MATH 308 Notes

Contents

1 Section One

1.1 Linear Systems

A linear system is a system of equations made up of linear equations

- A consistent linear system has solutions
- An inconsistent linear system has no solutions
- A linear system with n variables requires $\geq n$ equations to obtain a unique solution
- A system of equations is homogeneous if all the constant terms are zero
- A homogeneous system is never consistent
- A homogeneous system has infinitely many solutions when its matrix in echelon form has more columns than non-zero rows
- A homogeneous system has only the trivial solution when its matrix in echelon form has the same number of columns as non-zero rows

1.2 Equivalent Systems

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Systems of equations that have the same solution are called equivalent systems

1.3 Echelon Form

$$\begin{cases} x_1 - 2x_2 + x_3 + 2x_4 & = 0 \\ 2x_2 - 2x_3 & = 2 \\ x_4 & = -1 \end{cases}$$

A system is in echelon form if (assuming variables are written in ascending index order $x_1, x_2, ..., x_n$):

- Every variable is the leading variable of no more than one equation
- The system is organized in a descending stair step pattern so that the index of the leading variables increases from top to bottom
- Equations without variables are at the bottom

Variables that are not leading variables in any equation are called independent/free variables

1.4 Triangular System

$$\begin{cases} x_1 - 2x_2 + x_3 &= 0 \\ \quad\quad 2x_2 - x_3 &= 1 \\ \quad\quad\quad\quad x_3 &= -1 \end{cases}$$

A triangular system is an echelon system with no free variables

- A triangular system has the same number of variables and equations
- A triangular system has exactly one solution

$$m \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ ... &... \\ a_{mn}x_n &= b_m \end{cases}$$

Where: $a_{mn} = 0$ when $m > n$

1.5 Augmented Matrix

$$\begin{cases} x_1 + x_2 + x_3 &= 0 \\ x_1 - x_2 &= 1 \\ x_1 + \quad\quad x_3 &= 3 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{array}\right)$$

An augmented matrix represents the coefficients of a linear system

- The leading term of a row is the leftmost non-zero element in the row
- Pivot positions are those that contain a leading term
- Pivot columns are those that contain pivot positions

1.6 Matrix in Echelon Form

$$\left(\begin{array}{ccc|c} -1 & 0 & 0 & 2 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

A matrix is in echelon form if:

- The pivot of a non-zero row is always to the right of the pivot of the row above it
- All rows consisting of only zeros are at the bottom

In a matrix in echelon form:

- x_i is a dependent variable of the system if column i contains a pivot
- x_j is an independent/free variable of the system if it is not a dependent variable
- A consistent system has a unique solution if it has no independent variables
- A consistent system has infinite solutions if it has independent variables
- Different sequences of row operations for the same matrix can lead to different echelon forms that are equivalent to one another

1.7 Matrix in Reduced Echelon Form

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right) \; or \; \left(\begin{array}{ccc|c} 1 & 1 & 0 & -5 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

A matrix is in reduced echelon form if:

- It is in echelon form
- All leading terms are 1
- The only non-zero term in a pivot column is in the pivot position

In a matrix in reduced echelon form, all sequences of row operations for the same matrix will lead to the same reduced echelon form

2 Section Two

2.1 Vector Span

The span of vectors $v_1, v_2, ..., v_m$ in $\mathbb{R}^n$ is the set of all linear combinations of the vectors, denoted by $\mathrm{Span}(v_1, v_2, ..., v_m)$

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ 5 \\ 5 \end{pmatrix}$$

Checking if (x, y, z) is in $\mathrm{Span}(v_1, v_2, v_3)$ corresponds to checking if the system below has solutions

$x_1(1, 1, 1) + x_2(2, 3, 3) + x_3(3, 5, 5) = (x, y, z)$

$$\begin{cases} x_1 + 2x_2 + 3x_3 & = & x \\ x_1 + 3x_2 + 5x_3 & = & y \\ x_1 + 3x_2 + 5x_3 & = & z \end{cases}$$

