MATH 324 Notes

Contents

1 General

1.1 Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

1.2 Polar Coordinates

$$x = r\cos\theta$$

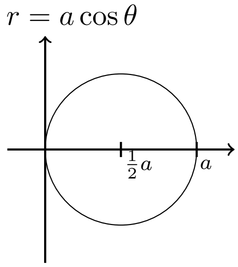
$$y = r\sin\theta$$

$$r = \sqrt{x^2+y^2}$$

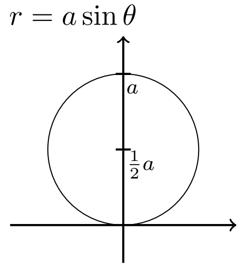
$$\theta = \tan^{-1}(\tfrac{y}{x})$$

1.3 Polar Curves

$r = a\cos\theta$



$r = a\sin\theta$



1.4 Curve Parametrization

$$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$$

$$\vec{r}\,'(t) = \left\langle \left(\frac{dx}{dt}\right)^2, \left(\frac{dy}{dt}\right)^2, \left(\frac{dz}{dt}\right)^2 \right\rangle$$

$$s = \int_a^b |\vec{r}\,'(t)|\,du = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

$$\frac{ds}{dt} = |\vec{r}\,'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

$$ds = |\vec{r}\,'(t)|\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}dt$$

2 Section Fourteen

2.1 Chain Rule of Single Variable Functions

Given that $y = f(x)$, if $x = x(t)$ and $y = f(x(t))$ then

$$\text{if } x = x(t),\ y = f(x(t)) \xrightarrow{\text{chain rule}} \frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

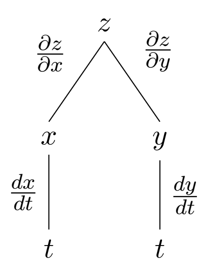
2.2 Chain Rule of Multivariable Functions

Given that $z = f(x, y)$, if $x = x(t)$, $y = y(t)$ and $z = f(x, y)$ then

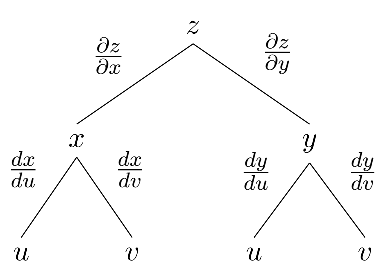
$$\begin{aligned} \text{if } \begin{cases} x = x(t) \\ y = y(t) \end{cases} \xrightarrow{\text{chain rule}} \frac{dz}{dt} &= \frac{dz}{dx} + \frac{dz}{dy} \\ &= \frac{\partial z}{\partial x} \times \frac{dx}{dt} + \frac{\partial z}{\partial y} \times \frac{dy}{dt} \end{aligned}$$

2.3 Chain Rule via Tree Diagram

Multiply the partial derivatives along each branch, then sum up along all branches



$$\frac{dz}{dt} = \left[\frac{\partial z}{\partial x} \times \frac{dx}{dt}\right] + \left[\frac{\partial z}{\partial y} \times \frac{dy}{dt}\right]$$



$$\frac{dz}{dt} = \left[\frac{\partial z}{\partial x} \times \frac{dx}{du} + \frac{\partial z}{\partial x} \times \frac{dx}{dv}\right] + \left[\frac{\partial z}{\partial y} \times \frac{dy}{du} + \frac{\partial z}{\partial y} \times \frac{dy}{dv}\right]$$

2.4 Directional Derivative

$D_{\vec{u}}f(x,y)$ is the rate of change of z along the direction of the unit vector $\vec{u} = \langle a, b \rangle$

$$\begin{aligned} D_{\vec{u}}f(x,y) &= \nabla f(x,y) \cdot \vec{u} \\ &= \left\langle \frac{\partial f}{\partial x}(x,y), \frac{\partial f}{\partial y}(x,y) \right\rangle \cdot \vec{u} \end{aligned}$$

