ELEC 441: Control Systems

Lecture 12: Kalman Decomposition

Dr. Abdullah Al-Digs
Department of Electrical and Computer Engineering
The University of British Columbia

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Course Roadmap

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/Observability	$\rightarrow \bullet \leftarrow$	$\rightarrow \bullet \leftarrow$
Realization		
State Feedback/Observers		
LQR/Kalman Filter		

Why is Decomposition Important?

- We can determine what is possible with control input $u(t)$ and output $y(t)$. We cannot affect the uncontrollable or unobservable parts
- If uncontrollable part is unstable, then we cannot stabilize the system using feedback (More about this later in the course!)
- This assessment may suggest addition of actuators or change of actuator locations
- Similarly, this assessment may suggest addition of sensors or change of sensor locations

Coordinate Transformation

- Consider the CT state-space model

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

- Every state-space model can be transformed with coordinate transformation as

$$z(t) := Tx(t), \text{ where } T \text{ is any nonsingular matrix}$$

such that we can write

$$\begin{cases} \dot{z}(t) = \underbrace{TAT^{-1}}_{\overline{A}} z(t) + \underbrace{TB}_{\overline{B}} u(t) \\ y(t) = \underbrace{CT^{-1}}_{\overline{C}} z(t) + Du(t) \end{cases}$$

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- Note: coordinate transformations do not change system transfer function, stability, controllability, or observability

Controllability Decomposition

- If (A,B) is uncontrollable with $\operatorname{rank}(\mathcal{C}) = m < n$, then there exists a coordinate transformation (i.e. nonsingular T) that decomposes system states into controllable and uncontrollable parts

- Given a state equation $\dot{x}(t) = Ax(t) + Bu(t)$, we can define

$$\begin{bmatrix} z_c(t) \\ z_{\overline{c}}(t) \end{bmatrix} := Tx(t)$$

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$$\begin{bmatrix} \dot{z}_c(t) \\ \dot{z}_{\overline{c}}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_c & A_{12} \\ \mathbb{0} & A_{\overline{c}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_c(t) \\ z_{\overline{c}}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B_c \\ \mathbb{0} \end{bmatrix}}_{TB} u(t)$$

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- In the above, $A_c \in \mathbb{R}^{m \times m}$ and (A_c,B_c) is controllable
- Note: rank of controllability matrix indicates the number of controllable states

Image (Column) Space

- Key Question: how to find the coordinate transformation matrix T?

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- Image Space: the space spanned by the column vectors of a matrix
- For a matrix $T \in \mathbb{R}^{q\times p}$, the image space is defined as

$$\mathrm{Im}(T) := \{z \in \mathbb{R}^q \mid z = Tx,\ x \in \mathbb{R}^p\}$$

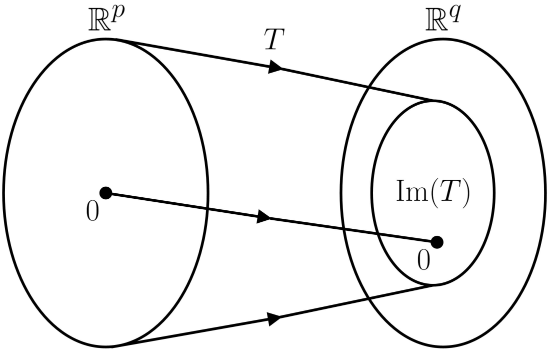

Image (Column) Space Example

- Consider the matrix $T = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \end{bmatrix}$, we compute $\mathrm{Im}(T)$ as

$$
\begin{aligned}
\mathrm{Im}(T) = \{Tx \mid x \in \mathbb{R}^3\} &= \left\{ \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ x_i \in \mathbb{R}\ \forall i \right\} \\
&= \left\{ x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -4 \end{bmatrix},\ x_i \in \mathbb{R}\ \forall i \right\} \\
&= \left\{ (x_1 + 2x_3) \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ x_i \in \mathbb{R}\ \forall i \right\} \\
&= \left\{ \alpha \begin{bmatrix} 1 \\ -2 \end{bmatrix},\ \alpha \in \mathbb{R} \right\}
\end{aligned}
$$

