ELEC 441 Assignment 2

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February 18, 2025

P1

a) Continuous-time system:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & -1 \end{bmatrix} x(t)$$

i) Eigenvalue Criteria:
We begin by computing the eigenvalues of A

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} -1-\lambda & 0 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & -2 & -1-\lambda \end{bmatrix} = 0$$

$$\begin{aligned} &= (-1-\lambda)\big((-1-\lambda)^2 - (-2)\big) - 0 + 0 = 0 \\ &= (-1-\lambda)(\lambda^2 + 2\lambda + 3) = 0 \end{aligned}$$

$$\lambda_1 = -1, \quad \lambda_2 = -1 + j\sqrt{2}, \quad \lambda_3 = -1 - j\sqrt{2}$$

$\boxed{\text{This system is asympototically stable as } \mathrm{Re}\{\lambda_i\} < 0, \forall i}$

ii) Lyapunov Theorem:
For the CT system, we solve for P in the following equation

$$A^T P + PA = -Q$$

Taking Q to be the identity marix

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 & p_3 \\ p_2 & p_4 & p_5 \\ p_3 & p_5 & p_6 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -2 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2p_1 & -2p_2 - 2p_3 & p_2 - 2p_3 \\ -2p_2 & -2p_4 - 4p_5 & p_4 - 2p_5 - 2p_6 \\ p_2 - 2p_3 & p_4 - 2p_5 & 2p_5 - 2p_6 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$p_1 = \boxed{\frac{1}{2}}$$

$$\begin{aligned} -2p_4 - 4p_5 &= -1 \\ -2p_4 &= -1 + 4p_5 \\ p_4 &= \frac{1}{2} - 2p_5 \end{aligned}$$

$$
\begin{aligned}
2p_5 - 2p_6 &= -1 \\
-2p_6 &= -1 - 2p_5 \\
p_6 &= \frac{1}{2} + p_5
\end{aligned}
$$

$$
\begin{aligned}
p_4 - 2p_5 - 2p_6 &= 0 \\
\frac{1}{2} - 2p_5 - 2p_5 - 2\left(\frac{1}{2} + p_5\right) &= 0 \\
\frac{1}{2} - 4p_5 - 1 - 2p_5 &= 0 \\
\frac{1}{2} - 6p_5 - 1 &= 0 \\
-\frac{1}{2} - 6p_5 &= 0
\end{aligned}
$$

$$p_5 = \boxed{-\frac{1}{12}}$$

$$p_4 = \boxed{\frac{2}{3}}$$

$$p_6 = \boxed{\frac{5}{12}}$$

$$p_2 = p_3 = \boxed{0}$$

$$P = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{12} \\ 0 & -\frac{1}{2} & \frac{5}{12} \end{bmatrix}$$

All diagonal entries are > 0 so P may be positive definite
Check principle minors

$$
\begin{aligned}
A_1 &= \frac{1}{2} > 0 \\
A_2 &= \det \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} = \frac{1}{2} \times \frac{2}{3} > 0 \\
A_3 &= \det \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{12} \\ 0 & -\frac{1}{12} & \frac{5}{12} \end{bmatrix} = \frac{1}{2} \det \begin{bmatrix} \frac{2}{3} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{5}{12} \end{bmatrix} + 0 + 0 \\
&= \frac{1}{2}\left(\frac{2}{3} \times \frac{5}{12} - \frac{1}{12} \times \frac{1}{12}\right) > 0
\end{aligned}
$$

All principle minors are greater than 0 so P is positive definite.

$\boxed{\text{The system is asympototically stable since } P > 0}$

b) Discrete-time system:

$$x[k+1] = \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & 0 \end{bmatrix} x[k]$$

i) Eivenvalue Criteria:
We begin by computing the eigenvalues of A

$$\det(A - \lambda I) = 0$$

$$\det \begin{bmatrix} -\lambda & -1 \\ \frac{1}{2} & -\lambda \end{bmatrix} = \lambda^2 + \frac{1}{2} = 0$$

$$\lambda_1 = j\frac{\sqrt{2}}{2}, \quad \lambda_2 = \frac{\sqrt{2}}{j2}$$

$$|\lambda_1| = |\lambda_2| = \frac{\sqrt{2}}{2} < 1$$

$\boxed{\text{This system is asympototically stable as } |\lambda_i| < 1, \forall i}$

ii) Lyapunov Theorem:
For the DT system, we solve for P in the following equation

$$A^T P A - P = -Q$$

Taking Q to be the identity matrix

$$\begin{bmatrix} 0 & \frac{1}{2} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ \frac{1}{2} & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4}p_3 - p_1 & -\frac{1}{2}p_3 - p_2 \\ -\frac{1}{2}p_2 - p + 2 & p_1 - p_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} p_1 - p_3 &= -1 \\ p_1 &= p_3 - 1 \end{aligned}$$

$$\begin{aligned} \frac{1}{4}p_3 - p_1 &= -1 \\ \frac{1}{4}p_3 - p_3 + 1 &= -1 \\ -\frac{3}{4}p_3 &= -2 \end{aligned}$$

$$p_3 = \boxed{\frac{8}{3}}$$

$$p_1 = \boxed{\frac{5}{3}}$$

$$p_2 = \boxed{0}$$

$$P = \begin{bmatrix} \frac{5}{3} & 0 \\ 0 & \frac{8}{3} \end{bmatrix}$$

P is diagonal with positive entries and is therefore positive definite.

