The Quantum Harmonic Oscillator

The Hamiltonian for the Quantum Harmonic Oscillator (QHO) can be constructed as follows:

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}$$

Where $\hat{p}$ is the momentum operator $(\hat{p} \equiv -i\hbar\frac{\partial}{\partial x})$ and $\hat{x}$ is the position operator $(\hat{x} = x)$

The Commutator of 2 operators $\hat{A}$ and $\hat{B}$ are defined as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

- $\hat{A}$ and $\hat{B}$ are said to commute if $[\hat{A}, \hat{B}] = 0$
- $\hat{A}$ and $\hat{B}$ <u>do not</u> commute if $[\hat{A}, \hat{B}] \neq 0$

Let's operate the $[\hat{x}, \hat{p}]$ on some arbitrary function $f(x)$ to gain a better understanding of how these operators behave

$$\begin{aligned}
[\hat{x}, \hat{p}]f(x) &= (\hat{x}\hat{p} - \hat{p}\hat{x})f(x) \\
&= \hat{x}\hat{p}f(x) - \hat{p}\hat{x}f(x) \\
&= \hat{x}(\hat{p}f(x)) - \hat{p}(\hat{x}f(x)) \\
&= \hat{x}\left(-i\hbar\frac{\partial}{\partial x}f(x)\right) - \hat{p}(xf(x)) \\
&= -i\hbar(\hat{x}f'(x)) - (\hat{p}xf(x)) \\
&= -i\hbar xf'(x) - \left(-i\hbar\frac{\partial}{\partial x}[xf(x)]\right) \\
&= -i\hbar xf'(x) + i\hbar\frac{\partial}{\partial x}[xf(x)] \\
&= i\hbar\left(-xf'(x) + \frac{\partial}{\partial x}[xf(x)]\right) \\
&= i\hbar\left(\cancel{-xf'(x)} + f(x) + \cancel{xf'(x)}\right) \\
&= i\hbar f(x)
\end{aligned}$$

We find that $[\hat{x},\hat{p}]f(x) = i\hbar f(x)$

Therefore, $[\hat{x},\hat{p}] = i\hbar \neq 0 \Rightarrow \hat{x}$ and $\hat{p}$ do not commute

Let's analyze our QHO Hamiltonian:

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$\mathcal{H} = \frac{1}{2}\left[\frac{\hat{p}^2}{m} + m\omega^2\hat{x}^2\right]$$

$$\mathcal{H} = \frac{1}{2}m\left[\frac{\hat{p}^2}{m^2} + \omega^2\hat{x}^2\right]$$

$$\mathcal{H} = \frac{1}{2}m\omega^2\left[\frac{\hat{p}^2}{m^2\omega^2} + \hat{x}^2\right]$$

$$\mathcal{H} = \frac{1}{2}m\omega^2\underbrace{\left[\left(\frac{\hat{p}}{m\omega}\right)^2 + \left(\hat{x}\right)^2\right]}$$

Notice; The arguement within the parenthesis resembles a sum of squares, i.e. something of the form $\hat{A}^2 + \hat{B}^2$, where $\hat{B} = \frac{\hat{p}}{m\omega}$ and $\hat{A} = \hat{x}$

Question; Can we factor $(\hat{A}^2 + \hat{B}^2) = (\hat{A} + i\hat{B})(\hat{A} - i\hat{B})$? ... <u>No!</u>

The above expression is incorrect because $\hat{A}$ and $\hat{B}$ are both related to the position and momentum, which do not commute. Therefore, the factored terms also do not commute.

However, we can define new operators that resembles one of these factored terms, scaled by an arbitrary constant C; $C \in \mathbb{R}$.

$$\hat{a} = C\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$$

Interestingly, the Hermition conjugate of $\hat{a}$ resembles the other factored term

$$\hat{a}^\dagger = C\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)$$

Similar to the commutation analysis we did for $\hat{x}$ and $\hat{p}$, let's compute $[a,a^\dagger]f(x)$ to better understand the behavior of our new operators.

$$[a,a^\dagger]f(x) = (aa^\dagger - a^\dagger a)f(x)$$
$$= a(a^\dagger f(x)) - a^\dagger(af(x))$$

