems

- If the constraints do not give an upper limit to the entering variable, then the problem is unbounded and the objective function can be made arbitrarily large

## 1.9 Simplex Algorithm on Problems With Infeasible Origins

- If the origin is not a feasible initial solution, then one can be found through the Phase I algorithm
  - If the optimal $P$ in the Phase I algorithm is $0$, then a feasible initial solution has been found
  - If the optimal $P$ in the Phase I algorithm is $< 0$, then there are no feasible initial solutions
- Given that the constraints are expressed in the form $\sum_{j=1}^{n} a_{ij}x_j \leq b_i$ for $i = 1, 2, ..., m$, if $b_i < 0$ for any $i$ then the origin is infeasible

*Example:*

Original problem:

$$\begin{array}{ll} \text{maximize} & x_1 + 3x_2 \\ \text{subject to} & -2x_1 + x_2 \leq -2 \\ & x_1 + 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$

Initial Phase I setup:

$$\begin{array}{ll} \text{maximize} & P = -x_0 \\ \text{subject to} & w_1 = -2 + 2x_1 - x_2 + x_0 \\ & w_2 = 2 - x_1 - 2x_2 + x_0 \\ & x_0, x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

We want to find an initial feasible solution for $x_0$, where $x_1, x_2 = 0$. The equation with $w_1$ implies $x_0 \geq 2$ and the equation with $w_2$ implies $x_0 \geq -2$. We can take $x_0 = 2$; keep $x_1, x_2$ the same. Now we find an initial feasible solution:

$$\begin{array}{lllllll} w_1 & = & -2 + 2x_1 - x_2 + x_0 & = & -2 + 2(0) - 0 + 2 & = & 0 \\ w_2 & = & 2 - x_1 - 2x_2 + x_0 & = & 2 - 0 - 2(0) + 2 & = & 4 \end{array}$$

The equation with $w_1$ gave us the bound $x_0 \geq 2$, so solve for $x_0$:

$$\begin{array}{lllll} x_0 & = & 2 + w_1 - 2x_1 + x_2 & & \\ w_2 & = & 2 - x_1 - 2x_2 + (2 + w_1 - 2x_1 + x_2) & = & 4 + w_1 - 3x_1 - x_2 \end{array}$$

New Phase I setup:

$$\begin{array}{ll} \text{maximize} & P = -2 - w_1 + 2x_1 - x_2 \\ \text{subject to} & x_0 = 2 + w_1 - 2x_1 + x_2 \\ & w_2 = 4 + w_1 - 3x_1 - x_2 \\ & x_0, x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

Initial Phase I feasible solution: $(x_0, x_1, x_2, w_1, w_2) = (2, 0, 0, 0, 4)$
Initial Phase I $P$ value: $-2$

We want to increase $x_1$. The equation with $x_0$ implies $x_1 \leq 1$ and the equation with $w_2$ implies $x_1 \leq 1\frac{1}{3}$. We can take $x_1 = 1$; keep $x_2, w_1$ the same. Now we find a new feasible solution:

$$\begin{array}{lllllll} x_0 & = & 2 + w_1 - 2x_1 + x_2 & = & 2 + 0 - 2(1) + 0 & = & 0 \\ w_2 & = & 4 + w_1 - 3x_1 - x_2 & = & 4 + 0 - 3(1) - 0 & = & 1 \end{array}$$

The equation with $x_0$ gave us the bound $x_1 \leq 1$, so solve for $x_1$:

$$\begin{array}{lllll} x_1 & = & 1 + 0.5w_1 - 0.5x_0 + 0.5x_2 & & \\ w_2 & = & 4 + w_1 - 3(1 + 0.5w_1 - 0.5x_0 + 0.5x_2) - x_2 & = & 1 - 0.5w_1 + 1.5x_0 - 2.5x_2 \end{array}$$

