MATH 307 Notes

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1 First Order ODE

1.1 Integral of a function

$$\begin{aligned}
y'(x) &= g(x), \quad y(x_0) = y_0 \\
y(x) &= \int_{x_0}^{x} y'(t)\ dt \\
&= \int_{x_0}^{x} g(t)\ dt
\end{aligned}$$

1.2 Integral of a product of functions

$$\begin{aligned}
(m(x)y(x))' &= g(x), \quad y(x_0) = y_0 \\
(m(x)y(x))' &= m(x)y'(x) + m'(x)y(x) \\
&= g(x) \\
m(x)y(x) &= \int_{x_0}^{x} (m(t)y(t))'\ dt \\
&= \int_{x_0}^{x} g(t)\ dt
\end{aligned}$$

1.3 Solving ordinary differential equations

$$\begin{aligned}
y' + py &= g, \quad y(x_0) = y_0 \\
my' + mpy &= mg \\
mp &= m' \\
p &= \frac{m'}{m} \\
ln|m| &= \int_{x_0}^{x} p(t)\ dt \leftarrow \text{no constant needed} \\
|m| &= e^{\int_{x_0}^{x} p(t)\ dt} \\
(my)' &= mg \\
my &= \int_{x_0}^{x} mg\ dt \\
y &= \frac{1}{m} \int_{x_0}^{x} mg\ dt
\end{aligned}$$

2 Second Order ODE

2.1 Solving for characteristic equation

$$
\begin{aligned}
ny'' + py' + qy &= 0 \\
y &= e^{rt} \leftarrow \text{guess for y} \\
n(r^2e^{rt}) + p(re^{rt}) + q(e^{rt}) &= 0 \\
nr^2 + pr + q &= 0 \\
r &= \frac{-p \pm \sqrt{p^2 - 4nq}}{2n}
\end{aligned}
$$

- If $p^2 - 4nq > 0$, then there are 2 roots r_1, r_2

  $y = c_1e^{r_1t} + c_2e^{r_2t}$

- If $p^2 - 4nq < 0$, then there are 2 roots $r = a \pm ib$

  $y = c_1e^{(a+ib)t} + c_2e^{(a-ib)t}$

  $y = d_1e^{at}cos(bt) + d_2e^{at}sin(bt)$

- If $p^2 - 4nq = 0$, then there is 1 root $r = -\frac{p}{2n}$

  $y = c_1e^{rt} + c_2te^{rt}$

2.2 Solving for particular solution

With polynomial forcing function

$$
\begin{aligned}
&ny'' + py' + qy = t^n \\
&y = At^n + Bt^{n-1} + ... + C
\end{aligned}
$$

With exponential forcing function

$$ny'' + py' + qy = (a_0 + a_1t + ...)e^{rt}$$

- If r is not a root of the characteristic equation

  $y = (b_0 + b_1t + ...)e^{rt}$

- If r is one of two roots of the characteristic equation

  $y = t(b_0 + b_1t + ...)e^{rt}$

- If r is the only root of the characteristic equation

  $y = t^2(b_0 + b_1t + ...)e^{rt}$

2.3 Miscellaneous Identities

$$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

$$Re\left[e^{(a+ib)t}\right] = e^{at}\cos bt$$

$$Im\left[e^{(a+ib)t}\right] = e^{at}\sin bt$$

$$a\cos\theta t + b\sin\theta t = \sqrt{a^2+b^2}\cos\left(\theta t - \tan^{-1}\left(\frac{b}{a}\right)\right)$$

$$\frac{d}{dt}(te^{at}) = e^{at} + ate^{at}$$

$$\begin{aligned}\frac{d}{dt}\left(d_1 e^{at}\cos\theta t + d_2 e^{at}\sin\theta t\right) = {} & d_1\left(ae^{at}\cos\theta t - \theta e^{at}\sin\theta t\right) + \\ & d_2\left(ae^{at}\sin\theta t + \theta e^{at}\cos\theta t\right)\end{aligned}$$

$$\begin{aligned}\int e^{at}(\cos\theta + i\sin\theta)\,dt = {} & \frac{a}{a^2+\theta^2}\,e^{at}\cos\theta t + \frac{a}{a^2+\theta^2}\,e^{at}\sin\theta t + \\ & \left(\frac{a}{a^2+\theta^2}\,e^{at}\sin\theta t - \frac{a}{a^2+\theta^2}\,e^{at}\cos\theta t\right)i + C\end{aligned}$$

3 Laplace Transforms

3.1 Laplace Transformation

$$\begin{aligned}
\mathcal{L}(f(t))(s) &= \int_0^\infty e^{-st} f(t)\ dt \\
&= \lim_{A\to\infty} \int_0^A e^{-st} f(t)\ dt \\
\mathcal{L}(1)(s) &= \lim_{A\to\infty} \int_0^A e^{-st}\ dt \\
&= \lim_{A\to\infty} \left[-\frac{1}{s}\ e^{-st}\right]_0^A \\
&= \lim_{A\to\infty} -\frac{1}{s}\left(e^{-sA} - 1\right) \\
&= \frac{1}{s}
\end{aligned}$$