If the system is consistent, then the vector (x, y, z) is in span, and vice versa

- For m vectors $v_1, v_2, ..., v_m$ to span $\mathbb{R}^n$, $m \geq n$ must be true
- If the linear system is inconsistent, then m vectors will not span $\mathbb{R}^n$ even if $m \geq n$

2.2 Vector Span Notation

Let A be the matrix having $v_1, v_2, ..., v_m$ for columns

$Span(\{v_1, v_2, ..., v_m\})$ is the set of constant vectors $\vec{b}$ for which the system $A\vec{x} = \vec{b}$ is consistent

$$A = \begin{pmatrix} v_{11} & v_{12} & ... & v_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & v_{n2} & ... & v_{nm} \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$A\vec{x} = \vec{b} \;\; or \;\; [A : \vec{b}] \;\rightarrow\; \left(\begin{array}{cccc|c} v_{11} & v_{22} & ... & v_{1m} & b_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ v_{n1} & v_{n2} & ... & v_{nm} & b_n \end{array}\right)$$

2.3 Redundant Vector Span Equations

An equation is redundant if it is in the span of the other rows

- An equation is redundant if its constituent vectors are also in the span
- If an equation is redundant, it will correspond to a zero row in echelon form

$$\begin{cases} x+y &= 1 \\ x-y &= 0 \\ 2x &= 2 \end{cases}$$

The third row is redundant because it can be produced by a combination of the other two rows (it is in the span of the other rows)

2.4 Homogeneous System

The system $A\vec{x} = \vec{b}$ is homogeneous if $\vec{b}$ is a zero vector

- For any system $A\vec{x} = \vec{b}$, its associated homogeneous system is $A(\vec{v} - \vec{w}) = 0$
- $\vec{v}$ and $\vec{w}$ are both solutions to the system
- $\vec{z}$ (defined as $\vec{v} - \vec{w}$) is a solution to the associated homogeneous system

2.5 Linearly Independent Vectors

$x_1\vec{v_1}, x_2\vec{v_2}, ..., x_3\vec{v_m} = \vec{0}$

Vectors $v_1, v_2, ..., v_m$ in $\mathbb{R}^n$ are called linearly independent only when all the scalars $x_1, x_2, ..., x_3$ are equal to 0

- Vectors $v_1, v_2, ..., v_m$ are linearly independent only if no vector v_i can be written as a linear combination of any other vectors
- A system of linearly independent vectors equal to 0 can undergo Gauss-Jordan reduction into an identity matrix
- If $v_1, v_2, ..., v_m$ is a set of vectors in $\mathbb{R}^n$ and $n < m$, then the set is linearly dependent
- Any system of vectors that contains the zero vector is linearly dependent
- E.g. $(1, 0)$ and $(0, 1)$ are linearly independent while $(1, 0)$ and $(2, 0)$ are linearly dependent

2.6 Theorems

Columns of a matrix A are linearly independent if every column of A is a pivot column

- All variables are pivots, no free variables
- System only has the trivial solution

A set consisting of n vectors can span $\mathbb{R}^n$ if the set is linearly independent

- There will be n pivots, one for each column
- All variables are pivots, no free variables
- No zero rows, system is consistent for all values $\vec{b}$

If $\mathrm{Span}(u_1, u_2, u_3)$ is $\mathbb{R}^n$, then $\mathrm{Span}(u_1, u_2, u_3, u_4)$ will be also be $\mathbb{R}^n$

- If we bring in a new vector u_4, $\vec{b}$ is still a linear combination of the first three vectors, and hence a linear combination of all four vectors

If we have $S = \{v_1, v_2, ..., v_n\}$ as the set of vectors in $\mathbb{R}^n$, and let $A = [v_1, v_2, ..., v_n]$. Then the collective following are either true or false

