Numerical Solutions to the Wave Equation

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1 Abstract

The goal for this project is to numerically solve the wave equation for different situations. Our first situation, and the coding test-bed for the project, is to numerically solve the wave equation in 1 dimension. This could be a bound guitar string, for instance. Next, we can expand to 2 dimensions and numerically solve the wave equation for a bounded rectangular plate and a bounded circular drum. Then we can introduce curvature into our surfaces, and see how the solutions of the wave equation behave on a curved surface like a sphere.

Having analytical solutions is nice since we can compare them to our numerical solutions. However, sometimes it is impossible to obtain analytical solutions. That's why numerical solutions play such a big role. They describe phenomenon that we can't exactly describe with pen and paper.

Electromagnetic waves, mechanical waves, Chladni patterns, and membrane mechanics are all physics fields of study that essentially depend on the wave equation. Hopefully, by further studying more complicated and intricate scenarios of the wave equation, we can advance these research areas.

2 Results

Each wave equation scenario has infinitely many standing wave solutions. Listed below are Youtube video links that contain the first few modes of standing waves for each wave situation. (*These videos are my own work from my own code. Each solution is derived numerically*)

1D Wave Equation Solutions
`https://www.youtube.com/watch?v=QxEP6LINeR8`
Initial Position: $u(x,t=0) = Asin(x\frac{n\pi}{L})$

2D Rectangular Plate Wave Equation Solution
`https://www.youtube.com/watch?v=F7fg599uEfI`
Initial Position: $u(x,y,t=0) = Asin(x\frac{\pi n}{L_x})sin(y\frac{\pi m}{L_y})$

2D Circular Drum Wave Equation Solution
`https://www.youtube.com/watch?v=_V-VPZ5qxbk`

Initial Position: $u(\rho, \phi, t = 0) = [Asin(m\phi) + Bcos(m\phi)]J_m(\rho\frac{\mu_{m,n}}{L})$

Spherical Surface Wave Equation Solution
`https://www.youtube.com/watch?v=UUMtCsr4suw`
Initial Position: $u(\theta, \phi, t = 0) = P_l^m(cos\theta)[Asin(\sqrt{l(l+1)}\phi)+Bcos(\sqrt{l(l+1)}\phi)]$

Torus Surface Wave Equation solution
`https://www.youtube.com/watch?v=v7LfcGI7g5w`
Initial Position: $u(\phi, \theta, t = 0) = Asin(\phi)sin(\theta)$

3 Preliminary Mathematics

3.1 The Wave Equation

The wave equation

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{1}$$

Describes the behavior of vibrating medium. v is the wave speed, and u is the displacement of the vibration, normal to the surface.

If we parameterize a surface **S** as the surface we wish to solve the wave equation on, then we can describe the displacement of each particle as **W**

$$\mathbf{W} = \mathbf{S} + \mathbf{u}\hat{n} \tag{2}$$

where $\hat{n}$ is the normal unit vector to the surface **S**.

3.2 The Laplacian in Different Coordinate Systems

3.2.1 Cartesian

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{3}$$

3.2.2 Cylindrical

$$\nabla^2 u = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\frac{\partial u}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial z^2} \tag{4}$$

3.2.3 Spherical

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}(sin\theta\frac{\partial u}{\partial \theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} \tag{5}$$

3.2.4 Curvilinear Coordinates

$$\nabla^2 u = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \xi^i} (\sqrt{g} g^{ij} \frac{\partial u}{\partial \xi^j}) \tag{6}$$

where g_{ij} is the metric tensor

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j = \frac{\partial \mathbf{S}}{\partial \xi^i} \cdot \frac{\partial \mathbf{S}}{\partial \xi^j} \tag{7}$$

and g^{ij} is the inverse of g_{ij}.

$$g^{ij} = (g_{ij})^{-1} \tag{8}$$

and g is the determinant of gij

$$g = det(g_{ij}) \tag{9}$$

3.3 Special Functions

Special functions like the Bessel function and Legendre Polynomials appear in the solutions to the wave equation for the disk and the sphere (respectively).