First, lets take a detour and compute $a^\dagger f(x)$

$$a^\dagger f(x) = c\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)f(x)$$
$$= c\left(\hat{x}f(x) - i\frac{\hat{p}}{m\omega}f(x)\right)$$
$$= c\left(xf(x) - \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)f(x)\right)$$
$$= c\left(xf(x) - \frac{\hbar}{m\omega}f'(x)\right)$$

Next, let's perform an application of a on $a^\dagger f(x)$ to get $aa^\dagger f(x)$

$$aa^\dagger f(x) = a\,c\left(xf(x) - \frac{\hbar}{m\omega}f'(x)\right)$$
$$aa^\dagger f(x) = c\left[a(xf(x)) - \frac{\hbar}{m\omega}a(f'(x))\right]$$
$$aa^\dagger f(x) = c^2\left[\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)(xf(x)) - \frac{\hbar}{m\omega}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)f'(x)\right]$$
$$aa^\dagger f(x)/c^2 = \hat{x}(xf(x)) + i\frac{\hat{p}}{m\omega}(xf(x)) - \frac{\hbar}{m\omega}\hat{x}f'(x) - i\frac{\hbar}{(m\omega)^2}\hat{p}f'(x)$$
$$aa^\dagger f(x)/c^2 = x^2f(x) + \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)(xf(x)) - \frac{\hbar}{m\omega}xf'(x) - \frac{i\hbar}{(m\omega)^2}\left(-i\hbar\frac{\partial}{\partial x}\right)f'(x)$$
$$aa^\dagger f(x)/c^2 = x^2f(x) + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}(xf(x)) - \frac{\hbar}{m\omega}xf'(x) - \left(\frac{\hbar}{m\omega}\right)^2f''(x)$$
$$aa^\dagger f(x)/c^2 = x^2f(x) + \frac{\hbar}{m\omega}f(x) + \cancel{\frac{\hbar}{m\omega}xf'(x)} - \cancel{\frac{\hbar}{m\omega}xf'(x)} - \left(\frac{\hbar}{m\omega}\right)^2f''(x)$$
$$aa^\dagger f(x) = c^2\left[x^2f(x) + \frac{\hbar}{m\omega}f(x) - \left(\frac{\hbar}{m\omega}\right)^2f''(x)\right]$$

Now, let's compute $af(x)$

$$af(x) = C\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)f(x)$$

$$af(x) = C\left(\hat{x}f(x) + i\frac{\hat{p}}{m\omega}f(x)\right)$$

$$af(x) = C\left[xf(x) + \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)f(x)\right]$$

$$af(x) = C\left[xf(x) + \frac{\hbar}{m\omega}f'(x)\right]$$

Next, we want to perform an application of $a^\dagger$ onto $af(x)$ to obtain $a^\dagger af(x)$.

$$a^\dagger af(x) = a^\dagger C\left[xf(x) + \frac{\hbar}{m\omega}f'(x)\right]$$

$$a^\dagger af(x) = C\left[a^\dagger(xf(x)) + \frac{\hbar}{m\omega}a^\dagger(f'(x))\right]$$

$$a^\dagger af(x) = C^2\left[\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)(xf(x)) + \frac{\hbar}{m\omega}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right)f'(x)\right]$$

$$a^\dagger af(x) = C^2\left[\hat{x}xf(x) - \frac{i}{m\omega}\hat{p}(xf(x)) + \frac{\hbar}{m\omega}\hat{x}f'(x) - i\frac{\hbar}{(m\omega)^2}\hat{p}f'(x)\right]$$

$$a^\dagger af(x)/C^2 = x^2f(x) - \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)(xf(x)) + \frac{\hbar}{m\omega}xf'(x) - i\frac{\hbar}{(m\omega)^2}\left(-i\hbar\frac{\partial}{\partial x}\right)f'(x)$$

$$a^\dagger af(x)/C^2 = x^2f(x) - \frac{\hbar}{m\omega}\frac{\partial}{\partial x}(xf(x)) + \frac{\hbar}{m\omega}xf'(x) - \left(\frac{\hbar}{m\omega}\right)^2f''(x)$$

$$a^\dagger af(x)/C^2 = x^2f(x) - \frac{\hbar}{m\omega}f(x) - \cancel{\frac{\hbar}{m\omega}xf'(x)} + \cancel{\frac{\hbar}{m\omega}xf'(x)} - \left(\frac{\hbar}{m\omega}\right)^2f''(x)$$

$$a^\dagger af(x) = C^2\left[x^2f(x) - \frac{\hbar}{m\omega}f(x) - \left(\frac{\hbar}{m\omega}\right)^2f''(x)\right]$$