$\boxed{\text{The system is asympototically stable since } P > 0}$

P2

a) $M_1 = M_2 = k_1 = k_2 = 1$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

i) Controllability

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n \times nm}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 & 1 \\ 5 & 0 & -3 & 0 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & 2 & 0 \end{bmatrix}$$

We can determine from inspection, that pre-multiplying B with A^n will result in the second column of A^n

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

We can easily determine the rank of $\mathcal{C}$ by re-arranging the rows

$$\begin{bmatrix} 0 & 1 & 0 & -2 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{C}) = 4$$

$\boxed{\text{Since the Controllability matrix has full rank, this system is controllable}}$

ii) Observability

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$$CA = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

$$CA^2 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & -2 & 0 \\ 5 & 0 & -3 & 0 \\ -3 & 0 & 2 & 0 \end{bmatrix}$$

$$CA^3 = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 & 1 \\ 5 & 0 & -3 & 0 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 & 0 & -2 \\ 0 & 5 & 0 & -3 \\ 0 & -3 & 0 & 2 \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ -2 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -2 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 3 & 0 & -2 & 0 \\ 5 & 0 & -3 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & 0 & -3 \\ 0 & -3 & 0 & 2 \end{bmatrix}$$

We can conlcude the rank of $\mathcal{O}$ is 4 if we can find a subset of the rows with rank 4

$$\begin{bmatrix} 5 & 0 & -3 & 0 \\ -3 & 0 & 2 & 0 \\ 0 & 3 & 0 & -2 \\ 0 & 5 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{3}{5} & 0 \\ 1 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{3}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{3}{5} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & -\frac{3}{5} \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -\frac{3}{5} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{-3}{5} & 0 \\ 0 & 1 & 0 & -\frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \text{rank}(\mathcal{O}) = 4$$

$\boxed{\text{Since } \mathcal{O} \text{ has a rank of 4, the system is observable}}$

b) $R = C = 1$

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

i) Controllability

$$\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times nm}$$

$$A^2 = \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} -3 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 1 \\ -5 & 6 & -4 \\ 1 & -4 & 5 \end{bmatrix}$$

Pre-multiplying B with A^n will result in a matrix with twice the first column of A^n and twice the third column of A^n

$$AB = \begin{bmatrix} -6 & 0 \\ 2 & 2 \\ 0 & -6 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 20 & 2 \\ -10 & -8 \\ 2 & 10 \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} 2 & 0 & -6 & 0 & 20 & 2 \\ 0 & 0 & 2 & 2 & -10 & -8 \\ 0 & 2 & 0 & -6 & 2 & 10 \end{bmatrix}$$

$\mathcal{C}$ will have rank 3 if we can find 3 linearly independent columns

$$\begin{bmatrix} 2 & 0 & -6 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{C}) = 3$$

$\boxed{\text{Since the Controllability matrix has rank 3, this system is controllable}}$

P3

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} x(t) \end{cases}$$

We first find the controllability matrix, which will simply be B concatonated with the first column of A

$$\mathcal{C} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{C}) = 2$$

This system is fully controllable.
The observability matrix will be C and the second row of A subtracted from the first

$$\mathcal{O} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Rightarrow \text{rank}(\mathcal{O}) = 1$$

This system is not fully observable.
Since $\mathcal{C}$ has full rank, the Image space can be expressed as the span of the 2-D basis vectors

$$\text{Im}(\mathcal{C}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

We will next find the Kernel space of $\mathcal{O}$ which can easily be seen to be the line where $x_1 = x_2$

$$\text{Ker}(\mathcal{O}) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

We can now begin finding the bases of the subspaces needed to construct T^{-1}

$$T^{-1} = \begin{bmatrix} T_{co} & T_{c\bar{o}} & T_{\bar{c}o} & T_{\bar{c}\bar{o}} \end{bmatrix}$$

$T_{c\bar{o}}$ will be a basis for the intersection of $\mathrm{Im}(\mathcal{C})$ and $\mathrm{Ker}(\mathcal{O})$

$$T_{c\bar{o}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$T_{\bar{c}o}$ will be the intersection of the orthogonal compliment of $\mathrm{Im}(\mathcal{C})$ and the orthgonal compliment of $\mathrm{Ker}(\mathcal{O})$, which is the zero subspace

$$T_{\bar{c}o} = \mathbf{0}$$

$T_{\bar{c}\bar{o}}$ will be the intersection of the orthogonal compliment of $\mathrm{Im}(\mathcal{C})$ and $\mathrm{Ker}(\mathcal{O})$, which again is the zero subspace

$$T_{\bar{c}\bar{o}} = \mathbf{0}$$

T_{co} will be the intersection of $\mathrm{Im}(\mathcal{C})$ and the orthgonal compliment of $\mathrm{Ker}(\mathcal{O})$

$$T_{co} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow T = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

We can now perform the coordinate transformation

$$\begin{aligned} \bar{A} &= TAT^{-1} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \bar{B} &= TB \\ &= \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \bar{C} &= CT^{-1} \\ &= \begin{bmatrix} 2 & 0 \end{bmatrix} \end{aligned}$$

The new system after the coordinate transformation is

$$\begin{cases} \dot{z}(t) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} z(t) + \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \\ y(t) = \begin{bmatrix} 2 & 0 \end{bmatrix} z(t) \end{cases}$$

The new state vector z is

$$
\begin{aligned}
z &= Tx \\
&= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} x_1 - x_2 \\ x_1 + x_2 \end{bmatrix}
\end{aligned}
$$

The state which is controllable and observable is $\frac{1}{2}(x_1 - x_2)$
The state which is controllable and not observable is $\frac{1}{2}(x_1 + x_2)$
No states are not controllable and observable/not observable.