The Schrodinger-Newton System with Self Coupling

V. Gopalaswamy, A. McNutt, A. Morgan, C. Proepper

September 18, 2013

What/Why?: Quantum Gravity

- Ideally the unification of quantum mechanics and general relativity.
- Fundamental understanding of the universe.
- Lots of complicated models (String Theory, Supergravity, Loop Quantum Gravity...)

A Semi-Classical Approach: Newton-Schrodinger

- Our system:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + \Phi\Psi \qquad \nabla^2\Phi = 4\pi G m^2 \Psi^*\Psi$$

- Motivation

Semi-Classical with a correction

- From Electrodynamics

$$W = \int \left[\frac{\epsilon_0}{2}\nabla V \cdot \nabla V\right] d\tau \qquad u_{em} = \frac{\epsilon_0}{2}\nabla V \cdot \nabla V$$

- Gravitational Analogy: $\epsilon_0 = -1/(4\pi G)$

$$u_g = \frac{-1}{8\pi G}\nabla V \cdot \nabla V$$

- Yielding a new set of equations:

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi + \Phi\Psi \quad \nabla^2\Phi = 4\pi G m^2\Psi^*\Psi - \frac{1}{2mc^2}\nabla\Phi\cdot\nabla\Phi$$

A Computer Friendly Modification

- Assumptions
- Non-Dimensionalizations

$$r_0 = \frac{\hbar^2}{8\pi G M_p^3} \quad t_0 = \frac{\hbar^2}{32\pi^2 G^2 M_p^5} \quad \Phi_0 = \frac{32\pi^2 G^2 M_p^5}{\hbar^2} \quad \mu = \frac{m}{M_p}$$

- Non-Dimensionalized System

$$i\frac{\partial S}{\partial T} = -\frac{1}{\mu R}\frac{\partial^2}{\partial R^2}(RS) + VS$$

$$\frac{1}{R}\frac{d^2}{dR^2}(RV) = \mu^2 S^* S - \frac{(4\pi)^2}{\mu}\left(\frac{dV}{dR}\right)^2$$

- Bound States
- Time Evolution

Introduction to E_{NSP}

- **Objective:** Investigate how the ground state energy changes as μ changes and compare the Newton-Schrodinger-Peters (NSP) and Newton-Schrodinger (NS) results.

- It has been established that the ground state energy for the NS system is $\boxed{E_{NS} = -.163\mu^5 \frac{G^2 M_P^5}{\hbar^2}}$ [1].

- Now I will proceed to derive E_{NSP} as a function of μ. Then I will show results of E_{NSP} and E_{NS} for varying μ.

[1] D.H. Bernstein, E. Giladi, K. R. W. Jones, *Mod. Phys. Lett.*, **A13**, 2327-2336, 1998.

Derivation of E_{NSP}

- Consider $S(R,T) = e^{-ieT}S(R)$ (where $e = Et_0/\hbar$):
  $$(Rs)'' = -\mu R(e-V)s \;\&\; (RV)'' = R(\mu^2 s^* s - \frac{(4\pi)^2}{\mu}(\frac{dV}{dR})^2)$$
- Define $v \equiv e - V$:
  $$(Rs)'' = -\mu Rvs \;\&\; (Rv)'' = -R(\mu^2 s^* s - \frac{(4\pi)^2}{\mu}(\frac{dv}{dR})^2)$$
- Initially, $s'(0) = v'(0) = 0$, $v(0) = 1$, and $s(0) = s_0$. Also we demand that as $r \to \infty$, $s(r) \to 0$.
- We used a shooting method (bisection) to determine which s_0 meets these boundary conditions.

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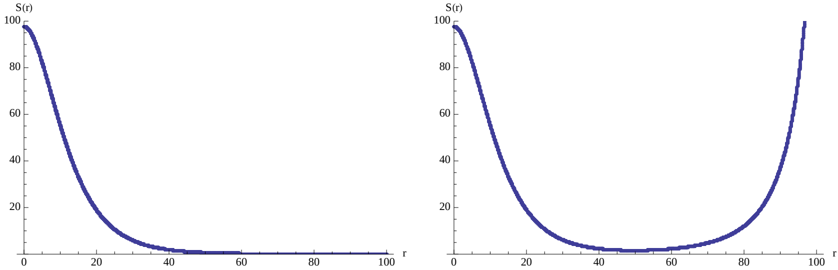
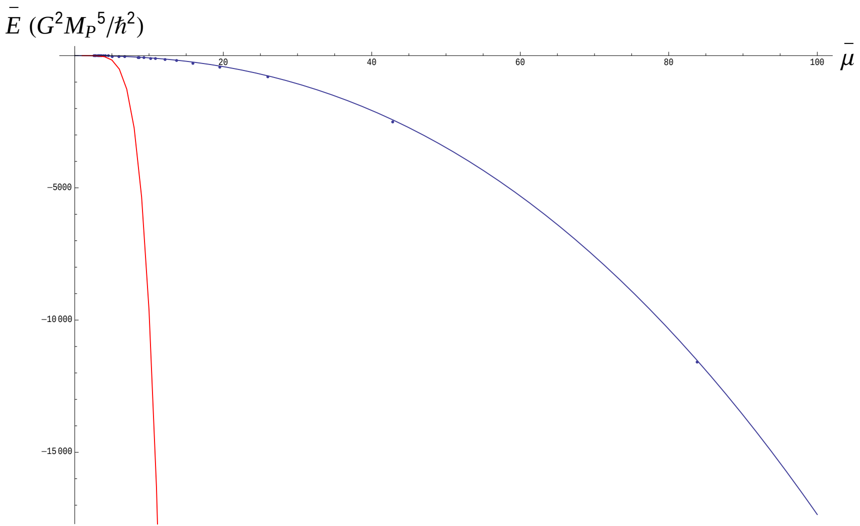


