MATH 327 Notes

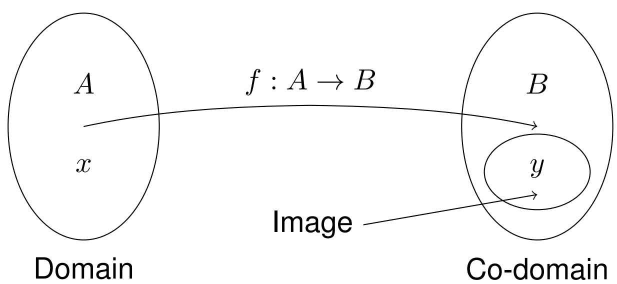
Contents

1 Section One

1.1 Sets

- A — universal set
- $a \in A$ — elements
- $E \subset A$ — subset
- E^c — set complement
- $E \backslash A$ — set difference
- $E_\lambda \subset A$ — indexed subset, for all λ in the indexing set Λ
- $\bigcap_{\lambda \in \Lambda} E_\lambda = \{a \in A \mid a \in E_\lambda \mid \forall \lambda \in \Lambda\}$ — intersection of indexed set
- $\bigcup_{\lambda \in \Lambda} E_\lambda = \{a \in A \mid a \in E_\lambda \mid \exists \lambda \in \Lambda\}$ — union of indexed set
- $A^k = A \times A \times ... \times A = \{(a_1, a_2, ..., a_k) \mid a_1, a_2, ..., a_k \in A\}$ — set power
- $A_1 \times A_2 \times ... \times A_k = \{(a_1, a_2, ..., a_k) \mid a_1 \in A_1, a_2 \in A_2, ..., a_k \in A_k\}$ — direct product

1.2 Functions



- $f(A) = \{f(a) \mid a \in A\} = \{b \in B \mid \exists a \in A \text{ such that } f(a) = b\}$

1.3 Injective Function

Given a function $f : A \to B$, f is injective if it is one-to-one, that is $\forall x_1, x_2 \in A,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

1.4 Surjective Function

Given a function $f : A \to B$, f is surjective if it is onto, that is $\forall y \in B,\ \exists x \in A,\ y = f(x)$

1.5 Bijective Function

Given a function $f : A \to B$, f is bijective if it is injective (one-to-one) and surjective (onto)

1.6 Identity Map

$I_A = Id_A$ is the identity map from $A \to A$

1.7 Composition

If $f : A \to B$ and $g : B \to C$, then the composition of $g \circ f = g(f) = A \to C$

1.8 Invertible Functions

$f : A \to B$ is invertible if and only if there exists $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$

- A function is invertible if and only if it is bijective

1.9 Partitions

A partition of A is a family of disjoint subsets of A such that their union is A

If E_λ is an indexed family of partitions of A

- $\bigcup\limits_{\lambda \in \Lambda} E_\lambda = A$
- $E_\lambda \cap E_\mu = \varnothing$ when $\lambda \neq \mu$

1.10 Relations

A relation R from set A to set B is a subset of $A \times B$

- If $(a, b) \in R$, where R is a relation from some set A to some set B, we can write $a \sim b$
- A relation R on a set A is a subset of $A \times A$

1.11 Equivalence Relations

An equivalence relation is a relation that has the properties

- Reflexive: $\forall a \in A,\ a \sim a$
- Symmetric: $\forall a, b \in A,\ a \sim b \Leftrightarrow b \sim a$
- Transitive: $\forall a, b, c \in A,\ (a \sim b \wedge b \sim C) \Rightarrow a \sim c$

1.12 Equivalence Classes

Given an equivalence relation on A and $a \in A$, the equivalence class of a is $[a] = \{b \in A \mid a \sim b\}$

- Equivalence classes in A form a partition of A
- There is a 1-to-1 correspondence between equivalence relations and partitions

1.13 Set Notations

- $\mathbb{Z}^+$: set of positive integers $1, 2, 3, ...$
- $\mathbb{Z}$: set of integers $0, 1, 2, -1, -2, ...$
- $\mathbb{N}$: set of natural numbers $0, 1, 2, ...$
- $\mathbb{Q}$: set of rational numbers $\frac{1}{2}, 1, 0, -\frac{3}{4}, \frac{m \in \mathbb{Z}}{n \neq 0 \in \mathbb{Z}}, ...$
- $\mathbb{R}$: set of real numbers $\pi, \sqrt{2}, e, 0, -1, 2, ...$

1.14 Rings

A ring is a nonempty set R that can undergo two operations, usually written as addition and multiplication