The gradient vector $\nabla f(x,y)$ maximizes the directional derivative of $f(x,y)$ and $\nabla f(x,y) \perp \text{level curves}$

$$\begin{aligned} D_{\vec{u}}f(x,y) &= \nabla f(x,y) \cdot \vec{u} \longleftarrow \vec{u} \text{ is a unit vector} \\ &= |\nabla f(x,y)||\vec{u}| \cos\theta \longleftarrow 0 \leq \theta \leq \pi \\ &= |\nabla f(x,y)| \cos\theta \begin{cases} \text{is maximized at } \theta = 0, & D_{\vec{u}}f(x,y) = \nabla f(x,y) \\ \text{is minimized at } \theta = \pi, & D_{\vec{u}}f(x,y) = -\nabla f(x,y) \\ \text{is 0 at } \theta = \frac{\pi}{2}, & D_{\vec{u}}f(x,y) = 0 \end{cases} \end{aligned}$$

Properties of the gradient vector

- $\nabla(f \pm g) = \nabla f \pm \nabla g$
- $\nabla(kf) = k \cdot \nabla f$
- $\nabla(fg) = g * \nabla f + f * \nabla g$
- $\nabla(\frac{f}{g}) = \frac{g * \nabla f - f * \nabla g}{g^2}$

3 Section Fifteen

3.1 Riemann Sum in $\mathbb{R}^3$

$$\partial x = \frac{b-a}{m}$$
$$\partial y = \frac{d-c}{n}$$
$$\partial A = \partial x \partial y$$

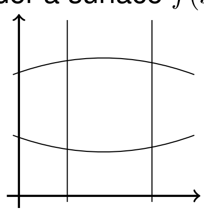
Sample point: (x_i^*, y_j^*)

$$V_{ij} \approx f(x_i^*, y_j^*) \cdot \partial A$$
$$V \approx \sum_{j=1}^{n} \sum_{i=1}^{m} V_{ij}$$

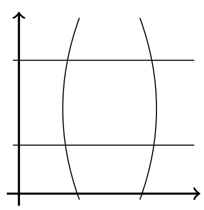
3.2 Double Integrals

Represents the volume that lies under a surface $f(x, y)$ and above the area D

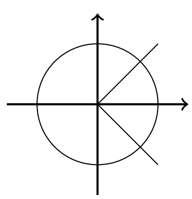
Type I iterated integral
$\iint_D f(x, y)\ dydx$



Type II iterated integral
$\iint_D f(x, y)\ dxdy$



Polar iterated integral
$\iint_D f(x, y)(r)\ drd\theta$



3.3 Applications of Double Integrals: Lamina

Given a density function $\rho(x, y)$

$$
\begin{aligned}
m &= \iint_D \rho(x,y)\,dA \\
\overline{x} &= \frac{1}{m} M_y = \frac{1}{m} \iint_D x\rho(x,y)\,dA \longrightarrow M_y = \text{moment about y-axis} \\
\overline{y} &= \frac{1}{m} M_x = \frac{1}{m} \iint_D y\rho(x,y)\,dA \longrightarrow M_x = \text{moment about x-axis} \\
I_x &= \iint_D y^2\rho(x,y)\,dA \longrightarrow I_x = \text{moment of inertia about } x\text{-axis} \\
I_y &= \iint_D x^2\rho(x,y)\,dA \longrightarrow I_y = \text{moment of inertia about } y\text{-axis} \\
I_o &= \iint_D r^2\rho(x,y)\,dA \longrightarrow I_o = \text{moment of inertia about the origin} \\
&= I_x + I_y
\end{aligned}
$$

3.4 Integrals and Probability

Given a random variable x with probability density function $f(x)$

$$
\begin{aligned}
& f(x) \geq 0 \longrightarrow P(X) \text{ cannot be smaller than } 0 \\
& \int_{-\infty}^{\infty} f(x)\,dx = 1 \longrightarrow \text{ total probability is } 1 \\
& P(a \leq X \leq b) = \int_a^b f(x)\,dx \\
& \overline{x} = \int_{-\infty}^{\infty} x f(x)\,dx
\end{aligned}
$$

3.5 Joint Probability

The joint probability density function $f(x, y)$ represents the probability of both x and y occurring together

- x and y are independent if their joint probability density function is equal to the product of their individual probability density functions
- $f(x,y) = f_1(x)f_2(y) \;\Leftrightarrow\; P(x \cap y) = 0$