3.2 Laplace Transformation Properties

- $\mathcal{L}(cf) = c\ \mathcal{L}(f)$
- $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$
- $\mathcal{L}(f) = \mathcal{L}(g) \to f = g$
- $\mathcal{L}(e^{rt} f)(s) = \mathcal{L}(f)(s-r)$
- $\mathcal{L}(tf(t)) = -\frac{d}{ds}\mathcal{L}(f(t))$
- $\mathcal{L}(f') = s\ \mathcal{L}(f) - f(0)$
- $\mathcal{L}(f'') = s\ \mathcal{L}(f') - f'(0)$

  $= s\ (s\ \mathcal{L}(f) - f(0)) - f'(0)$

  $= s^2\mathcal{L}(f) - sf(0) - f'(0)$
- $\mathcal{L}(ay'' + by' + cy) = \mathcal{L}(f)(as^2 + bs + c) - f(0)(as+b) - f'(0)(a)$

3.3 Laplace Transformation Identities

$$\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}(e^{at})(s) = \mathcal{L}(1)(s-a) = \frac{1}{s-a}$$

$$\mathcal{L}(e^{ibt})(s) = \mathcal{L}(1)(s-ib) = \frac{1}{s-ib} = \frac{s+ib}{s^2+b^2}$$

$$\mathcal{L}(\sin(bt))(s) = Im\left[\mathcal{L}\left(e^{ibt}\right)(s)\right] = \frac{b}{s^2+b^2}$$

$$\mathcal{L}(\cos(bt))(s) = Re\left[\mathcal{L}\left(e^{ibt}\right)(s)\right] = \frac{s}{s^2+b^2}$$

$$\mathcal{L}(\sin(a+bt))(s) = \mathcal{L}(\sin a\cos bt + \cos a\sin bt)(s) = \frac{s\sin(a)+b\cos(a)}{s^2+b^2}$$

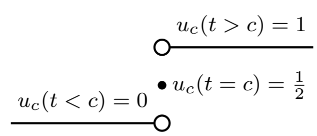
$$\mathcal{L}(\cos(a+bt))(s) = \mathcal{L}(\cos a\cos bt - \sin a\cos bt)(s) = \frac{s\cos(a)-b\sin(a)}{s^2+b^2}$$

$$\mathcal{L}(\sinh(bt))(s) = Im\left[\mathcal{L}\left(\frac{e^{bt}-e^{-bt}}{2}\right)(s)\right] = \frac{b}{s^2-b^2}$$

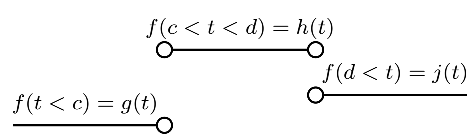
$$\mathcal{L}(\cosh(bt))(s) = Re\left[\mathcal{L}\left(\frac{e^{bt}+e^{-bt}}{2}\right)(s)\right] = \frac{s}{s^2-b^2}$$

3.4 Heaviside Step Function

$$u_c(t) = \begin{cases} 0 & \text{when } t < c \\ \frac{1}{2} & \text{when } t = c \\ 1 & \text{when } t > c \end{cases}$$



$$f(t) = \begin{cases} g(t) & \text{when } 0 < t < c \\ h(t) & \text{when } c < t < d \\ j(t) & \text{when } d < t \end{cases}$$



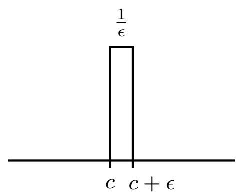
$$f(t) = (u_0 - u_c)g(t) + (u_c - u_d)h(t) + (u_d)j(t)$$

For calculation purposes:

- $u_c(t-d) = u_{c+d}(t)$
- $\mathcal{L}(u_c(t)) = \frac{e^{-cs}}{s}$
- $\mathcal{L}(u_c(t)f(at+b)) = e^{-cs}\mathcal{L}(f(a(t+c)+b))$

3.5 Delta Function

$$\delta_c = \begin{cases} \frac{1}{\epsilon} & \text{when } c < t < c + \epsilon \\ 0 & \text{when otherwise} \end{cases}$$



$$\delta_c = \frac{u_c - u_{c+\epsilon}}{\epsilon}$$
$$\mathcal{L}(\delta_c) = e^{-cs} \left[\frac{1 - e^{-\epsilon s}}{\epsilon s} \right]$$
$$\lim_{\epsilon \to 0} \mathcal{L}(\delta_c) = e^{-cs}$$

For calculation purposes:

- $\delta_c(t - d) = \delta_{c+d}(t)$
- $\mathcal{L}(\delta_c(t)) = e^{-cs}$
- $\mathcal{L}(\delta_c(t) f(t)) = e^{-cs} f(c)$

4 Convolutions

4.1 Convolution

$$(f * g)(t) = \int_0^t f(t-x)g(x)\,dx$$

4.2 Convolution Properties

- $f * g = g * f$
- $f * (g + h) = f * g + f * h$
- $f * cg = c(f * g)$
- $(f * g) * h = f * (g * h)$
- $f * 0 = 0 * f = 0$
- $f * \delta_0 = \delta_0 * f = f$
- $f * 1 \neq f$

4.3 Convolution and Laplace

$$\begin{aligned}&ay'' + by' + cy = g(t)\\&ay_0'' + yb_0' + cy_0 = 0\\&Y_0 = \frac{1}{as^2 + bs + c}\end{aligned}$$

y_0 is the impulse response function

$y = y_0 * g(t)$

Y_0 is the transfer function

$\mathcal{L}(y) = Y_0 \cdot \mathcal{L}(g)$