MATH 424 Notes

Contents

1 Section One

1.1 Continuous Functions

A function $f : D \to \mathbb{R}$ is continuous at a point $x_0 \in D$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in D$ where $|x - x_0| < \delta$

- All polynomials $f(x) = a_0 + a_1 x + ... + a_n x^n$ are continuous
- Exponential functions $f(x) = b^x$ are continuous when $b > 0$
- Monomial functions $f(x) = x^n$ are continuous for any $n \in \mathbb{Z}$
  - If $n < 0$, then $x = 0$ is not in the domain

1.2 Continuous Functions and Convergence

A function $f : D \to \mathbb{R}$ is continuous at a point $x_0 \in D$ if and only if $f(x_n) \to f(x_0)$ for all sequences $x_n \in D$ where $x_n \to x_0$

- A function f is continuous at x_0 if and only if $\lim\limits_{x \to x_0} f(x) = f(x_0)$

1.3 Continuous Functions Arithmetic

If the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are continuous at $x_0 \in D$, then

- $(f + g)(x)$ is continuous at x_0
- $(fg)(x)$ is continuous at x_0
- $\left(\frac{f}{g}\right)(x)$ is continuous at x_0 when $g(x_0) \neq 0$

1.4 Accumulation Points

A point $x_0 \in D$ is an accumulation point of D if for all $\delta > 0$, there exists $x \in D \backslash \{x_0\}$ such that $|x - x_0| < \delta$

- An accumulation point is also known as a limit point or a cluster point
- $\mathbb{Z}$ has no accumulation points
- $\mathbb{R}$ represents the set of all accumulation points of $\mathbb{Q}$
- $\mathbb{R}$ represents the set of all accumulation points of $\mathbb{R} \backslash \mathbb{Q}$
- 0 is the only accumulation point of $\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$

1.5 Accumulation Points and Convergence

A point $x_0 \in D$ is an accumulation point of D if there exists a sequence $(x_n) \subset D \backslash \{x_0\}$ such that $x_n \to x_0$

1.6 Limits

The limit of a function $\lim_{x \to x_0} f(x) = \ell$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in D \backslash \{x_0\}$ where $|x - x_0| < \delta$

- f is continuous at $x_0 \in D$ if and only if $\lim_{x \to x_0} f(x) = f(x_0)$
- If a limit $\lim_{x \to x_0} f(x)$ exists, then it is unique

1.7 Limits and Convergence

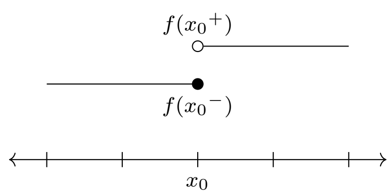
The limit of a function $\lim_{x \to x_0} f(x) = \ell$ if and only if $f(x_n) \to \ell$ for all $x \in D$ where $x_n \neq x_0$ and $x_n \to x_0$

1.8 Limits Arithmetic

Given functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ and an accumulation point $x_0 \in D$

- If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, then $\lim_{x \to x_0} (f + g)(x) = A + B$
- If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$, then $\lim_{x \to x_0} (fg)(x) = AB$
- If $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$ with $B \neq 0$, then $\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \frac{A}{B}$

1.9 One-Sided Limits



The left-hand limit $\lim_{x \to x_0{}^-} f(x) = \ell$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in (x_0 - \delta, x_0)$

- The left-hand limit is written as $f(x_0{}^-) = \lim_{x \to x_0{}^-} f(x)$

The right-hand limit $\lim_{x \to x_0{}^+} f(x) = \ell$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in (x_0, x_0 + \delta)$

- The right-hand limit is written as $f(x_0{}^+) = \lim_{x \to x_0{}^+} f(x)$

Theorems

- $\lim_{x \to x_0} f(x) = \ell$ if and only if $f(x_0{}^-) = f(x_0{}^+) = \ell$
- A function f is continuous at x_0 if and only if $f(x_0{}^-) = f(x_0{}^+) = f(x)$

