Quantization of Mean Curvature to Describe Energy Levels of a Vibrating 2D Spherical Membrane

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***Abstract*—The wave number, k_l of a vibrating 2D spherical membrane is quantized in integer multiples of the mean curvature in the high l limit, where l is the lth mode of vibration. Also, in the high l limit, we obtain a solution for the energy states of the manifold analogous to those of the quantum harmonic oscillator. In particular, we find that $E_l = \daleth\omega_l$ where $\daleth = \frac{1}{4}E_k^{eff}H/v$, E_k^{eff} is the total stored energy in the membrane, v is the wave velocity, and H is the mean curvature of the surface. We see that $\daleth$ is the classical analog of $\hbar$ in the case of the quantum harmonic oscillator. We will also show that k_l is also quantized in multiples of H for a cylindrical surface, but whether or not this is a general rule for any axisymmetric surface is to be determined by future research. The sphere and cylinder are examples of shapes with constant H where this property appears to be valid.**

I. The Wave Equation

A. Formalism

We begin with the wave equation,

$$\nabla^2 u(\mathbf{r},t) = \frac{1}{v^2}\frac{\partial^2 u(\mathbf{r},t)}{\partial t^2} \tag{1}$$

where $u = u(\mathbf{r},t)$ is the displacement normal to the surface.

We impose the initial conditions:

$$\begin{aligned} u(\mathbf{r},t=0) &= \quad f(\mathbf{r}) \quad (2)\\ \partial_t u(\mathbf{r},t) &= \quad g(\mathbf{r}) \quad (3)\end{aligned}$$

We use the common approach of separation of variables,

$$u(\mathbf{r},t) = \alpha(\mathbf{r})T(t) \tag{4}$$

assuming that $\alpha(\mathbf{r}) \neq 0$ and $T(t) \neq 0$.
Grouping functions of the same variables together,

$$\frac{\nabla^2\alpha(\mathbf{r})}{\alpha(\mathbf{r})} = \frac{1}{v^2}\frac{1}{T(t)}\partial_t^2 T(t) = -k^2 \tag{5}$$

Here, we define the angular frequency $\omega \equiv vk$.
Now, we can write a system of differential equations

$$\begin{aligned} T''(t) + \omega^2 T(t) &= 0 \quad (6)\\ \nabla^2\alpha(\mathbf{r}) + k^2\alpha(\mathbf{r}) &= 0 \quad (7)\end{aligned}$$

We see that (6) is the time-dependent part of the partial differential equation (PDE), and that (7) is the Helmholtz equation.

B. The Time-Dependent part

The Eigenfunction $\alpha_l(\mathbf{r})$ retrieved from the Helmholtz equation is determined by the surface-dependent operator ∇^2 that we choose to utilize. However, the time-dependent part of the PDE can be solved for any arbitrary ∇^2.

Solving the time-dependent part of the PDE (6) using standard ordinary differential equation techniques, we obtain the solution:

$$T(t) = Acos(\omega t) + Bsin(\omega t) \tag{8}$$

If we assume that $f(\mathbf{r})$ and $g(\mathbf{r})$ are scalar multiples of the eigenfunction $\alpha_l(\mathbf{r})$, then A and B are constant coefficients. We will solve for A and B directly by applying our initial conditions

$$\begin{aligned} u(\mathbf{r},t=0) &= \alpha(\mathbf{r})T(t=0) = f(\mathbf{r}) \quad (9)\\ \alpha(\mathbf{r})A &= f(\mathbf{r}) \quad (10)\\ A &= f(\mathbf{r})/\alpha(\mathbf{r}) \quad (11)\end{aligned}$$

and

$$\begin{aligned} \partial_t u(\mathbf{r},t=0) &= \alpha(\mathbf{r})T'(t=0) = g(\mathbf{r}) \quad (12)\\ \alpha(\mathbf{r})\omega B &= g(\mathbf{r}) \quad (13)\\ B &= \tfrac{g(\mathbf{r})}{\omega\alpha(\mathbf{r})} \quad (14)\end{aligned}$$

We define $\lambda^2 \equiv A^2 + B^2$ where λ is a *displacement parameter*. Meaning, λ is a parameter at which we effectively scale $\alpha_l(\mathbf{r})$ to put elastic energy into the system. The scale of the of energy encapsulated in the membrane is determined by λ.

C. The Helmholtz Equation

To solve the time-independent part of the PDE (7), we must select a Laplacian that describes the geometry of our chosen 2D manifold.

In our case, we will examine a spherical 2D surface and assume azimuthal symmetry.

$$\nabla^2\alpha = \frac{1}{R^2 sin\theta}\frac{\partial}{\partial\theta}\big(sin\theta\frac{\partial\alpha}{\partial\theta}\big) \tag{15}$$

Upon substituting (15) into (7), we recognize that this is an Associated Legendre Differential Equation where the solutions are the Associated Legendre Polynomials.

$$\alpha(\theta) \quad \propto P_l(cos\theta) \tag{16}$$
$$k_l \quad = \tfrac{1}{R}\sqrt{l(l+1)} \; ; l\epsilon\mathbb{Z}^+ \tag{17}$$

As a consequence of solving this differential equation, we obtain a quantization constraint on the wave number k. Furthermore, we notice that $1/R$ just so happens to be the mean curvature H of a sphere.

So,

$$k_l = H\sqrt{l(l+1)} \tag{18}$$

D. PDE Solution

Thus, the solution of this PDE under our defined constraints is

$$u_l(\theta, t) = P_l(cos\theta)\big(Acos(\omega_l t) + Bsin(\omega_l t)\big) \tag{19}$$

II. Energy

Next, we compute the total elastic energy of the spherical membrane for each mode l.