3.3.1 Bessel Functions

Bessel's Equation

$$x^2 y'' + xy' + (x^2 - v^2)y = 0 \tag{10}$$

Bessel Function of the First Kind
For an integer value of n,

$$J_n(x) = \frac{1}{\pi} \int_0^\pi cos(n\theta - xsin\theta) d\theta \tag{11}$$

Bessel Function of the Second Kind

$$Y_n(x) = \frac{J_n(x) cos(n\pi) - J_{-n}(x)}{sin(n\pi)} \tag{12}$$

3.3.2 Legendre Polynomials

Legendre's differential equation is:

$$(1 - x^2)y'' - 2xy' + l(l+1)y = 0 \tag{13}$$

The Rodrigues' Formula for the Legendre Polynomials is

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l \tag{14}$$

3.3.3 Associated Legendre Polynomials

Associated Legendre Equation

$$(1-x^2)y'' - 2xy' + [l(l+1) - \frac{m^2}{1-x^2}]y = 0 \tag{15}$$

The solution is $y = c_1 P_l^m(x) + c_2 Q_l^m(x)$

For $m \leq |l|$, $m\epsilon\mathbb{Z}$, and $l\epsilon\mathbb{Z}$, we can generate the Associated Legendre Polynomials from the Legendre Polynomials.

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m P_l(x)}{dx^m}$$
$$Q_l^m(x) = (1-x^2)^{m/2}\frac{d^m Q_l(x)}{dx^m}$$

3.3.4 Derivative Approximations

We will approximate derivatives using a central difference.

First Derivatives

$$\frac{\partial u(x,y,t)}{\partial x} \approx \frac{u(x+a,y,t) - u(x-a,y,t)}{2a} = \frac{u_{i+1,j}^k - u_{i-1,j}^k}{2a}$$

$$\frac{\partial u(x,y,t)}{\partial y} \approx \frac{u(x,y+b,t) - u(x,y-b,t)}{2b} = \frac{u_{i,j+1}^k - u_{i,j-1}^k}{2b}$$

$$\frac{\partial u(x,y,t)}{\partial t} \approx \frac{u(x,y,t+h) - u(x,y,t-h)}{2h} = \frac{u_{i,j}^{k+1} - u_{i,j}^{k-1}}{2h}$$

Second Derivatives

$$\frac{\partial^2 u(x,y,t)}{\partial x^2} \approx \frac{u(x+a,y,t) - 2u(x,y,t) + u(x-a,y,t)}{a^2} = \frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k}{a^2}$$

$$\frac{\partial^2 u(x,y,t)}{\partial y^2} \approx \frac{u(x,y+b,t) - 2u(x,y,t) + u(x,y-b,t)}{b^2} = \frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{b^2}$$

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} \approx \frac{u(x,y,t+h) - 2u(x,y,t) + u(x,y,t-h)}{h^2} = \frac{u_{i,j}^{k+1} - 2u_{i,j}^k + u_{i,j}^{k-1}}{h^2}$$

Note: Although I represent the derivative approximations in terms of a Cartesian Coordinate system, the approximations hold true for any coordinate system. If we make the substitution $x \to \xi^1$ and $y \to \xi^2$, where ξ^1 and ξ^2 are arbitrary coordinates of some coordinate system, the derivative approximations above still hold. This means we are able to use the above approximations for cylindrical coordinates, spherical coordinates, etc.

4 1D Wave Equation: Guitar String

4.1 Analytical Solution

We can use (1) in 1D Cartesian coordinates to model the bound guitar string.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{16}$$

We can use separation of variables. i.e. $u(x,t) = X(x)T(t)$ and isolate functions of x and functions of t.

$$\frac{X''(x)}{X(x)} = \frac{1}{v^2}\frac{T''(t)}{T(t)} = -\lambda^2 \tag{17}$$

We are allowed to equate the expression to some arbitrary constant $-\lambda^2$, since taking a partial derivative with respect to either position or time on both sides of (17) will give us 0. i.e. $\frac{\partial}{\partial x}(\frac{X''(x)}{X(x)}) = 0$ or $\frac{\partial}{\partial t}(\frac{1}{v^2}\frac{T''(t)}{T(t)}) = 0$. If the derivative of some expression is 0 then that expression must be equivalent to some constant eq (17).