Derivation of E_{NSP} Continued

- In order to solve for e, we want to find v ($e = v + V$) at spatial infinity where e is constant and $V(\infty) = 0$.
- The solution for v as $R \longrightarrow \infty$ is:
$v(\infty) = v(0) - \int_0^\infty (\mu^2 s(R)^2 - \frac{(4\pi)^2}{\mu}(\frac{dv}{dR})^2)RdR$
- The normalized energy is $e = \sqrt{4\pi \int_0^\infty s^2 R^2 dR}[1 + \int_0^\infty (-\mu^2 s(R)^2 + \frac{(4\pi)^2}{\mu}(\frac{dv}{dR})^2)RdR]$
- And the dimensionful energy is $\boxed{\bar{E}_{NSP} = 32\pi^2 e \frac{G^2 M_P^5}{\hbar^2}}$

Results



$$\bar{E}_{NS} = -.163\mu^5 \frac{G^2 M_P^5}{\hbar^2} \qquad \bar{E}_{NSP} = -.399\mu^{2.32} \frac{G^2 M_P^5}{\hbar^2}$$

Discussion

- $|E_{NS}|$ is typically larger than $|E_{NSP}|$ because the self-sourcing term in NSP makes the "effective" density less than for the NS system.
- Given the constants G, m, and $\hbar$ it makes sense that $E_{NS} \propto m^5$, but by introducing c, there are many more combinations of these constants that can be made to make E_{NSP} have dimensions of energy

Crank-Nicolson

We can discretize Schrodinger's Equation:

$$\mathbb{H}(\phi)\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

where $\Psi = \psi/r$ by using a Crank-Nicolson method:

$$\left[\mathbb{I} + \frac{i\Delta t}{2}\mathbb{H}(\phi^{n+1})\right]\boldsymbol{\Psi}^{n+1} = \left[\mathbb{I} - \frac{i\Delta t}{2}\mathbb{H}(\phi^{n})\right]\boldsymbol{\Psi}^{n}$$

which is stable and preserves norm.

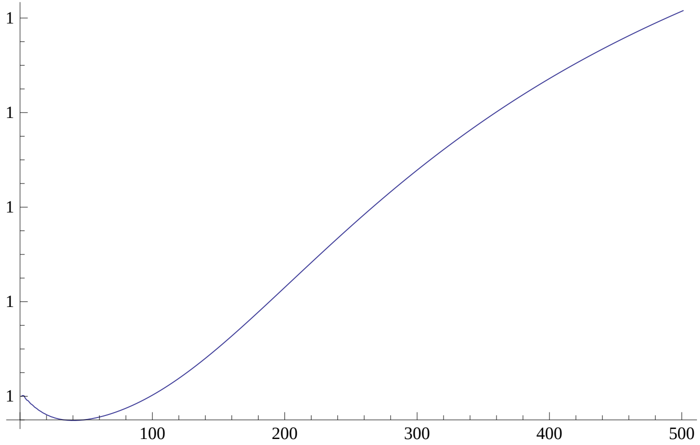
Potential

We now have

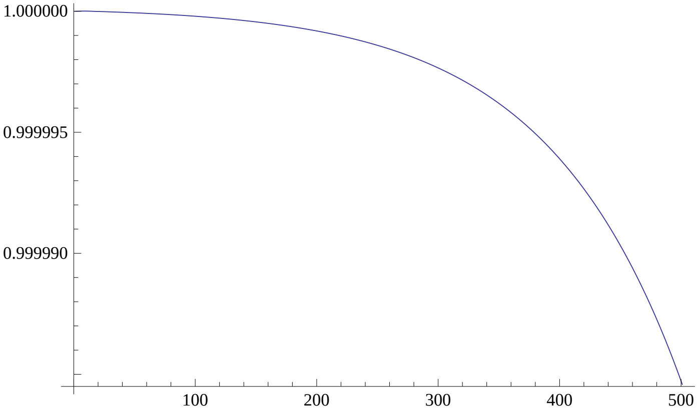
$$(R\phi^n)'' = \vec{f}\left(\mathbf{\Psi}^n, \phi'^n\right)$$

where $\vec{f}$ is a non-linear function of $\mathbf{\Psi}^n$ and ϕ'^n. We therefore use Verlet's method to generate a ϕ^n, which results in a chicken and egg scenario with Crank-Nicolson. We resolve this by estimating $\mathbf{\Psi}^{n+1}$ using Euler's method, and iterating that estimate through $\vec{f}$ and Crank-Nicolson.

