ELEC 441: Control Systems

Lecture 11: Observability

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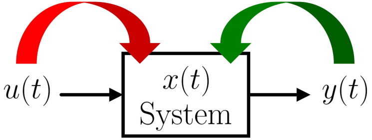
Course Roadmap

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/Observability	$\rightarrow$ ● $\leftarrow$	$\rightarrow$ ● $\leftarrow$
Realization		
State Feedback/Observers		
LQR/Kalman Filter		

Controllability and Observability

- Consider a system with state $x(t)$, input $u(t)$, and output $y(t)$:



  - We control or actuate using the input $u(t)$
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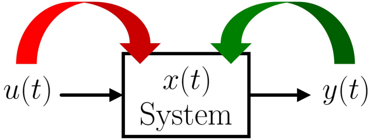

  - We control or actuate using the input $u(t)$
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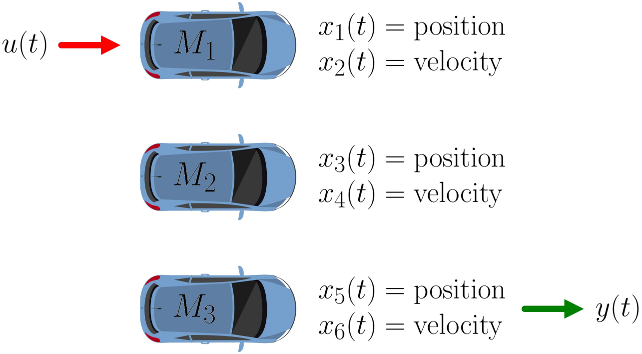
- Controllability: how much can we *control* $x(t)$ by controlling $u(t)$

- Observability: how much can we *observe* $x(t)$ by observing $y(t)$

Simplistic Example Revisited

- Consider three cars with one input and one output:



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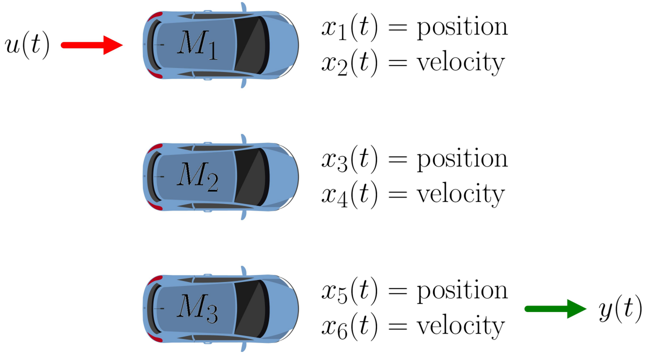

- Intuitively, we cannot control $x_3(t)$–$x_6(t)$
- Similarly, we cannot observe $x_1(t)$–$x_4(t)$

Simplistic Example State-space Model

- The three car example has the following state-space model

$$
\begin{cases}
\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{M_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\
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- Key Question: how can we determine system controllability and observability from the mathematical model (i.e. A, B, C, & D)?

Observability for CT LTI Systems

- Consider the CT state-space model with no input

$$\begin{cases} \dot{x}(t) = Ax(t), \ A \in \mathbb{R}^{n \times n} \\ y(t) = Cx(t), \ C \in \mathbb{R}^{q \times n} \end{cases}$$

- Key Assumptions: $y(t)$ is measurable, $x(0)$ is unknown

- Observability: a CT system or (A,C) is observable if there is a finite $t_f > 0$ such that $y(t)$ over interval $[0, t_f]$ determines uniquely $x(0)$

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- Condition for Observability: a CT system is observable if the observability matrix

$$\mathcal{O} := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{R}^{nq\times n}$$

has full column rank (i.e. $\operatorname{rank}(\mathcal{O}) = n$)

Remarks on Observability

- Observability depends only on A and C matrices
- If a system is observable, we can determine $x(0)$ and in turn $x(t)$, $t > 0$. However, its only possible after time t_f which is not practical
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- We can use `obsv.m` in MATLAB to compute observability matrix $\mathcal{O}$
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- Hence, we can determine the observability of a system using `rank(obsv(A,C))` in MATLAB
- Note: be careful when computing the rank of a matrix numerically!

Examples

- Example: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{3\times 3}$, $C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \operatorname{rank}(\mathcal{O}) = 1 < 3$$

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- Observability matrix is computed as

$$\mathcal{O} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \implies \text{rank}(\mathcal{O}) = 2 < 6 \rightarrow \text{ Unobservable!}$$

Observability for DT LTI Systems

- Consider the DT state-space model with no input

$$\begin{cases} x[k+1] = Ax[k], \ A \in \mathbb{R}^{n\times n} \\ y[k] = Cx[k], \ C \in \mathbb{R}^{q\times n} \end{cases}$$

- Key Assumptions: $y[k]$ is measurable, $x[0]$ is unknown

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has full column rank (i.e. $\mathrm{rank}(\mathcal{O}_d) = n$)

Derivation of DT Observability Condition

- Solve the state equation with no input recursively: $x[k+1] = Ax[k]$

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- Substitute this into the output equation with no input: $y[k] = Cx[k]$

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Then, given output measurements $y[k]$, we can write

$$\underbrace{\begin{bmatrix} y[0] \\ \vdots \\ y[n-1] \end{bmatrix}}_{Measured} = \underbrace{\begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{\mathcal{O}_d} x[0]$$