Now that we have computed expressions for $aa^\dagger f(x)$ and $a^\dagger af(x)$, let's substitute them into $[a,a^\dagger]f(x) = (aa^\dagger - a^\dagger a)f(x)$

$$\begin{cases} aa^\dagger f(x) = C^2\left[x^2f(x) + \frac{\hbar}{m\omega}f(x) - \left(\frac{\hbar}{m\omega}\right)^2f''(x)\right] \\ a^\dagger af(x) = C^2\left[x^2f(x) - \frac{\hbar}{m\omega}f(x) - \left(\frac{\hbar}{m\omega}\right)^2f''(x)\right] \end{cases}$$

$$aa^\dagger f(x) - a^\dagger af(x) = C^2\cdot\frac{2\hbar}{m\omega}f(x)$$

Therefore, $[a,a^\dagger]f(x) = C^2\cdot\frac{2\hbar}{m\omega}f(x)$

Dropping the $f(x)$ on either side,

$$[a, a^\dagger] = c^2 \cdot \frac{2\hbar}{m\omega}$$

Since c is an arbitrary constant, let's select $c = \sqrt{\frac{m\omega}{2\hbar}}$ such that

$$\boxed{[a, a^\dagger] = 1.}$$

With this new definition of c, a and $a^\dagger$ become:

$$\boxed{\begin{cases} \hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right) \\ \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - i\frac{\hat{p}}{m\omega}\right) \end{cases}}$$

We can invert these equations to write $\hat{x}$ and $\hat{p}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$

$$\hat{a} + \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(2\hat{x}\right)$$

$$\sqrt{\frac{\cancel{2}\hbar}{m\omega}}\,(a + a^\dagger) \cdot \frac{1}{\cancel{\sqrt{2}}\sqrt{2}} = \hat{x}$$

$$\Rightarrow \boxed{\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)}$$

$$a - a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\frac{2i}{m\omega}\hat{p}\right)$$

$$a - a^\dagger = \sqrt{\frac{\cancel{m\omega}}{\cancel{2}\hbar}}\;\frac{\cancel{\sqrt{2}}\sqrt{2}\,i}{\cancel{\sqrt{m\omega}}\sqrt{m\omega}}\;\hat{p}$$

$$a - a^\dagger = \sqrt{\frac{2}{m\omega\hbar}}\; i\hat{p}$$

$$-i\sqrt{\frac{m\omega\hbar}{2}}\,(a - a^\dagger) = \hat{p}$$

$$\Rightarrow \boxed{\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\,(a^\dagger - a)}$$

Next, we can substitute $\hat{x}$ and $\hat{p}$ into $\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ to write $\mathcal{H}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)$$

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}\,(a^\dagger - a)$$

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$\mathcal{H} = \frac{1}{2m}\left(i\sqrt{\frac{m\omega\hbar}{2}}\,(a^\dagger - a)\right)^2 + \frac{1}{2}m\omega^2\left(\sqrt{\frac{\hbar}{2m\omega}}\,(a + a^\dagger)\right)^2$$

$$\mathcal{H} = \frac{-1}{2\cancel{m}}\left(\frac{\cancel{m}\omega\hbar}{2}\right)(a^\dagger - a)^2 + \frac{1}{2}\cancel{m}\omega^{\cancel{2}}\left(\frac{\hbar}{2\cancel{m}\cancel{\omega}}\right)(a + a^\dagger)^2$$

$$\mathcal{H} = -\frac{\hbar\omega}{4}(a^\dagger - a)^2 + \frac{\hbar\omega}{4}(a + a^\dagger)^2$$

$$\mathcal{H} = \frac{\hbar\omega}{4}\left[(a + a^\dagger)^2 - (a^\dagger - a)^2\right]$$

$$\mathcal{H} = \frac{\hbar\omega}{4}\left[(a + a^\dagger)(a + a^\dagger) - (a^\dagger - a)(a^\dagger - a)\right]$$