- $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is a basis of $\mathrm{Im}(T)$

Finding T for Controllability Decomposition

- We use the image space of controllability matrix as

$$T^{-1} := \begin{bmatrix} T_c & T_{\overline{c}} \end{bmatrix},$$

where T_c is basis of $\mathrm{Im}(\mathcal{C})$ & $T_{\overline{c}}$ is any complement of T_c in $\mathbb{R}^n$

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- T_c is constructed as m linearly independent columns from $\mathcal{C}$
- $T_{\bar{c}}$ is any linearly independent matrix such that T^{-1} invertible
- Controllable Subspace $\mathrm{Im}(\mathcal{C})$: subspace where the controllable states trajectories cannot escape

Example $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- We begin by computing controllability matrix as

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \implies \mathrm{rank}(\mathcal{C}) = 1 < 2 \implies \text{Uncontrollable!}$$

- Next, we find the image space of $\mathcal{C}$ as

$$\mathrm{Im}(\mathcal{C}) = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \left\{ (x_1 + x_2) \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{T_c} \right\}$$

- Then, we construct the transformation matrix as

$$T^{-1} = \left[\begin{array}{c|c} T_c & T_{\overline{c}} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ 1 & 1 \end{array}\right]$$

- Finally, we can write

$$TAT^{-1} = \left[\begin{array}{c|c} A_c & A_{12} \\ \hline \mathbb{0} & A_{\overline{c}} \end{array}\right] = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right], \quad TB = \left[\begin{array}{c} B_c \\ \hline \mathbb{0} \end{array}\right] = \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right]$$

Example $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 8 \\ 1 & 2 & 4 \end{bmatrix} \Longrightarrow \operatorname{rank}(\mathcal{C}) = 2 < 3 \Longrightarrow \text{Uncontrollable!}$$

$$\operatorname{Im}(\mathcal{C}) = \left\{ \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 8 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + 2x_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 4x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$$

- Only two basis vectors are needed, we will go with first two

$$T^{-1} = \left[\; T_c \;\middle|\; T_{\overline{c}} \; \right] = \left[\begin{array}{cc|c} 1 & 1 & 1 \\ -2 & 0 & 0 \\ 1 & 1 & 0 \end{array} \right]$$

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Observability Decomposition

- If (A,C) is unobservable with $\operatorname{rank}(\mathcal{O}) = m < n$, then there exists a coordinate transformation (i.e. nonsingular T) that decomposes system states into observable and unobservable parts

- Given a state-space model $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$, we can define

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- Note: rank of observability matrix indicates the number of observable states

Kernel (Null) Space

- Key Question: how to find the coordinate transformation matrix T?

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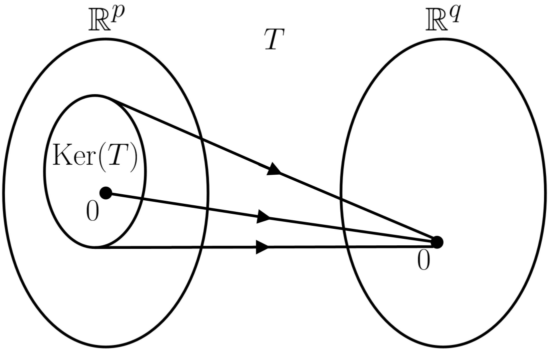
- Kernel Space: the subspace which is mapped to the zero vector
- For a matrix $T \in \mathbb{R}^{q \times p}$, the kernel space is defined as

$$\mathrm{Ker}(T) := \{x \in \mathbb{R}^p \mid Tx = \mathbb{0}\}$$



Kernel (Null) Space Example

- Consider the matrix $T = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \end{bmatrix}$, we compute $\mathrm{Ker}(T)$ as

$$\mathrm{Ker}(T) = \{x \in \mathbb{R}^3 \mid Tx = \mathbb{0}\} = \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbb{0} \right\}$$

$$= \left\{ x \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} x_1 + 2x_3 \\ -2x_1 - 4x_3 \end{bmatrix} = \mathbb{0} \right\}$$

$$= \left\{ x \in \mathbb{R}^3 \;\middle|\; x_1 = -2x_3 \right\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \; x_2, x_3 \in \mathbb{R} \right\}$$