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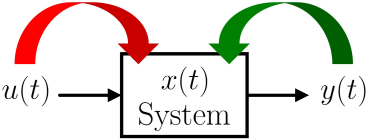
- There is unique $x[0]$ only if $\mathrm{rank}(\mathcal{O}_d) = n$

Duality Between Control and Estimation



- There is mathematical duality between:
  - Controllability & Observability
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  - Linear-quadratic Regulator & Kalman Filter

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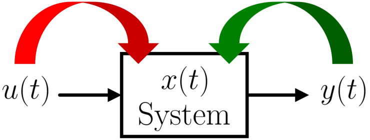


- There is mathematical duality between:
  - Controllability & Observability
  - State Feedback & Observers
  - Linear-quadratic Regulator & Kalman Filter
- Importance of duality is that results for one, will lead to, and will be led by, results for the other

Duality Between Controllability and Observability

- (A,B) is controllable $\iff$ $(A^{\mathrm{T}},B^{\mathrm{T}})$ is observable
- (A,C) is observable $\iff$ $(A^{\mathrm{T}},C^{\mathrm{T}})$ is controllable

Duality Between Controllability and Observability

- (A,B) is controllable $\iff$ $(A^{\mathrm{T}},B^{\mathrm{T}})$ is observable
- (A,C) is observable $\iff$ $(A^{\mathrm{T}},C^{\mathrm{T}})$ is controllable
- Proof: we know that $\mathrm{rank}(A) = \mathrm{rank}(A^{\mathrm{T}})$

$$(A, B) \text{ is controllable} \iff \mathrm{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$$

$$(A, B) \text{ is controllable} \iff \mathrm{rank}\begin{bmatrix} B^{\mathrm{T}} \\ B^{\mathrm{T}}A^{\mathrm{T}} \\ \vdots \\ B^{\mathrm{T}}\left(A^{\mathrm{T}}\right)^{n-1} \end{bmatrix} = n$$

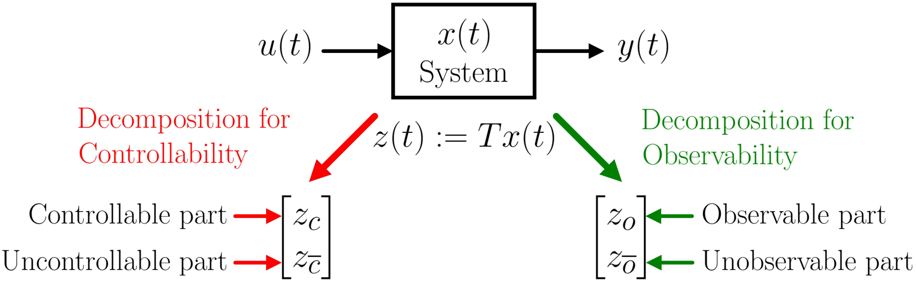
$$(A, B) \text{ is controllable} \iff (A^{\mathrm{T}}, B^{\mathrm{T}}) \text{ is observable}$$

- Note: we can check the observability of $(A^{\mathrm{T}},B^{\mathrm{T}})$ and controllability of $(A^{\mathrm{T}},C^{\mathrm{T}})$ to determine system controllability and observability, respectively

Next Controllability/Observability Decompositions

- Consider the following state-space model

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$



Kalman Decomposition Definition

- Kalman Decomposition: is the combination of decompositions for controllability and observability
- Consider the following state-space model

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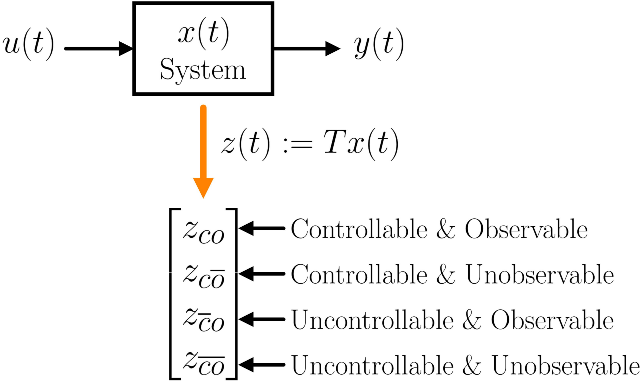
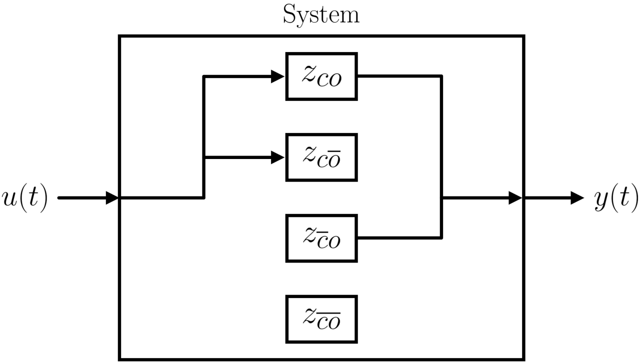


Kalman Decomposition Conceptual Diagram



- Note: this is not a block diagram!

Summary

- Observability for CT and DT LTI systems
- Necessary and sufficient conditions
- Duality between controllability and observability
- Examples
- Next, Kalman decomposition