ELEC 441: Control Systems

Lecture 6: Discretization & Discrete-time State-space Models Solution

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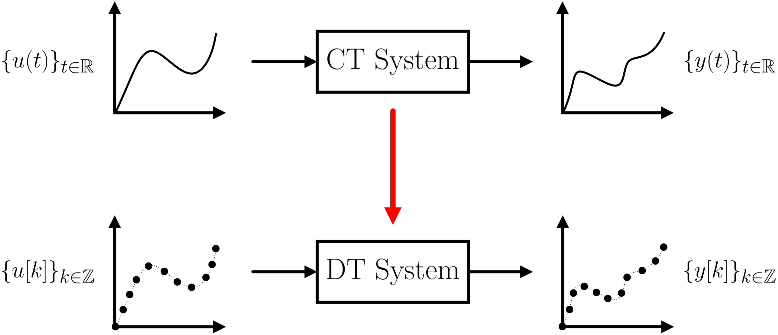
Course Roadmap

Topics	**CT**	**DT**
Modeling	✓	$\rightarrow \bullet \leftarrow$
Stability		
Controllability/Observability		
Realization		
State Feedback/Observers		
LQR/Kalman Filter		

What is Discretization?

Discretization: is the approximation of a CT system by a DT system



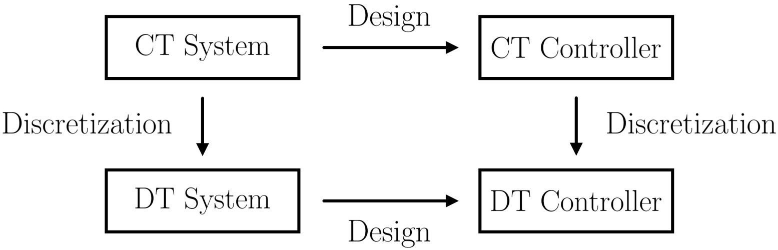
Why Discretization?

- Digital Control: to realize a controller in a digital computer, we need a DT controller

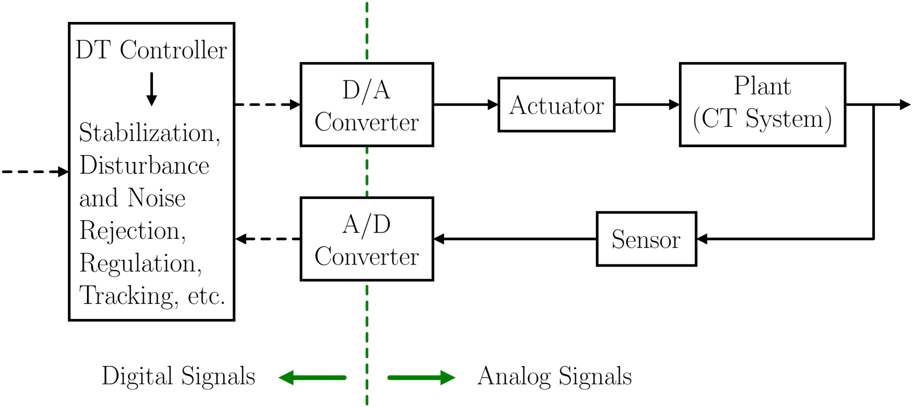
- Digital Simulations: simulation of a CT system is done in discrete-time (e.g. MATLAB)

Why Discretization?

- Digital Control: to realize a controller in a digital computer, we need a DT controller
- Digital Simulations: simulation of a CT system is done in discrete-time (e.g. MATLAB)
- We can discretize before or after the controller is designed (i.e. in modelling or implementation stage)



Digital Control System

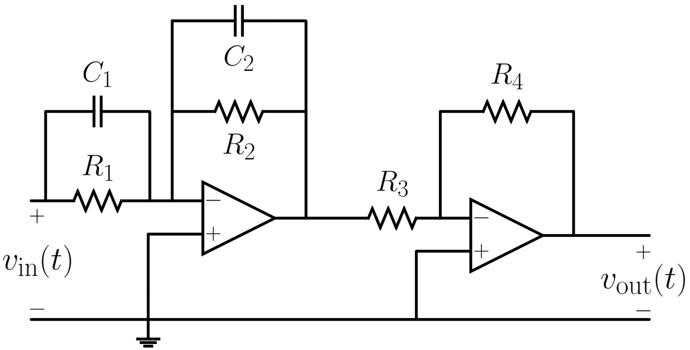


Advantages of Digital Control

- Reduced cost
  - A single digital computer can replace numerous analog controllers
- Flexibility in response to design changes
  - Future required modifications can be implemented with simple software updates, rather than expensive hardware modifications
  - Complex control algorithms can be realized easily
- Examples of microcontrollers: Arduino, Raspberry Pi, LabVIEW, etc.