- $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are the basis of $\mathrm{Ker}(T)$

Finding T for Observability Decomposition

- We use the kernel space of observability matrix as

$$T^{-1} := \begin{bmatrix} T_o & T_{\overline{o}} \end{bmatrix},$$

where $T_{\overline{o}}$ is basis of $\mathrm{Ker}(\mathcal{O})$ & T_o is any complement of $T_{\overline{o}}$ in $\mathbb{R}^n$

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- $T_{\overline{o}}$ is constructed as $n - m$ linearly independent basis from $\mathrm{Ker}(\mathcal{O})$
- T_o is any linearly independent matrix such that T^{-1} invertible
- Unobservable Subspace $\mathrm{Ker}(\mathcal{O})$: subspace where the unobservable states trajectories cannot escape

Example $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & -1 \\ 3 & 0 & 1 \end{bmatrix} \implies \operatorname{rank}(\mathcal{O}) = 2 < 3 \implies \text{Unobservable!}$$

$$\operatorname{Ker}(\mathcal{O}) = \left\{ \begin{bmatrix} 0 & 0 & 1 \\ 3 & 0 & -1 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbb{0} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{T_{\overline{o}}} \right\}$$

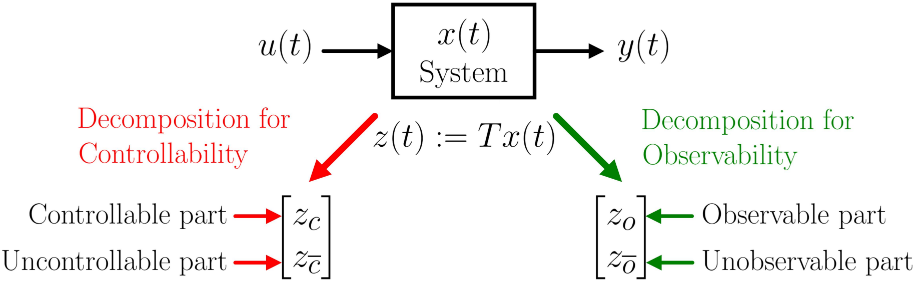
$$T^{-1} = \left[\begin{array}{c|c} T_o & T_{\overline{o}} \end{array} \right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{array} \right]$$

$$TAT^{-1} = \left[\begin{array}{c|c} A_o & \mathbb{0} \\ \hline A_{21} & A_{\overline{o}} \end{array} \right] = \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 3 & -1 & 0 \\ \hline 2 & 2 & 2 \end{array} \right], \quad CT^{-1} = \left[\begin{array}{c|c} C_0 & \mathbb{0} \end{array} \right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \end{array} \right]$$

Two Decompositions

- Consider the following state-space model

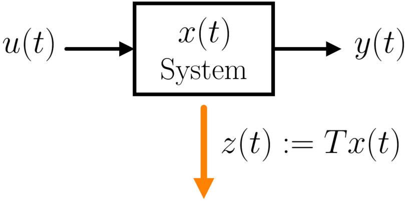
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$



Kalman Decomposition Definition

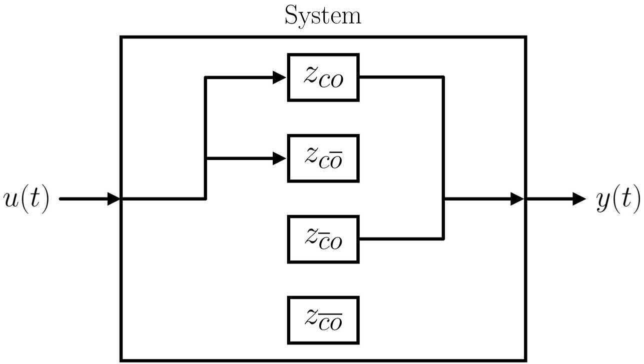
- Kalman Decomposition: is the combination of decompositions for controllability and observability
- Consider the following state-space model