- Additive operations satisfy the following axioms
  1. Closed under addition: if $a \in R$ and $b \in R$, then $a + b \in R$
  2. Associative: $a + (b + c) = (a + b) + c$
  3. Commutative: $a + b = b + c$
  4. Additive identity: there is one 0_R in R such that $a + 0_R = a$ for all a
  5. Additive inverse: there is one x in R such that $a + x = 0_R$
- Multiplicative operations satisfy the following axioms
  6. Closed under multiplication: if $a \in R$ and $b \in R$, then $ab \in R$
  7. Associative: $a(bc) = (ab)c$
  8. Distributive: $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$

Multiplicative operations are not necessarily commutative, i.e. $ab \neq ba$

Multiplicative operations do not necessarily have a multiplicative identity, i.e. $a1_R = 1_R a = a$ for all a

1.15 Commutative Rings

A commutative ring is a ring R in which multiplication is commutative, i.e. $ab = ba$

1.16 Ring With Identity

A ring with identity is a ring R that contains one multiplicative identity, i.e. $a1_R = 1_R a = a$ for all a

1.17 Fields

A field is a commutative ring with identity where all non-zero elements are have multiplicative inverses

- i.e. $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}_p$
- All fields are integral domains

1.18 Order

An order on a set S is a less-than relation, denoted as $<$, that satisfies the following properties

- If $x, y \in S$, then precisely one of $x < y$, $x = y$, or $y < x$ is true
- If $x, y, z \in S$ and $x < y$ and $y < z$, then $x < z$

1.19 Ordered Set

An ordered set is a set S with an order on S

- $y > x \Leftrightarrow x < y$
- $x \leq y \Leftrightarrow x < y$ or $x = y$
- $x \geq y \Leftrightarrow x > y$ or $x = y$

$\mathbb{Z}, \mathbb{N}, \mathbb{Q}$ are examples of ordered sets

1.20 Ordered Field

An ordered field is a field F with an order that satisfies the following properties

- If $y < z$, then $x + y < x + z$
- If $x > 0$ and $y > 0$, then $xy > 0$
- Given $x \neq 0$, x is positive if and only if $x > 0$
- Given $x \neq 0$, x is negative if and only if $x < 0$

The following are true in every ordered field

- $x > 0$ if and only if $-x < 0$
- If $x > 0$ and $y < z$, then $xy < xz$
- If $x < 0$ and $y < z$, then $xy > xz$
- If $x \neq 0$, then $x^2 > 0$
- If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$

$\mathbb{Q}$ is an example of an ordered field

1.21 Supremum

If S is an ordered set, the subset E of S is bounded above if there exists an upper bound $\alpha \in S$ such that $x \leq \alpha$ for all elements $x \in E$. The element α is the least upper bound of E if

- α is an upper bound of E
- If $\beta \in S$ and $\beta < \alpha$, then there exists $x \in E$ such that $x > \beta$

The supremum is unique, denoted as $\sup(E)$

1.22 Supremum of Combined Sets

If A, B are bounded above then $A + B = \{a + b \mid a \in A, b \in B\}$

- $A + B$ is bounded above
- $\sup(A + B) = \sup(A) + \sup(B)$

1.23 Infimum

If S is an ordered set, the subset E of S is bounded below if there exists a lower bound $\alpha \in S$ such that $\alpha \leq x$ for all elements $x \in E$. The element α is the greatest lower bound of E if

- α is a lower bound of E
- If $\beta \in S$ and $\beta > \alpha$, then there exists $x \in E$ such that $x < \beta$

The infimum is unique, denoted as $\inf(E)$

- $\inf(E) = -\sup(-E)$

1.24 Set of Real Numbers $\mathbb{R}$

There exists a unique ordered field $\mathbb{R}$ such that

- $\mathbb{R}$ extends $\mathbb{Q}$ as an ordered field
- Any non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound in $\mathbb{R}$

1.25 Archimedean Properties of Real Numbers

- If $x, y \in \mathbb{R}$ and $x > 0$, then there exists $n \in \mathbb{N}$ such that $nx > y$
- If $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$, then there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \varepsilon$

1.26 Dense Subsets

A subset $E \subset \mathbb{R}$ is dense in $\mathbb{R}$ if for every $x, y \in \mathbb{R}$ with $x < y$, there exists $z \in E$ such that $x < z < y$