New Phase I setup:

$$\begin{array}{ll} \text{maximize} & P = -x_0 \\ \text{subject to} & x = 1 + 0.5w_1 - 0.5x_0 + 0.5x_2 \\ & w_2 = 1 - 0.5w_1 + 1.5x_0 - 2.5x_2 \\ & x_0, x_1, x_2, w_1, w_2 \geq 0 \end{array}$$

New Phase I feasible solution: $(x_0, x_1, x_2, w_1, w_2) = (0, 1, 0, 0, 1)$
New Phase I $P$ value: $0$

We cannot increase any more variables as increasing them would cause $P$ to decrease. We stop the algorithm here

An initial feasible solution to our original problem is $(x_1, x_2, w_1, w_2) = (1, 0, 0, 1)$

## 1.10 Simplex Algorithm Termination

- If the simplex algorithm terminates after a finite number of iterations, then it either
  - Returns a basic optimal solution
  - Identifies the problem as infeasible
  - Identifies the problem as unbounded
- If the simplex algorithm does not terminate after a finite number of iterations, then it has entered a cycle

## 1.11 Cycling

- Cycling occurs when some dictionary repeats after multiple iterations
  - Every iteration results in a degenerate pivot
  - The algorithm is guaranteed to cycle indefinitely
- If the algorithm does not terminate, it must cycle
- To show that the algorithm cycles, show that some dictionary repeats after multiple iterations

  *An example of a cycling problem can be found in **MATH 407 HW 3**, page 6*

## 1.12 Bland's Rule

- Select the entering variable $x_j$ by choosing the smallest index variable that still increases the objective function
- Select the leaving variable based on which one gives the strongest restriction on how much $x_j$ can increase. If there is a tie, choose the variable with the smallest index
- The simplex algorithm never cycles and always terminates when the entering and leaving variables are chosen using Bland's rule
- One disadvantage is that Bland's rule might take more iterations before terminating

## 1.13 Fundamental Theorem of Linear Programming

- If a linear programming problem does not have an optimal solution, then it is either infeasible or unbounded
- If a feasible solution exists, a basic feasible solution exists
- If an optimal solution exists, a basic optimal solution exists

## 1.14 Perturbation Method

- Add tiny offsets to the dictionary constraints to avoid cycling
  - Cycling problems are rare, so adding offsets to the dictionary will likely result in a non-cycling problem
  - Possible that the perturbed problem still cycles
- Solution is approximate
  - Too much perturbation may result in an entirely different solution

### 1.15 Lexicographical Method

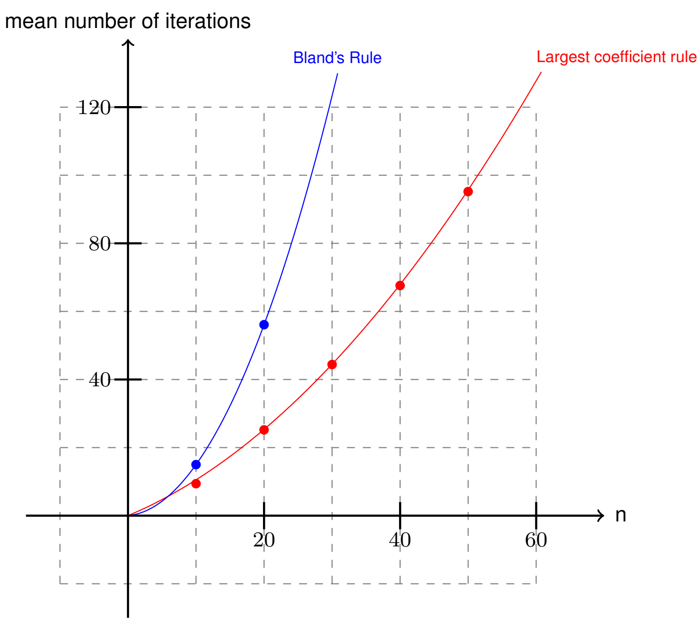
- Treats perturbations symbolically
- Offsets are labeled $\varepsilon_1, \varepsilon_2, ..., \varepsilon_m$ with $0 < \varepsilon_1 << \varepsilon_2 << ... << \varepsilon_m << 1$
  - Offsets $\varepsilon_i$ are constants
  - $<<$ indicates orders of magnitude smaller (i.e. $0 << 0.001 << 0.1 << 1$)
  - At the end take $\varepsilon_i = 0$ to remove perturbations
- The simplex algorithm never cycles and always terminates when combined with the lexicographical method