We can solve (17) for $X(x)$ and $T(t)$ individually.

$$T''(t) + \lambda^2 v^2 T(t) = 0 \tag{18}$$

This can be solved with the characteristic equation ansatz. We get:

$$T(t) = C_{1t}sin(v\lambda t) + C_{2t}cos(v\lambda t) \tag{19}$$

We don't have to worry about enforcing any boundary conditions on $T(t)$, but we must consider boundary conditions for $X(x)$.

$$X''(x) + \lambda^2 X(x) = 0 \tag{20}$$

$$X(x) = C_{1x}sin(\lambda x) + C_{2x}cos(\lambda x) \tag{21}$$

Remember, we're modeling a guitar string of length L that is bounded at both ends such that $u(x = 0, t) = u(x = L, t) = 0$. This means our boundary condition for $X(x)$ is $X(0) = X(L) = 0$.

$X(0) = C_{2x} = 0$

$X(L) = C_{1x}sin(\lambda L) \rightarrow \lambda L = \pi, 2\pi, 3\pi ... = \pi n \ ; \ n \epsilon \mathbb{N}$

We can write our final solution as $u(x,t) = X(x)T(t)$

$$u(x,t) = sin(\lambda x)(Acos(v\lambda t) + Bsin(v\lambda t)) \ ; \ \lambda = \frac{\pi n}{L} \text{ where } \lambda \epsilon \mathbb{N} \tag{22}$$

We could expand upon (22) as a sum over n and solve for A and B as Fourier coefficients to form the general solution, but we will ignore that step since we want to compare different modes rather than superpositions of modes to easily verify our computational solution in Python.

4.2 Numerical Solution

We can start with the 1D wave equation (16) and substitute in the derivative approximation for $\frac{\partial^2 u}{\partial t^2}$.

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= v^2 \frac{\partial^2 u}{\partial x^2} \\ \frac{u_i^{k+1} - 2u_i^k + u_i^{k-1}}{h^2} &= v^2 \frac{\partial^2 u}{\partial x^2}\end{aligned}$$

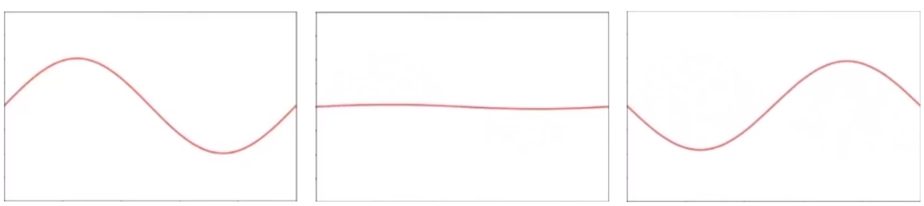
Solving for u_i^{k+1},

We now have a numerical scheme for solving the 1D wave equation.

$$\begin{aligned}u_i^{k+1} &= h^2 v^2 \frac{\partial^2 u}{\partial x^2} + 2u_i^k - u_i^{k-1} \\ \frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{a^2}\end{aligned}$$

4.3 Time Evolution

The images listed below visually describe the time evolution of the wave from its initial position.



5 Pseudocode

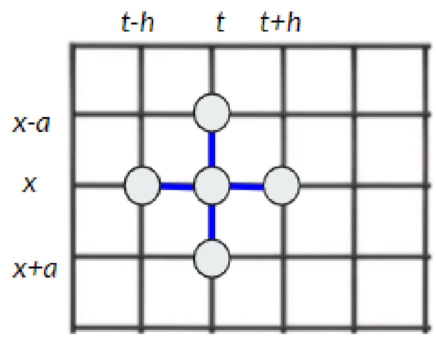
Here, I outline the generalized algorithm to solve any wave equation given a the appropriate numerical scheme for the situation.

1) Prompt $u_{i,j}^0$ at $t = 0$ with an initial position and appropriate boundary conditions.

2) Approximate u at the 2nd time step $t = h$ by using Euler's Method. If ψ is the initial velocity of $u_{i,j}^0$, then $u_{i,j}^1 \approx u_{i,j}^0 + \psi h$

3) Finally, we loop over our numerical scheme (in 3.2, for instance) as many times as we'd like using appropriate time and spacial steps while respecting our boundary conditions.

