ELEC 441: Control Systems

Lecture 9: Lyapunov Theorem

Dr. Abdullah Al-Digs
Department of Electrical and Computer Engineering
The University of British Columbia

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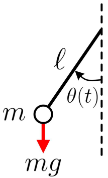
Course Roadmap

Topics	CT	DT
Modeling	✓	✓
Stability	$\rightarrow \bullet \leftarrow$	$\rightarrow \bullet \leftarrow$
Controllability/Observability		
Realization		
State Feedback/Observers		
LQR/Kalman Filter		

Review and Examples

- Eigenvalue criteria for internal stability:
  - CT system: asymptotically stable $\iff$ $\mathrm{Re}\{\lambda_i\} < 0,\ \forall i$
  - DT system: asymptotically stable $\iff$ $|\lambda_i| < 1,\ \forall i$

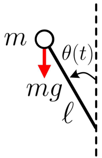
Example: Pendulum



$$\begin{bmatrix} \dot{\theta}(t) \\ \ddot{\theta}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{\ell} & 0 \end{bmatrix} \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$$

$\mathrm{eig}(A) = \pm \mathrm{j}\sqrt{\frac{g}{\ell}}$ (Marginally Stable)

Example: Inverted Pendulum



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$\mathrm{eig}(A) = \pm \sqrt{\frac{g}{\ell}}$ (Unstable)

Lyapunov Theorem

- Lyapunov theorem is another method for assessing the internal stability of state-space models (equivalent to Eigenvalue criterion)
- Lyapunov theorem topic outline:
  - Positive definiteness concept
  - Lyapunov theorem
  - Examples
  - Idea behind Lyapunov theorem

Positive Definiteness

- A square and symmetric matrix P is positive definite if

$$x^{\mathrm{T}}Px > 0,\ \forall x \in \mathbb{R}^n,\ x \neq 0$$

- Notation: $P > 0$ means P is positive definite
- Note that $x^{\mathrm{T}}Px$ yields a scalar. Hence, three possibilities:
  - $x^{\mathrm{T}}Px > 0 \rightarrow P$ is Positive Definite
  - $x^{\mathrm{T}}Px < 0 \rightarrow P$ is Negative Definite
  - $x^{\mathrm{T}}Px > 0$ and $x^{\mathrm{T}}Px < 0$ depending on $x \rightarrow P$ is Indefinite

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  - $x^{\mathrm{T}} P x > 0$ and $x^{\mathrm{T}} P x < 0$ depending on $x \rightarrow P$ is Indefinite
- Example: $P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Longrightarrow\ x^{\mathrm{T}} P x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2x_1^2 + x_2^2 > 0,\ \forall x \neq 0$$

Positive Definite Matrix Properties

1. Property 1: positive definite matrices have real & positive Eigenvalues

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2. Property 2: Sylvester's criterion (positive leading principal minor)

$$P > 0 \iff \det\left(P[1:i, 1:i]\right) > 0, \ \forall i$$

- Example: $P = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$$\implies \det(P[1,1]) = 2 > 0$$
$$\implies \det(P[1:2, 1:2]) = \det(P) = 1 > 0$$

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- Note: positive entries do NOT mean positive definite

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- Example: $P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \implies P \not> 0$
- Note: always check diagonal elements first
  - If there are negative diagonal entries, then P is NOT positive definite
  - If all diagonal entries are positive, then P may or may not be positive definite

Lyapunov's Theorem for CT Systems

- All the Eigenvalues of a matrix A lie in the open left half plane (i.e. $\mathrm{Re}\{\lambda_i\} < 0$, $\forall i$) if and only if the solution P to the following Lyapunov equation

$$A^{\mathrm{T}}P + PA = -Q$$

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- Note: for LTI state-space models, Lyapunov's theorem has no advantage over Eigenvalue criteria

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- We study the idea behind Lyapunov's theorem, which is useful for advanced control (nonlinear, time-varying, and robust control)

Examples $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

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$$\implies P = \begin{bmatrix} 4p_3 + 1 & 2p_3 \\ 2p_3 & p_3 \end{bmatrix} \implies \text{Not unique!}$$

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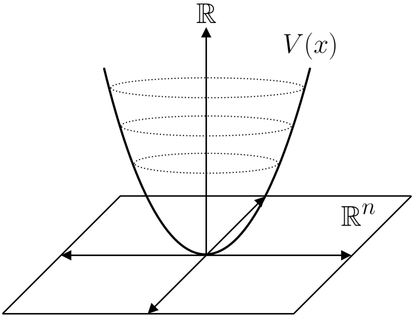
$$\implies P = \begin{bmatrix} 4p_3 + 1 & 2p_3 \\ 2p_3 & p_3 \end{bmatrix} \implies \text{Not asymptotically stable!}$$