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$



$$\begin{array}{ll} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\overline{co}} \end{bmatrix} & \begin{array}{l} \leftarrow \text{Controllable \& Observable} \\ \leftarrow \text{Controllable \& Unobservable} \\ \leftarrow \text{Uncontrollable \& Observable} \\ \leftarrow \text{Uncontrollable \& Unobservable} \end{array} \end{array}$$

Kalman Decomposition Conceptual Diagram



- Note: this is not a block diagram!

Kalman Decomposition

- Every state-space model can be transformed using some T into

$$\begin{cases} \begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{c\bar{o}} \\ \dot{z}_{\bar{c}o} \\ \dot{z}_{\overline{co}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & \mathbb{0} & A_{13} & \mathbb{0} \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ \mathbb{0} & \mathbb{0} & A_{\bar{c}o} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & A_{43} & A_{\overline{co}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\overline{co}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ B_{c\bar{o}} \\ \mathbb{0} \\ \mathbb{0} \end{bmatrix}}_{TB} u(t) \\ y(t) = \underbrace{\begin{bmatrix} C_{co} & \mathbb{0} & C_{c\bar{o}} & \mathbb{0} \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co} \\ z_{c\bar{o}} \\ z_{\bar{c}o} \\ z_{\overline{co}} \end{bmatrix} + Du(t) \end{cases}$$

- Note: this is the decomposition structure for controllability

Alternative Kalman Decomposition

- Exchanging the second and third states, we get an alternative form

$$\begin{cases} \begin{bmatrix} \dot{z}_{co} \\ \dot{z}_{\bar{c}o} \\ \dot{z}_{c\bar{o}} \\ \dot{z}_{\overline{co}} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{co} & A_{13} & \mathbb{0} & \mathbb{0} \\ \mathbb{0} & A_{\bar{c}o} & \mathbb{0} & \mathbb{0} \\ A_{21} & A_{23} & A_{c\bar{o}} & A_{24} \\ \mathbb{0} & A_{43} & \mathbb{0} & A_{\overline{co}} \end{bmatrix}}_{TAT^{-1}} \begin{bmatrix} z_{co} \\ z_{\bar{c}o} \\ z_{c\bar{o}} \\ z_{\overline{co}} \end{bmatrix} + \underbrace{\begin{bmatrix} B_{co} \\ \mathbb{0} \\ B_{c\bar{o}} \\ \mathbb{0} \end{bmatrix}}_{TB} u(t) \\ y(t) = \underbrace{\begin{bmatrix} C_{co} & C_{\bar{c}o} & \mathbb{0} & \mathbb{0} \end{bmatrix}}_{CT^{-1}} \begin{bmatrix} z_{co} \\ z_{\bar{c}o} \\ z_{c\bar{o}} \\ z_{\overline{co}} \end{bmatrix} + Du(t) \end{cases}$$

- Note: this is the decomposition structure for observability

Remarks on Kalman Decomposition

- When a system is both controllable and observable, then the system is minimal with no redundant states. The Kalman decomposition of this system is the same system itself
- In some systems, some states may be missing and do not belong to any of the following categories z_{co}, $z_{c\bar{o}}$, $z_{\bar{c}o}$, $z_{\overline{co}}$. For example, a controllable system would not have $z_{\bar{c}o}$ and $z_{\overline{co}}$
- (A_{co}, B_{co}) is controllable and (A_{co}, C_{co}) is observable
- System transfer function is determined by only the controllable and observable parts

$$\left(CT^{-1}\right)\left(sI - TAT^{-1}\right)^{-1}(TB) + D = C_{co}\left(sI - A_{co}\right)^{-1} B_{co} + D$$

- We can perform Kalman decomposition in MATLAB using `minreal.m` (yields a different order for the states than that in slide 20)

