ELEC 441: Control Systems

Lecture 5: Continuous-time State-space Models Solution

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Course Roadmap

Topics	**CT**	**DT**
Modeling	$\rightarrow \bullet \leftarrow$	
Stability		
Controllability/Observability		
Realization		
State Feedback/Observers		
LQR/Kalman Filter		

Objective and Motivation

- Objective: analytically solve CT state-space model equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t),\ x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

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- Why would we want to solve state-space models analytically?
  - Understand behaviour of state and output trajectories (without running simulations)
  - Derive theoretical analysis and design results from explicit solution
  - Useful in discretization (next lecture!)
  - Interpret (or debug) simulation results (e.g. MATLAB)

Analytical Solutions to CT State-space Models

- Continuous-time state-space models are described by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \ x(0) = x_0 \\ y(t) = Cx(t) + Du(t) \end{cases}$$

- Analytical solution to CT state-space models is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau,$$

$$y(t) = Ce^{At}x_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau + Du(t),$$

where the matrix exponential e^{At} is given by

$$e^{At} := I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

Verification of Analytical Solution

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$$x(0) = e^{A\cdot(0)}x_0 + \int_0^0 e^{A(0-\tau)}Bu(\tau)d\tau = x_0$$

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- State evolution verification: $\dot{x}(t) = Ax(t) + Bu(t)$

$$\begin{aligned}\dot{x}(t) &= Ae^{At}x_0 + \frac{d}{dt}\left(e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau\right)\\ &= Ae^{At}x_0 + Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau + e^{At}e^{-At}Bu(t)\\ &= Ax(t) + Bu(t)\end{aligned}$$

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- Linear time-varying analytical solution looks similar, but more complicated (Not covered in this course!)

Matrix Exponentials

- Definition: $e^{At} := I + At + \dfrac{(At)^2}{2!} + \dfrac{(At)^3}{3!} + \cdots$
- Property: $\dfrac{d}{dt}\left(e^{At}\right) = Ae^{At} = e^{At}A$

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- Matrix exponentials are computed analytically using:
  - Definition of matrix exponential
  - Laplace transform
  - Diagonal form or Jordan form $\leftarrow$ Not covered in this course!
  - Cayley-Hamilton theorem $\leftarrow$ Not covered in this course!

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- Matrix exponentials are computed numerically using:
  - `expm.m` in MATLAB

Definitions for Matrix Exponentials

- Matrix exponential definitions are used for special matrices
- Nilpotent matrices: $A^n = 0$ for some n

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow A^3 = A^4 = \cdots = 0$$

$$e^{At} := I + At + \frac{(At)^2}{2!} + \underbrace{\cdots}_{=0} = \begin{bmatrix} 1 & t & \frac{t^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

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- Diagonal matrices:

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \rightarrow A^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$$

$$e^{At} := I + At + \frac{(At)^2}{2!} + \cdots = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix}$$

Shift Matrices

- Consider the following A matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Every time we increase the exponent, we shift the off-diagonal of 1s

$$A^2 = \begin{bmatrix} 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ etc.}$$

Laplace Transform

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- Example: $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

$$(sI - A)^{-1} = \frac{1}{(s-\sigma)^2 + \omega^2}\begin{bmatrix} s-\sigma & \omega \\ -\omega & s-\sigma \end{bmatrix}$$

hence,

$$e^{At} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \boxed{e^{\sigma t}\begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}}$$

Laplace Transform Example $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$(sI - A)^{-1} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}^{-1} = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

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Using partial fraction expansion, we get

$$(sI - A)^{-1} = \frac{1}{s+1} K_1 + \frac{1}{s+2} K_2 \;\&\; \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = (s+2)K_1 + (s+1)K_2$$

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Then,

$$(sI - A)^{-1} = \frac{1}{s+1} \underbrace{\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}}_{K_1} + \frac{1}{s+2} \underbrace{\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}}_{K_2}$$

$$e^{At} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

$$= \boxed{e^{-t} \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + e^{-2t} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}}$$

Diagonal Form (Similarity Transform)

- Suppose that we have distinct Eigenvalues and corresponding Eigenvectors as

$$Ax_i = \lambda_i x_i,\ i = 1, \cdots, n$$

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- Suppose that we have distinct Eigenvalues and corresponding Eigenvectors as

$$Ax_i = \lambda_i x_i,\ i = 1, \cdots, n$$

- Then, the matrix exponential can be computed as

$$e^{At} = Te^{Dt}T^{-1},$$

where

$$T := \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \in \mathbb{C}^{n\times n}$$
$$D := \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \in \mathbb{C}^{n\times n}$$

- Note: diagonalization of A by nonsingular (i.e. invertible) matrix T is not always possible!