- i.e. $\mathbb{Q}$

1.27 Bounded Set

A set $E \subset S$ is bounded if it is bounded both above and below

1.28 Intervals in $\mathbb{R}$

A subset $I \subset \mathbb{R}$ is an interval if I has the property that if $x, y \in I$ and $x < z < y$, then $z \in I$

1.29 Bounded Intervals in $\mathbb{R}$

For $a, b \in \mathbb{R}$ and $a \leq b$

- Closed interval
  - $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$
- Open interval
  - $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$
- Half open / half closed
  - $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$
  - $(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$

If I is a bounded interval and $a = \inf(I)$ and $b = \sup(I)$, then $(a, b) \subset I \subset [a, b]$

1.30 Unbounded Intervals in $\mathbb{R}$

- $[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$
- $(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$
- $(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$
- $(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$
- $(-\infty, \infty) = \{x \in \mathbb{R}\}$

1.31 Powers

If $c > 0$ and $n \in \mathbb{N}$ then

- $c^n = c \times c \times ... \times c$
- $c^{-n} = \frac{1}{c^n}$
- $c^0 = 1$
- $c^{\frac{1}{n}} = x$ where x is the unique positive number such that $x^n = c$

Additional theorems

- If $n \in \mathbb{N}$ and $c > 0$ then there exists a unique $x > 0$ such that $x^n = c$
  - *More information on this theorem can be found in Rudin 1.21*
- $\{r \in \mathbb{Q} \mid r > 0, r^2 < 2\}$ does not have a least upper bound in $\mathbb{Q}$

1.32 Absolute Value

For $c \in \mathbb{R}$, define

$$|c| = \begin{cases} c & \text{if } c \geq 0 \\ -c & \text{if } c < 0 \end{cases}$$

For $d > 0$ we have $|c| \leq d$ if and only if $-d \leq c \leq d$

- If $x, y \in \mathbb{R}$, $|x - y|$ is the distance between x and y
- $E \in \mathbb{R}$ is bounded if and only if there exists K such that $|x| \leq K$ for all $x \in E$

1.33 Triangle Inequality

If $a, b \in \mathbb{R}$, then $|a + b| \leq |a| + |b|$

- In a triangle, the sum of lengths of two sides is greater than or equal to the length of the remaining side

2 Section Two

2.1 Sequences

A sequence in $\mathbb{R}$ is a map $f : \mathbb{N} \to \mathbb{R}$, denoted as (a_n) where $a_n = f(n)$ for $n \in \mathbb{N}$

2.2 Convergent Sequences

A sequence (a_n) converges to a point $a \in \mathbb{R}$ if given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$

- a is the limit of (a_n)
  - $a = \lim\limits_{n\to\infty} a_n$
  - $a_n \to a$ as $n \to \infty$
- Examples of convergent sequences
  - $a_n = \frac{1}{n}$
  - $a_n = (-1)^n \frac{1}{n^2}$

2.3 Divergent Sequences

A sequence (a_n) diverges if it is not convergent

- Examples of divergent sequences
  - $a_n = 2^n$
  - $a_n = (-1)^n$

2.4 Convergent and Divergent Theorems

- If (a_n) converges, then its limit is unique
- If (a_n) converges, then it is bounded
  - The set $\{a_n \mid n \in \mathbb{N}\}$ is bounded
- If $a_n \to a$, then $|a_n| \to |a|$
  - $||a_n| - |a|| \leq |a_n - a|$
- If $a_n \to a$ and $b_n \to b$, then
  - $a_n + b_n \to a + b$
  - $a_n b_n \to ab$
  - $\frac{1}{a_n} \to \frac{1}{a}$
  - $\frac{b_n}{a_n} \to \frac{b}{a}$ when $a \neq 0$
- If $a_n \to a$ and $p(x) = c_0 + c_1 x + ... + c_d x^d$ is a polynomial, then $p(a_n) \to p(a)$

2.5 Squeeze Theorem

If $a_n, b_n, c_n \in \mathbb{R}$ such that $a_n \leq b_n \leq c_n$ for all n and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = a$, then $\lim\limits_{n\to\infty} b_n = a$

2.6 Dense Subsets and Sequences

A set $S \subset \mathbb{R}$ is dense in $\mathbb{R}$ if and only if every $x \in \mathbb{R}$ is the limit of a sequence in S

- Every real number is the limit of a sequence of rational numbers

2.7 Standard Sequences

- For any $c > 0$, $c^{\frac{1}{n}} \to 1$ as $n \to \infty$
- $n^{\frac{1}{n}} \to 1$ as $n \to \infty$
- For $p > 0$, $\frac{1}{n^p} \to 0$ as $n \to \infty$
- If $|x| < 1$, then $x^n \to 0$ as $n \to \infty$