1.10 One-Sided Limits in Monotonic Functions

Let $f : (a, b) \to \mathbb{R}$ be a monotonically increasing function. Then $f(x^-)$ and $f(x^+)$ exist at every point $x \in (a, b)$ where

$$\sup_{a<t<x} f(t) = f(x^-) \leq f(x) \leq f(x^+) = \inf_{x<t<b} f(t)$$

- If $a < x < y < b$, then $f(x^+) \leq f(y^-)$

Let $f : (a, b) \to \mathbb{R}$ be a monotonically decreasing function. Then $f(x^-)$ and $f(x^+)$ exist at every point $x \in (a, b)$ where

$$\inf_{a<t<x} f(t) = f(x^-) \geq f(x) \geq f(x^+) = \sup_{x<t<b} f(t)$$

- If $a < x < y < b$, then $f(x^+) \geq f(y^-)$

1.11 Discontinuity in Monotonic Functions

Let $f : (a, b) \to \mathbb{R}$ be a monotonic function. Then the set of points at which f is discontinuous is at most countable

- Analogues hold for $[a, b)$, $(a, b]$, $[a, b]$ and for all unbounded intervals
- An increasing function $f : I \to \mathbb{R}$ defined on an interval is continuous if and only if $f(I)$ is an interval
- An increasing function $f : I \to \mathbb{R}$ is invertible on its range $f(I)$ if and only if it is strictly increasing
  - In this case, $f^{-1} : f(I) \to I$ is also strictly increasing
- If $f : I \to \mathbb{R}$ is strictly increasing and continuous, then $f(I)$ is an interval and $f^{-1} : f(I) \to I$ is strictly increasing and continuous
- Analogues of the above hold for decreasing functions

1.12 Pointwise Convergent Sequences of Functions

A sequence of functions $\{f_n : D \to \mathbb{R}\}$ converges pointwise to a function $f : D \to \mathbb{R}$ if for each point $x \in D$, given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$

- $\{f_n\}$ converges pointwise to f if $\lim_{n\to\infty} f_n(x) = f(x)$ for each point $x \in D$

1.13 Uniformly Convergent Sequences of Functions

A sequence of functions $\{f_n : D \to \mathbb{R}\}$ converges uniformly to a function $f : D \to \mathbb{R}$ if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $n \geq N$ and for all $x \in D$

- $\{f_n\}$ is said to converge uniformly on D to f

1.14 Uniformly Convergent Sequences of Continuous Functions

If $\{f_n : D \to \mathbb{R}\}$ is a sequence of continuous functions that converges uniformly to the function $f : D \to \mathbb{R}$, then the limit function f is also continuous

2 Section Two

2.1 Differentiable Functions

A function $f : I \to \mathbb{R}$ is differentiable at x_0 if the following limit exists

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

- The limit $f'(x_0)$ is called the derivative of f at x_0
- If $f : I \to \mathbb{R}$ is differentiable at every point in I, then f is differentiable
- If f is differentiable at x_0, then f is continuous at x_0
  - The converse does not hold
- If f is differentiable at x_0 and $f'(x_0)$ exists, then

  $$F(x) = \begin{cases} \frac{f(x)-f(x_0)}{x-x_0} & \text{if } x \neq x_0 \text{ and } x \in I \\ f'(x_0) & \text{if } x = x_0 \end{cases}$$

  is continuous at x_0
- If $f'(x_0)$ exists, then

  $$u(h) = \begin{cases} \frac{f(x+h)-f(x_0)}{h} - f'(x_0) & \text{if } h \neq 0 \\ 0 & \text{if } h = 0 \end{cases}$$

  is continuous at 0
- All polynomials $f(x) = a_0 + a_1 x + ... + a_n x^n$ are differentiable everywhere
- Exponential functions $f(x) = b^x$ are differentiable everywhere when $b > 0$
- Monomial functions $f(x) = x^n$ are differentiable everywhere for any $n \in \mathbb{Z}$
  - If $n < 0$, then $x = 0$ is not in the domain

2.2 Differentiable Functions Arithmetic

If the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are differentiable at x_0, then

- $(f + g)(x)$ is differentiable at x_0 with $(f + g)'(x_0) = f'(x_0) + g'(x_0)$
- $(fg)(x)$ is differentiable at x_0 with $(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0)$
- $\left(\frac{f}{g}\right)'(x)$ is differentiable at x_0 with $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}$ when $g(x) \neq 0$ for all $x \in I$