Figure 1: Numerical Scheme For Solving Wave Equations



New FTCS
Skeleton Structure

6 2D Wave Equation: Rectangular Plate

6.1 Analytical Solution

The most natural progression from the 1D wave equation is to tackle the 2D wave equation. In a similar manner to the guitar string, we will fix the edges of the plate such that they will not oscillate. For a $L_x \times L_y$ rectangle, our boundary conditions are $u(L_x, y, t) = u(x, L_y, t) = 0$.

Let's start with the general form of the wave equation (1)

$$\begin{aligned} \nabla^2 u &= \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \end{aligned}$$

We can use separation of variables $u(x, y, t) = X(x)Y(y)T(t)$.

$$\begin{aligned} &\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{1}{v^2}\frac{T''(t)}{T(t)} = -\lambda_1^2 \\ &T''(t) + \lambda_1^2 v^2 T(t) = 0 \\ &\quad T(t) = C_{1t}sin(\lambda_1 vt) + C_{2t}cos(\lambda_1 vt) \end{aligned}$$

Now, we can solve for $X(x)$ and $Y(y)$ by splitting our constant $\lambda_1^2 = \lambda_2^2 + \lambda_3^2$.

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda_2^2 - \lambda_3^2$$
$$\frac{X''(x)}{X(x)} = -\lambda_2^2$$
$$X''(x) = \lambda_2^2 X(x) = 0$$

$$X(x) = C_{1x} sin(\lambda_2 x) + C_{2x} cos(\lambda_2 x) \tag{23}$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\lambda_2^2 - \lambda_3^2$$
$$\frac{Y''(y)}{Y(y)} = -\lambda_3^2$$
$$Y''(y) + \lambda_3^2 Y(y) = 0$$

$$Y(y) = C_{1y} sin(\lambda_3 y) + C_{2y} cos(\lambda_3 y) \tag{24}$$

Let's enforce our boundary condition: $u(L_x, y, t) = u(x, L_y, t) = 0$ on (23) and (24).

$$X(0) = X(L_x) = 0 \to C_{2x} = 0 \text{ and } \lambda_2 = \tfrac{\pi n}{L_x}$$
$$Y(0) = Y(L_y) = 0 \to C_{2y} = 0 \text{ and } \lambda_3 = \tfrac{\pi m}{L_y}$$

Also, since $\lambda_1^2 = \lambda_2^2 + \lambda_3^2$, $\lambda_1 = \pi\sqrt{(\frac{n}{L_x})^2 + (\frac{m}{L_y})^2}$

$$X(x) = C_{1x} sin(x\frac{\pi n}{L_x}) \; ; \; n \epsilon \mathbb{N}$$
$$Y(y) = C_{1y} sin(y\frac{\pi m}{L_y}) \; ; \; m \epsilon \mathbb{N}$$
$$T(t) = C_{1t} sin(\lambda_1 v t) + C_{2t} cos(\lambda_1 v t)$$

We can write $u(x, y, t) = X(x)Y(y)T(t)$

$$u(x, y, t) = sin(x\frac{\pi n}{L_x}) sin(y\frac{\pi m}{L_y})[A sin(\lambda_1 v t) + B cos(\lambda_1 v t)]$$
$$\lambda_1 = \pi\sqrt{(\frac{n}{L_x})^2 + (\frac{m}{L_y})^2}$$
$$n \epsilon \mathbb{N}$$
$$m \epsilon \mathbb{N}$$

6.2 Numerical Solution

Again, let's start with the 2D wave equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

And substitute the approximation for $\frac{\partial^2 u}{\partial t^2}$

$$(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})v^2 h^2 = u_{i,j}^{k+1} - 2u_{i,j}^{k} + u_{i,j}^{k-1}$$

