ELEC 441: Control Systems

Lecture 13: Realization

Dr. Abdullah Al-Digs
Department of Electrical and Computer Engineering
The University of British Columbia

February 27, 2025

Course Roadmap

Topics	**CT**	**DT**
Modeling	✓	✓
Stability	✓	✓
Controllability/Observability	✓	✓
Realization	$\rightarrow \bullet \leftarrow$	$\rightarrow \bullet \leftarrow$
State Feedback/Observers		
LQR/Kalman Filter		

Motivation

- We now know two ways to describe LTI systems
  1. State-space models

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

  2. Transfer functions

$$Y(s) = G(s)U(s)$$

- To use analysis and design techniques from this course, we need to transform transfer functions to an equivalent state-space model

Motivation

- We now know two ways to describe LTI systems
  1. State-space models

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

  2. Transfer functions

$$Y(s) = G(s)U(s)$$

- To use analysis and design techniques from this course, we need to transform transfer functions to an equivalent state-space model

- Key Question: how can we convert transfer functions to state-space models?

- More generally, what is the relationship between the two models?

Transfer Functions from State-space Models

- Given a CT state-space model,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

we can get a transfer function as

$$\boxed{G(s) := \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

- Given a DT state-space model,

$$\begin{cases} x[k+1] = Ax[k] + Bu[k] \\ y[k] = Cx[k] + Du[k] \end{cases}$$

we can get a transfer function as

$$\boxed{G(z) := \frac{Y(z)}{U(z)} = C(zI - A)^{-1}B + D}$$

Realization

- Realization: conversion of transfer functions to state-space models

Realization

- Realization: conversion of transfer functions to state-space models
- Given a rational and proper transfer matrix $G(s)$, we can find the corresponding state-space matrices (A, B, C, D) such that

$$G(s) = C(sI - A)^{-1}B + D$$

- Rationality: polynomial over polynomial

$$\frac{s+1}{s+2} \Rightarrow \text{Rational}, \quad e^{-s} \Rightarrow \text{Not rational}$$

- Properness: degree of numerator $\leq$ degree of denominator

$$\frac{s+1}{s+2} \Rightarrow \text{Proper}, \quad \frac{s^2+2s+1}{s+1} \Rightarrow \text{Not proper}$$

- Strictly Proper: degree of numerator $<$ degree of denominator

Important Remarks

- Conversion from state-space models to transfer function is unique
- Conversion from transfer functions to state-space models is not unique, there are in fact infinitely many equivalent state-space models (e.g. coordinate transformations)
- We can always verify our realization by computing the transfer function and checking if we can recover the original transfer matrix
- We can use `ss.m` in MATLAB for realization

Steps for Realization

1. Extract D-matrix (always do this first!)

$$G(s) = G(\infty) + G_{\mathrm{SP}}(s)$$

where $G(\infty) = D =$ constant and $G_{\mathrm{SP}}(s) = C(sI - A)^{-1}B$

- Example:

$$\begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \frac{s}{\frac{s}{2}+1} & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{G(\infty)} + \underbrace{\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+2} \\ \frac{-4}{s+2} & 0 \end{bmatrix}}_{G_{\mathrm{SP}}(s)}$$

2. Find A, B, and C such that

$$G_{\mathrm{SP}}(s) = G(s) - G(\infty) = C(sI - A)^{-1}B$$

Controllable Canonical Form (SISO Case)

- Consider a single-input-single-output (SISO) system with transfer function

$$G(s) = \frac{\beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$

- The corresponding state-space model in controllable canonical form is

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_3 & -\alpha_2 & -\alpha_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \beta_3 & \beta_2 & \beta_1 \end{bmatrix} x(t) \end{cases}$$

- Note: controllable canonical realization is always controllable

Alternate Controllable Canonical Form (SISO Case)

- Consider a single-input-single-output (SISO) system with transfer function

$$G(s) = \frac{\beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$

- Equivalently, we can reverse order of states to get

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 \end{bmatrix} x(t) \end{cases}$$

- Note: controllable canonical realization is always controllable

Companion Matrix

- The following matrix form (and its transpose) are referred to as companion matrices

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix} \in \mathbb{R}^{n \times n}$$

Companion Matrix

- The following matrix form (and its transpose) are referred to as companion matrices

$$A := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 1 \\ -\alpha_n & -\alpha_{n-1} & \cdots & -\alpha_2 & -\alpha_1 \end{bmatrix} \in \mathbb{R}^{n\times n}$$