3.6 Applications of Double Integrals: Joint Probability

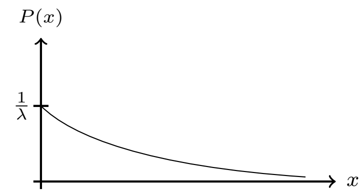
Given random variables x, y with joint probability density function $f(x, y)$

$$
\begin{aligned}
& f(x,y) \geq 0 \longrightarrow P(X,Y) \text{ cannot be smaller than } 0 \\
& \iint_{\mathbb{R}^2} f(x,y)\,dA = 1 \longrightarrow \text{total probability is } 1 \\
& P((X,Y) \in D) = P(a \leq X \leq b, c \leq Y \leq d) = \iint_D f(x,y)\,dA
\end{aligned}
$$

3.7 Examples of Probability Density Functions

Exponential probability density function

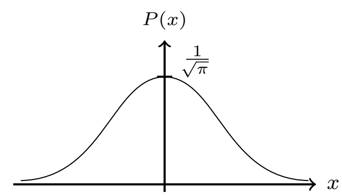
$$f(x) = \begin{cases} \frac{1}{\lambda}e^{-\frac{x}{\lambda}} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



- Used when a random variable x has a mean value equal to λ

Normal distribution probability density function

$$f(x) = \frac{1}{\sqrt{\pi}} \underbrace{e^{-x^2}}_{\text{Gaussian function}}$$



- Used when a random variable x has a mean value equal to λ

3.8 Applications of Double Integrals: Surface Area

$$\vec{a} = \langle \partial x,\ 0,\ f_x(x_i, y_i)\partial x \rangle$$

$$\vec{b} = \langle 0,\ \partial y,\ f_y(x_i, y_i)\partial y \rangle$$

$$\begin{aligned} S_{ij} &\approx |\vec{a} \times \vec{b}| \\ &\approx \langle -f_x(x,y) \cdot \partial x \partial y,\ -f_y(x,y) \cdot \partial x \partial y,\ \partial x \partial y \rangle \\ &\approx \langle -f_x(x,y),\ -f_y(x,y),\ 1 \rangle\, \partial x \partial y \end{aligned}$$

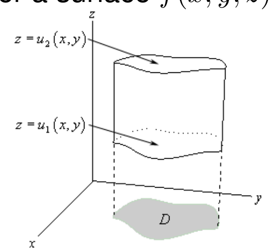
$$S = \lim_{\substack{m \to \infty \\ n \to \infty}} \sum_{j=1}^{n} \sum_{i=1}^{m} S_{ij}$$

$$\begin{aligned} S &= \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\ dA \\ &= \iint_D (r) \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\ drd\theta \end{aligned}$$

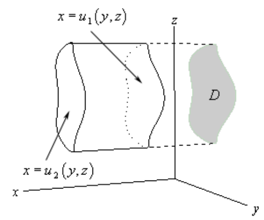
3.9 Triple Integrals

Represents the 4D object that lies under a surface $f(x, y, z)$ and above the 3D solid E

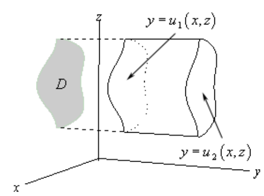
Type I triple integral
$\iint_D (\int_{u_1}^{u_2} f(x, y, z)\ dz)\ dA$



Type II triple integral
$\iint_D (\int_{u_1}^{u_2} f(x, y, z)\ dx)\ dA$



Type III triple integral
$\iint_D (\int_{u_1}^{u_2} f(x, y, z)\ dy)\ dA$



3.10 Applications of Triple Integrals: Solids

Given density function $\rho(x, y, z)$

$$m = \iiint_E \rho(x, y, z)\ dV$$
$$\overline{x} = \frac{1}{m} \iiint_E x\rho(x, y, z)\ dV$$
$$\overline{y} = \frac{1}{m} \iiint_E y\rho(x, y, z)\ dV$$
$$\overline{z} = \frac{1}{m} \iiint_E z\rho(x, y, z)\ dV$$
$$I_x = \iiint_E (y^2 + z^2)\rho(x, y, z)\ dV \longrightarrow I_x = \text{moment of inertia about } x\text{-axis}$$
$$I_y = \iiint_E (x^2 + z^2)\rho(x, y, z)\ dV \longrightarrow I_y = \text{moment of inertia about } y\text{-axis}$$
$$I_z = \iiint_E (x^2 + y^2)\rho(x, y, z)\ dV \longrightarrow I_z = \text{moment of inertia about } z\text{-axis}$$

