ELEC 441: Control Systems

Lecture 14: Minimal Realization

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Course Roadmap

Topics	**CT**	**DT**
Modeling	✓	✓
Stability	✓	✓
Controllability/Observability	✓	✓
Realization	$\rightarrow \bullet \leftarrow$	$\rightarrow \bullet \leftarrow$
State Feedback/Observers		
LQR/Kalman Filter		

Motivation

- In the last lecture, we considered the realizations of the following transfer matrix

$$G(s)\left[\frac{1}{s^2+4s+3} \quad \frac{1}{s+3}\right] = \frac{1}{s^2+4s+3}\left\{\begin{bmatrix}0 & 1\end{bmatrix}s + \begin{bmatrix}1 & 1\end{bmatrix}\right\}$$

- Controllable Canonical Form Realization: (4 states)

$$\dot{x}(t) = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -3 & 0 & -4 & 0 \\ 0 & -3 & 0 & -4 \end{array}\right] x(t) + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u(t)$$

$$y(t) = \left[\begin{array}{cc|cc} 1 & 1 & 0 & 1 \end{array}\right] x(t)$$

- Observable Canonical Form Realization: (2 states)

$$\dot{x}(t) = \left[\begin{array}{c|c} 0 & -3 \\ \hline 1 & -4 \end{array}\right] x(t) + \left[\begin{array}{cc} 1 & 1 \\ \hline 0 & 1 \end{array}\right] u(t)$$

$$y(t) = \left[\begin{array}{c|c} 0 & 1 \end{array}\right] x(t)$$

Minimal Realization

- Minimal Realization of $G(s)$: realization (A,B,C,D) of $G(s)$ with the smallest dimension of A-matrix
  - How to characterize such a realization?
  - How to obtain such a realization?

Minimal Realization

- Minimal Realization of $G(s)$: realization (A,B,C,D) of $G(s)$ with the smallest dimension of A-matrix
  - How to characterize such a realization?
  - How to obtain such a realization?
- McMillan Degree of $G(s)$: the dimension of A of the minimal realization.
  - McMillan degree indicates the complexity of the system

Why Minimal Realization?

- Easier to analyze/understand the system
- Simpler designs for controllers
- Simpler implementation of controllers
- Reduced computational expense in both design and implementation
- Improved reliability
  - Fewer parts to malfunction in hardware
  - Fewer bugs to fix in software

Important Remarks on Minimal Realization

1. A realization (A,B,C,D) is minimal if and only if (A,B) is controllable and (A,C) is observable
2. All minimal realizations of $G(s)$ are related by coordinate transformations

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We can check if systems are minimal in two ways:

1. Check if controllable canonical form realization (always controllable) is observable
2. Check if observable canonical form realization (always observable) is controllable

Examples of Non-minimal Realizations

- Consider the following transfer function

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$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} x(t) \end{cases} \Rightarrow \operatorname{rank}(\mathcal{O}) = 2 < 3$$

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- Observable canonical form realization: (Uncontrollable!)

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x(t) \end{cases} \Rightarrow \operatorname{rank}(\mathcal{C}) = 2 < 3$$

Obtaining Minimal Realizations

- Single-Input Single-Output (SISO) Case: $G(s)$ is single fraction
  1. Remove any common factors from numerator and denominator of $G(s)$
  2. Realize $G(s)$ using controllable or observable canonical form
- Example: $G(s) = \dfrac{s^2 - 1}{s^3 + 1}$
  1. Lets factorize the transfer function as
  $$G(s) = \frac{s^2 - 1}{s^3 + 1} = \frac{(s+1)(s-1)}{(s+1)(s^2 - s + 1)} = \frac{s-1}{s^2 - s + 1}$$
  2. Realizing $G(s)$ using controllable canonical form, we get
  $$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} -1 & 1 \end{bmatrix} x(t) \end{cases}$$
- Since above is observable, system is minimal with McMillan degree 2

Obtaining Minimal Realizations

- Single-Input Multi-Output (SIMO) Case: $G(s)$ is a column vector
  1. Realize $G(s)$ using controllable canonical form

- Example: $G(s) = \dfrac{1}{s^2+4s+3}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
  1. Realizing $G(s)$ using controllable canonical form, we get

