ELEC 441: Control Systems

Lecture 10: Controllability

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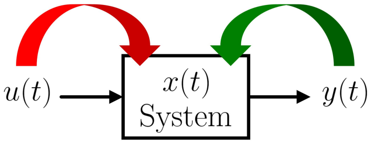
Course Roadmap

Topics	CT	DT
Modeling	✓	✓
Stability	✓	✓
Controllability/Observability	$\rightarrow \bullet \leftarrow$	$\rightarrow \bullet \leftarrow$
Realization		
State Feedback/Observers		
LQR/Kalman Filter		

Controllability and Observability

- Consider a system with state $x(t)$, input $u(t)$, and output $y(t)$:



  - We control or actuate using the input $u(t)$
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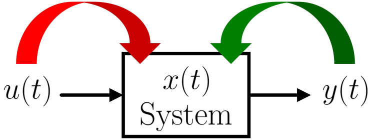

  - We control or actuate using the input $u(t)$
  - We observe or measure using the output $y(t)$
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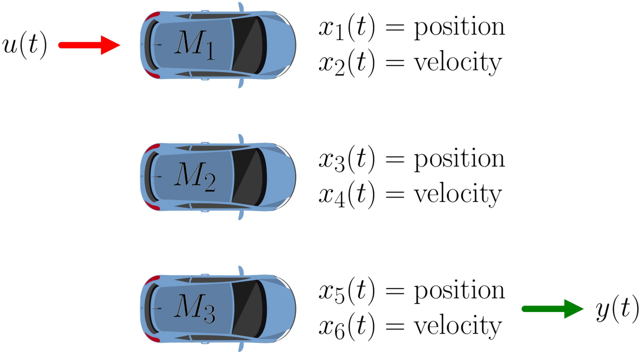
- Controllability: how much can we *control* $x(t)$ by controlling $u(t)$
- Observability: how much can we *observe* $x(t)$ by observing $y(t)$

Simplistic Example

- Consider three cars with one input and one output:



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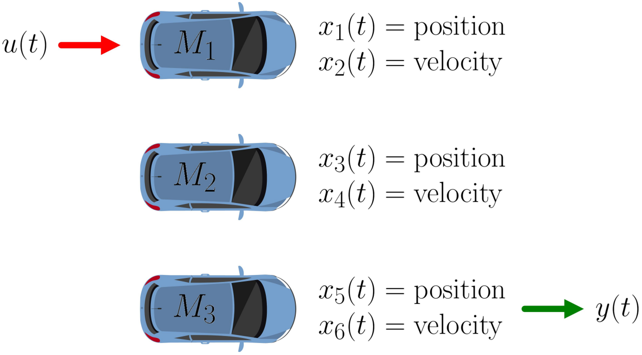

- Intuitively, we cannot control $x_3(t)$–$x_6(t)$
- Similarly, we cannot observe $x_1(t)$–$x_4(t)$

Simplistic Example State-space Model

- The three car example has the following state-space model

$$
\begin{cases}
\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \dfrac{1}{M_1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t) \\
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- Key Question: how can we determine system controllability and observability from the mathematical model (i.e. A, B, C, & D)?

Why Are Controllability & Observability Important?

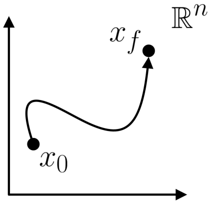
- Identify essential/redundant actuators and/or sensors
- Clarify the structure of the system:
  - Determine which states are controllable, uncontrollable, observable, and unobservable
  - Identify redundant states from input-output viewpoint (Minimal realization)
- Clarify possibilities and limitations in control & estimation
  - Control: state-feedback, linear-quadratic regulator, etc.
  - Estimation: observers, Kalman filter, etc.
- Note that stability, controllability, observability are independent properties and can be analyzed separately (i.e. in any order).

Controllability for CT LTI Systems

- Consider the CT state equation

$$\dot{x}(t) = Ax(t) + Bu(t),\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}$$

- Controllability: a CT system or (A,B) is controllable if, for any initial state x_0 and any final state x_f, there is input $u(t)$ that transfers the system from x_0 to x_f in finite time



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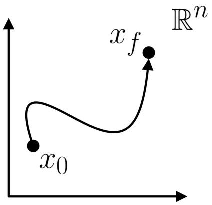

- Condition for Controllability: a CT system is controllable if the controllability matrix

$$\mathcal{C} := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \in \mathbb{R}^{n\times nm}$$

has full row rank (i.e. $\text{rank}(\mathcal{C}) = n$)

Remarks on Controllability

- Controllability depends only on A and B matrices
- We can use `ctrb.m` in MATLAB to compute the controllability matrix $\mathcal{C}$
- We can compute the rank of a matrix using `rank.m` in MATLAB

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- Hence, we can determine the controllability of a system using `rank(ctrb(A,B))` in MATLAB

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- We can use `ctrb.m` in MATLAB to compute the controllability matrix $\mathcal{C}$
- We can compute the rank of a matrix using `rank.m` in MATLAB
- Hence, we can determine the controllability of a system using `rank(ctrb(A,B))` in MATLAB
- Note: be careful when computing the rank of a matrix numerically!

$$\text{rank}(A) = \text{rank}\begin{bmatrix} 1 & 0 \\ 0 & 0.000001 \end{bmatrix} \leftarrow \text{rank}(A) = 1 \text{ or } 2?$$

Examples

- Example: $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathbb{R}^{2\times 2}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \Longrightarrow \mathrm{rank}(\mathcal{C}) = 1 < 2$$

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$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 & -2 \\ 1 & 2 & -2 & -4 \end{bmatrix} \implies \operatorname{rank}(\mathcal{C}) = 2$$