$$\mathcal{H} = \frac{\hbar\omega}{4}\left[aa + a^\dagger a^\dagger + a^\dagger a + aa^\dagger - (a^\dagger a^\dagger + aa - aa^\dagger - a^\dagger a)\right]$$

$$\mathcal{H} = \frac{\hbar\omega}{4}\left[\cancel{aa} + \cancel{a^\dagger a^\dagger} + a^\dagger a + aa^\dagger - \cancel{a^\dagger a^\dagger} - \cancel{aa} + aa^\dagger + a^\dagger a\right]$$

$$\mathcal{H} = \frac{\hbar\omega}{4}\left[2a^\dagger a + 2aa^\dagger\right]$$

$$\boxed{\mathcal{H} = \frac{\hbar\omega}{2}(a^\dagger a + aa^\dagger)}$$

Which is the Hamiltonian of the system in terms of a and $a^\dagger$

We can also take advantage of the fact that $[a, a^\dagger] = 1$ to rewrite our Hamiltonian in a more compact form

$$\mathcal{H} = \frac{\hbar\omega}{2}\left(a^\dagger a + a a^\dagger\right)$$

$$[a, a^\dagger] = a a^\dagger - a^\dagger a = 1$$

$$\Rightarrow \quad a a^\dagger = 1 + a^\dagger a$$

$$\mathcal{H} = \frac{\hbar\omega}{2}\left[a^\dagger a + (1 + a^\dagger a)\right]$$

$$\mathcal{H} = \frac{\hbar\omega}{2}\left[2a^\dagger a + 1\right]$$

$$\boxed{\mathcal{H} = \hbar\omega\left(\tfrac{1}{2} + a^\dagger a\right)} \Leftarrow$$ The Hamiltonian of the QHO expressed in terms of $\hat{a}$ and $\hat{a}^\dagger$.

Raising the Energy

The reason why we even bother writing $\mathcal{H}$ as a function of a and $a^\dagger$ is because of what these operators actually do. We will examine the meaning of $a^\dagger$ in this section.

Let's start with the Time-Independent Schrodinger Equation (TIDSE)

$$\underbrace{\mathcal{H}\psi(x) = \mathcal{E}\psi(x)}$$

This equation says that a wavefunction $\psi(x)$ solves the TIDSE with eigenenergy $\mathcal{E}$.

Suppose we had a new wavefunction $a^\dagger\psi(x)$ act on $\mathcal{H}$ such that

$$\underbrace{\mathcal{H}(a^\dagger\psi(x)) = \mathcal{E}'(a^\dagger\psi(x))}$$

This equation says that $a^\dagger\psi(x)$ solves the TIDSE with eigenenergy $\mathcal{E}'$

Let's substitute in $\mathcal{H} = \hbar\omega\left(\frac{1}{2} + a^\dagger a\right)$

$$\hbar\omega\left(\tfrac{1}{2} + a^\dagger a\right)\left(a^\dagger\psi(x)\right) = \mathcal{E}' a^\dagger\psi(x)$$

$$\hbar\omega\left(\tfrac{1}{2}a^\dagger + a^\dagger a a^\dagger\right)\psi(x) = \mathcal{E}' a^\dagger\psi(x)$$

$$\hbar\omega\, a^\dagger\left(\tfrac{1}{2} + a a^\dagger\right)\psi(x) = \mathcal{E}' a^\dagger\psi(x)$$

$$\hbar\omega a^\dagger\left(\tfrac{1}{2}+aa^\dagger\right)\psi(x)=\varepsilon' a^\dagger\psi(x)$$

Recall, $[a,a^\dagger]=1 \Rightarrow aa^\dagger = 1+a^\dagger a$. Substituting this in,

$$\hbar\omega a^\dagger\left(\tfrac{1}{2}+(1+a^\dagger a)\right)\psi(x)=\varepsilon' a^\dagger\psi(x)$$

$$a^\dagger\Big(\hbar\omega+\underbrace{\hbar\omega\left(\tfrac{1}{2}+a^\dagger a\right)}_{=\mathcal{H}}\Big)\psi(x)=\varepsilon' a^\dagger\psi(x)$$

$$a^\dagger(\hbar\omega+\mathcal{H})\psi(x)=\varepsilon' a^\dagger\psi(x)$$

$$\hbar\omega a^\dagger\psi(x)+a^\dagger\underbrace{\mathcal{H}\psi(x)}=\varepsilon' a^\dagger\psi(x)$$