Diagonal Form Proof

- Suppose that we have Eigenvalues/Eigenvectors as

$$Ax_i = \lambda_i x_i,\ i = 1, \cdots, n$$

which can be written in matrix form as

$$A \underbrace{\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}}_{T} = \underbrace{\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}}_{D}$$

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- If T is invertible, we get that

$$A = TDT^{-1}, \quad A^2 = TD^2T^{-1}, \quad A^3 = TD^3T^{-1}, \text{ etc.}$$

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$$A = TDT^{-1}, \quad A^2 = TD^2T^{-1}, \quad A^3 = TD^3T^{-1}, \text{ etc.}$$

- More generally, we have

$$A^n = (TDT^{-1})(TDT^{-1}) \cdots (TDT^{-1})(TDT^{-1}) = TD^nT^{-1}$$

Diagonal Form Proof (Continued)

- Recall the matrix exponential definition, given by

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Diagonal Form Proof (Continued)

- Recall the matrix exponential definition, given by

$$e^{At} := I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

- Then, substituting $A = TDT^{-1}$ into the above, we can compute matrix exponentials using diagonal form as

$$\begin{aligned} e^{At} &= I + TDT^{-1}t + \frac{TD^2T^{-1}t^2}{2!} + \frac{TD^3T^{-1}t^3}{3!} + \cdots \\ &= TT^{-1} + TDT^{-1}t + \frac{TD^2T^{-1}t^2}{2!} + \frac{TD^3T^{-1}t^3}{3!} + \cdots \\ &= T\underbrace{\left(I + Dt + \frac{(Dt)^2}{2!} + \frac{(Dt)^3}{3!} + \cdots\right)}_{=e^{Dt} \text{ by definition above}}T^{-1} \\ &= \boxed{Te^{Dt}T^{-1}} \end{aligned}$$

Diagonal Form Example $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

- Eigenvalues are computed as

$$\det(\lambda I - A) = 0 \;\rightarrow\; \lambda_1 = -1,\; \lambda_2 = -2$$

- Eigenvectors are computed as

$$\lambda_1 = -1 \;\rightarrow\; (\lambda_1 I - A)x_1 = 0 \;\rightarrow\; x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = -2 \;\rightarrow\; (\lambda_2 I - A)x_2 = 0 \;\rightarrow\; x_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

- Finally, the matrix exponential is computed using diagonal form as

$$e^{At} = Te^{Dt}T^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1}$$

$$= \boxed{e^{-t} \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} + e^{-2t} \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}}$$

Summary

- Solution to CT linear time-invariant systems
- Analytical computation of matrix exponentials
  - Definition of matrix exponential
  - Laplace transform
  - Diagonal form or Jordan form
  - Cayley-Hamilton theorem
- Next, discretization and solution to DT state-space models

$$\begin{cases} x[k+1] = A[k]x[k] + B[k]u[k] \\ y[k] = C[k]x[k] + D[k]u[k] \end{cases} \quad \rightarrow \quad \text{How to compute } x[k] \ \& \ y[k]?$$

Appendix A: Laplace Transform Table

$F(s) = \mathcal{L}\{f(t)\}$ and $f(t) = \mathcal{L}^{-1}\{F(s)\}$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$t \cdot u(t)$	$\frac{1}{s^2}$
$t^n \cdot u(t)$	$\frac{n!}{s^n + 1}$
$e^{-at} \cdot u(t)$	$\frac{1}{s+a}$
$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$
$te^{-at} \cdot u(t)$	$\frac{1}{(s+a)^2}$

Note: $u(t)$: unit step function

Appendix B: Linear Algebra Review

Matrix Determinants:

- 2×2 Case:

$$\det(A) = \det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- 3×3 Case:

$$\det(A) = \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13}$$
$$- a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}$$

Matrix Eigenvalues λ and Eigenvectors x:

- Definition: $Ax = \lambda x$, where $\lambda \in \mathbb{C}$ and $x \neq 0$, $x \in \mathbb{C}^{n \times 1}$
- Eigenvalues are computed as: $\det(\lambda I - A) = 0$

Appendix B: Linear Algebra Review

Matrix Inversion:

- A square matrix A is invertible or nonsingular if and only if $\det(A) \neq 0$

$$AA^{-1} = A^{-1}A = I$$

- Inverse of a matrix is computed as

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)},$$

  where $\text{adj}(A)$ is the adjoint matrix of A

- 2×2 Case:

$$A^{-1} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{\det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$