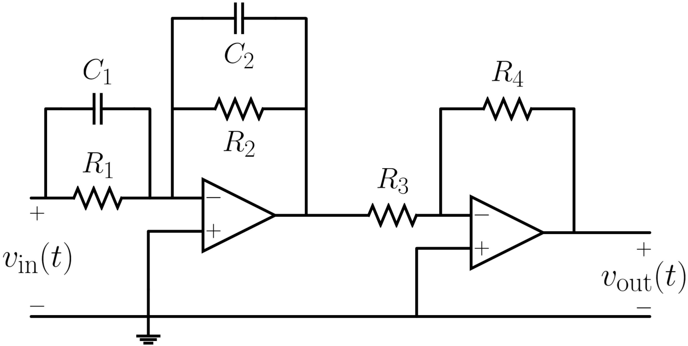
Analog Controllers Inflexibility

- Consider the below analog lead compensator which utilizes operational amplifiers (Opamps)



Analog Controllers Inflexibility

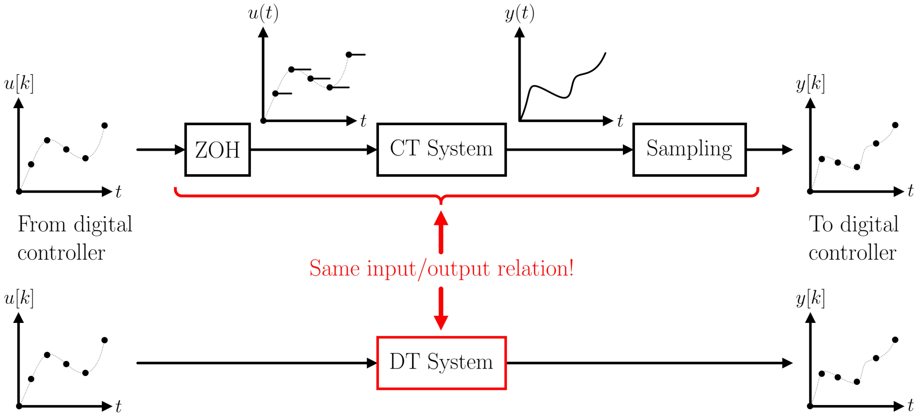
- Consider the below analog lead compensator which utilizes operational amplifiers (Opamps)



- To modify the controller, we need to physically replace electrical elements

Zero-order Hold (ZOH) Discretization

Given a CT system and sampling time T, we obtain a DT system as



ZOH Descritization of State-space Models

- Recall that CT state-space models are described as

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

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$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

- DT state-space models are obtained analytically using ZOH with sampling time T as

$$\begin{cases} x[k+1] = A_d x[k] + B_d u[k] \\ y[k] = C_d x[k] + D_d u[k] \end{cases}$$

where

$$A_d := e^{AT}, \quad B_d := \left(\int_0^T e^{A\tau} d\tau \right) \cdot B, \quad C_d = C, \quad D_d = D$$

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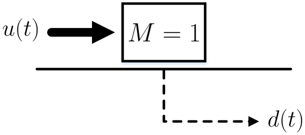
- DT state-space models are obtained numerically using `c2d.m` in MATLAB

Discretization Example

- Mass with a driving force SS model

$$
\begin{cases}
\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\[2em]
y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}
\end{cases}
$$

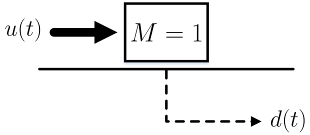


where $x_1(t) := d(t)$ and $x_2(t) := \dot{d}(t)$

Discretization Example

- Mass with a driving force SS model

$$\begin{cases} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{cases}$$



where $x_1(t) := d(t)$ and $x_2(t) := \dot{d}(t)$

- Discretization by ZOH with sampling time T yields

$$A_d := e^{AT} = I + AT + \underbrace{\cdots}_{=0} = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} \leftarrow \; A \text{ is a Nilpotent matrix!}$$

$$B_d := \left(\int_0^T e^{A\tau} d\tau \right) \cdot B = \left(\int_0^T \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix} d\tau \right) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{T^2}{2} \\ T \end{bmatrix}$$