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(t) \end{cases}$$

- Since above is observable, system is minimal with McMillan degree 2

Obtaining Minimal Realizations

- Multi-Input Single-Output (MISO) Case: $G(s)$ is a row vector
  1. Realize $G(s)$ using observable canonical form
- Example: $G(s) = \dfrac{1}{s^2+4s+3}\left\{\begin{bmatrix}0 & 1\end{bmatrix}s + \begin{bmatrix}1 & 1\end{bmatrix}\right\}$
  1. Realizing $G(s)$ using observable canonical form, we get

$$\begin{cases}\dot{x}(t) = \begin{bmatrix}0 & -3\\ 1 & -4\end{bmatrix}x(t) + \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}u(t)\\ y(t) = \begin{bmatrix}0 & 1\end{bmatrix}x(t)\end{cases}$$

- Since above is controllable, system is minimal with McMillan degree 2

Obtaining Minimal Realizations

- Multi-Input Multi-Output (MIMO) Case:
  1. Realize $G(s)$ using controllable or observable canonical form
  2. Use Kalman decomposition to remove uncontrollable and unobservable parts

Obtaining Minimal Realizations

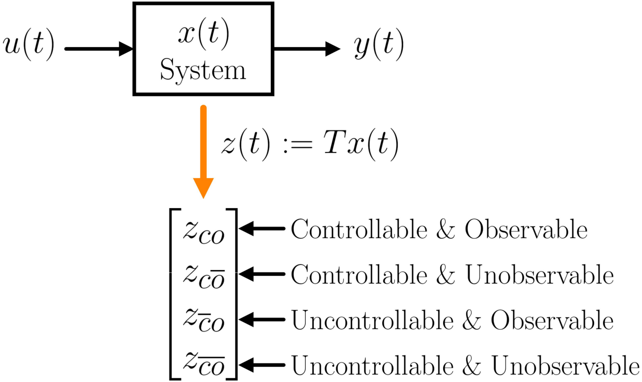
- Multi-Input Multi-Output (MIMO) Case:
  1. Realize $G(s)$ using controllable or observable canonical form
  2. Use Kalman decomposition to remove uncontrollable and unobservable parts
- Unfortunately, it is generally hard to compute Kalman decompositions by hand
- We can perform Kalman decompositions using `minreal.m` in MATLAB
- There is another famous algorithm, namely Ho's Algorithm, to compute a minimal realization (not covered in this course!)

Kalman Decomposition (Review)

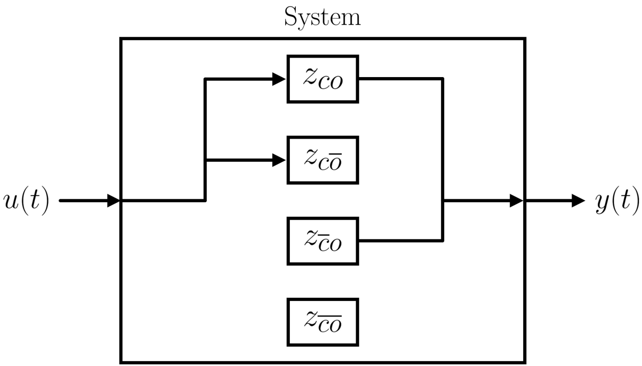
- Kalman Decomposition: is the combination of decompositions for controllability and observability

- Consider the following state-space model

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$



Kalman Decomposition Conceptual Diagram (Review)



- Note: this is not a block diagram!

Kalman Decomposition (Review)

- Every state-space model can be transformed using some T into

$$\begin{cases} \begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{c\bar{o}} \\ \dot{z}_{\bar{c}o} \\ \dot{z}_{\overline{co}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & \mathbb{0} & A_{13} & \mathbb{0} \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ \mathbb{0} & \mathbb{0} & A_{\bar{c}o} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & A_{43} & A_{\overline{co}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\overline{co}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ \mathbb{0} \\ \mathbb{0} \end{bmatrix}}_{TB} u(t) \\ y(t) = \underbrace{\begin{bmatrix} C_{co} & \mathbb{0} & C_{c\bar{o}} & \mathbb{0} \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\overline{co}} \end{bmatrix} + Du(t) \end{cases}$$

- (A_{co}, B_{co}) is controllable and (A_{co}, C_{co}) is observable
- System transfer function is determined by only the controllable and observable parts

$$\left(CT^{-1}\right)\left(sI - TAT^{-1}\right)^{-1}(TB) + D = C_{co}\left(sI - A_{co}\right)^{-1} B_{co} + D$$

Summary

- Minimal realization
- Realization of DT systems is the same as CT systems
  - Replace Laplace transforms with z-transforms
- Next design for control and estimation
  - State feedback (eventually linear-quadratic regulator (LQR))
  - Observers (eventually Kalman filter)