TIDSE says that $\mathcal{H}\psi(x)=\varepsilon\psi(x)$

$$\hbar\omega a^\dagger\psi(x)+a^\dagger\varepsilon\psi(x)=\varepsilon' a^\dagger\psi(x)$$

$$\underline{(\varepsilon+\hbar\omega)}\,a^\dagger\psi(x)=\underline{\varepsilon}' a^\dagger\psi(x)$$

Therefore, $\varepsilon' = \varepsilon + \underbrace{\hbar\omega}$

ε': Energy of $a^\dagger\psi(x)$

ε: Energy of $\psi(x)$

$\hbar\omega$: Energy Shift

Therefore, we can say that an application of the $a^\dagger$ operator on $\psi(x)$ raises the energy by a factor of $\hbar\omega$.

We define $a^\dagger$ as the 'raising operator' for having this property.

<u>Lowering the energy</u>

Suppose that once again, we have a wavefunction $\psi(x)$ that solves the TIDSE with eigenenergy $\mathcal{E}$.

$$\mathcal{H}\psi(x) = \mathcal{E}\psi(x)$$

Suppose we have a new wavefunction, $a\psi(x)$ that acts on $\mathcal{H}$ such that $a\psi(x)$ solves the TIDSE with eigenenergy $\mathcal{E}'$

$$\mathcal{H}(a\psi(x)) = \mathcal{E}'(a\psi(x))$$

Again, substituting in $\mathcal{H} = \hbar\omega(\frac{1}{2} + a^\dagger a)$

$$\hbar\omega(\tfrac{1}{2} + a^\dagger a)a\psi(x) = \mathcal{E}' a\psi(x)$$

$$\hbar\omega(\tfrac{1}{2}a + a^\dagger aa)\psi(x) = \mathcal{E}' a\psi(x)$$

Recall, $[a, a^\dagger] = 1 \Rightarrow aa^\dagger - a^\dagger a = 1 \Rightarrow -a^\dagger a = 1 - aa^\dagger$
$\Rightarrow \underline{a^\dagger a = aa^\dagger - 1}$ ← substituting this in

$$\hbar\omega\left[\tfrac{1}{2}a + (aa^\dagger - 1)a\right]\psi(x) = \mathcal{E}' a\psi(x)$$

$$\hbar\omega\left[\tfrac{1}{2}a + aa^\dagger a - a\right]\psi(x) = \mathcal{E}' a\psi(x)$$

$$\hbar\omega\left[-\tfrac{1}{2}a + aa^\dagger a\right]\psi(x) = \mathcal{E}' a\psi(x)$$

$$a\hbar\omega\left[-\tfrac{1}{2} + a^\dagger a\right]\psi(x) = \mathcal{E}' a\psi(x)$$

$$a\hbar\omega\left[\tfrac{1}{2} - \tfrac{1}{2} - \tfrac{1}{2} + a^\dagger a\right]\psi(x) = \mathcal{E}' a\psi(x)$$

$$a\hbar\omega\left(\tfrac{1}{2} - 1 + a^\dagger a\right)\psi(x) = \mathcal{E}' a\psi(x)$$

$$a\left(-\hbar\omega + \underbrace{\hbar\omega\left(\tfrac{1}{2} + a^\dagger a\right)}_{= \mathcal{H}}\right)\psi(x) = \mathcal{E}' a\psi(x)$$

$$a\left(-\hbar\omega + \mathcal{H}\right)\Psi(x) = \mathcal{E}' a \Psi(x)$$

$$-\hbar\omega a\Psi(x) + a\underbrace{\mathcal{H}\Psi(x)} = \mathcal{E}' a\Psi(x)$$

$\Longrightarrow$ TIDSE says that $\mathcal{H}\Psi(x) = \mathcal{E}\Psi(x)$

$$-\hbar\omega a\Psi(x) + a\mathcal{E}\Psi(x) = \mathcal{E}' a\Psi(x)$$

$$(-\hbar\omega + \mathcal{E})\, a\Psi(x) = \mathcal{E}' a\Psi(x)$$

Therefore, $\mathcal{E}' = \mathcal{E} \underbrace{- \hbar\omega}$

Energy of $a\Psi(x)$ Energy of $\Psi(x)$ Energy Shift

Therefore, an application of a on $\Psi(x)$ Lowers the energy by $\hbar\omega$.