  *An example of the lexicographical method can be found in **MATH 407 HW 3**, page 9*

### 1.16 Algorithm Efficiency

- Ways to measure efficiency
  - Measure time taken to solve random problems
    * Different computers have different speeds
    * Computers run at different speeds based on external conditions
    * Computers change significantly every year
  - Measure average/maximum number of iterations needed to solve random problems
    * Harder to compare different algorithms

### 1.17 Comparing Algorithm Efficiency




- The mean number of iterations it takes to solve a problem using Bland's Rule increases much faster with respect to $n$ than using the largest coefficient rule
- For very small values of $n$, Bland's rule tends to comes very close to the largest coefficient rule
- Bland's rule tends to take fewer iterations than the largest coefficient rule for approximately $n < 5$

## 1.18 Klee-Minty Problem

The Klee-Minty problem is a general problem with $n$ variables and $n$ constraints where the largest coefficient rule requires $2^n - 1$ iterations

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{j=1}^{n} 10^{n-j} x_j \\ \text{subject to} & 2 \cdot \displaystyle\sum_{j=1}^{i-1} 10^{i-j} x_j + x_i \leq 100^{i-1} \text{ for } i = 1, 2, ..., n \\ & x_j \geq 0 \text{ for } j = 1, 2, ..., n \end{array}$$

Klee-Minty problem for $n = 4$

$$\begin{array}{ll} \text{maximize} & 1000x_1 + 100x_2 + 10x_3 + x_4 \\ \text{subject to} & x_1 \leq 1 \\ & 20x_1 + x_2 \leq 100 \\ & 200x_1 + 20x_2 + x_3 \leq 10000 \\ & 2000x_1 + 200x_2 + 20x_3 + x_4 \leq 1000000 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

## 1.19 Duality

Given the primal $\mathscr{P}$ where $A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^{n \times 1}$, $c \in \mathbb{R}^{n \times 1}$, and $b \in \mathbb{R}^{m \times 1}$

$$\begin{array}{ll} \text{maximize} & c^\top x \\ \text{subject to} & Ax \leq b \\ & x \geq 0 \end{array}$$

The dual $\mathscr{D}$ is

$$\begin{array}{ll} \text{minimize} & b^\top y \\ \text{subject to} & y^\top A \geq c^\top \\ & y \geq 0 \end{array}$$

where $y \in \mathbb{R}^{m \times 1}$

If primal $\mathscr{P}$ is

$$\begin{array}{ll} \text{maximize} & c_1x_1 + c_2x_2 + c_3x_3 \\ \text{subject to} & A_1x_1 + B_1x_2 + C_1x_3 \leq b_1 \\ & A_2x_1 + B_2x_2 + C_2x_3 \leq b_2 \\ & A_3x_1 + B_3x_2 + C_3x_3 \leq b_3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$$\begin{array}{ll} \text{minimize} & b_1y_1 + b_2y_2 + b_3y_3 \\ \text{subject to} & A_1y_1 + A_2y_2 + A_3y_3 \geq c_1 \\ & B_1y_1 + B_2y_2 + B_3y_3 \geq c_2 \\ & C_1y_1 + C_2y_2 + C_3y_3 \geq c_3 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Then dual $\mathscr{D}$ is