Stability of Uncontrollable and/or Unobservable Systems

- If uncontrollable and/or unobservable parts are unstable, then we need to change actuator/sensor structure of system because we cannot stabilize the system with feedback

- Eigenvalues of decomposed A matrix

$$\begin{aligned}
\mathrm{eig}(TAT^{-1}) &= \mathrm{eig}\left(\left[\begin{array}{cc|cc} A_{co} & \mathbb{0} & A_{13} & \mathbb{0} \\ A_{21} & A_{c\bar{o}} & A_{23} & A_{24} \\ \hline \mathbb{0} & \mathbb{0} & A_{\bar{c}o} & \mathbb{0} \\ \mathbb{0} & \mathbb{0} & A_{43} & A_{\overline{co}} \end{array}\right]\right) \\
&= \mathrm{eig}\left(\begin{bmatrix} A_{co} & \mathbb{0} \\ A_{21} & A_{c\bar{o}} \end{bmatrix}\right) \cup \mathrm{eig}\left(\begin{bmatrix} A_{\bar{c}o} & \mathbb{0} \\ A_{43} & A_{\overline{co}} \end{bmatrix}\right) \\
&= \mathrm{eig}(A_{co}) \cup \underbrace{\mathrm{eig}(A_{c\bar{o}}) \cup \mathrm{eig}(A_{\bar{c}o}) \cup \mathrm{eig}(A_{\overline{co}})}_{\text{Cannot influence this part!}}
\end{aligned}$$

Finding T for Kalman Decomposition

We use both image space of $\mathcal{C}$ and kernel space of $\mathcal{O}$ matrices

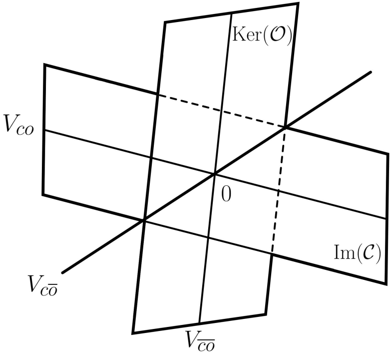
$$T^{-1} := \begin{bmatrix} T_{co} & T_{c\bar{o}} & T_{\bar{c}o} & T_{\overline{co}} \end{bmatrix}$$

$T_{c\bar{o}}$ is basis for subspace $V_{c\bar{o}} := \mathrm{Im}(\mathcal{C}) \cap \mathrm{Ker}(\mathcal{O})$

T_{co} is basis for any complement of $V_{c\bar{o}}$ in $\mathrm{Im}(\mathcal{C}) \to \mathrm{Im}(\mathcal{C}) = V_{c\bar{o}} \oplus V_{co}$

$T_{\overline{co}}$ is basis for any complement of $V_{c\bar{o}}$ in $\mathrm{Ker}(\mathcal{O}) \to \mathrm{Ker}(\mathcal{O}) = V_{c\bar{o}} \oplus V_{\overline{co}}$

$T_{\bar{c}o}$ is basis for subspace $V_{\bar{c}o}$ s.t. $\mathbb{R}^n = V_{co} \oplus V_{c\bar{o}} \oplus V_{\overline{co}} \oplus V_{\bar{c}o}$



Example $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{rank}(\mathcal{C}) = 1 < 3 \implies \text{Uncontrollable!}$$

$$\text{Im}(\mathcal{C}) = \left\{ \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right\} = \left\{ x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Longrightarrow \operatorname{rank}(\mathcal{O}) = 2 < 3 \Longrightarrow \text{Unobservable!}$$

$$\operatorname{Ker}(\mathcal{O}) = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbb{0} \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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$$T^{-1} = T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} z_{c\bar{o}} \\ z_{\bar{c}o} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_3 \\ x_2 \end{bmatrix}$$

Summary

- Controllability decomposition
  - Image space
  - Controllable subspace
  - Examples
- Observability decomposition
  - Kernel space
  - Unobservable subspace
  - Examples
- Kalman decomposition $\rightarrow$ combination of the two decompositions!
- Next, realization & minimal realization