expectation_values.pdf

Calculating the expectation value of $\hat{M}$, where $\langle \hat{M} \rangle$ is prepared to be measured in the Z-basis.

The expectation value rule for Z-transformed operators in a 2-qubit Hilbert space is

$$\langle M \rangle = \mathbb{P}(|00\rangle) + \mathbb{P}(|11\rangle) - \mathbb{P}(|01\rangle) - \mathbb{P}(|10\rangle)$$

Why? $\Rightarrow$ Let's compute the expectation value of $Z^{\otimes 2}$

$$Z^{\otimes 2} = Z \otimes Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & -1\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}$$

$$Z^{\otimes 2} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & 1 \end{pmatrix} \leftarrow$$

We see that $Z^{\otimes 2}$ has eigenvalues $\{1, -1, -1, 1\}$ associated with the eigenbasis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

Calculating the expectation value *in general* is done by

$$\langle \hat{O} \rangle = \sum_i \lambda_i \mathbb{P}(\lambda_i)$$

where $\lambda_i$ = ith eigenvalue
$\mathbb{P}(\lambda_i)$ = probability of $\lambda_i$ occurring

In particular for $Z^{\otimes 2}$,

$$\langle Z^{\otimes 2} \rangle = \sum_i \lambda_i \mathbb{P}(\lambda_i) = (+1)\mathbb{P}(|00\rangle) + (-1)\mathbb{P}(|01\rangle) + (-1)\mathbb{P}(|10\rangle) + (+1)\mathbb{P}(|11\rangle)$$

$$\Rightarrow \boxed{\langle Z^{\otimes 2} \rangle = \mathbb{P}(|00\rangle) + \mathbb{P}(|11\rangle) - \mathbb{P}(|01\rangle) - \mathbb{P}(|10\rangle)}$$