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- Example: $A = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 2 & 2 \\ 3 & 0 & -1 \end{bmatrix} \in \mathbb{R}^{3\times 3}$, $B = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ -2 & 0 & 8 \\ 1 & 2 & 4 \end{bmatrix} \Longrightarrow \operatorname{rank}(\mathcal{C}) = 2 < 3$$

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Simplistic Example Revisited

- State equation with $M_1 = 1$ is given by ($A \in \mathbb{R}^{6\times 6}$)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t)$$

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- Controllability matrix is computed as

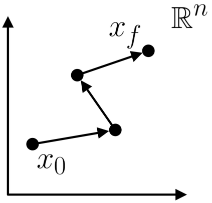
$$\mathcal{C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \mathrm{rank}(\mathcal{C}) = 2 < 6 \rightarrow \text{ Uncontrollable!}$$

Controllability for DT LTI Systems

- Consider the DT state equation

$$x[k+1] = Ax[k] + Bu[k],\ A \in \mathbb{R}^{n \times n},\ B \in \mathbb{R}^{n \times m}$$

- Controllability: a DT system or (A,B) is controllable if, for any initial state x_0 and any final state x_f, there is input sequence $u[0]$, $u[1]$, $u[2]$, ... of finite length that transfers the system from x_0 to x_f



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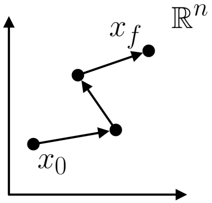

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has full row rank (i.e. $\operatorname{rank}(\mathcal{C}_d) = n$)

Derivation of DT Controllability Condition

- Solve the state equation recursively: $x[k+1] = Ax[k] + Bu[k]$

$$x[1] = Ax[0] + Bu[0]$$

$$x[2] = Ax[1] + Bu[1] = A^2x[0] + \begin{bmatrix} B & AB \end{bmatrix} \begin{bmatrix} u[1] \\ u[0] \end{bmatrix}$$

$$x[3] = Ax[2] + Bu[2] = A^3x[0] + \begin{bmatrix} B & AB & A^2B \end{bmatrix} \begin{bmatrix} u[2] \\ u[1] \\ u[0] \end{bmatrix}$$

$$\vdots$$

$$x[n] = A^nx[0] + \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix}$$

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$$x[n] = A^n x[0] + \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} u[n-1] \\ u[n-2] \\ \vdots \\ u[0] \end{bmatrix}$$

- For any $x[n]$ and $x[0]$, this set of equations have a solution $u[0], \ldots, u[n-1]$ if and only if $\text{rank} \begin{bmatrix} B & \cdots & A^{n-1}B \end{bmatrix} = n$

Remark on Controllability of DT LTI Systems

- Recall that for CT systems, if (A,B) is controllable, then any state transfer is possible for any finite time
- However, this does not apply for DT systems. Meaning, discretizing a controllable CT system does NOT necessarily result in a controllable DT system

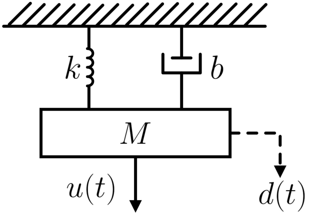
Remark on Controllability of DT LTI Systems

- Recall that for CT systems, if (A,B) is controllable, then any state transfer is possible for any finite time
- However, this does not apply for DT systems. Meaning, discretizing a controllable CT system does NOT necessarily result in a controllable DT system
- For discrete time systems, the sampling time matters significantly!
- Depending on the sampling time, we get a different number of steps which dictates whether we can transfer the system from one state to another in a specific number of steps

Mass-spring-damper System Example

- Set $M = 1$, $k = 4$, $b = 0$, to get

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}$$



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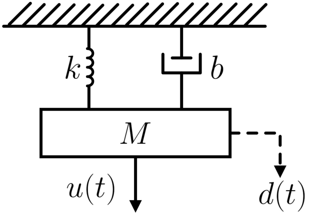

- Discretize the controllable system with sampling period T:

$$A_d = e^{AT} = \mathcal{L}^{-1}\left\{(sI - A)^{-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\begin{bmatrix} s & 1 \\ -4 & s \end{bmatrix}\right\}$$

$$= \begin{bmatrix} \cos(2T) & \frac{1}{2}\sin(2T) \\ -2\sin(2T) & \cos(2T) \end{bmatrix}$$

$$B_d = \int_0^T e^{A\tau} d\tau \cdot B = \begin{bmatrix} -\frac{1}{4}\cos(2T) + \frac{1}{4} \\ \frac{1}{2}\sin(2T) \end{bmatrix}$$

Mass-spring-damper System Example (Continued)

- Is the discretized system controllable?

$$\mathcal{C}_d = \begin{bmatrix} -\frac{1}{4}\cos(2T) + \frac{1}{4} & -\frac{1}{4}\left(\cos^2(2T) - \sin^2(2T) - \cos(2T)\right) \\ \frac{1}{2}\sin(2T) & \sin(2T)\cos(2T) - \frac{1}{2}\sin(2T) \end{bmatrix}$$

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- Looking closely at this matrix, we find

$$\det(\mathcal{C}_d) = 0, \text{ if } T = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$$

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- Looking closely at this matrix, we find

$$\det(\mathcal{C}_d) = 0, \text{ if } T = \frac{\pi}{2}, \pi, \frac{3\pi}{2}, \ldots$$

- Hence, the discretized system is uncontrollable for some T!

Summary

- Controllability for CT and DT LTI systems
- Necessary and sufficient conditions
- Rank computation (be careful when doing it numerically!)
- Examples
- Next, observability for CT and DT LTI systems