ELEC 441: Control Systems

Lecture 7: BIBO Stability

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Course Roadmap

Topics	CT	DT
Modeling	✓	✓
Stability	$\rightarrow \bullet \leftarrow$	$\rightarrow \bullet \leftarrow$
Controllability/Observability		
Realization		
State Feedback/Observers		
LQR/Kalman Filter		

Stability

- Stability is the utmost important specification in control design
  - After stability, we look at performance (tracking, disturbance and noise rejection, etc.)
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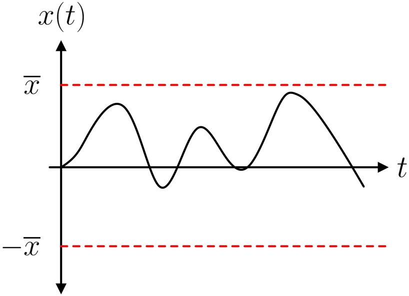
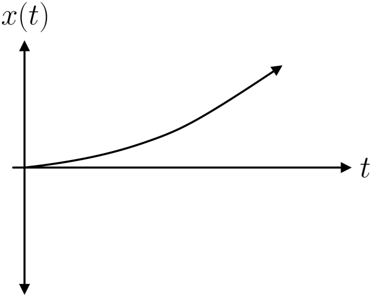
- In this course, we will discuss two types of stability:
  - Bounded-input bounded-output (BIBO) stability (today's lecture)
  - Internal stability (next lecture)

Bounded Signals

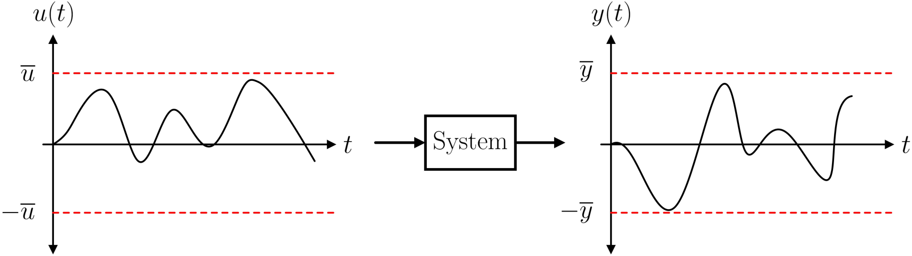
- Bounded Signal: a signal $x(t)$ is bounded if there exists a positive scalar $\overline{x}$ such that $|x(t)| \leq \overline{x} < \infty$, $\forall t \geq 0$
- A vector of signals $x(t)$ is bounded if every entry is bounded





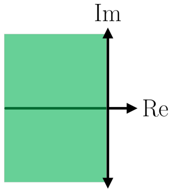
Bounded-Input Bounded-Output (BIBO) Stability

- BIBO Stability: a system is BIBO stable if every bounded input $u(t)$ excites a bounded output $y(t)$



BIBO Stability for CT LTI Systems

- A CT LTI system described by transfer function $G(s)$ is BIBO stable if and only if all the poles of $G(s)$ are in the open left half of the complex plane (OLHP)



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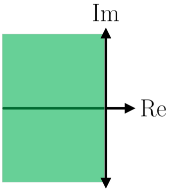

- Example:

$$G(s) = \frac{s-1}{(s+3)(s+1)} \rightarrow \text{Poles are } \begin{cases} s = -1 \\ s = -3 \end{cases}$$

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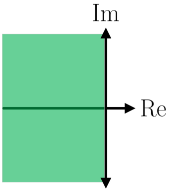

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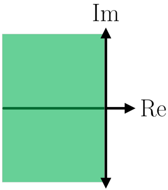

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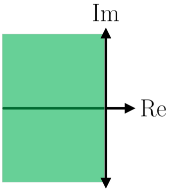

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$$G(s) = \frac{s+2}{(s+3)(s-1)} \rightarrow \text{Poles are } \begin{cases} s = 1 \\ s = -3 \end{cases} \rightarrow \text{Not BIBO stable!}$$

Transfer Functions (Review)

- Transfer functions are defined by

$$G(s) := \frac{Y(s)}{U(s)} = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}}$$

under the assumption of zero initial conditions

$$U(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s) = G(s)U(s)$$

Transfer Functions from CT State-space Models

- Recall that CT LTI state-space models are described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

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- Laplace transform with $x(0) = 0$

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$$\rightarrow \begin{cases} X(s) = (sI - A)^{-1}BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$\rightarrow \boxed{G(s) := \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D} \leftarrow \text{ Memorize this!}$$

Simple BIBO Stability Criteria

- First-order polynomial $F(s) = a_1 s + a_0$

  All roots are in OLHP $\iff$ a_1 and a_0 have the same sign

- Second-order polynomial $F(s) = a_2 s^2 + a_1 s + a_0$

  All roots are in OLHP $\iff$ a_2, a_1, and a_0 have the same sign

- Higher order polynomials $F(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$

  All roots are in OLHP $\implies$ all a_k $\forall$ k have the same sign

Examples

Denominator of $G(s)$	$G(s)$ BIBO Stable?
$3s + 5$	Yes / No
$-2s^2 - 5s - 100$	Yes / No
$523s^2 - 57s + 189$	Yes / No
$(s^2 + s - 1)(s^2 + s + 1)$	Yes / No
$s^3 + 5s^2 + 10s - 3$	Yes / No

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BIBO Stability for DT LTI Systems

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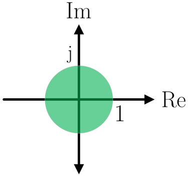

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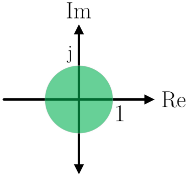

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$$G(z) = \frac{2}{z - 0.5} \rightarrow \text{Pole is } z = 0.5$$

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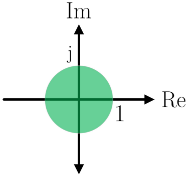

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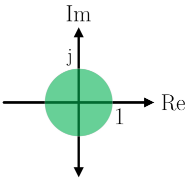

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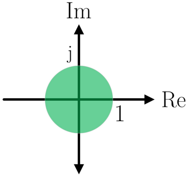

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Z-Transform (Review)

- For a sequence $\{f[k] : k = 0, 1, 2, ...\}$

$$F(z) = \mathcal{Z}\{f[k]\} := \sum_{k=0}^{\infty} f[k] z^{-k}$$

- Shift property of Z-transforms

$$\begin{aligned} \mathcal{Z}\{f[k+1]\} &= \sum_{k=0}^{\infty} f[k+1] z^{-k} \\ &= z \sum_{k=0}^{\infty} f[k+1] z^{-(k+1)} \\ &= z\left(F(z) - f[0]\right) \end{aligned}$$

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- Consider the following CT LTI system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} x(t) \end{cases}$$

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- System is BIBO stable after pole/zero cancellation $\leftarrow$ PROBLEM!
  This is problematic for non-zero initial conditions!

BIBO Stability Limitations

- Pole/zero cancellation in unstable region is DANGEROUS!
  - Plant model is NEVER exact! (cancellation does not occur in reality)
- Neither nonzero initial conditions nor input disturbances are considered (very fragile!)
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$\leftarrow$ Internal Stability!

Summary

- BIBO stability for CT and DT systems
  - Definition
  - Conditions and criteria
  - Examples
- Transformation from state-space models to transfer functions
- BIBO stability is not good enough for assessing stability of state-space models
- Next, internal stability