2.3 Derivative Chain Rule

Let $f : (a, b) \to \mathbb{R}$ and $g : (c, d) \to \mathbb{R}$ be functions with $f((a, b)) \subseteq (c, d)$. If f is differentiable at $x_0 \in (a, b)$ and g is differentiable at $f(x_0)$, then

- $g \circ f : (a, b) \to \mathbb{R}$ is differentiable at x_0
- $(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

2.4 Derivative of the Inverse Function

Let $f : (a, b) \to \mathbb{R}$ be strictly increasing and continuous. Let $g = f^{-1} : f((a, b)) \to (a, b)$. If f is differentiable at $x_0 \in (a, b)$ and $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$ and

$$g'(y_0) = \frac{1}{f'(x_0)}$$

- $g : f((a, b)) \to (a, b)$ is also strictly increasing
- $f((a, b))$ is the open interval $(f(a), f(b))$
- Analogues of the above hold for strictly decreasing functions

2.5 Local Maximum

A function $f : I \to \mathbb{R}$ has a local maximum at $x_0 \in I$ if there exists $\delta > 0$ such that $f(x_0) \geq f(x)$ for all $x \in I$ where $|x - x_0| < \delta$

2.6 Local Minimum

A function $f : I \to \mathbb{R}$ has a local minimum at $x_0 \in I$ if there exists $\delta > 0$ such that $f(x_0) \leq f(x)$ for all $x \in I$ where $|x - x_0| < \delta$

2.7 Derivatives at Local Vertices

Let $f : I \to \mathbb{R}$ be differentiable at x_0. If x_0 is a local maximum or minimum of f, then $f'(x_0) = 0$

2.8 Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous over $[a, b]$ and differentiable over (a, b). If $f(a) = f(b)$, then there exists a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$

2.9 Cauchy Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continuous over $[a, b]$ and differentiable over (a, b). Then there exists a point $x \in (a, b)$ at which

$$[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$$

- Also known as the generalized mean value theorem

2.10 Mean Value Theorem

Let $f : [a, b] \to \mathbb{R}$ be continuous over $[a, b]$ and differentiable over (a, b). Then there exists a point $x_0 \in (a, b)$ at which

$$f'(x_0) = \frac{f(b) - f(a)}{b - a}$$

- Given two points on a curve with slope m, there exists a point in between such that the tangent also has slope m

2.11 Intermediate Value Theorem for Derivatives

Let $f : [a, b] \to \mathbb{R}$ be differentiable and suppose that $f'(a) < \lambda < f'(b)$. Then there exists $\alpha \in (a, b)$ such that $f'(\alpha) = \lambda$

- If a function is differentiable everywhere, then intermediate values are assumed
- If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$

2.12 Derivatives and Monotonicity

Let $f : (a, b) \to \mathbb{R}$ be differentiable over (a, b)

- $f'(x) \geq 0$ for all $x \in (a, b)$ if and only if f is monotone increasing on (a, b)
- $f'(x) = 0$ for all $x \in (a, b)$ if and only if f is constant
- If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b)
  - The converse does not hold

2.13 Taylor's Theorem

Let $I = (a, b)$ and $f : I \to \mathbb{R}$ such that $f^{(n)}(x)$ exists for every $x \in I$ for some $n \in \mathbb{N}$. If α, β are distinct points in I, then there exists a point $c \in (\alpha, \beta)$ such that

$$f(\alpha) = \sum_{k=0}^{n-1} \left[\frac{f^{(k)}(\beta)}{k!} (\alpha - \beta)^k \right] + \frac{f^{(n)}(c)}{n!} (\alpha - \beta)^n$$

$$f(\beta) = \sum_{k=0}^{n-1} \left[\frac{f^{(k)}(\alpha)}{k!} (\beta - \alpha)^k \right] + \frac{f^{(n)}(c)}{n!} (\beta - \alpha)^n$$

3 Section Three

3.1 Partitions

A partition P of $[a, b]$ is a finite set of points $x_0, x_1, ..., x_n$ where

$$a = x_0 \leq x_1 \leq ... \leq x_{n-1} \leq x_n = b$$

- A partition is a subdivision of $[a, b]$ into finitely many closed subintervals