- S spans $\mathbb{R}^n$
- S is linearly independent
- $A\vec{x} = \vec{b}$ has a unique solution for all b in $\mathbb{R}^n$

3 Section Three

3.1 Matrix Multiplication

If A and B are a $m \times k$ and $k \times t$ matrix respectively, then they can be multiplied where AB is a $m \times t$ matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ax + by + cz \\ dx + ey + fz \end{pmatrix}$$

$$2 \times 3 \qquad\qquad 3 \times 1 \qquad\qquad\qquad 2 \times 1$$

- It is possible that $AB \neq BA$
- It is possible that $AB = 0$, but $A \neq 0$ and $B \neq 0$
- It is possible that $AB = AC$ and $B \neq C$ and $A \neq 0$
- $(AB)C = A(BC)$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$

3.2 Special Matrices

The zero matrix 0_{mn} and identity matrix I_{mn} are special as they are the additive and multiplicative identities respectively

$$0_{mn} = \begin{pmatrix} 0 & 0 & ... & 0 \\ 0 & 0 & ... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & 0 \end{pmatrix}$$ equivalent to the integer 0

$$I_{mn} = \begin{pmatrix} 1 & 0 & 0 & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & 1 \end{pmatrix}$$ equivalent to the integer 1, always a square matrix

3.3 Transpose of a Matrix

If $A = (a_{ij})$ is a $m \times n$ matrix, then its transpose $A^T = (a_{ji})$ is a $n \times m$ matrix. The elements of the matrix are mirrored along the diagonal

Transpose rules:

- $(A + B)^T = A^T + B^T$
- $(cA)^T = cA^T$
- $(AB)^T = B^T A^T$

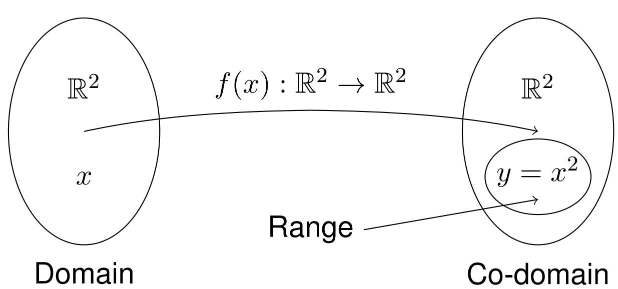
3.4 Square Matrices

A square matrix has the same number of rows as columns

A square matrix $A = (a_{ij})$ is:

- Diagonal if $a_{ij} = 0$ whenever $i \neq j$
- Upper triangular if $a_{ij} = 0$ whenever $i > j$ (all elements 0 below the diagonal)
- Lower triangular if $a_{ij} = 0$ whenever $i < j$ (all elements 0 above the diagonal)
- Triangular if its either upper or lower triangular

3.5 Functions



- The domain is the set of values that can enter a function (span of input $\mathbb{R}^2$)
- The co-domain is the set of values that **can** exit a function (span of output $\mathbb{R}^2$)
- The range/image is the set of values that **do** exit a function (span of $y = x^2$)
- A function $f : R \to R$ is one-to-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all x_1, x_2 in x
- A function $f : R \to R$ is onto if $\mathrm{Range}(f)$ spans R

3.6 Linear Transformation

A function $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if:

- $T(v + w) = T(v) + T(w)$
- $T(cv) = cT(v)$, where c is a scalar
- $T(0) = 0$

Linear transformation can transform:

- Lines into lines
- Planes into lines
- Lines into points

Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(v) = Mv$

- b is in $\mathrm{Range}(T)$ if and only if the system $Mv = b$ has at least one solution
- b is in $\mathrm{Range}(T)$ if and only if it belongs to the span of the columns of M

3.7 Linear Transformation Matrix Notation

If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, then there exists an $n \times m$ matrix M such that

$$T((x_1, ..., x_m)) = \underset{n\times m}{M} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$$