2.8 Monotone Sequences

A sequence (a_n) is monotonic if it is monotone increasing and monotone decreasing

A sequence (a_n) is monotone increasing if it is increasing such that $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$

A sequence (a_n) is monotone decreasing if it is decreasing such that $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$

- A sequence is monotone increasing and decreasing if and only if it is a constant sequence
- A monotone sequence converges if and only if it is bounded
- A bounded monotone increasing sequence converges to its supremum
- A bounded monotone decreasing sequence converges to its infimum

2.9 Subsequences

If $(a_n)_{n\in\mathbb{N}}$ is a sequence, and $n_1 < n_2 <$ ... is a strictly increasing sequence of natural numbers, then the sequence $(a_{n_k})_{k\in\mathbb{N}} = a_{n_1}, a_{n_2},$ ... is a subsequence of $(a_n)_{n\in\mathbb{N}}$

- If $a_n \to a$, then every subsequence of (a_n) converges to a
- A sequence is a subsequence of itself
- A subsequence must be infinite
- Every sequence has a monotone subsequence
- Every bounded sequence of real numbers has a convergent subsequence
  - Every bounded sequence of real numbers has a bounded monotone subsequence
  - Every bounded monotone subsequence is convergent

2.10 Sequentially Compact

A subset $S \subset \mathbb{R}$ is sequentially compact if every sequence $s_n \in S$ has a subsequence that converges to a point in S

- $[a, b]$ is sequentially compact

2.11 Cauchy Sequences

A sequence (a_n) of real numbers is Cauchy if for every ε there exists $N \in \mathbb{N}$ such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$

- Every convergent sequence is a Cauchy sequence
- Every Cauchy sequence is bounded
- A sequence of real numbers converges in $\mathbb{R}$ if and only if it is Cauchy

2.12 Completeness

A set $S \subset \mathbb{R}$ is complete if every Cauchy sequence in S converges to a point in S

- Completeness of $\mathbb{R}$ is equivalent to the least upper bound property of $\mathbb{R}$
- $\mathbb{R}$ is complete since every Cauchy sequence in $\mathbb{R}$ converges to a point in $\mathbb{R}$
- $\mathbb{Q}$ is incomplete since there are Cauchy sequences in $\mathbb{Q}$ that converge to a point in $\mathbb{R}$
- $[a, b]$ is complete
- $[a, b)$ is incomplete
- $[a, \infty)$ is complete

2.13 Limit Inferior

Let (x_n) be a bounded sequence in $\mathbb{R}$. For $n \in \mathbb{N}$, let

$$\ell_n = \inf_{i \geq n} x_i = \inf\{x_i \mid i \geq n\}$$

$$\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \ell_n = \sup \ell_n$$

- Let $S_n = \{x_i \mid i \geq n\}$ and $S_{n+1} = \{x_i \mid i \geq n+1\}$. Since $S_{n+1} \subset S_n$, the infimum of S_n might not be in S_{n+1} such that $\ell_n \leq \ell_{n+1}$
- ℓ_n is monotone increasing

2.14 Limit Superior

Let (x_n) be a bounded sequence in $\mathbb{R}$. For $n \in \mathbb{N}$, let

$$u_n = \sup_{i \geq n} x_i = \sup\{x_i \mid i \geq n\}$$

$$\lim_{n\to\infty} \sup x_n = \lim_{n\to\infty} u_n = \inf u_n$$

- Let $S_n = \{x_i \mid i \geq n\}$ and $S_{n+1} = \{x_i \mid i \geq n+1\}$. Since $S_{n+1} \subset S_n$, the supremum of S_n might not be in S_{n+1} such that $u_n \geq u_{n+1}$
- u_n is monotone decreasing

2.15 Convergence and Limit Superior/Inferior

- $\lim_{n\to\infty} \inf x_n \leq \lim_{n\to\infty} \sup x_n$
- $\lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \sup x_n$ if and only if (x_n) converges
  - If (x_n) converges, then $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \inf x_n = \lim_{n\to\infty} \sup x_n$

2.16 Conventions for $+\infty$

For a subset $E \subset \mathbb{R}$, $\sup E = \infty$ if and only if E is not bounded above

For a sequence $x_n \in \mathbb{R}$, $\lim_{n\to\infty} x_n = \infty$ if and only if for all $K \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $x_n \geq K$ for all $n \geq N$