Discretization in MATLAB

- Code for discretizing CT state-space models:

```
>> A = [0 1; 0 0]; % System matrix A
>> B = [0; 1]; % System matrix B
>> C = [1 0]; % System matrix C
>> D = 0; % System matrix D
>> sys = ss(A,B,C,D); % CT state-space model
>> T = 0.1; % Sampling time
>> sysd = c2d(sys,T); % Discretization
```

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```

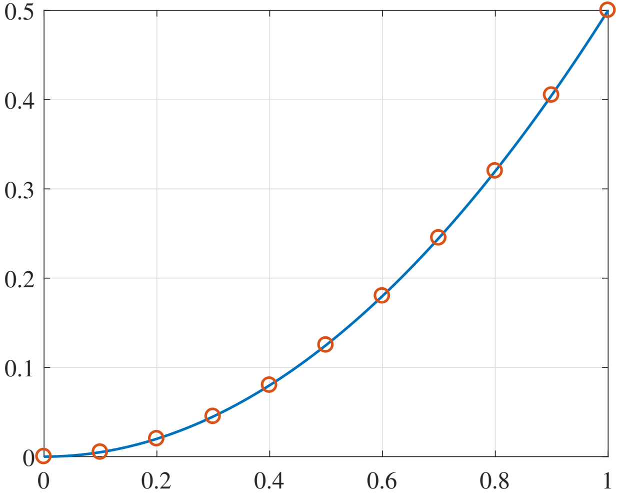
- Discretization result:

```
>> sysd.a
ans =
    1.0000    0.1000
         0    1.0000
```

```
>> sysd.b
ans =
    0.0050
    0.1000
```

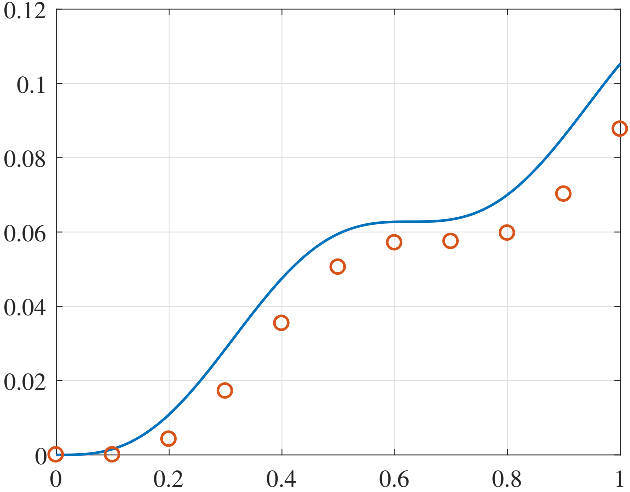
Simulations in MATLAB

- Step input response (i.e. $u(t) = 1$)



Simulations in MATLAB

- Sinusoidal input response (i.e. $u(t) = \sin(10t)$)



Analytical Solutions to DT State-space Models

- Discrete-time state-space models are described by

$$\begin{cases} x[k+1] = Ax[k] + Bu[k], \ x[0] = x_0 \\ \quad y[k] = Cx[k] + Du[k] \end{cases}$$

- Analytical solution to DT state-space models is given by

$$x[k] = A^k \underbrace{x[0]}_{=x_0} + \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \begin{bmatrix} u[k-1] \\ u[k-2] \\ \vdots \\ u[0] \end{bmatrix} \tag{1}$$

$$y[k] = \underbrace{Cx[k] + Du[k]}_{\text{substituted from (1)}} \tag{2}$$

Verification of Analytical Solution

- Solving the state equation recursively, we get
  - $k = 0$: $x[1] = Ax[0] + Bu[0]$ $\leftarrow$ Substituting initial conditions
  - $k = 1$: $x[2] = Ax[1] + Bu[1] = A^2x[0] + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$
  - $k = 2$: $x[3] = Ax[2] + Bu[2] = A^3x[0] + \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix}$

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- By induction, we get

$$x[k] = A^k x[0] + \begin{bmatrix} B & AB & \cdots & A^{k-1}B \end{bmatrix} \begin{bmatrix} u[k-1] \\ u[k-2] \\ \vdots \\ u[0] \end{bmatrix}$$

Summary

- Discretization using (Zero-order Hold)
  - Digital control
  - Formulas for discretizing state-space models
  - Example with MATLAB code/simulations
- Solution to DT linear time-invariant systems
- Next, stability $\leftarrow$ VERY IMPORTANT!

Note: now you can solve all the problems in Assignment 1, which is due on January 31, 2025 at 11:59 PM