If $e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), e_3 = (0, 0, 1, ..., 0), e_m = (0, 0, ..., 1)$, then we can describe the transformation of each unit vector that make up $\mathbb{R}^m$ as follows

$$T(v) = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) & ... & T(e_m) \\ \downarrow & \downarrow & \downarrow & & \downarrow \end{pmatrix}$$

3.8 Linear Transformation Composition

Given the linear transformations

$$\begin{aligned} &S : \mathbb{R}^m \to \mathbb{R}^k && Sv = Bv \\ &T : \mathbb{R}^k \to \mathbb{R}^n && Tw = Aw \\ &T(S(v)) = Aw = A(Bv) = ABv \end{aligned}$$

3.9 Null Space

Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(v) = Mv$, its null space $N(T)$ is the set of all vectors in $\mathbb{R}^m$ such that $T(v) = 0$

- w is in $N(T)$ if w is a solution to the homogeneous system $Mx = 0$
- If the null space contains non-zero vectors, then the transformation cannot be one-to-one and vice-versa

3.10 Linear Transformation Theorems

Given a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, $T(v) = Mv$, the collective following are either true or false

- The columns of M are independent
- The columns of M span $\mathbb{R}^n$
- $Mv = b$ has a unique solution for all b
- $Mv = 0$ has only the trivial solution
- T is onto
- T is one-to-one

Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(v) = Mv$, if $m < n$, then T cannot be onto

- T is onto if $\mathrm{Span}(c_1, c_2, ..., c_m) = \mathbb{R}^n$, where $c_1, c_2, ..., c_m$ are the columns of M
- If $m < n$, then T cannot span $\mathbb{R}^n$

Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$, $T(v) = Mv$, if $m > n$, then T cannot be one-to-one

- The columns of $\underset{n \times m}{M}$ will not be linearly dependent
- There will be multiple combinations for each b in $\mathbb{R}^n$

3.11 Inverse Functions

The inverse of a function $f : X \to Y$ is a function $f^{-1} : Y \to X$ that has the property $f^{-1}(y) = x$ when $f(x) = y$

- A function f has an inverse only if it is one-to-one and onto

3.12 Invertible Matrices

An $n \times n$ matrix A is invertible if there is a matrix B such that $AB = I_n$. B is the inverse of A and denoted as A^{-1}, where $AA^{-1} = A^{-1}A = I_n$

- Only square matrices are invertible
- The inverse of an inverted matrix is the matrix itself
- Zero matrices have no inverse
- An invertible matrix is called non-singular
- A non-invertible matrix is called singular

$$A = \left(\underbrace{\begin{matrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{matrix}}_{A} \middle| \underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3}\right) \xrightarrow{\text{reduced row echelon}} \left(\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{I_3} \middle| \underbrace{\begin{matrix} -\frac{9}{2} & 7 & -\frac{3}{2} \\ -2 & 4 & -1 \\ \frac{3}{2} & -2 & \frac{1}{2} \end{matrix}}_{A^{-1}}\right)$$

Assuming A and B are $n \times n$ invertible matrices

- $(A^{-1})^{-1} = A$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $AC = 0$ implies $C = 0$
- $AC = AD$ implies $C = D$, when A is invertible
- $(kA)^{-1} = \frac{1}{k}(A)^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$

3.13 Invertible Matrix Theorems

If A is an $n \times n$ matrix then the collective following are either true or false

- A is invertible
- $Ax = b$ has a unique solution for every b
- $Ax = 0$ has only the trivial solution
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- The rows of A are linearly independent
- The reduced echelon form of A is I_n
- The linear transformation $T(v) = Av$ is one-to-one
- The linear transformation $T(v) = Av$ is onto
- The linear transformation $T(v) = Av$ is invertible
- $\det(A) \neq 0$

4 Section Four

4.1 Subspaces

A subspace of $\mathbb{R}^n$ is a subset S of $\mathbb{R}^n$ that satisfies the following conditions