Note that the dual of a dual is the primal

## 1.20 Weak Duality Theorem

If $x$ is a feasible solution for the primal problem and $y$ is a feasible solution for the dual problem, then $c^\top x \leq b^\top y$

## 1.21 Strong Duality Theorem

The primal problem has an optimal solution $x^*$ if and only if the dual has an optimal solution $y^*$ and $c^\top x^* = b^\top y^*$

## 1.22 Corollaries of the Duality Theorems

| | $\mathscr{D}$ has optimal solution | $\mathscr{D}$ is infeasible | $\mathscr{D}$ is unbounded |
|---|---|---|---|
| $\mathscr{P}$ has optimal solution | possible | impossible | impossible |
| $\mathscr{P}$ is infeasible | impossible | possible | possible |
| $\mathscr{P}$ is unbounded | impossible | possible | impossible |

*A full proof of these corollaries can be found in **MATH 407 HW 4**, page 6*

## 1.23 Duality With the Simplex Algorithm

- We can solve for the optimal solution of the dual to obtain the optimal solution of the primal
  - The coefficients of the slack variables $a_1 z_1 + a_2 z_2 + ...$ in the dual correspond to the optimal solution $(x_1, x_2, ...) = (a_1, a_2, ...)$ of the primal
- If the dual is unbounded, then the primal is infeasible

## 1.24 Dual Phase I Algorithm

If the origin is not feasible in the primal or dual, then the Dual Phase I Algorithm is used to obtain a feasible solution

If primal $\mathscr{P}$ is

$$\begin{array}{ll} \text{maximize} & c_1x_1 + c_2x_2 + c_3x_3 \\ \text{subject to} & A_1x_1 + B_1x_2 + C_1x_3 \leq b_1 \\ & A_2x_1 + B_2x_2 + C_2x_3 \leq b_2 \\ & A_3x_1 + B_3x_2 + C_3x_3 \leq b_3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

And dual $\mathscr{D}$ is

$$\begin{array}{ll} \text{minimize} & b_1y_1 + b_2y_2 + b_3y_3 \\ \text{subject to} & A_1y_1 + A_2y_2 + A_3y_3 \geq c_1 \\ & B_1y_1 + B_2y_2 + B_3y_3 \geq c_2 \\ & C_1y_1 + C_2y_2 + C_3y_3 \geq c_3 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

Then the Phase I primal $\mathscr{P}'$ is

$$\begin{array}{ll} \text{maximize} & -x_1 - x_2 - x_3 \\ \text{subject to} & A_1x_1 + B_1x_2 + C_1x_3 \leq b_1 \\ & A_2x_1 + B_2x_2 + C_2x_3 \leq b_2 \\ & A_3x_1 + B_3x_2 + C_3x_3 \leq b_3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

And the Phase I dual $\mathscr{D}'$ is

$$\begin{array}{ll} \text{minimize} & b_1y_1 + b_2y_2 + b_3y_3 \\ \text{subject to} & A_1y_1 + A_2y_2 + A_3y_3 \geq -1 \\ & B_1y_1 + B_2y_2 + B_3y_3 \geq -1 \\ & C_1y_1 + C_2y_2 + C_3y_3 \geq -1 \\ & y_1, y_2, y_3 \geq 0 \end{array}$$

We can then run the simplex algorithm on $\mathscr{P}'$ or $\mathscr{D}'$

- The optimal solution of the Phase I algorithm is a feasible solution in $\mathscr{P}'$ and $\mathscr{D}'$
- If $\mathscr{P}'/\mathscr{D}'$ is unbounded, then $\mathscr{D}'/\mathscr{P}'$ is infeasible
- If $\mathscr{P}'/\mathscr{D}'$ is infeasible, then $\mathscr{P}/\mathscr{D}$ is infeasible