- Important property of companion matrices is that

$$\det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_{n-1}s + \alpha_n$$

which looks very similar to the transfer functions we looked at so far

Controllable Canonical Form (MIMO Case)

- Consider a multiple-input-multiple-output (MIMO) system with transfer function

$$G(s) = \underbrace{\frac{\beta_1 s^{r-1} + \beta_2 s^{r-2} + \cdots + \beta_{r-1} s + \beta_r}{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1} s + \alpha_r}}_{\text{least common denominator}}, \ \beta_i \in \mathbb{R}^{q \times p}$$

- The corresponding state-space model in controllable canonical form is

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} \mathbb{0} & I_p & \mathbb{0} & \cdots & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & I_p & \cdots & \mathbb{0} \\ \vdots & \vdots & \vdots & & \vdots \\ \mathbb{0} & \mathbb{0} & \mathbb{0} & \cdots & I_p \\ -\alpha_r I_p & -\alpha_{r-1} I_p & \cdots & -\alpha_2 I_p & -\alpha_1 I_p \end{bmatrix} x(t) + \begin{bmatrix} \mathbb{0} \\ \mathbb{0} \\ \vdots \\ \mathbb{0} \\ I_p \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \beta_r & \beta_{r-1} & \cdots & \beta_2 & \beta_1 \end{bmatrix} x(t) \end{cases}$$

Example (SISO Case)

- Example: $G(s) = \dfrac{5s^2 + 3s + 2}{s^3 + 11s^2 + 14s + 6}$

1. $G(\infty) = \lim_{s\to\infty} G(s) = 0 \implies D = 0$
2. $G_{\text{SP}}(s) = G(s) - G(\infty) = G(s)$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -14 & -11 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 3 & 5 \end{bmatrix}$$

Example (SISO Case)

- Example: $G(s) = \dfrac{4s^2 + 3s + 2}{2s^2 + 5s + 6}$

1. $G(\infty) = \lim_{s\to\infty} G(s) = \dfrac{4}{2} \implies D = 2$

2. $$G_{\mathrm{SP}}(s) = G(s) - G(\infty) = \frac{4s^2 + 3s + 2}{2s^2 + 5s + 6} - \frac{4s^2 + 10s + 12}{2s^2 + 5s + 6}$$
$$= \frac{-3.5s - 5}{s^2 + 2.5s + 3}$$

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} -5 & -3.5 \end{bmatrix}$$

Example (MIMO Case)

- Example: $G(s) = \begin{bmatrix} \dfrac{1}{s^2+4s+3} & \dfrac{1}{s+3} \end{bmatrix}$

$$\begin{aligned} G(s) &= \frac{1}{s^2+4s+3}\begin{bmatrix} 1 & s+1 \end{bmatrix} \\ &= \frac{1}{s^2+4s+3}\left\{ \begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix} \right\} \end{aligned}$$

1. $G(\infty) = \lim_{s\to\infty} G(s) = 0 \implies D = 0$

2. $G_{\mathrm{SP}}(s) = G(s) - G(\infty) = G(s)$

$$A = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{array}\right], \ B = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right], \ C = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \end{array}\right]$$

- Note: MIMO transfer function was realized with 4 states

Observable Canonical Form (SISO Case)

- Consider a single-input-single-output (SISO) system with transfer function

$$G(s) = \frac{\beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$

- The corresponding state-space model in observable canonical form is

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 0 & -\alpha_3 \\ 1 & 0 & -\alpha_2 \\ 0 & 1 & -\alpha_1 \end{bmatrix} x(t) + \begin{bmatrix} \beta_3 \\ \beta_2 \\ \beta_1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t) \end{cases}$$

- Note: observable canonical realization is always observable

Alternate Observable Canonical Form (SISO Case)

- Consider a single-input-single-output (SISO) system with transfer function

$$G(s) = \frac{\beta_1 s^2 + \beta_2 s + \beta_3}{s^3 + \alpha_1 s^2 + \alpha_2 s + \alpha_3}$$

- Equivalently, we can reverse order of states to get

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -\alpha_1 & 1 & 0 \\ -\alpha_2 & 0 & 1 \\ -\alpha_3 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) \end{cases}$$

- Note: observable canonical realization is always observable

Observable Canonical Form (MIMO Case)

- Consider a multiple-input-multiple-output (MIMO) system with transfer function

$$G(s) = \underbrace{\frac{\beta_1 s^{r-1} + \beta_2 s^{r-2} + \cdots + \beta_{r-1}s + \beta_r}{s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r}}_{\text{least common denominator}}, \ \beta_i \in \mathbb{R}^{q\times p}$$