- $\vec{0} \in S$
- if $\vec{v}$ and $\vec{w} \in S$ then $\vec{v} + \vec{w} \in S$
- if $\vec{v} \in S$ then $k\vec{v} \in S$ for any scalar k

Subspaces associated with a $m \times n$ matrix A

- The null space of a matrix $N(A)$ (solution set of the homogeneous system $Ax = 0$ or $ax + by + cz = 0$) is a subspace of $\mathbb{R}^n$
- The range of a matrix $\mathrm{Range}(A)$ (span of the columns of A, denoted $\mathrm{col}(A)$) is a subspace of $\mathbb{R}^n$

4.2 Bases

A set $v_1, v_2, ..., v_n$ of vectors that is linearly independent and spans $\mathbb{R}^n$, is called a basis for $\mathbb{R}^n$

- $\mathbb{R}^n$ can have multiple bases
- Any vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors in the basis of $\mathbb{R}^n$
- All bases for $\mathbb{R}^n$ have n vectors. This is called the dimension of $\mathbb{R}^n$
- All bases for the subspace W of $\mathbb{R}^n$ have the same number of vectors. This is called the dimension of the subspace W

4.3 Finding Bases

$$A = \left(\begin{array}{cc|c} a_1 & b_1 & x_1 \\ a_2 & b_2 & x_2 \\ \downarrow & \downarrow & \downarrow \\ a_n & b_n & x_n \end{array}\right)$$

To find a basis for $\mathbb{R}^n$ containing $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$ choose any vector $(x_1, x_2, ..., x_n)$ where the columns of matrix A are linearly independent

$$A = \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 3 & b \\ 3 & 8 & c \end{array}\right) \longrightarrow \mathrm{rref}(A) = \left(\begin{array}{cc|c} 1 & 2 & a \\ 0 & 3 & b \\ 0 & 0 & c + 3a - \frac{14}{3}b \end{array}\right)$$

Any vector (a, b, c) where $c + 3a - \frac{14}{3}b \neq 0$ is a valid third basis

4.4 Nullity

Given a matrix A, the nullity of A is equal to the dimension of $N(A)$

- $\text{nullity}(A) = \dim N(A)$
- Equal to the number of independent vectors in the null space of A
- Equal to the number of free variables in A
- If the columns of A are linearly independent, then the nullity of A must be 0

4.5 Row and Column Space

Given a $n \times m$ matrix A

- The row space of A denoted $\text{row}(A)$ is the span of the rows of A, a subspace of $\mathbb{R}^m$
- The column space of A denoted $\text{col}(A)$ is the span of the columns of A, a subspace of $\mathbb{R}^n$
- If A is in echelon form, the non-zero rows are linearly independent
- If A is in echelon form, the columns containing pivots are linearly independent
- The dimension of the row space is always equivalent to the dimension of the column space
- $\text{rank}(A) = \dim \text{col}(A) = \dim \text{row}(A)$
- If A is a $n \times m$ matrix then $\text{rank}(A) + \text{nullity}(A) = m$
  - $\text{rank}(A) = \text{number of pivots}$
  - $\text{nullity}(A) = \text{number of free variables}$
  - $m = \text{number of columns}$
- If A and B are equivalent matrices
  - $\text{row}(A)$ will equal $\text{row}(B)$
  - $\text{col}(A)$ may not equal $\text{col}(B)$

4.6 Change of Basis

$$B_1 = (1,1), (1,-1)$$

$$[(a,b)]_{B_1} = (k_1, k_2) \;\rightarrow\; k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \;\rightarrow\; \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$U_{B_1}^{B_c} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- (a,b) is the vector represented in B_c (canonical base) space
- (k_1, k_2) is the vector represented in B_1 space

