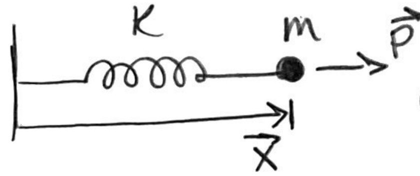
The LC Tank Oscillator

Before, in my "Quantum Harmonic Oscillator Derivation" Notes, we investigated a harmonic oscillator based on a mechanical system with well-defined position and momentum.

$$\omega = \sqrt{K/m}$$



(Mass on a spring system)

We wrote down a Hamiltonian

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

Applied the First Quantization with

$$\begin{cases} \hat{x} = x \\ \hat{p} = -i\hbar \, \partial/\partial x \end{cases}$$

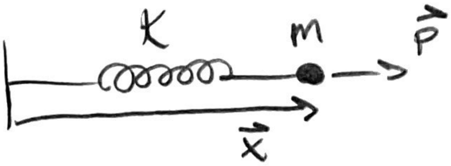
Eventually, we wrote down the second quantization of the QHO Hamiltonian as

$$\mathcal{H} = \hbar\omega\left(\frac{1}{2} + a^\dagger a\right)$$

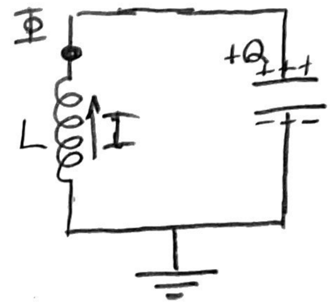
Unfortunately, this is not enough to make a qubit. We can't make a qubit with a mass $ spring system since #1) A Harmonic Oscillator is a bad qubit, we need an anharmonic oscillator and #2) We don't have any quantum ingredients... quantum technology cannot be built by only classical ingredients; we need at least one quantum ingredient.

Luckily for us, we can meet both criteria #1 and #2 in the superconducting circuit world. The LC Tank circuit is the electrical analog of a mass on a spring system

Mechanical Harmonic Oscillator



$\Longleftrightarrow$



Electrical Harmonic Oscillator.

$$\mathcal{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

$$\mathcal{H} = \underbrace{\tfrac{1}{2}CV^2}_{\text{Energy of a capacitor}} + \underbrace{\tfrac{1}{2}LI^2}_{\text{Energy of an Inductor}}$$

$$\mathcal{H} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$$

Where Q is the charge on a capacitor plate ; $Q = CV$

And Φ is the Flux through an inductor ; $\Phi = LI$

It's more appropriate to express $\mathcal{H} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L}$ in terms of superconducting circuit language.

We define:

- Reduced Flux : $\phi \equiv 2\pi\Phi/\Phi_0$; $\underbrace{\Phi_0}_{\text{"Magnetic Flux Quantum"}} = h/2e$
- Reduced Charge : $n \equiv Q/2e$ (Excess number of Cooper pairs on the island)
- Charging Energy : $E_C \equiv e^2/2C$ (Energy required to add a Cooper pair to the island)
- Inductive Energy : $E_L = \frac{(\Phi_0/2\pi)^2}{L}$ (Proportional to the energy of an inductor)

Utilizing these definitions, we can derive

$$\mathcal{H} = 4E_C n^2 + \tfrac{1}{2}E_L\phi^2$$

Proof: Show that $\mathcal{H} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} = 4E_C n^2 + \frac{1}{2}E_L \phi^2$

$$\mathcal{H} = 4E_C n^2 + \frac{1}{2}E_L \phi^2$$

$$\mathcal{H} = 4\left(\frac{e^2}{2C}\right)\left(\frac{Q}{2e}\right)^2 + \frac{1}{2}\left[\frac{(\Phi_0/2\pi)^2}{L}\right]\left(\frac{2\pi\Phi}{\Phi_0}\right)^2$$

$$\mathcal{H} = \cancel{4}\left(\frac{\cancel{e^2}}{2C}\right)\left(\frac{Q^2}{\cancel{4e^2}}\right) + \frac{1}{2}\left(\frac{\cancel{\Phi_0^2}}{\cancel{4\pi^2}L}\right)\left(\frac{\cancel{4\pi^2}\Phi^2}{\cancel{\Phi_0^2}}\right)$$

$$\mathcal{H} = \frac{Q^2}{2C} + \frac{1}{2}\frac{\Phi^2}{L} \quad \blacksquare$$

Let's compare the mechanical – electrical Hamiltonian analogs

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \quad \text{(Mechanical)}$$

$$\mathcal{H} = \frac{Q^2}{2C} + \frac{\Phi^2}{2L} \quad \text{(Electrical)}$$

$$= 4E_C n^2 + \frac{1}{2}E_L \phi^2 \quad \text{(Superconducting Circuit)}$$

Here, the analog between the mechanical & electrical variables seems to be

	Electrical		Mechanical	
Charge	Q	$\longleftrightarrow$	p	momentum
flux	Φ	$\longleftrightarrow$	x	position
Capacitance	C	$\longleftrightarrow$	m	mass
Inductance	L	$\longleftrightarrow$	$\frac{1}{m\omega^2} = \frac{1}{k}$	Inverse Spring Constant

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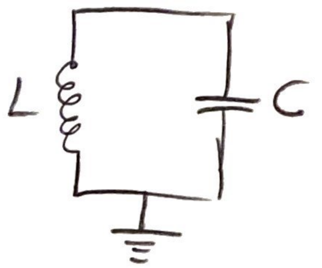
Things to do:
- write a and $a^\dagger$ in terms of electrical variables
- Make a note that $[\hat{\Phi}, \hat{Q}] = i\hbar$

The resonant frequency of an LC circuit is



$$\omega = \sqrt{\frac{1}{LC}}$$

$\omega = \frac{1}{\sqrt{LC}}$ is written in terms of electrical circuit language.

To translate to superconducting circuit language we should write L and C in terms of E_L and E_C, respectively.

$$E_C = \frac{e^2}{2C} \qquad E_L = \frac{(\Phi_0/2\pi)^2}{L}$$

$$C = \frac{e^2}{2E_C} \qquad L = \frac{(\Phi_0/2\pi)^2}{E_L}$$

$$LC = \left(\frac{e^2}{2E_C}\right)\left(\frac{1}{E_L}\left(\frac{\Phi_0}{2\pi}\right)^2\right)$$

$$LC = \left(\frac{e^2}{2E_C}\right)\left(\frac{\Phi_0^2}{4\pi^2 E_L}\right)$$

$$LC = \frac{e^2\Phi_0^2}{8\pi^2 E_C E_L}$$

$$\omega^2 = \frac{1}{LC} = \frac{\pi^2}{e^2\Phi_0^2} E_C E_L$$

$$\omega = \frac{\pi}{e\Phi_0}\sqrt{8E_C E_L}$$

Substituting in $\Phi_0 = \frac{h}{2e}$

$$\omega = \frac{\pi}{\cancel{e}} \cdot \frac{2\cancel{e}}{h}\sqrt{8E_C E_L}$$

$$\boxed{\omega = \sqrt{8E_C E_L}/\hbar}$$

Resonant Frequency of the LC circuit in terms of charging and inductance energies.