3.11 Applications of Triple Integrals: Joint Probability

Given random variables x, y, z with joint probability density function $f(x, y, z)$

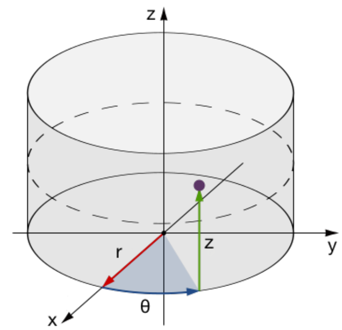
$$f(x,y,z) \geq 0 \longrightarrow P(X,Y,Z) \text{ cannot be smaller than } 0$$

$$\iiint_{\mathbb{R}^3} f(x,y,z)\, dV = 1 \longrightarrow \text{total probability is } 1$$

$$\begin{aligned} P((X,Y,Z) \in E) &= P(a \leq X \leq b, c \leq Y \leq d, e \leq Z \leq f) \\ &= \iiint_E f(x,y,z)\, dV \end{aligned}$$

3.12 Cylindrical Coordinates for $\mathbb{R}^3$

Points in $\mathbb{R}^3$ are located by $\underbrace{(r, \theta}_{x,y \text{ converted to Polar coordinates}}, z)$



3.13 Triple Integrals in Cylindrical Coordinates

$$\iiint_E f(x,y,z)\, dV = \iint_D \int_{u_1}^{u_2} f(r\cos\theta, r\sin\theta, z) \cdot r \;\; dz dr d\theta$$

3.14 Spherical Coordinates for $\mathbb{R}^3$

Points in $\mathbb{R}^3$ are located by (ρ, θ, ϕ)

$$r = OP' = \rho \sin\phi$$

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$

$$y = r\sin\theta = \rho\sin\phi\sin\theta$$

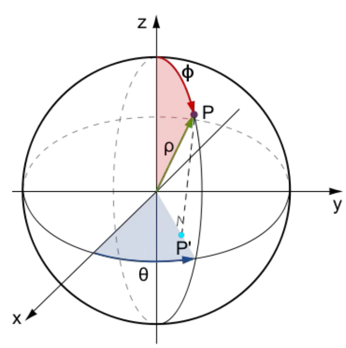
$$z = \rho\cos\phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

$$\rho^2 = r^2 + z^2$$

$$\tan\theta = \frac{y}{x}$$

$$\tan\phi = \frac{r}{z}$$



3.15 Triple Integrals in Spherical Coordinates

$$\iiint_E f(x,y,z)\, dV = \iiint_{g_1}^{g_2} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \cdot \rho^2 \sin\phi \;\; d\rho d\theta d\phi$$

3.16 Change of Variables in Multiple Integrals

A change of variable is a transformation function $T : \mathbb{R}^n \to \mathbb{R}^n$

$$\iint\limits_{R \text{ in } xy \text{ plane}}{}_{R} f(x,y)\, dA = \iint\limits_{S \text{ in } uv \text{ plane}}{}_{S} f(x(u,v), y(u,v)) \left| \det \left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right) \right| dudv$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right) \leftarrow \text{Jacobian of the transformation}$$

$$\iiint\limits_{R \text{ in } xyz \text{ plane}}{}_{R} f(x,y,z)\, dV = \iiint\limits_{S \text{ in } uvt \text{ plane}}{}_{S} f(x(u,v,t), y(u,v,t), z(u,v,t)) \left| \det \left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{bmatrix} \right) \right| dudvdt$$

$$\frac{\partial(x,y,z)}{\partial(u,v,t)} = \det \left(\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial t} \end{bmatrix} \right) \leftarrow \text{Jacobian of the transformation}$$

4 Section Sixteen

4.1 Vector Fields

A vector field $\vec{F}$ is a vector-valued function that assigns any point in the domain $D \subseteq \mathbb{R}^n$ an n dimensional vector