3.2 Intervals

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$ and let α be a monotonically increasing function on $[a, b]$. Then the interval of the partition is $\Delta x_i = x_i - x_{i-1}$ and the function interval of the partition is $\Delta\alpha_i = \Delta\alpha(x_i) = \alpha(x_i) - \alpha(x_{i-1})$ for $i = 1, ..., n$

3.3 Riemann Integrals

Let $f : [a, b] \to \mathbb{R}$ be a bounded function, let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$, and let α be a monotonically increasing function of $[a, b]$. Then

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \qquad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

$$\bar{\int_a^b} f \, d\alpha = \inf_{P \in \Omega_P} U(P, f, \alpha) \qquad \underline{\int_a^b} f \, d\alpha = \sup_{P \in \Omega_P} L(P, f, \alpha)$$

$\bar{\int}_a^b f \, d\alpha$ and $\underline{\int}_a^b f \, d\alpha$ are the upper and lower Riemann-Stieltjes integrals of f over $[a, b]$ respectively

- If $\bar{\int}_a^b f \, d\alpha = \underline{\int}_a^b f \, d\alpha = \int_a^b f \, d\alpha$, then f is integrable with respect to α in the Riemann sense
- $\mathscr{R}(\alpha)$ represents the set of functions integrable with respect to α in the Riemann sense
- If $\bar{\int}_a^b f \, dx = \underline{\int}_a^b f \, dx = \int_a^b f \, dx$, then f is Riemann-integrable on $[a, b]$
- $\mathscr{R}$ represents the set of Riemann-integrable functions
- If the upper and lower integrals are equal, then the common integral is denoted as

$$\int_a^b f \, d\alpha = \int_a^b f(x) \, d\alpha(x)$$

- $m_i \leq M_i$ for all $i \in \{0, 1, ..., n\}$ such that $L(P, f, \alpha) \leq U(P, f, \alpha)$
- α is not necessarily continuous

3.4 Composite Riemann-Stieltjes Integrals

Let $f \in \mathscr{R}(\alpha)$ be a bounded function on $[a, b]$ with $m \leq f \leq M$. If ϕ is continuous on $[m, M]$ and $h(x) = \phi(f(x))$, then $h \in \mathscr{R}(\alpha)$ on $[a, b]$

3.5 Conditions for Riemann-Stieltjes Integrals

Let f be a function on the interval $[a,b]$

- A function $f \in \mathscr{R}(\alpha)$ if and only if for every $\varepsilon > 0$, there exists a partition P such that
$$U(P,f,\alpha) - L(P,f,\alpha) < \varepsilon$$
- If f is continuous, then $f \in \mathscr{R}(\alpha)$
- If f is monotonic and α is continuous, then $f \in \mathscr{R}(\alpha)$
- If f is bounded, f has only finitely many points of discontinuity, and α is continuous at every point at which f is discontinuous, then $f \in \mathscr{R}(\alpha)$

3.6 Properties of Riemann-Stieltjes Integrals

Let f, g be functions on the interval $[a,b]$ and $c \in \mathbb{R}$

- If $f, g \in \mathscr{R}(\alpha)$, then $\int_a^b (f+g)\ d\alpha = \int_a^b f\ d\alpha + \int_a^b g\ d\alpha$
- If $f \in \mathscr{R}(\alpha)$, then $\int_a^b cf\ d\alpha = c\int_a^b f\ d\alpha$
- If $f \leq g$, then $\int_a^b f\ d\alpha \leq \int_a^b g\ d\alpha$
- If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $c \in (a,b)$, then $f \in \mathscr{R}(\alpha)$ on $[a,c]$ and $[c,b]$ with $\int_a^c f\ d\alpha + \int_c^b f\ d\alpha = \int_a^b f\ d\alpha$
- If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and $|f(x)| \leq M$ on $[a,b]$, then $\left|\int_a^b f\ d\alpha\right| \leq M\,[\alpha(b) - \alpha(a)]$
- If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $\int_a^b f\ d(\alpha_1 + \alpha_2) = \int_a^b f\ d\alpha_1 + \int_a^b f\ d\alpha_2$
- If $f \in \mathscr{R}(\alpha)$, then $|f| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b f\ d\alpha\right| \leq \int_a^b |f|\ d\alpha$