### 1.25 Complementary Slackness Theorem

The primal feasible solution $x^* = (x_1{}^*, x_2{}^*, ..., x_n{}^*)$ and the dual feasible solution $y^* = (y_1{}^*, y_2{}^*, ..., y_m{}^*)$ are simultaneously optimal if and only if

$$x_j{}^* z_j{}^* = 0 \text{ for } j = 1, 2, ..., n$$
$$w_i{}^* y_i{}^* = 0 \text{ for } i = 1, 2, ..., m$$

where $(w_1{}^*, w_2{}^*, ..., w_m{}^*)$ are the primal slack variable values and $(z_1{}^*, z_2{}^*, ..., z_n{}^*)$ are the dual slack variable values

### 1.26 Block Matrices

A block decomposition of $A$ is a decomposition of $A$ into submatrices

$$A = \begin{pmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & ... & A_{mn} \end{pmatrix}$$

- All elements $A_{ij}$ are matrices
- All matrices $A_{ij_1}, A_{ij_2}, ..., A_{ij_n}$ on the same row have the same number of rows
- All matrices $A_{i_1j}, A_{i_2j}, ..., A_{i_nj}$ on the same column have the same number of columns

$$A = \left(\begin{array}{ccc:ccc} 3 & -4 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 0 & 3 \end{array}\right) = \begin{pmatrix} \begin{pmatrix} 3 & -4 & 1 \\ 0 & 2 & 2 \\ 1 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} 2 & 1 & 4 \\ 1 & 0 & 3 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} C & I \\ 0 & D \end{pmatrix}$$

$$B = \left(\begin{array}{cc} 1 & 2 \\ 0 & 4 \\ -1 & -1 \\ \hdashline 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{array}\right) = \begin{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ -1 & -1 \end{pmatrix} \\ \begin{pmatrix} 2 & -1 \\ 4 & 3 \\ -2 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

$$AB = \begin{pmatrix} C & I \\ 0 & D \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} CF + IG \\ 0F + DG \end{pmatrix}$$

- Addition and multiplication of block matrices $A$ and $B$ follows the standard rule of matrix addition and multiplication
- Multiplication of block matrices $A$ and $B$ is possible if and only if $A$ and $B$ are conformal

## 1.27 Conformal Block Matrices

Block matrices $M$ and $N$ are conformal with respect to multiplication if and only if $MN$ can be formed from the submatrices in the decomposition of $M$ and $N$

The following matrices are conformal

$$\begin{pmatrix} \mathbb{R}^{3\times3} & \mathbb{R}^{3\times3} \\ \mathbb{R}^{2\times3} & \mathbb{R}^{2\times3} \end{pmatrix} \begin{pmatrix} \mathbb{R}^{3\times2} \\ \mathbb{R}^{3\times2} \end{pmatrix} = \begin{pmatrix} \mathbb{R}^{3\times3}\mathbb{R}^{3\times2} + \mathbb{R}^{3\times3}\mathbb{R}^{3\times2} \\ \mathbb{R}^{2\times3}\mathbb{R}^{3\times2} + \mathbb{R}^{2\times3}\mathbb{R}^{3\times2} \end{pmatrix} = \begin{pmatrix} \mathbb{R}^{3\times2} \\ \mathbb{R}^{2\times2} \end{pmatrix} = \begin{pmatrix} \mathbb{R}^{5\times2} \end{pmatrix}$$

The following matrices are non-conformal

$$\begin{pmatrix} \mathbb{R}^{3\times3} & \mathbb{R}^{3\times2} \end{pmatrix} \begin{pmatrix} \mathbb{R}^{2\times2} \\ \mathbb{R}^{3\times2} \end{pmatrix} = \begin{pmatrix} \mathbb{R}^{3\times3}\mathbb{R}^{2\times2} + \mathbb{R}^{3\times2}\mathbb{R}^{3\times2} \end{pmatrix} \leftarrow \mathbb{R}^{3\times3}\mathbb{R}^{2\times2} \text{ and } \mathbb{R}^{3\times2}\mathbb{R}^{3\times2} \text{ are not defined}$$

