ELEC 441: Control Systems

Lecture 8: Internal Stability

Dr. Abdullah Al-Digs
Department of Electrical and Computer Engineering
The University of British Columbia

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Course Roadmap

Topics	CT	DT
Modeling	✓	✓
Stability	$\rightarrow \bullet \leftarrow$	$\rightarrow \bullet \leftarrow$
Controllability/Observability		
Realization		
State Feedback/Observers		
LQR/Kalman Filter		

Stability Recap

- Last lecture we discussed BIBO stability
- BIBO stability cannot determine stability of state-space models with non-zero initial conditions
- Today, we study internal stability for:
  - LTI CT Systems: $\dot{x}(t) = Ax(t)$, $x(0) = x_0$
  - LTI DT Systems: $x[k+1] = Ax[k]$, $x[0] = x_0$
  - Notice that there is no input in the above systems
- Internal stability is assessed in two ways:
  - Eigenvalue criteria (today's lecture)
  - Lyapunov theorem (next lecture)

BIBO Stability Example (Review)

- Consider the following CT LTI system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} x(t) \end{cases}$$

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- The transfer function for the above SS model is given by

$$\begin{aligned} G(s) &:= C(sI - A)^{-1}B + D \\ &= \begin{bmatrix} -2 & 3 \end{bmatrix} \left(sI - \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} \right)^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix} \\ &= \frac{4(s-1)}{(s+1)(s-1)} = \frac{4}{(s+1)} \rightarrow \text{Pole is } s = -1 \end{aligned}$$

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- System is BIBO stable after pole/zero cancellation $\leftarrow$ PROBLEM! This is problematic for non-zero initial conditions!

BIBO Stability Limitations (Review)

- Pole/zero cancellation in unstable region is DANGEROUS!
  - Plant model is NEVER exact! (cancellation does not occur in reality)
- Neither nonzero initial conditions nor input disturbances are considered (very fragile!)
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$\} \leftarrow$ Internal Stability!

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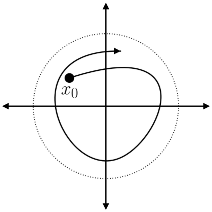
$$||x(t)|| \leq M < \infty, \; \forall t > 0$$

where $||x(t)||$ is the norm of the state vector



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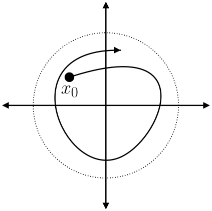

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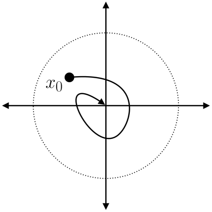

Eigenvalue Criteria for CT LTI Systems

Denote the i-th Eigenvalues of A matrix with λ_i

- Asymptotically Stable: $\mathrm{Re}\{\lambda_i\} < 0$, $\forall i$

- Marginally Stable: $\mathrm{Re}\{\lambda_i\} \leq 0$, $\forall i$
  - For $\mathrm{Re}\{\lambda_i\} = 0 \rightarrow \mathrm{rank}(\lambda_i I - A) = n - m_i$,
    where n is the number of states and m_i is the multiplicity of λ_i
  - Note: if $m_i = 1$, the rank condition always holds!

- Unstable: two possibilities:
  - $\mathrm{Re}\{\lambda_i\} > 0$ for some i
  - For $\mathrm{Re}\{\lambda_i\} = 0 \rightarrow \mathrm{rank}(\lambda_i I - A) \neq n - m_i$

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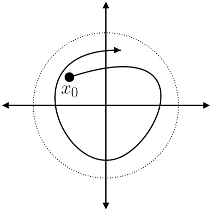

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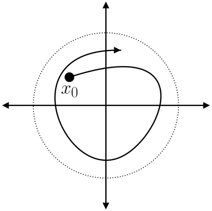

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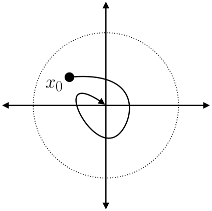

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Computing Matrix Eigenvalues

How do we compute the eigenvalues of a matrix?

- We can compute the eigenvalues of matrix A by solving

$$\det(\lambda I - A) = 0$$

- The above equation is referred to as the characteristic equation

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Eigenvalues for special matrices:

- Eigenvalues of diagonal matrices are the diagonal elements
- Eigenvalues of triangular matrices are the diagonal elements
- Eigenvalues of block diagonal matrices are the eigenvalues of the diagonal matrices

Idea Behind Internal Stability

Continuous-time (CT)

$$\begin{cases} \dot{x}(t) = Ax(t) \\ x(0) = x_0 \end{cases} \rightarrow x(t) = e^{At}x_0$$

- Example $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Example $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \rightarrow \begin{bmatrix} \infty & 0 \\ 0 & 0 \end{bmatrix}$$

Discrete-time (DT)

$$\begin{cases} x[k+1] = Ax[k] \\ x[0] = x_0 \end{cases} \rightarrow x[k] = A^k x_0$$

- Example $A = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.3 \end{bmatrix}$

$$A^k = \begin{bmatrix} 0.1^k & 0 \\ 0 & -0.3^k \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- Example $A = \begin{bmatrix} 1.1 & 0 \\ 0 & -0.3 \end{bmatrix}$

$$A^k = \begin{bmatrix} 1.1^k & 0 \\ 0 & -0.3^k \end{bmatrix} \rightarrow \begin{bmatrix} \infty & 0 \\ 0 & 0 \end{bmatrix}$$

Important Examples

- Consider the following system

$$\dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(t)$$

Eigenvalues: $\lambda_1 = \lambda_2 = 0$, $m = 2$
$\mathrm{rank}(\lambda I - A) = n - m = 0 \rightarrow$ Marginally Stable!

Important Examples

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Eigenvalues: $\lambda_1 = \lambda_2 = 0$, $m = 2$
$\underbrace{\mathrm{rank}(\lambda I - A)}_{=1} \neq \underbrace{n - m}_{=0} \rightarrow$ Unstable!

Remark on Stability of LTV Systems

- Consider the following system

$$\dot{x}(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} x(t),\ x(0) = x_0$$

- Eigenvalues of $A(t)$ matrix are $\lambda_1 = \lambda_2 = -1$, $\forall t$.
  However, this system is unstable!

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- Why? Solve the differential equations:

$$x(t) = \begin{bmatrix} e^{-t} & 0.5(e^{t} - e^{-t}) \\ 0 & e^{-t} \end{bmatrix} x_0$$

- For an initial state

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \implies x(t) = \begin{bmatrix} 0.5(e^{t} - e^{-t}) \\ e^{-t} \end{bmatrix} \rightarrow \begin{bmatrix} \infty \\ 0 \end{bmatrix}$$

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- Eigenvalue criteria does not work for LTV systems!

Summary

- Internal stability and definitions of:
  - Asymptotic stability
  - Marginal stability
  - Instability
- Internal stability for CT and DT systems:
  - Eigenvalue criteria (today's lecture)
  - Lyapunov theorem (next lecture)
- Next, Lyapunov theorem for internal stability

Appendix A: Linear Algebra Review

Linear Independence:

- Vectors $\{v_1, \cdots, v_n\}$ are linearly independent if:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 \implies \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

Matrix Rank:

- Definition: maximum number of linearly independent row (and column) vectors of a matrix

- Example:

$$\operatorname{rank}(A) = \operatorname{rank} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = 2$$

- Example:

$$\operatorname{rank}(A) = \operatorname{rank} \begin{bmatrix} 1 & -1 \\ -5 & 5 \end{bmatrix} = 1$$