Recall the kinetic and potential energies of a simple harmonic oscillator but of infinitesimal mass dm are

$$d(KE) \quad = \tfrac{1}{2}\big(\tfrac{\partial u}{\partial t}\big)^2 dm \tag{20}$$
$$d(PE) \quad = \tfrac{1}{2}\omega^2 u^2 dm \tag{21}$$

Combining $dE = d(KE) + d(PE)$,

$$dE = \frac{1}{2}[\big(\frac{\partial u}{\partial t}\big)^2 + \omega^2 u^2]dm \tag{22}$$

Factoring out the position-dependent part of u, and inserting our solutions for $T(t)$ (8), we find that

$$dE = \frac{1}{2}\omega^2\lambda^2[\alpha(\mathbf{r})]^2 dm \tag{23}$$

Assuming that the membrane has uniform mass density per unit area $\sigma = m/A$ at rest, we may expand the differential mass dm according to the following steps:

$$dm \quad = \sigma dA \tag{24}$$
$$dm \quad = \sigma|J|d\theta d\phi \tag{25}$$
$$dm \quad = \sigma R^2 sin\theta d\theta d\phi \tag{26}$$

To find the *total energy*, we insert (26) and (16) into (23) and integrate over the entire surface of the sphere.

Upon evaluating this integral, we end up with

$$E = \pi\omega^2\lambda^2 R^2\sigma \int_{-1}^{1}[P_l(x)]^2 dx \tag{27}$$

Acknowledging the mathematical fact that

$$\int_{-1}^{-1}[P_l(x)]^2 dx = \frac{2}{2l+1} \tag{28}$$

and substituting this result (28) into (27) to solve the integral to completion, we find

$$E_l = \frac{1}{R^2} \cdot \frac{1}{2}mv^2\lambda^2 \cdot \frac{l(l+1)}{2l+1} \tag{29}$$

Let's dissect (29)!

- The $1/R^2$ suggests the presence of a curvature dependence on energy. Whether or not it is the Gaussian Curvature K or the Mean Curvature Squared H^2 is unclear because both are equivalent for a sphere.
- $\frac{1}{2}mv^2\lambda^2$ suggests a kinetic energy term, defined by our displacement parameter λ. Let's define $E_k^{eff} \equiv \frac{1}{2}mv^2\lambda^2$ to be our *effective* kinetic energy
- Finally, the $\frac{l(l+1)}{2l+1}$ implies a quantization condition on the energy states of a spherical membrane.

Rewriting (29) with these new definitions,

$$E_l = H^2 E_k^{eff}\frac{l(l+1)}{2l+1} \tag{30}$$

Also, recall the quantization condition we found earlier in (18),

$$\omega_l = vH\sqrt{l(l+1)} \tag{31}$$

In the large l limit, the following approximations are valid

$$\sqrt{l(l+1)} \quad \approx l \tag{32}$$
$$\frac{l(l+1)}{2l+1} \quad \approx \tfrac{1}{2}l \tag{33}$$

Thus, as we explore higher modes of l, the relationship between E_l and ω_l becomes more linear. In the high l limit, we can formulate a classical analog to the energy-frequency relationship found in the quantum harmonic oscillator.

In particular, we find for large l

$$E_l = \eth\omega_l \tag{34}$$

where $\eth$ is the classical analog of $\hbar$ for the quantum harmonic oscillator.

We solve $\eth$ to be

$$\eth = \frac{1}{2}E_k^{eff} H/v \tag{35}$$

Besides the amount of effective kinetic energy contained within the membrane which is arbitrarily chosen based on the scale of λ, this $\eth$ parameter is intrinsically defined by the mean curvature of the spherical surface H.

III. Interpretation of Results

A. Discussing the Sphere

Claiming that k is a quantized multiple of H is nothing but the result of an observation I noted using the sphere as a test bed case. Determining whether or not this is a generalized rule (or at the very least the foundations of one) is a task left for future research.

B. Discussing the Infinitely Long Cylinder

There are only a few surfaces with curvature that we are able to analytically investigate. The sphere and an infinitely long cylinder are two good candidates. Doing a similar computation as above in Section I but for an infinite cylinder suggests a similar quantization condition on H.

We substitute in the Laplacian for an infinitely long cylinder to solve the Helmholtz equation (7)

$$\nabla^2 \alpha = \frac{1}{R^2} \frac{\partial^2 \alpha}{\partial \phi^2} \tag{36}$$

and solve for $\alpha(\phi)$ under the continuity constraint of the cylinder that $\alpha(\phi = 0) = \alpha(\phi = 2\pi)$ to obtain the following result

$$l = Rk \tag{37}$$

The mean curvature H for the cylinder just so happens to be $\frac{1}{2R}$

Thus, the quantization condition for the infinitely long cylinder is

$$k_l = 2Hl \tag{38}$$

k is *even* integer multiples of H for all $l \epsilon \mathbb{Z}^+$. It's interesting to note that the Gaussian curvature of a cylinder is 0. So, it would be impossible to formulate a quantization rule of k_l in terms of the Gaussian Curvature for a cylinder.

C. Concluding Remarks

The exploration between k_l being quantized in terms of the mean curvature H for both the cylinder and the sphere is an interesting result. Perhaps, it's a result of coincidence, and not anything particularly special about mean curvature. Future research investigating a generalized axisymmetric surface to derive this rule should be done to confirm these two specific test case results.