The Quantum Harmonic Oscillator (Part 2)

What do a and $a^\dagger$ do to $\psi_n(x)$?

Before, we found that the operators $a^\dagger$ and a raise and lower the energy level respectively i.e.

$$\begin{cases} \mathcal{H} a\psi_n(x) = (\mathcal{E}_n - \hbar\omega) a\psi_n(x) \\ \mathcal{H}\psi_n(x) = \mathcal{E}_n\psi_n(x) \\ \mathcal{H} a^\dagger\psi_n(x) = (\mathcal{E}_n + \hbar\omega)\psi_n(x) \end{cases}$$

This is a great description of the a and $a^\dagger$ operators in the context of the TIDSE, but what do these operators do when applied directly to the wavefunction? i.e. We want to find out

$$a\psi_n(x) = ?$$
$$a^\dagger\psi_n(x) = ?$$

We assume that the raising operator transforms $\psi_n(x)$ to the $\psi_{n+1}(x)$ times some constant λ.

Notation: $\psi_n(x) \equiv |\psi_n(x)\rangle$ and $[\psi_n(x)]^\dagger \equiv \langle\psi(x)|$

We propose: $$a^\dagger|\psi_n(x)\rangle = \lambda|\psi_{n+1}(x)\rangle$$

Taking the Hermitian conjugate of both sides,

$$\langle\psi_n(x)|a = \langle\psi_{n+1}(x)|\lambda^*$$

Combining these 2 equations,

$$\langle\psi_n(x)|aa^\dagger|\psi_n(x)\rangle = \langle\psi_{n+1}(x)|\lambda^*\lambda|\psi_{n+1}(x)\rangle$$
$$= |\lambda|^2 \underbrace{\langle\psi_{n+1}(x)|\psi_{n+1}(x)\rangle}_{=1\text{, orthogonality \& normalization rules}}$$

$$\underbrace{\langle\psi_n(x)|aa^\dagger|\psi_n(x)\rangle}_{} = |\lambda|^2$$

This is just an expectation value: $\langle\psi_n(x)|aa^\dagger|\psi_n(x)\rangle = \langle aa^\dagger\rangle$

$$\langle \psi_n(x) | a a^\dagger | \psi_n(x) \rangle = |\lambda|^2$$

Utilizing the commutation relation $[a, a^\dagger] = aa^\dagger - a^\dagger a = 1$, we can re-write the expression above

$$\langle \psi_n(x) | (1 + a^\dagger a) | \psi_n(x) \rangle = |\lambda|^2$$

$$\underbrace{\langle \psi_n(x) | \psi_n(x) \rangle}_{\substack{=1 \\ \text{(Normalization)}}} + \langle \psi_n(x) | a^\dagger a | \psi_n(x) \rangle = |\lambda|^2$$

$$1 + \langle \psi_n(x) | a^\dagger a | \psi_n(x) \rangle = |\lambda|^2$$

Here, we will take a small detour to investigate the $a^\dagger a$ term, which is known as the 'Number Operator', $\hat{N}$. The reason it's called the number operator is because when it is applied to $\psi_n(x)$, its eigenvalue is n, the oscillator mode.

In particular $\hat{N} = a^\dagger a$ (Number Operator)

$$\hat{N} \psi_n(x) = n \psi_n(x)$$

(We will prove this expression).

Number Operator, Wavefunction, Oscillator Mode number, wavefunction

The number operator can be found in the Hamiltonian.

$$H = \hbar\omega(\underbrace{a^\dagger a}_{=N} + \tfrac{1}{2})$$

$$H = \hbar\omega(N + \tfrac{1}{2})$$

We can use the TIDSE and the QHO energy spectrum to prove that $N\psi_n(x) = n\psi(x)$.

Starting with the TIDSE

$$\mathcal{H}\psi_n(x) = \mathcal{E}_n\psi_n(x)$$

Substituting in $\mathcal{H} = \hbar\omega(N+\frac{1}{2})$ and $\mathcal{E}_n = \hbar\omega(n+\frac{1}{2})$

$$\cancel{\hbar\omega}(N+\tfrac{1}{2})\psi_n(x) = \cancel{\hbar\omega}(n+\tfrac{1}{2})\psi_n(x)$$

$$(N+\tfrac{1}{2})\psi_n(x) = (n+\tfrac{1}{2})\psi_n(x)$$

$$\therefore \boxed{\hat{N}\psi_n(x) = n\psi_n(x)} \blacksquare$$

Now that we know what the number operator is, what it does, and where it comes from, let's return to the equation from before

$$1 + \langle\psi_n(x)|\underbrace{a^\dagger a}_{=\hat{N}}|\psi(x)\rangle = |\lambda|^2$$

$$1 + \langle\psi_n(x)|\hat{N}|\psi_n(x)\rangle = |\lambda|^2$$

Note, $\hat{N}|\psi_n(x)\rangle = n|\psi_n(x)\rangle$

$$1 + \langle\psi_n(x)|n|\psi_n(x)\rangle = |\lambda|^2$$

$$1 + n\underbrace{\langle\psi_n(x)|\psi_n(x)\rangle}_{=1} = |\lambda|^2$$

$$|\lambda|^2 = 1 + n$$

$$\Rightarrow |\lambda| = \sqrt{n+1}$$

Therefore, following our assumption of $a^\dagger|\psi_n(x)\rangle = \lambda|\psi_{n+1}(x)\rangle$,

$$\boxed{a^\dagger|\psi_n(x)\rangle = \sqrt{n+1}\,|\psi_{n+1}(x)\rangle}$$

A similar analysis can be done for the lowering operator a to show that

$$\boxed{a|\psi_n(x)\rangle = \sqrt{n}\,|\psi_{n-1}(x)\rangle}$$

Eigenfunctions of the Harmonic Oscillator

Previously, we found that we can utilize a and $a^\dagger$ to explore different energy states

$$\begin{cases} a^\dagger \psi_n(x) = \sqrt{n+1}\, \psi_{n+1}(x) & \text{(Raising)} \\ a\, \psi_n(x) = \sqrt{n}\, \psi_{n-1}(x) & \text{(Lowering)} \end{cases}$$

Furthermore, we've already derived the ground state wave function

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}$$

Applying $a^\dagger$ to $\psi_0(x)$ would give us something proportional to $\psi_1(x)$. And, applying $a^\dagger$ to $\psi_1(x)$ would give us something proportional to $\psi_2(x)$, and so on. Therefore, we can apply iterative applications of $a^\dagger$ to $\psi_0(x)$ to generate the eigenfunctions of the Harmonic Oscillator.

This treatment directly yields

$$\psi_n(x) = e^{-x^2/(2\alpha^2)} \frac{H_n(x/\alpha)}{\sqrt{\alpha 2^n n! \sqrt{\pi}}}$$

with a corresponding eigenenergy $\varepsilon_n = \hbar\omega(n+\frac{1}{2})$; $n \in \{0,1,2,\ldots\}$

Where $\alpha \equiv \sqrt{\hbar/(m\omega)}$ is the 'characteristic length' of the QHO system and $H_n(x)$ are the 'Hermite Polynomials'.