- (x_n) is said to diverge to ∞
- Given increasing sequence (x_n), either (x_n) is convergent or $\lim_{n\to\infty} (x_n) = \infty$

2.17 Conventions for $-\infty$

For a subset $E \subset \mathbb{R}$, $\inf E = -\infty$ if and only if E is not bounded below

For a sequence $x_n \in \mathbb{R}$, $\lim_{n\to\infty} x_n = -\infty$ if and only if for all $K \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $x_n \leq K$ for all $n \geq N$

- (x_n) is said to diverge to $-\infty$
- Given decreasing sequence (x_n), either (x_n) is convergent or $\lim_{n\to\infty} (x_n) = -\infty$

2.18 Infinity and Limit Superior/Inferior

- $\lim_{n\to\infty} \sup x_n = \infty$ if and only if (x_n) is not bounded above
- $\lim_{n\to\infty} \inf x_n = \infty$ if and only if $\lim_{n\to\infty} x_n = \infty$

3 Section Three

3.1 Continuous Functions

A function $f : D \to \mathbb{R}$ is continuous at a point $x_0 \in D$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in D$ where $|x - x_0| < \delta$

3.2 Continuous Functions and Convergence

A function $f : D \to \mathbb{R}$ is continuous at a point $x_0 \in D$ if and only if $f(x_n) \to f(x_0)$ for all sequences $x_n \in D$ where $x_n \to x_0$

3.3 Function Arithmetic

Given functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$

- The sum $f + g : D \to \mathbb{R}$ is defined by $(f + g)(x) = f(x) + g(x)$ for all $x \in D$
- The product $fg : D \to \mathbb{R}$ is defined by $(fg)(x) = f(x)g(x)$ for all $x \in D$
- The quotient $\frac{f}{g} : D \to \mathbb{R}$ is defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for all $x \in D$, given that $g(x) \neq 0$ for all $x \in D$

3.4 Continuity of Function Arithmetic

Given functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ continuous at the point $x_0 \in D$

- $f + g$ is continuous at x_0
- fg is continuous at x_0
- $\frac{f}{g}$ is continuous at x_0 when $g(x) \neq 0$ for all $x \in D$

3.5 Continuity of Polynomial Functions

Every polynomial is continuous. Given polynomials p and q

- $p + q$ is continuous on the domain
- pq is continuous on the domain
- $\frac{p}{q}$ is continuous on D, where $D = \{x \in \mathbb{R} \mid g(x) \neq 0\}$

3.6 Continuity of Composition of Functions

Let $f : D \to \mathbb{R}$ and $g : U \to \mathbb{R}$ with $f(D) \subset U$. If f is continuous at the point $x_0 \in D$ and g is continuous at the point $f(x_0) \in U$, then $g \circ f : D \to \mathbb{R}$ is continuous at x_0

3.7 Extreme Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists $\alpha, \beta \in [a, b]$ such that $f(\alpha) = \sup(f([a, b]))$ and $f(\beta) = \inf(f([a, b]))$

3.8 Intermediate Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and c lies between $f(a)$ and $f(b)$, then there exists $\alpha \in (a, b)$ such that $f(\alpha) = c$

Alternately, if I is an interval and $f : I \to \mathbb{R}$ is continuous, then $f(I)$ is an interval

3.9 Uniform Continuity

$f : D \to \mathbb{R}$ is uniformly continuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(\tilde{x})| < \varepsilon$ whenever $x, \tilde{x} \in D$ and $|x - \tilde{x}| < \delta$

- If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous

3.10 Unifying Theorem

For a function $f : D \to \mathbb{R}$, the following are equivalent

- f is uniformly continuous
- For all sequences $u_n, v_n \in D$, if $u_n - v_n \to 0$, then $f(u_n) - f(v_n) \to 0$

4 Section Four

4.1 Series

A series is an infinite sum of the terms of a sequence $S = \sum_{n=1}^{\infty} a_n$

4.2 Series Theorems

Let S_n be the n^{th} partial sum $S_n = \sum_{i=1}^{n} a_i$

- If the sequence (S_n) converges, then S converges to $\lim_{n\to\infty} S_n$
- If S converges, then a_i converges to 0
- S converges if and only if (S_n) is a Cauchy sequence

4.3 Series Arithmetic

Given the series $A = \sum_{n=1}^{\infty} a_n$ and $B = \sum_{n=1}^{\infty} b_n$

- $A + B = \sum_{n=1}^{\infty} (a_n + b_n)$
- $cA = \sum_{n=1}^{\infty} (ca_n)$

4.4 Standard Series

For $x \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ if $|x| < 1$ and diverges to infinity if $|x| \geq 1$