## 1.28 Matrix Form of Dictionary

Given the initial dictionary and the current basic and non-basic variables, we can determine the current dictionary

Initial dictionary:

$$\begin{aligned} &\text{maximize} \quad P = \begin{bmatrix} c_1 & c_2 & ... & c_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \\ &\text{subject to} \quad \begin{bmatrix} A_{1,1} & A_{1,2} & ... & A_{1,m} \\ A_{2,1} & A_{2,2} & ... & A_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & ... & A_{n,m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \end{aligned}$$

Current dictionary:

$$\begin{aligned} &\text{maximize} \quad P = {c_B}^\top {A_B}^{-1} b + ({c_N}^\top - {c_B}^\top {A_B}^{-1} A_N) x_N \\ &\text{subject to} \quad x_B = {A_B}^{-1} b - {A_B}^{-1} A_N x_N \end{aligned}$$

- $B$ is the set of basic variables $x_b$ in the current dictionary
- $N$ is the set of non-basic variables $x_n$ in the current dictionary
- $A_S$ is the block matrix $\begin{bmatrix} A_{\bullet,j} & ... \end{bmatrix}$ for all $j$ where $x_j$ are variables in set $S$
- $c_S$ is the block matrix $\begin{bmatrix} c_j \\ \vdots \end{bmatrix}$ for all $j$ where $x_j$ are variables in set $S$
- $x_S$ is the block matrix $\begin{bmatrix} x_j \\ \vdots \end{bmatrix}$ for all $j$ where $x_j$ are variables in set $S$

## 1.29 Matrix Form of the Simplex Algorithm

Given the current dictionary

$$\begin{aligned} &\text{maximize} && P = {c_B}^\top {A_B}^{-1} b + ({c_N}^\top - {c_B}^\top {A_B}^{-1} A_N) x_N \\ &\text{subject to} && x_B = {A_B}^{-1} b - {A_B}^{-1} A_N x_N \end{aligned}$$

Starting the simplex algorithm:

- If the origin is feasible, start by defining $B = \{\text{slack variables}\}$
- If the origin is infeasible, adapt the Phase I algorithm or Dual Phase I algorithm to find a set $B$

Rule for pivot:

- Calculate ${c_N}^\top - {c_B}^\top {A_B}^{-1} A_N$ and find an entry at the $n^{\text{th}}$ column that is positive
  - If no such entry exists, then the optimal solution has been reached and the algorithm can be terminated
  - The optimal solution is given by $P^* = {c_B}^\top {A_B}^{-1} b,\ {x_B}^* = {A_B}^{-1} b,\ {x_N}^* = 0$
- Select the entering variable $x_{n'}$ by choosing the non-basic variable that corresponds to the entry at the $n^{\text{th}}$ column of ${c_N}^\top - {c_B}^\top {A_B}^{-1} A_N$
- Calculate ${A_B}^{-1} b$ and $({A_B}^{-1} A_N)_n$, that is the $n^{\text{th}}$ column of ${A_B}^{-1} A_N$ corresponding to $x_{n'}$
- Find the largest $t$ such that ${A_B}^{-1} b - t({A_B}^{-1} A_N)_n \geq 0$
  - If $t$ is unbounded, then the problem is unbounded
- Select the leaving variable $x_b$ by choosing the basic variable that corresponds to the row that has value $0$ in ${A_B}^{-1} b - t({A_B}^{-1} A_N)_{n'}$ for this largest value of $t$
- Move $x_{n'}$ from set $N$ to $B$ and move $x_b$ from set $B$ to $N$

## 1.30 Hyperplanes

A hyperplane in $\mathbb{R}^n$ is any set of the form $H(\alpha, \beta) = \{x \in \mathbb{R}^n \mid \alpha^\top x = \beta\} = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n = \beta\}$ where $\alpha \in \mathbb{R}^{n \times 1}$ and $\beta \in \mathbb{R}$