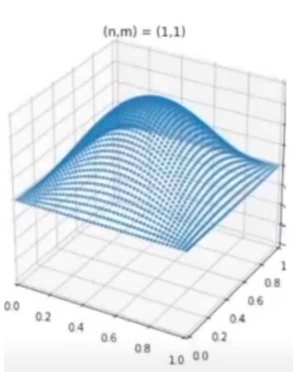
And solving for $u_{i,j}^{k+1}$, we get

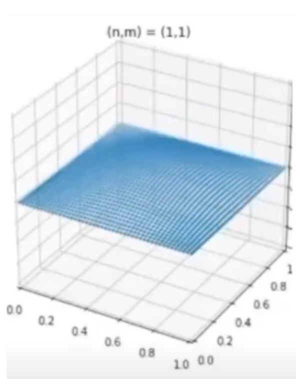
$$u_{i,j}^{k+1} = (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})v^2 h^2 + 2u_{i,j}^{k} - u_{i,j}^{k-1} \tag{25}$$

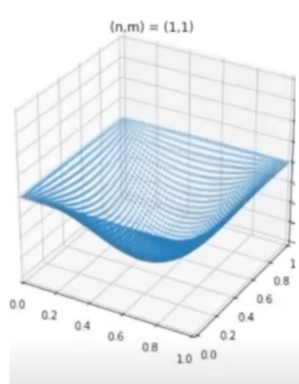
Now, we have a numerical scheme for solving the 2D wave equation on a rectangular plate:

$$\begin{aligned}
u_{i,j}^{k+1} &= (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})v^2 h^2 + 2u_{i,j}^{k} - u_{i,j}^{k-1} \\
\frac{\partial^2 u}{\partial x^2} &= \frac{u_{i+1,j}^{k} - 2u_{i,j}^{k} + u_{i-1,j}^{k}}{a^2} \\
\frac{\partial^2 u}{\partial y^2} &= \frac{u_{i,j+1}^{k} - 2u_{i,j}^{k} + u_{i,j-1}^{k}}{b^2}
\end{aligned}$$

6.3 Time Evolution







7 2D Wave Equation: Circular Drum

7.1 Analytical Solution

Starting from (1),

$$\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}$$

We can use the gradient in cylindrical coordinates, but we can set $\frac{\partial^2 u}{\partial z^2} = 0$ for 2D polar coordinates.

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\frac{\partial u}{\partial \rho}) + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{26}$$

First, we can separate the time part from the polar part. $u(\rho, \phi, t) = f(\rho, \phi)T(t)$

$$\frac{1}{\rho f(\rho,\phi)}\frac{\partial}{\partial \rho}(\rho\frac{\partial f(\rho,\phi)}{\partial \rho}) + \frac{1}{\rho^2 f(\rho,\phi)}\frac{\partial^2 f(\rho,\phi)}{\partial \phi^2} = \frac{1}{v^2 T(t)}\frac{\partial^2 T(t)}{\partial t^2} = -\lambda^2 \tag{27}$$

Solving for $T(t)$,

$$\begin{aligned} T''(t) + \lambda_1^2 v^2 T(t) = 0 \\ T(t) = C_{1t} sin(v\lambda t) + C_{2t} cos(v\lambda t) \end{aligned}$$

We can do a second separation. $f(\rho, \phi) = R(\rho)\Phi(\phi)$

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial \rho}(\rho R'(\rho)) + \lambda^2\rho^2 = -\frac{\Phi''(\phi)}{\Phi(\phi)} = m^2 \tag{28}$$

We can solve for $\Phi(\phi)$ and $R(\rho)$.

$$\begin{aligned} \Phi''(\phi) + m^2\Phi(\phi) = 0 \\ \Phi(\phi) = C_{1\phi} sin(m\phi) + C_{2\phi} cos(m\phi) \end{aligned}$$

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial \rho}(\rho R'(\rho)) + \lambda^2\rho^2 = m^2 \tag{29}$$

This equation is Bessel's equation (10), and the solution are the Bessel Functions (11 , 12).

$$R(\rho) = C_{1\rho} J_m(\rho\lambda) + C_{2\rho} Y_m(\rho\lambda)\ ;\ m\epsilon\mathbb{N} \tag{30}$$