2D vector field

$$\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$$

3D vector field

$$\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$$

4.2 Conservative Vector Fields

A vector field $\vec{F}$ is conservative if $\vec{F}$ is the vector gradient of a function f. In this case we call the function f a potential function of the vector field $\vec{F}$

- Any vector field $\vec{F}(x,y) = \nabla f$ is conservative
- If D is an open connected region and $\vec{F}$ is a vector field that is continuous in D, then $\vec{F}$ is conservative if and only if the line integral of $\vec{F}$ is path-independent in D

2D conservative vector field $\vec{F} = \langle P(x,y), Q(x,y) \rangle$

$\vec{F}(x,y)$ is conservative if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

3D conservative vector field $\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

$\vec{F}(x,y,z)$ is conservative if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ and $\dfrac{\partial Q}{\partial z} = \dfrac{\partial R}{\partial y}$ and $\dfrac{\partial R}{\partial x} = \dfrac{\partial P}{\partial z}$

4.3 Fundamental Theorem of Line Integrals

$$\int_C \nabla f \; d\vec{r} = \int_C \nabla f \cdot r'(t) \; d\vec{t} = \underbrace{f(\vec{r}(b))}_{\text{ending pt. B}} - \underbrace{f(\vec{r}(a))}_{\text{starting pt. A}}$$

$$\int_C \nabla f \; d\vec{r} = \int_C P \; dx + Q \; dy$$

- If $\vec{F}$ is a conservative vector field, then the line integral of $\vec{F}$ along any curve C is path-independent
- If $\vec{F}$ is a conservative vector field, then the line integral of $\vec{F}$ is equal to 0 along any closed curve

4.4 Open Regions

A region D in $\mathbb{R}^n$ is open if for any point P in D, there exists a disk centered at P that lies entirely within the borders of D

- D is open if it does not contain any boundary points
- The region is defined by an inequality containing only $<$ and $>$
- i.e. $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 4\}$

4.5 Closed Regions

A region D in $\mathbb{R}^n$ is closed if its complement $\mathbb{R}^n - D$ is open

- D is closed if it contains boundary points
- The region is defined by an inequality containing $\leq$ or $\geq$
- i.e. $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$

4.6 Connected Regions

A region D in $\mathbb{R}^n$ is connected if any two points in D can be joined by a path that lies in D

4.7 Simply Connected Regions

A region D in $\mathbb{R}^n$ is simply-connected if D is connected, and the region enclosed by any closed curve C in D is contained entirely within D

- Does not contain a hole

4.8 Unifying Theorems

- If D is an open connected region and $\vec{F}$ is a vector field that is continuous on D, then $\vec{F}$ is conservative if and only if the line integral of $\vec{F}$ is path-independent in D

4.9 Boundary Curve Orientation in $\mathbb{R}^2$

- The boundary curve(s) C of a 2D region D ($\partial D = C_1 \cup C_2 ...$) is positively oriented if C is traversed in a way that the region D is always on the left side of C
- The boundary curve(s) C of a 2D region D ($\partial D = C_1 \cup C_2 ...$) is negatively oriented if C is traversed in a way that the region D is always on the right side of C

4.10 Green's Theorem

Let C be a positively oriented simple closed piecewise-smooth curve in $\mathbb{R}^2$ that bounds the region D

Let $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ be a 2D vector field where $P(x,y)$ and $Q(x,y)$ both have continuous partial derivatives on an open region containing D

Then $\displaystyle\int_C \vec{F}\, d\vec{r} = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$ (multiply by -1 if C is negatively oriented)

Uses:

- Alternative way to calculate line integrals
- Calculate the area of a region

4.11 Curl of a Vector Field

Given a 3D vector field $\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$, the curl of $\vec{F}$ is the vector field:

$$\begin{aligned}
\operatorname{rot}\vec{F} &= \nabla \times \vec{F} \\
&= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle \\
&= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle
\end{aligned}$$