- The corresponding state-space model in observable canonical form is

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} & -\alpha_r I_q \\ I_q & \mathbb{0} & \cdots & \mathbb{0} & -\alpha_{r-1} I_q \\ \mathbb{0} & \ddots & \ddots & \vdots & \vdots \\ \vdots & \mathbb{0} & I_q & \mathbb{0} & -\alpha_2 I_q \\ \mathbb{0} & \cdots & \mathbb{0} & I_q & -\alpha_1 I_q \end{bmatrix} x(t) + \begin{bmatrix} \beta_r \\ \beta_{r-1} \\ \vdots \\ \beta_2 \\ \beta_1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \mathbb{0} & \mathbb{0} & \cdots & \mathbb{0} & I_q \end{bmatrix} x(t) \end{cases}$$

Duality Between Controllable and Observable Canonical Forms

- There is a duality between CCF and OCF as follows

$$A_{\mathrm{CCF}} = (A_{\mathrm{OCF}})^{\mathrm{T}}, \quad B_{\mathrm{CCF}} = (C_{\mathrm{OCF}})^{\mathrm{T}},$$
$$C_{\mathrm{CCF}} = (B_{\mathrm{OCF}})^{\mathrm{T}}, \quad D_{\mathrm{CCF}} = D_{\mathrm{OCF}}$$

- The D-matrix is the same in both because it is unique and independent of the various state selections that we can choose from

- Both CCF and OCF are in companion form and its transpose

Example (SISO Case)

- Example: $G(s) = \dfrac{5s^2 + 3s + 2}{s^3 + 11s^2 + 14s + 6}$

1. $G(\infty) = \lim_{s\to\infty} G(s) = 0 \implies D = 0$
2. $G_{\text{SP}}(s) = G(s) - G(\infty) =$

$$A = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -14 \\ 0 & 1 & -11 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

Example (SISO Case)

- Example: $G(s) = \dfrac{4s^2 + 3s + 2}{2s^2 + 5s + 6}$

1. $G(\infty) = \lim_{s\to\infty} G(s) = \dfrac{4}{2} \implies D = 2$

2. $G_{\mathrm{SP}}(s) = G(s) - G(\infty) = \dfrac{4s^2 + 3s + 2}{2s^2 + 5s + 6} - \dfrac{4s^2 + 10s + 12}{2s^2 + 5s + 6} = \dfrac{-3.5s - 5}{s^2 + 2.5s + 3}$

$$A = \begin{bmatrix} 0 & -3 \\ 1 & -2.5 \end{bmatrix}, \quad B = \begin{bmatrix} -5 \\ -3.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Example (MIMO Case)

- Example: $G(s) = \begin{bmatrix} \dfrac{1}{s^2+4s+3} & \dfrac{1}{s+3} \end{bmatrix}$

$$G(s) = \frac{1}{s^2+4s+3}\begin{bmatrix} 1 & s+1 \end{bmatrix}$$
$$= \frac{1}{s^2+4s+3}\left\{\begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix}\right\}$$

1. $G(\infty) = \lim_{s\to\infty} G(s) = 0 \implies D = 0$
2. $G_{\mathrm{SP}}(s) = G(s) - G(\infty) = G(s)$

$$A = \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right],\ B = \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right],\ C = \left[\begin{array}{c|c} 0 & 1 \end{array}\right]$$

- Note: MIMO transfer function was realized with 2 states only

Example (MIMO Case)

- Example: $G(s) = \begin{bmatrix} \dfrac{1}{s^2+4s+3} & \dfrac{1}{s+3} \end{bmatrix}$

$$\begin{aligned} G(s) &= \frac{1}{s^2+4s+3}\begin{bmatrix} 1 & s+1 \end{bmatrix} \\ &= \frac{1}{s^2+4s+3}\left\{ \begin{bmatrix} 0 & 1 \end{bmatrix} s + \begin{bmatrix} 1 & 1 \end{bmatrix} \right\} \end{aligned}$$

1. $G(\infty) = \lim_{s\to\infty} G(s) = 0 \implies D = 0$
2. $G_{\text{SP}}(s) = G(s) - G(\infty) = G(s)$

$$A = \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right],\ B = \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right],\ C = \left[\begin{array}{c|c} 0 & 1 \end{array}\right]$$

- Note: MIMO transfer function was realized with 2 states only
- Key Question: what is the smallest or minimal realization size?

Summary

- Realization
- Controllable canonical form
- Observable canonical from
- Next, minimal realization