Norms (Newton-Schrodinger)



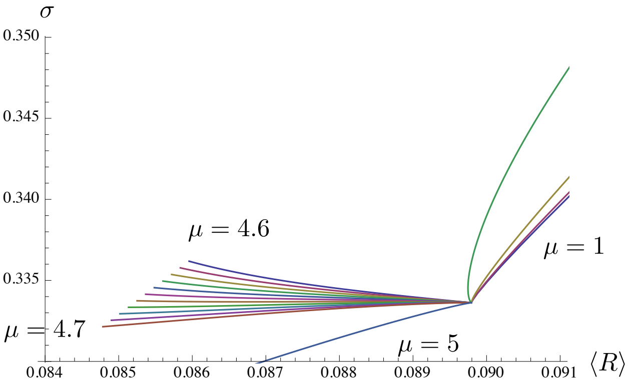
Norms (Peters)



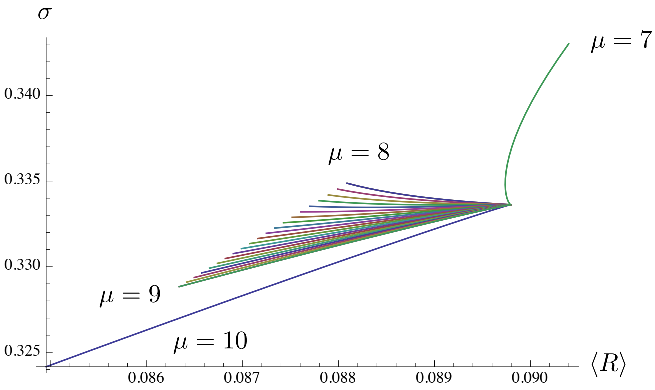
What is collapse?

We then compute the expectation value $\langle R \rangle$ and standard devation σ of $\mathbf{\Psi}^n$, and define a collapse as a trajectory on the $\langle R \rangle$-σ graph that approaches the origin.

$\langle R \rangle$-σ (Newton-Schrodinger)



$\langle R \rangle$-σ (Peters)



Newton-Klein-Gordon Equations

The Newton-Klein Gordon equations:

$$[(mc^2)^2 - \hbar^2 c^2 \nabla^2]\Psi = (i\hbar \frac{\partial}{\partial t} + m\Phi)^2 \Psi \tag{1}$$

$$\nabla^2 \Phi = 4\pi m G \Psi^* \Psi \tag{2}$$

The major distinction in solving these equations comes down to the square on the right hand side.

Finding Solutions

To numerically find a solution we used Euler's Method on the nondimensionalized equations:

$$U'' - U = \ddot{U} - im(\Phi \dot{U} + \dot{\Phi} U) - m^2 \Phi^2 U \tag{3}$$

$$\frac{\partial^2}{\partial q^2}(q\Phi) = \frac{\Psi^* \Psi}{q} \tag{4}$$

We use Euler's Method here rather than Crank-Nicholson because Klein-Gordon is not norm preserving, so we can accept the methods lack of norm preservation.

Uncoupled results

Over the course of the project, we were not able to achieve coupling of the potential with the wavefunction. However, we did get results for an independent potential.

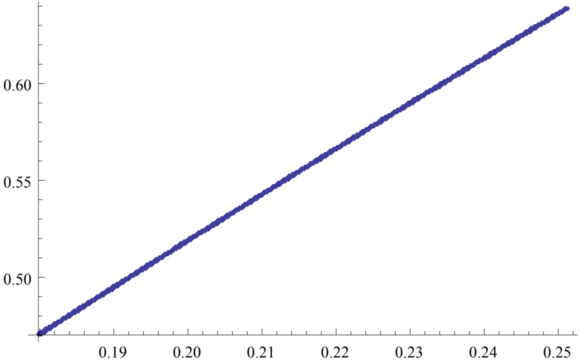


Figure : $\langle R \rangle$ vs σ for the 0 mass case

Results Cont.

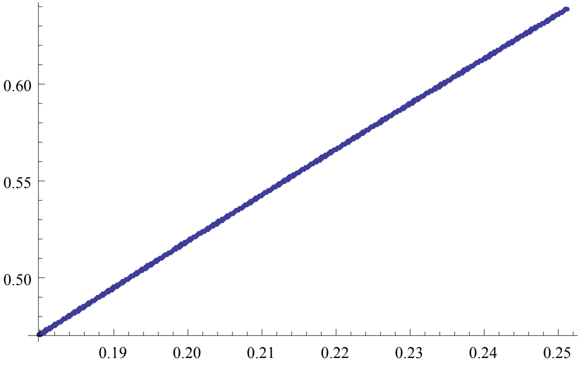


Figure : $\langle R \rangle$ vs σ for the 100 mass case

Analysis

- We can see, then that the two norm functions for very different masses appear identical, and indeed they are, numerically.
- Additionally, this result tells us that in this case the wave function borders on collapse, yet does not collapse under our definition.

Acknowledgements

- Joel Franklin
- Physics Department