- The curl of $\vec{F}$ measures the rotation of the vector field $\vec{F}$
  - Thinking of the velocity field as a fluid flow, $\operatorname{rot}\vec{F}$ measures the rotation of an object caused by the fluid
- Magnitude of the curl of $\vec{F}$ measures the intensity of rotation
- Direction of the curl of $\vec{F}$ gives the axis of rotation by the right-hand grip rule
- To detect the rotational effect of a 2D vector field $\vec{F} = \langle P(x,y), Q(x,y) \rangle$, we may extend the 2D vector field to $\vec{F} = \langle P(x,y), Q(x,y), 0 \rangle$
- The curl of $\vec{F}$ is zero if and only if $\vec{F}$ is conservative

4.12 Divergence of a Vector Field

Given a vector field $\vec{F}$, the divergence of $\vec{F}$ is the scalar function:

$$\begin{aligned}\mathrm{div}\vec{F} &= \nabla \cdot \vec{F} \\ &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle \\ &= \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\end{aligned}$$

- The divergence of $\vec{F}$ measures the "outgoing-ness" of $\vec{F}$
  - Thinking of the velocity field as a fluid flow, $\mathrm{div}\vec{F}$ measures the net rate at which the fluid flows through a point
  - If $\mathrm{div}\vec{F} < 0$, there is more fluid going in than out
  - If $\mathrm{div}\vec{F} > 0$, there is more fluid going out than in
- Applies to 2D and 3D vector fields
- If $\mathrm{div}\vec{F} = 0$, then $\vec{F}$ is incompressible
- $\mathrm{div}(\mathrm{curl}\vec{F})$ is always zero
  - Can be used to check if a curl is valid

4.13 Curve Parametrization

A curve C in $\mathbb{R}^3$ can be parametrized by a vector function of one parameter

$$C : \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \qquad t \in [a,b] \subseteq \mathbb{R}$$

- Consists of one parameter t
- C represents a 1D object in $\mathbb{R}^3$

4.14 Surface Parametrization

A surface S in $\mathbb{R}^3$ can be parametrized by a vector function of two parameters

$$S : \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \qquad (u,v) \in D \subseteq \mathbb{R}^2$$

- Consists of two parameters u, v
- S represents a 2D object in $\mathbb{R}^3$

4.15 Surface Parametrization of Plane

Plane through point $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ and is spanned by vectors $\vec{a}$ and $\vec{b}$

$$\vec{r}(u,v) = r_0 + u\vec{a} + v\vec{b}$$

4.16 Surface Parametrization of Cylinder

Cylinder centered at origin with radius r and axis of rotation z

$$\vec{r}(\theta,z) = \langle r\cos\theta, r\sin\theta, z \rangle$$

4.17 Surface Parametrization of Revolution

Revolution centered at origin with radius $f(z)$ and axis of rotation z

$$\vec{r}(\theta,z) = \langle f(z)\cos\theta, f(z)\sin\theta, z \rangle$$

4.18 Surface Parametrization of Sphere

Sphere centered at origin with radius ρ

$$\vec{r}(\phi,\theta) = \langle \rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi \rangle$$

4.19 Surface Area of Parametrized Surface

Given $S : \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

$$dS = |\vec{r}_u \times \vec{r}_v| \; dA \qquad \leftarrow \text{if } S \text{ is parametrized by } \vec{r}(u,v)$$

$$dS = \sqrt{{f_x}^2 + {f_y}^2 + 1} \; dA \qquad \leftarrow \text{if } S \text{ is the graph } z = f(x,y)$$

$$\iint_S f(x,y,z) \; ds = \iint_D f(\vec{r}(u,v)) \; |\vec{r}_u \times \vec{r}_v| \; dA$$

$$\iint_S f(x,y,z) \; ds = \iint_D f(\vec{r}(u,v)) \; \sqrt{{f_x}^2 + {f_y}^2 + 1} \; dA$$