3.7 Change of Variable of Integration

- Let α be a monotonically increasing function with derivative $\alpha' \in \mathscr{R}$ and let f be a bounded real function on $[a,b]$. Then
$$\int_a^b f\ d\alpha = \int_a^b f(x)\alpha'(x)\ dx$$
such that $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$
- Suppose that φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$, α is a monotonically increasing function on $[a,b]$, and $f \in \mathscr{R}(\alpha)$ on $[a,b]$. Let β and g be functions on $[A,B]$ such that $\beta(y) = \alpha(\varphi(y))$ and $g(y) = f(\varphi(y))$. Then
$$\int_A^B g\ d\beta = \int_a^b f\ d\alpha$$
and $g \in \mathscr{R}(\beta)$

3.8 Refinements

A partition P^* is a refinement of P if $P \subset P^*$, that is every point of P is a point of P^*

- If P_1, P_2 are partitions of $[a, b]$ and $P^* = P_1 \cup P_2$, then P^* is the common refinement of P_1, P_2

3.9 Partition Subsets

Let P^* be a refinement of P. Then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \leq U(P^*, f, \alpha) \leq U(P, f, \alpha)$$

$$L(P_1, f, \alpha) \leq L(P_1 \cup P_2, f, \alpha) \leq U(P_1 \cup P_2, f, \alpha) \leq U(P_2, f, \alpha)$$

- If P_1, P_2 are partitions of $[a, b]$, then $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$
- If P_1, P_2 range over all partitions of $[a, b]$, then $\sup\limits_{P_1 \in \Omega_P} L(P_1, f, \alpha) \leq \inf\limits_{P_2 \in \Omega_P} U(P_2, f, \alpha)$

3.10 Fundamental Theorem of Calculus

Let $f \in \mathscr{R}$ be a function on $[a, b]$ and let $F(x) = \displaystyle\int_a^x f(t)\, dt$ for some $a \leq x \leq b$. Then

- F is continuous on $[a, b]$
- If f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$
- $\displaystyle\int_a^b f(x)\, dx = F(b) - F(a)$

3.11 Mean Value Theorem for Integrals

If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $x_0 \in (a, b)$ such that $\displaystyle\int_a^b f(t)\, dt = f(x_0)(b - a)$

4 Section Four

4.1 Uniformly Cauchy Sequences

A sequence of functions $\{f_n : D \to \mathbb{R}\}$ is uniformly Cauchy if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n(x) - f_m(x)| < \varepsilon$ for all $m, n \geq N$ and for all $x \in D$

- A sequence of functions is uniformly convergent if and only if it is uniformly Cauchy

4.2 Uniform Convergence and Differentiation

Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a, b]$ and such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n{}'\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$ to a function f and

$$f'(x) = \lim_{n\to\infty} f_n{}'(x) \qquad (a \leq x \leq b)$$

4.3 Analytic Functions

An analytic function is a function of the form $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ where $c_n \in \mathbb{R}$

4.4 Uniform Convergence of Power Series

Suppose that the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges for all $|x| < R$. Then the analytic function $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n x^n$ over the domain $(-R, R)$ converges uniformly for all $|x| < R - \varepsilon$ and for any $\varepsilon > 0$

- The function f is continuous and differentiable for all $|x| < R$
- The derivative of f over the domain $(-R, R)$ is given by $f'(x) = \displaystyle\sum_{n=1}^{\infty} n c_n x^{n-1}$
- The radius of convergence is $R = \left(\limsup |c_n|^{\frac{1}{n}}\right)^{-1}$
  - If the series does not converge for any $x \in \mathbb{R}$, then $R = 0$
  - If the series converges for all $x \in \mathbb{R}$, then $R = \infty$

4.5 Absolute Convergence of Power Series

- If $|x| < R$, then $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ converges absolutely
- If $|x| > R$, then $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ diverges

4.6 Weierstrass M-test

Let $\{f_n\}$ be a sequence of functions defined on a set D and let $|f_n(x)| \leq M_n$ for all $x \in D$ and for all $n \in \mathbb{N}$. If $\displaystyle\sum_{n\in\mathbb{N}} M_n$ converges, then $\displaystyle\sum_{n\in\mathbb{N}} f_n$ converges uniformly on D

4.7 Root Test

Let $r = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$

- If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges
- If $r = 1$, then the root test is inconclusive