4.7 Change of Basis Matrix

Given bases $B_1 = v_1, v_2, ..., v_n$ and $B_2 = w_1, w_2, ..., w_n$ in $\mathbb{R}^n$, the matrix U has the property that for every v in $\mathbb{R}^n$, $U[v]_{B_1} = [v]_{B_2}$

$$\underbrace{\begin{pmatrix} [v_1]_{B_2} & [v_2]_{B_2} & \dots & [v_n]_{B_2} \\ \downarrow & \downarrow & & \downarrow \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} k_1 \\ k_2 \end{pmatrix}}_{B_1} = \underbrace{\begin{pmatrix} a \\ b \end{pmatrix}}_{B_2}$$

- $U_{B_1}^{B_2}$ is called the change of basis matrix from B_1 to B_2
- U^{-1} is the change of basis matrix from B_2 to B_1
- $(v_1\ v_2\ ...\ v_n)$ represents a change of base $U_{B_1}^{B_c}$
- $(w_1\ w_2\ ...\ w_n)$ represents a change of base $U_{B_2}^{B_c}$
- $U_{B_1}^{B_2} = U_{B_c}^{B_2} \times U_{B_1}^{B_c} = (U_{B_2}^{B_c})^{-1} \times U_{B_1}^{B_c}$

$$\begin{pmatrix} [v_1]_{B_2} & [v_2]_{B_2} & \dots & [v_n]_{B_2} \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix} = (w_1\ w_2\ ...\ w_n)^{-1}(v_1\ v_2\ ...\ v_n)$$

5 Section Five

5.1 Determinant

The determinant is a function that takes an $n \times n$ matrix and returns a real number

- Represents the factor by which a linear transformation scales a unit space (the space bound by the unit vectors)
- The determinant is zero if the matrix squeezes a space to a lower dimension

5.2 Calculating the Determinant

Determinant of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc$$

Determinant of an $n \times n$ matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1j} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2j} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & ... & a_{ij} & ... & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & ... & a_{nj} & ... & a_{nn} \end{pmatrix}$$

Expansion along the i^{th} row

- $\det(M) = (-1)^{1+i}\ a_{i1}\ \det(M_{i1}) + (-1)^{2+i}\ a_{i2}\ \det(M_{i2}) + ... + (-1)^{n+i}\ a_{in}\ \det(M_{in})$

Expansion along the j^{th} column

- $\det(M) = (-1)^{1+j}\ a_{1j}\ \det(M_{1j}) + (-1)^{2+j}\ a_{2j}\ \det(M_{2j}) + ... + (-1)^{n+j}\ a_{nj}\ \det(M_{nj})$

Given an expansion of $\det(M)$

- M_{ij} represents the $n-1 \times n-1$ matrix obtained by removing the i^{th} and j^{th} rows
- $C_{ij} = (-1)^{i+j}\ \det(M_{ij})$ is called the cofactor of a_{ij}
- $\det(M_{ij})$ is called the minor of a_{ij}

Determinant of a triangular matrix

$$M = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ 0 & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & ... & a_{nn} \end{pmatrix}$$

$$\det(M) = a_{11} \times a_{22} \times ... \times a_{nn}$$

5.3 Determinant Theorems

- $\det(AB) = \det(A) \times \det(B)$
- $\det(A^n) = \det(A)^n$
- If A is a square matrix then $\det(A^T) = \det(A)$
- If A is invertible then $\det(A^{-1}) = \frac{1}{\det(A)}$
- If A has no inverse (rows/columns are not linearly independent), then $\det(A) = 0$
- If B is obtained from A by interchanging any two rows or columns of A then $\det(A) = -\det(B)$
- If B is obtained from A by multiplying one row of A by a non-zero scalar c then $\det(A) = \frac{1}{c}\det(B)$
- If B is obtained from A by replacing row_i with $\mathrm{row}_i + c \times \mathrm{row}_j$ where $i \neq j$, then $\det(A) = \det(B)$

Given a $n \times n$ matrix A and $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(v) = Av$, the collective following are either true or false