4.5 Non-Negative Series

A series $S = \sum_{n=1}^{\infty} a_n$ is non-negative if $a_n \geq 0$ for all $n \in \mathbb{R}$

- If S is a non-negative series, then (S_n) is monotonically increasing
- S converges if and only if (S_n) is bounded above

4.6 Absolute Series

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges

4.7 Absolutely Convergent Series

A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n|$ is convergent

- If a series is absolutely convergent, then it is convergent
- All rearrangements of an absolutely convergent series converges to the same sum

4.8 Conditionally Convergent Series

A series is conditionally convergent if it converges but not absolutely

- If a series converges conditionally, then the positive terms converge to ∞ and the negative terms converge to $-\infty$
- A conditionally convergent series can be rearranged to converge to any value in $[-\infty, \infty]$
- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is conditionally convergent

4.9 Rearrangement of a Series

A rearrangement of a series $\displaystyle\sum_{n=1}^{\infty} a_n$ is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$

$$\sum_{n=1}^{\infty} a_{\sigma(n)} = a_{\sigma(1)} + a_{\sigma(2)} + ...$$

4.10 Partial Series Test

A series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges if and only if $\displaystyle\sum_{n=k}^{\infty} a_n$ converges, where $k \geq 1$

4.11 Direct Comparison Test

If $0 \leq a_n \leq b_n$ for all $n \in \mathbb{R}$, then

- If $\displaystyle\sum_{n=1}^{\infty} b_n$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_n$ converges
- If $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges, then $\displaystyle\sum_{n=1}^{\infty} b_n$ diverges

4.12 Limit Comparison Test

Let $\rho = \displaystyle\lim_{n\to\infty} \frac{a_n}{b_n}$ and $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$

- If ρ is finite and positive, then $\displaystyle\sum_{n=1}^{\infty} a_k$ converges if and only if $\displaystyle\sum_{n=1}^{\infty} b_k$ converges
- If $\rho = 0$ and $\displaystyle\sum_{n=1}^{\infty} b_k$ converges, then $\displaystyle\sum_{n=1}^{\infty} a_k$ converges
- If $\rho = \infty$ and $\displaystyle\sum_{n=1}^{\infty} b_k$ diverges, then $\displaystyle\sum_{n=1}^{\infty} a_k$ diverges

4.13 p-Test

- If $p > 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges
- If $p \leq 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges

4.14 Root Test

Let $r = \lim_{n\to\infty} \sup |a_n|^{\frac{1}{n}}$

- If $r < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely
- If $r > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges
- If $r = 1$, then the root test is inconclusive

4.15 Ratio Test

- If $\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely
- If $\lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges

4.16 Cauchy Condensation Test

If (a_n) decreases to 0, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges

4.17 Alternating Series Test

If (a_n) decreases to 0, then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges

- Even partial sums S_{2n} increase and converge to $\sum_{n=1}^{\infty} -a_n$
- Odd partial sums S_{2n+1} decrease and converge to $\sum_{n=1}^{\infty} a_n$

5 Appendix

5.1 Rings

Ring	Properties			
$\mathbb{R}$	infinite	commutative	has identity	field
$\mathbb{Q}$	infinite	commutative	has identity	field
$\mathbb{E}$	infinite	commutative	no identity	not field
$\mathbb{Z}$	infinite	commutative	has identity	not field
$\mathbb{Z}_n$	finite	commutative	has identity	not field
$\mathbb{Z}_p$	finite	commutative	has identity	field
$\mathbb{C}$	infinite	commutative	has identity	field
$\mathbb{Q}(\sqrt{2})$	infinite	commutative	has identity	field
$\mathbb{Z}_3[i]$	finite	commutative	has identity	field
$M_2(\mathbb{Z})$	infinite	not commutative	has identity	not field
$M_2(\mathbb{E})$	infinite	not commutative	no identity	not field
$M_2(\mathbb{Z}_n)$	finite	not commutative	has identity	not field
$M_2(\mathbb{Z}_p)$	finite	not commutative	has identity	not field
$\left\{\begin{pmatrix} 0 & r \\ 0 & 0 \end{pmatrix} \mid r \in \mathbb{Z}_n\right\}$	finite	commutative	no identity	not field

$\mathbb{E}$ denotes the ring containing all integers divisible by 2, i.e. $-2, 0, 4$