Since the Bessel function of the second kind has a singularity at $\rho = 0$, let's get rid of $Y_m(\rho\lambda)$ completely by setting its neighboring integration constant to 0. $C_{2\rho} = 0$

$$R(\rho) = C_{1\rho} J_m(\rho\lambda) \tag{31}$$

Now let's enforce boundary conditions. The position at the boundary of the drum should always be 0. If the drum has a radius L, R(L) = 0.

$$J_m(L\lambda) = 0$$

$$L\lambda = \mu_{m,n}$$

Where $\mu_{m,n}$ are the zeros of the Bessel Function. In particular, $\mu_{m,n}$ is the nth zero of the mth order Bessel function.

$$\lambda = \frac{\mu_{m,n}}{L} \tag{32}$$

$$T(t) = C_{1t} sin(v\lambda t) + C_{2t} cos(v\lambda t)$$
$$\Phi(\phi) = C_{1\phi} sin(m\phi) + C_{2\phi} cos(m\phi)$$
$$R(\rho) = C_{1\rho} J_m(\rho \frac{\mu_{m,n}}{L})$$

We can write $u(\rho, \phi, t) = R(\rho)\Phi(\phi)T(t)$

Our solution is:

$$u(\rho, \phi, t) = [Asin(\frac{v\mu_{m,n}}{L} t) + Bcos(\frac{v\mu_{m,n}}{L} t)][Csin(m\phi) + Dcos(m\phi)] J_m(\rho \frac{\mu_{m,n}}{L}) \tag{33}$$

7.2 Numerical Solution

Let's start with the wave equation in polar coordinates (26).

$$\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} \tag{34}$$

Let i correspond to the ρ coordinate, and let j correspond to the ϕ coordinate.

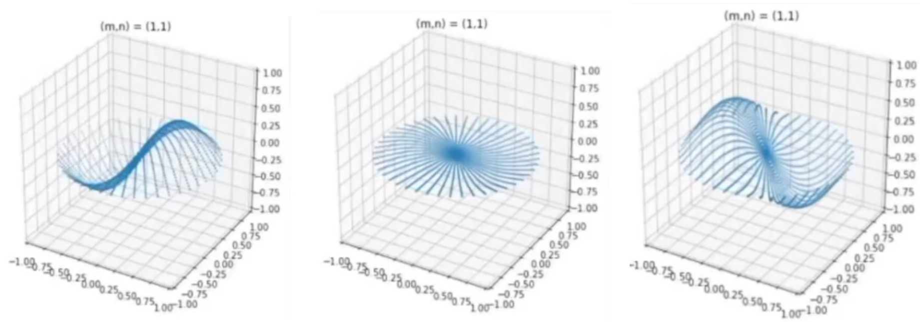
Let's substitute in the approximation for $\frac{\partial^2 u}{\partial t^2}$

$$u_{i,j}^{k+1} = v^2 h^2 [\frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2}] + 2u_{i,j}^k - u_{i,j}^{k-1} \tag{35}$$

We can index ρ as $\rho = ia$

$$u_{i,j}^{k+1} = v^2 h^2 [\frac{1}{ia}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{(ia)^2}\frac{\partial^2 u}{\partial \phi^2}] + 2u_{i,j}^k - u_{i,j}^{k-1} \tag{36}$$

7.3 Time Evolution



8 2D Wave Equation: The Sphere

8.1 Analytical Solution

We can start with the wave equation in spherical coordinates.

$$\frac{1}{R^2 sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{R^2 sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \tag{37}$$

We can separate the angular part from the time part $u(\theta,\phi,t) = f(\theta,\phi)T(t)$

$$\frac{1}{f(\theta,\phi)}\frac{1}{sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial f(\theta,\phi)}{\partial\theta}) + \frac{1}{f(\theta,\phi)}\frac{1}{sin^2\theta}\frac{\partial^2 f(\theta,\phi)}{\partial\phi^2} = \frac{R^2}{v^2}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = -\lambda_1^2 \tag{38}$$

We can solve for $T(t)$,

$$T''(t) + \frac{\lambda_1^2 v^2}{R^2}T(t) = 0$$
$$T(t) = C_{1t}sin(\frac{v\lambda_1}{R}t) + C_{2t}cos(\frac{v\lambda_1}{R}t)$$