Idea Behind Lyapunov's Theorem

1. For a fixed $P > 0$, define a Lyapunov function

$$V(x) := x^{\mathrm{T}} P x, \; x \in \mathbb{R}^n$$

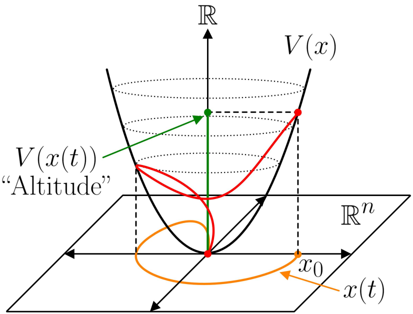


Idea Behind Lyapunov's Theorem

2. Consider the trajectory $V(x(t))$, where the behaviour of $x(t)$ is subject to

$$\dot{x}(t) = Ax(t)$$

We want to look at the altitude as $x(t)$ varies over time



Idea Behind Lyapunov's Theorem

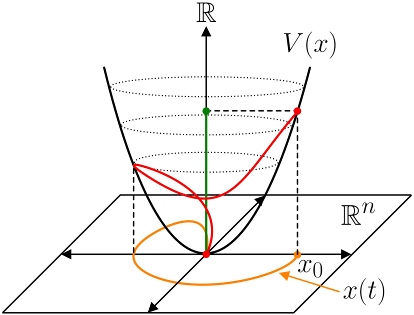
3. Take the derivative of $V(x(t))$ w.r.t. t, to get

$$\begin{aligned}\dot{V}(x(t)) &= \frac{d}{dt}\left(x(t)^{\mathrm{T}} P x(t)\right) \\ &= \dot{x}(t)^{\mathrm{T}} P x(t) \\ &\quad + x(t)^{\mathrm{T}} P \dot{x}(t)\end{aligned}$$

Recognizing that $\dot{x}(t) = Ax(t)$, we can rewrite the above as

$$\dot{V}(x(t)) = x(t)^{\mathrm{T}}\left(A^{\mathrm{T}} P + PA\right) x(t)$$

Note: for DT case, we take $V(x[k+1]) - V(x[k])$



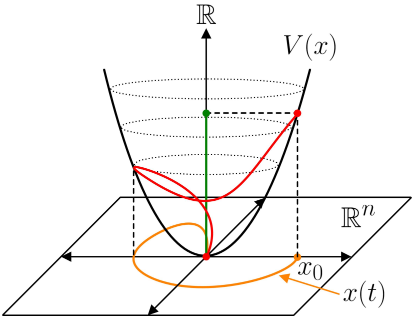
Idea Behind Lyapunov's Theorem

4. If $A^{\mathrm{T}}P + PA = -Q < 0$,
   then we have that

   $$\dot{V}(x(t)) = x(t)^{\mathrm{T}}(-Q)x(t) < 0,$$
   $$\forall x(t) \neq 0$$

   Meaning, $V(x(t))$ decreases as t increases

   Therefore,
   $x(t) \to 0$ as $t \to \infty$
   (i.e. asymptotically stable)



Lyapunov's Theorem Summary

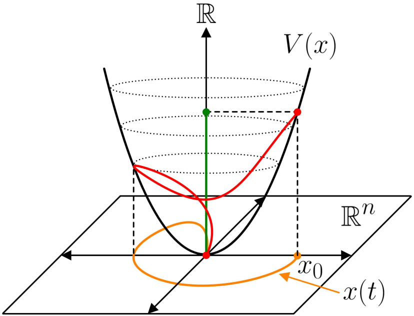
- Solving the Lyapunov equation w.r.t. P

$$A^{\mathrm{T}}P + PA = -Q$$

is equivalent to finding a Lyapunov function $V(x)$ such that

$$x(t) \to 0 \iff V(x(t)) \to 0$$

- Useful in advanced control!



Nonlinear System Example

- Nonlinear systems are difficult to solve analytically

$$\dot{x}_1(t) = x_2(t) - x_1(t)\left(x_1^2(t) + x_2^2(t)\right)$$
$$\dot{x}_2(t) = -x_1(t) - x_2(t)\left(x_1^2(t) + x_2^2(t)\right)$$

- Lyapunov function: $V(x(t)) := x_1^2(t) + x_2^2(t) > 0$, $\forall x \neq 0$

- Derivative of $V(x(t))$ w.r.t t, where $x(t)$ is the trajectory of the nonlinear system is

$$\begin{aligned}\dot{V}(x(t)) &= 2x_1(t)\dot{x}_1(t) + 2x_2(t)\dot{x}_2(t) \\ &= -2\left(x_1^2(t) + x_2^2(t)\right)^2 < 0,\ \forall x(t) \neq 0\end{aligned}$$

Summary

- Positive definiteness
- Lyapunov theorem
- Idea behind Lyapunov's theorem
- Next, controllability and observability