Writing the 2nd Quantization form of the Two Coupled Transmons as a Matrix

$$\mathcal{H} = \sum_{i\in\{1,2\}} \left[\hbar\omega_i\left(a_i^\dagger a_i + \frac{1}{2}\right) + \frac{\alpha_i}{2} a_i^\dagger a_i^\dagger a_i a_i\right] - \hbar g (a_1^\dagger - a_1)(a_2^\dagger - a_2)$$

For a qubit we can omit the anharmonic term by focusing only on the first two energy levels, $|0\rangle$ and $|1\rangle$

$$\mathcal{H} = \sum_{i\in\{1,2\}} \left[\hbar\omega_i\left(a_i^\dagger a_i + \frac{1}{2}\right)\right] - \hbar g (a_1^\dagger - a_1)(a_2^\dagger - a_2)$$

We make the transformation:

$$\begin{cases} a_i^\dagger \to \sigma_i^+ \\ a_i \to \sigma_i^- \end{cases} \quad \text{where } \sigma^\pm = \frac{\sigma_x \pm i\sigma_y}{2}$$

$$a^\dagger a \to \sigma^+\sigma^- = \left(\frac{\sigma_x + i\sigma_y}{2}\right)\left(\frac{\sigma_x - i\sigma_y}{2}\right)$$

$$\sigma^+\sigma^- = \frac{1}{4}\left[\sigma_x^2 + \sigma_y^2 + i\sigma_y\sigma_x - i\sigma_x\sigma_y\right]$$

$$\sigma^+\sigma^- = \frac{1}{4}\left(2\mathbb{I} - i[\sigma_x,\sigma_y]\right) \qquad [\sigma_x,\sigma_y] = -2i\sigma_z$$

$$\sigma^+\sigma^- = \frac{1}{4}\left(2\mathbb{I} - i(-2i\sigma_z)\right)$$

$$\sigma^+\sigma^- = \frac{1}{4}\left(2\mathbb{I} - 2\sigma_z\right)$$

$$\sigma^+\sigma^- = \frac{\mathbb{I} - \sigma_z}{2} = \frac{1}{2}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix} - \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right] = \begin{pmatrix}0&0\\0&1\end{pmatrix}$$

$$\boxed{\sigma^+\sigma^- = \frac{\mathbb{I} - \sigma_z}{2}}$$

$$\mathcal{H}_i^{QHO} = \hbar\omega_i\left(a_i^\dagger a_i + \frac{1}{2}\right)$$

$$\sigma^+\sigma^- = \frac{\mathbb{I} - \sigma_z}{2}$$

$$\mathcal{H}_i^{QHO} = \hbar\omega_i\left[\frac{1}{2} - \frac{\sigma_{z,i}}{2} + \frac{1}{2}\right]$$

$$\mathcal{H}_i^{QHO} = \hbar\omega_i\left[1 - \frac{\sigma_{z,i}}{2}\right]$$

Just an arbitrary scaling in the potential

$$\boxed{\mathcal{H}_i^{QHO} = -\frac{\hbar\omega_i}{2}\sigma_i^z}$$

The anharmonic part involves $(a_1^\dagger - a_1)(a_2^\dagger - a_2)$

$$(a_1^\dagger - a_1)(a_2^\dagger - a_2) \longrightarrow (\sigma_1^+ - \sigma_1^-)(\sigma_2^+ - \sigma_2^-)$$

$$= (\sigma^+ - \sigma^-) \otimes (\sigma^+ - \sigma^-)$$

$$\sigma^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sigma^+ - \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^+ - \sigma^- = i\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} = i\sigma_y$$

$$\Rightarrow (a_1^\dagger - a_1)(a_2^\dagger - a_2) \longrightarrow (i\sigma_1^y)(i\sigma_2^y) = -\sigma_1^y\sigma_2^y$$

Since the coupling part of the Hamiltonian is

$$-\hbar g \underbrace{(a_1^\dagger - a_1)(a_2^\dagger - a_2)}_{= -\sigma_1^y \sigma_2^y}$$

$$= \hbar g \sigma_1^y \sigma_2^y$$

Therefore the entire Hamiltonian of the two level system can be expressed as

$$\mathcal{H} = \mathcal{H}_1^{QHO} + \mathcal{H}_2^{QHO} + \hbar g \sigma_1^y \sigma_2^y$$

$$\mathcal{H} = -\frac{\hbar\omega_1}{2}\sigma_1^z - \frac{\hbar\omega_2}{2}\sigma_2^z + \hbar g \sigma_1^y \sigma_2^y$$

$$\boxed{\mathcal{H} = \sum_{i \in \{1,2\}} \left(-\frac{\hbar\omega_i}{2}\sigma_i^z\right) + \hbar g \sigma_1^y \sigma_2^y}$$