We can now separate $f(\theta,\phi) = \Theta(\theta)\Phi(\phi)$

$$\frac{sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}) + \lambda_1^2 sin^2\theta = -\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi} = \lambda_2^2 \tag{39}$$

Now solving for $\Phi(\phi)$,

$$\Phi''(\phi) + \lambda_2^2\Phi(\phi) = 0$$
$$\Phi(\phi) = C_{1\phi}sin(\lambda_2\phi) + C_{2\phi}cos(\lambda_2\phi)$$

When we solve for $\Theta(\theta)$, we get the Associated Legendre Polynomials in terms of $cos\theta$

$$\frac{sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}) + \lambda_1^2 sin^2\theta = \lambda_2^2 \tag{40}$$

Let $\lambda_1 = \sqrt{l(l+1)}$ where $l\epsilon\mathbb{N}$ and $m = \lambda_2$ where $m\epsilon\mathbb{N}$ also.

$$\Theta(\theta) = C_{1\theta}P_l^m(cos\theta) + C_{2\theta}Q_l^m(cos\theta) \tag{41}$$

In order to avoid singularities at the poles of the sphere, we must set $C_{2\theta} = 0$.

$$\Theta(\theta) = C_{1\theta}P_l^m(cos\theta) \tag{42}$$

We can write $u(\theta,\phi,t) = \Theta(\theta)\Phi(\phi)T(t)$

The solution to the wave equation on a spherical surface is:

$$u(\theta,\phi,t) = P_l^m(cos\theta)[Asin(\frac{v\sqrt{l(l+1)}}{R}t)+Bcos(\frac{v\sqrt{l(l+1)}}{R}t)][Csin(\sqrt{l(l+1)}\phi)+Dcos(\sqrt{l(l+1)}\phi)]$$

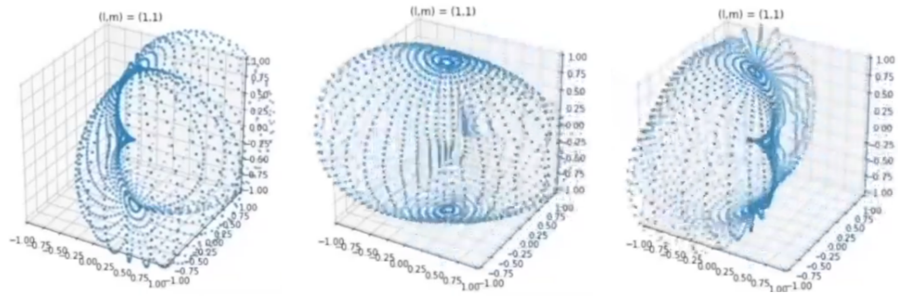
8.2 Numerical Solution

We can start with the wave equation in spherical coordinates

$$\begin{aligned}
\frac{1}{R^2 sin\theta}\frac{\partial}{\partial\theta}(sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{R^2 sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} &= \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} \\
cot\theta\frac{\partial u}{\partial\theta} + \frac{\partial^2 u}{\partial\theta^2} + csc^2\theta\frac{\partial^2 u}{\partial t^2} &= \frac{R^2}{v^2}\frac{\partial^2 u}{\partial t^2}
\end{aligned}$$

We can use the $\frac{\partial^2 u}{\partial t^2}$ approximation again, and we get our numerical scheme for the wave equation on a sphere.

$$\begin{aligned}
u_{i,j}^{k+1} &= \frac{h^2 v^2}{R^2}(cot\theta\frac{\partial u}{\partial\theta} + \frac{\partial^2 u}{\partial\theta^2} + csc^2\theta\frac{\partial^2 u}{\partial\phi^2}) + 2u_{i,j}^k - u_{i,j}^{k-1} \\
\frac{\partial u}{\partial\theta} &= \frac{u_{i+1,j}^k - u_{i-1,j}^k}{2a} \\
\frac{\partial^2 u}{\partial\theta^2} &= \frac{u_{i+1,j}^k - u_{i,j}^k + u_{i-1,j}^k}{a^2} \\
\frac{\partial^2 u}{\partial\phi^2} &= \frac{u_{i,j+1}^k - u_{i,j}^k + u_{i,j-1}^k}{b^2}
\end{aligned}$$



8.3 Time Evolution

9 Wave Equation on a Toroidal Surface

Unlike the previous cases for solving the wave equation, there is no obvious coordinate system we can choose to solve the wave equation on a torus. Perhaps *Toroidal Coordinates* may seem like a decent option, but the equations get too messy in that coordinate system. Let's parameterize the surface of a torus and then use the Laplacian in curvilinear coordinates instead to get a Laplacian that will be much easier to handle.