- A hyperplane generates two closed half-spaces on each side of the hyperplane
  - $H^+(\alpha, \beta) = \{x \in \mathbb{R}^n \mid \alpha^\top x \geq \beta\} = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \geq \beta\}$
  - $H^-(\alpha, \beta) = \{x \in \mathbb{R}^n \mid \alpha^\top x \leq \beta\} = \{\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_n x_n \leq \beta\}$

## 1.31 Convex Polyhedrons

A convex polyhedron in $\mathbb{R}^n$ is the intersection of finitely many half-spaces

- A polyhedron $C \subset \mathbb{R}^n$ is convex if and only if, for any two points $x, y \in C$, the line segment $[x, y] = \{\lambda x + (1 - \lambda) y \mid 0 \leq \lambda \leq 1\} \in C$
- If the convex polyhedron is bounded, then it is also a convex polytope

### 1.32 Vertices of Convex Polyhedrons

A point $z$ is a vertex of a convex polyhedron $C$ if $z \in [u, v] \subset C$ implies that $z = u$ or $z = v$

- If a vertex is on an interval, then it must be at the endpoint of the interval
- If $z$ is a vertex of a convex polyhedron $C$ described by $Ax \leq b$ and $x \geq 0$ with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, then there exists two index sets $I \subset \{1, 2, ..., m\}$ and $J \subset \{1, 2, ..., n\}$ where $|I| + |J| = n$ with $z$ representing the unique solution of $x_j = 0$ for $j \in J$ and $\sum_{j=1}^{n} a_{ij}x_j = b_i$ for $i \in I$
  - *An application of this corollary can be found in **MATH 407 HW 6**, page 11-12*
- If $z$ is a vertex of a convex polyhedron $C$ described by $Ax \leq b$ and $x \geq 0$ with $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, then $\mathrm{rank}(A_z) = n$, where $A_z$ is the submatrix consisting of the rows $i$ for which $A_i z = b_i$

### 1.33 Geometry of the Simplex Method

- Basic feasible solutions of the simplex method correspond to vertices of the polyhedron
- Unbounded problems correspond to unbounded polyhedrons where the vertices do not represent an optimal solution
- Degenerate pivots in $\mathbb{R}^n$ correspond to vertices where multiple sets of $n$ hyperplanes intersect at the same vertex
- Perturbations in $\mathbb{R}^n$ correspond to a shift in the position of hyperplanes such that only $n$ hyperplanes intersect at a given vertex
- Infeasible problems correspond to empty polyhedrons with no intersection of hyperplanes
- Phase I algorithms find a vertex of the polyhedron when the origin is infeasible

### 1.34 Sensitivity Analysis

Sensitivity analysis determines how much we can perturb the initial constraints and/or objective function and still obtain the same choice of basic and non-basic variables in our optimal solution

- Perturbing $c$, where we replace $c$ with $c + t \cdot \Delta c$:

  The coefficients in the **primal** objective function for non-basic variables are given by

  $$c_N{}^\top - c_B{}^\top A_B{}^{-1} A_N + t(\Delta c_N{}^\top - \Delta c_B{}^\top A_B{}^{-1} A_N)$$

  We can determine the values of $t$ where the primal coefficients $c_p$ are non-positive such that $c_p \leq 0$

- Perturbing $b$, where we replace $b$ with $b + t \cdot \Delta b$:

  Convert the problem to its dual where $c_{\,\text{dual}} = b_{\,\text{primal}}$

  The coefficients in the **dual** objective function for non-basic variables are given by

  $$c_N{}^\top - c_B{}^\top A_B{}^{-1} A_N + t(\Delta c_N{}^\top - \Delta c_B{}^\top A_B{}^{-1} A_N)$$

  We can determine the values of $t$ where the dual coefficients $c_d$ are non-positive such that $c_d \leq 0$