- $\det(A) \neq 0$
- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- The columns of A form a basis for $\mathbb{R}^n$
- T is onto
- T is one-to-one
- A is invertible
- $N(T) = \ker(T) = 0$
- $\mathrm{col}(A) = \mathbb{R}^n$
- $\mathrm{row}(A) = \mathbb{R}^n$
- $\mathrm{rank}(A) = n$

6 Section Six

6.1 Eigenvectors

Let A be a $n \times n$ matrix. A non-zero vector u is an eigenvector for A if there is a scalar λ called eigenvalue for A such that $Au = \lambda u$

- All the eigenvectors associated with λ plus the zero vector form a subspace of $\mathbb{R}^n$ called the eigenspace of λ
- An eigenvector is a vector that remains on its own span after a linear transformation
- An eigenvalue is the factor by which the eigenvector is stretched after a linear transformation
- The eigenspace is the set of vectors that remain on their own span after a linear transformation

6.2 Calculating Eigenvectors

$$\begin{aligned}
Av &= \lambda v \\
Av - \lambda v &= 0 \\
(A - \lambda I_n)v &= 0 \ \leftarrow \text{compute } \lambda \text{ for which there exists vectors transformed onto the zero vector} \\
\det(A - \lambda I_n) &= 0 \ \leftarrow \text{determinant of zero indicates transformation is not one-to-one} \\
(A - \lambda I_n)v &= 0 \ \leftarrow \text{compute } v \text{ by solving the homogeneous system, using the value(s) of } \lambda
\end{aligned}$$

6.3 Eigenvector Theorems

$n \times n$ matrix A has no eigenvectors or eigenvalues if any of the following is true

- The columns of A span $\mathbb{R}^n$
- The columns of A are linearly independent
- The columns of A form a basis for $\mathbb{R}^n$
- T is onto
- T is one-to-one
- A is invertible
- $N(T) = \ker(T) = 0$
- $\text{col}(A) = \mathbb{R}^n$
- $\text{row}(A) = \mathbb{R}^n$
- $\text{rank}(A) = n$
- $\det(A) \neq 0$
- $\lambda \neq 0 \ \leftarrow$ if $\lambda = 0$, then A represents a transformation into a lower dimension

6.4 Algebraic Multiplicity

The algebraic multiplicity of an eigenvalue is the number of times it appears as a root of the characteristic polynomial. For example:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\det(A) = (1-\lambda)^2(2-\lambda)$$

$$\lambda = 1, 1, 2$$

$\lambda = 1$ has a multiplicity of 2; $\lambda = 2$ has a multiplicity of 1

- The dimension of an eigenspace for λ cannot exceed its algebraic multiplicity

6.5 Geometric Multiplicity

The geometric multiplicity of A for λ is equal to $\text{nullity}(N(A - \lambda I))$, or the dimension of its eigenspace for λ

- $\text{nullity}(A) = \text{number of free variables}$
- $\text{number of columns} = \text{number of pivots} - \text{nullity}(A)$
- $1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}$
- If sum of geometric multiplicities for all values $\lambda = n$, then A is diagonalizable

6.6 Diagonalization

A matrix A is diagonalizable if there is an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. Given matrix A has eigenvectors $b_1, b_2, ..., b_n$

$$P = \begin{pmatrix} b_1 & b_2 & ... & b_n \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}$$

$$P^{-1}AP = D$$

- P is a change of basis matrix $U_B^{B_c}$
- D will have the corresponding eigenvalues along the diagonal
- A in $\mathbb{R}^n$ is guaranteed to be diagonalizable if A has n distinct eigenvalues
  - Each eigenvalue corresponds to a linearly independent eigenvector
- A is diagonalizable if and only if A has eigenvectors that form a basis $B = b_1, b_2, ..., b_n$ for $\mathbb{R}^n$
  - P has $b_1, b_2, ..., b_n$ for columns and D has the corresponding eigenvalues along the diagonal

6.7 General Theorems

- Eigenvectors corresponding to different eigenvalues are linearly independent
- Eigenvalues of any multiplicity may correspond to one or more eigenvectors