We can parameterize the the surface $\mathbf{S}$ of the Torus in terms of the two parameters θ (longitudinal direction) and ϕ (latitude direction).

$$\mathbf{S}(\phi,\theta) = (R + rcos\theta)cos\phi\hat{x} + (R + rcos\theta)sin\phi\hat{y} + rsin\theta\hat{z} \tag{43}$$

R is the major radius of the torus and r is the minor radius.

Now we can generate a Laplacian using $\nabla^2 u = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \xi^i}(\sqrt{g}g^{ij}\frac{\partial u}{\partial \xi^j})$. *(See the Mathematica File for this derivation).*

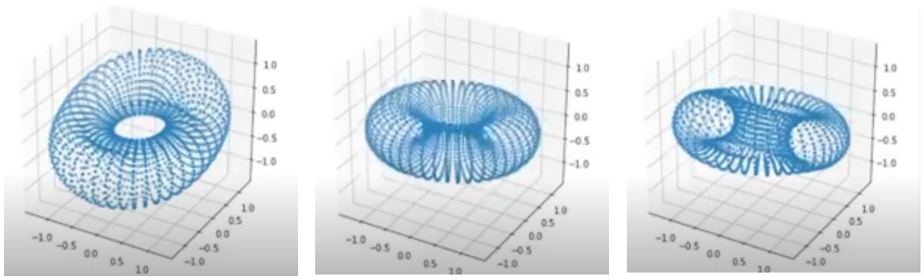
$$\nabla^2 u = \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{sin\theta}{r(R + rcos\theta)}\frac{\partial u}{\partial \theta} + \frac{1}{(R + rcos\theta)^2}\frac{\partial^2 u}{\partial \phi^2} \tag{44}$$

Using

$$u_{i,j}^{k+1} = h^2 v^2 \nabla^2 u + 2u_{i,j}^{k} - u_{i,j}^{k-1} \tag{45}$$

where i corresponds to the ϕ coordinate and j corresponds to the θ coordinate, we can numerically solve the wave equation on the surface of a torus.

9.1 Time Evolution



10 Forced Damped Harmonic Motion

Thus far, the wave equations we have dealt with obey conservation of energy. Let us consider this modified wave equation which includes a damping coefficient κ and an external driving force F

$$\frac{\partial^2 u}{\partial t^2} + \kappa \frac{\partial u}{\partial t} = v^2 \nabla^2 u + F(x, y, t) \tag{46}$$

Notice if we set $\kappa = 0$ and $F = 0$, we get the regular version of the wave equation.

Let's derive a numerical method for solving this. We can insert the approximation for $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial t^2}$.

$$\frac{1}{h^2}(u_{i,j}^{k+1} - 2u_{i,j}^{k} + u_{i,j}^{k-1}) + \frac{\kappa}{2h}(u_{i,j}^{k+1} - u_{i,j}^{k-1}) = v^2 \nabla^2 u + F_{i,j}^{k} \tag{47}$$

Solving for $u_{i,j}^{k+1}$,

$$u_{i,j}^{k+1} = \frac{1}{1 + \frac{\kappa h}{2}}(v^2 h^2 \nabla^2 u + h^2 F_{i,j}^{k} + 2u_{i,j}^{k} - u_{i,j}^{k-1}(1 - \frac{\kappa h}{2})) \tag{48}$$

This is our numerical scheme for solving Forced-damped problems given some Laplacian ∇^2 in a coordinate system.

11 Conclusion

After testing out all of our trial cases, we can conclude that as long as it is possible to parameterize a surface $\mathbf{S}(\xi^1, \xi^2)$, we can find a Laplacian for that surface and solve the wave equation numerically.

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