4.20 Lamina of Parametrized Surface

Given density function $\rho(x,y,z)$

$$m = \iint_S \rho(x,y,z) \; dS$$

$$\overline{x} = \frac{1}{m} \iint_S x\rho(x,y,z) \; dS$$

$$\overline{y} = \frac{1}{m} \iint_S y\rho(x,y,z) \; dS$$

$$\overline{z} = \frac{1}{m} \iint_S z\rho(x,y,z) \; dS$$

4.21 Orientation of Surfaces

A surface S is orientable if it has two sides

- If S is orientable, the tangent plane at any point on S has two unit normal vectors $\vec{n}_1$ and $\vec{n}_2$ where $\vec{n}_1 = -\vec{n}_2$
- The orientation of an orientable surface S is a choice between the two unit normal vectors of tangent plane at each point of the surface so that the choice varies continuously over S
- If S is a closed surface bounding solid E, S is positively oriented if $\vec{n}$ is chosen to be pointing outward from E, and negatively oriented if $\vec{n}$ is chosen to be pointing inward to E

4.22 Tangent Plane of Parametrized Surface

Given $S : \vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$

$$\left.\begin{aligned} \vec{r}_u &= \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \\ \vec{r}_v &= \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle \end{aligned}\right\}\text{tangent vectors}$$

$$\vec{n} = \vec{r}_u \times \vec{r}_v \qquad \leftarrow \text{normal vector } \vec{n} \text{ of tangent plane}$$

$$\vec{r} \cdot \vec{n} = \vec{a} \cdot \vec{n} \qquad \leftarrow \text{equation of tangent plane, given known point } \vec{a}$$

4.23 Unit Normal Vector

If S is parametrized by $\vec{r}(u,v)$ then

$$\widehat{n} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \qquad \leftarrow \pm \text{ depends on the desired orientation}$$

If S is the graph of $z = g(x,y)$ then

$$\widehat{n} = \pm \frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{{g_x}^2 + {g_y}^2 + 1}} \qquad \leftarrow \pm \text{ depends on the desired orientation}$$

4.24 Surface Integral of a Vector Field

The surface integral of $\vec{F}$ over an orientable surface S with unit normal vector $\widehat{n}$ is

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \widehat{n}\ dS \qquad \longleftarrow$$ represents the flux of $\vec{F}$ across S, the volume of fluid flowing through S per time unit

If S is parametrized by $\vec{r}(u, v)$ then

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_D (\vec{F} \cdot \pm\frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|})\ |\vec{r}_u \times \vec{r}_v|\ dA \\ &= \iint_D \vec{F} \cdot \pm(\vec{r}_u \times \vec{r}_v)\ dA\end{aligned}$$

If S is the graph of $z = g(x, y)$ then

$$\begin{aligned}\iint_S \vec{F} \cdot d\vec{S} &= \iint_D (\vec{F} \cdot \pm\frac{\langle -g_x, -g_y, 1\rangle}{\sqrt{g_x{}^2 + g_y{}^2 + 1}})\ \sqrt{g_x{}^2 + g_y{}^2 + 1}\ dA \\ &= \iint_D \vec{F} \cdot \pm\langle -g_x, -g_y, 1\rangle\ dA\end{aligned}$$

4.25 Boundary Curve Orientation in $\mathbb{R}^3$

- The boundary curve(s) C of a surface S ($\partial S = C_1 \cup C_2 ...$) is positively oriented if C is traversed in a way that the surface S is always on the left side of C from the perspective in the same direction of the normal vector of the surface
  - By the right hand rule, if the thumb is pointed towards $\vec{n}$, the fingers are curling along the orientation of C
- The boundary curve(s) C of a surface S ($\partial S = C_1 \cup C_2 ...$) is negatively oriented if C is traversed in a way that the surface S is always on the right side of C from the perspective in the same direction of the normal vector of the surface

4.26 Stokes' Theorem

Let S be an orientable piecewise-smooth surface whose boundary C is a positively oriented simple closed piecewise-smooth curve

Let $\vec{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$ be a 3D vector field where $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ each have continuous partial derivatives on an open region containing S

Then $\displaystyle\int_C \vec{F} \cdot d\vec{r} = \iint_S \mathrm{curl}\vec{F} \cdot d\vec{S}$ (multiply by -1 if C is negatively oriented)

4.27 Gauss' Theorem (Divergence Theorem)

Let E be a 3D solid that is bounded by a closed surface S where S is positively oriented

Let $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a 3D vector field where $P(x,y,z)$, $Q(x,y,z)$ and $R(x,y,z)$ each have continuous partial derivatives on an open region containing E

Then $\displaystyle\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \mathrm{div}\vec{F}\; dV$