We define a as the "lowering operator" in recognition of this property.

The ground state of the QHO.

We can raise the energy of the QHO with $a^\dagger$ and lower the energy by a. However, we cannot apply unlimited applications of a because eventually, we will reach energies equal to or less than 0, which are physically impossible. We will define $\Psi_0(x)$ as the ground state of the Harmonic oscillator with eigenenergy $\mathcal{E}_0$.

$$\mathcal{H}\Psi_0(x) = \mathcal{E}_0\Psi_0(x)$$

Since $\Psi_0(x)$ is the ground state, it is the lower band of QHO eigenstates. Therefore, applying a lowering operator a to $\Psi_0(x)$ will lower the energy to 0 which is indicative of the vacuum state (a state of 0 energy).

Therefore, $\underbrace{a\Psi_0(x) = 0}$

'a' annihilates Ψ_0 resulting in vacuum.

We can treat $a\Psi_0(x)=0$ as a boundary condition to $\mathcal{H}\Psi_0(x) = \mathcal{E}_0\Psi_0(x)$

$$\mathcal{H}\Psi_0(x) = \mathcal{E}_0\Psi_0(x)$$

$$\hbar\omega\left(\tfrac{1}{2} + a^\dagger a\right)\Psi_0(x) = \mathcal{E}_0\Psi_0(x)$$

$$\hbar\omega\left(\tfrac{1}{2}\Psi_0(x) + a^\dagger \underbrace{a\Psi_0(x)}_{=0}\right) = \mathcal{E}_0\Psi_0(x)$$

$$\tfrac{1}{2}\hbar\omega\Psi_0(x) = \mathcal{E}_0\Psi_0(x)$$

$$\Rightarrow \boxed{\mathcal{E}_0 = \tfrac{1}{2}\hbar\omega} \Leftarrow$$ Grand State energy of The QHO

We can use the 'raising operator' $a^\dagger$ to increase the grand state energy by a factor of $\hbar\omega$, and repeat this process to generate the energy spectrum of the QHO

$$\boxed{\mathcal{E}_n = \hbar\omega\left(n + \tfrac{1}{2}\right) \;;\; n \in \{0,1,2,\dots\}}$$

III In addition to our eigenenergies $\mathcal{E}_n$, we can also calculate the eigenstates of the QHO starting with $\psi_0(x)$

Again, starting with our boundary condition,

$$a\psi_0(x) = 0$$

Recall, $a = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)$ $\quad \begin{cases} \hat{x} = x \\ \hat{p} = -i\hbar\,\partial/\partial x \end{cases}$

$$\left(\hat{x} + i\frac{\hat{p}}{m\omega}\right)\psi_0(x) = 0$$

$$\left(x + \frac{\hbar}{m\omega}\partial_x\right)\psi_0(x) = 0$$

This is just a first order differential equation that can be solved via separation of variables

$$x\psi_0(x) + \frac{\hbar}{m\omega}\psi_0'(x) = 0$$

$$\frac{\hbar}{m\omega}\psi_0'(x) = -x\psi_0(x)$$

$$\frac{\psi_0'(x)}{\psi_0(x)} = -x\frac{m\omega}{\hbar}$$

$$\frac{d}{dx}\ln(\psi_0(x)) = -x\frac{m\omega}{\hbar}$$

$$\int\frac{d}{dx}\ln(\psi_0(x))dx = \int\left(-x\frac{m\omega}{\hbar}\right)dx$$

$$\ln(\psi_0(x)) = -\frac{m\omega}{2\hbar}x^2 + C$$

$$\Rightarrow \psi_0(x) = Ce^{-\frac{m\omega}{2\hbar}x^2}$$

Our constant of integration C can be determined by the constraint that wavefunctions must be normalized; i.e. $\int_{-\infty}^{+\infty}|\psi(x)|^2dx = 1$

under this normalization constraint, we find the ground state wavefunction of the QHO to be

$$\boxed{\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}e^{-\frac{m\omega}{2\hbar}x^2}}$$