Nonequivalent Lagrangian Mechanics

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Background

The main structure in Lagrangian mechanics is the Lagrangian,

$$L = T - U \tag{1}$$

Which comes from the definition of the action

$$S[q] = \int_{t_0}^{t_f} L(q, \dot{q}, t) dt \tag{2}$$

and is minimized by the Euler Lagrange Equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{3}$$

Together they yield equations of motion $\ddot{q} = \alpha(q, \dot{q}, t)$.

Noether's Theorem

- *For every symmetry of a certain type there exists a corresponding conservation law*
- A symmetry is something in a system that does not change under transformation
- $$\int L(Q, \dot{Q}, T)dT \to \int \left(L(q, \dot{q}, t) + \frac{dJ}{dt} \right) dt \qquad (4)$$

  If a Lagrangian is expressible in this way, then there exists an associated conserved quantity. For instance, if a system is invariant in time then it has conservation of energy.

Introduction

- What is Nonequivalent Lagrangian Mechanics?
- What are Nonequivalent Lagrangians?
  - $$\tilde{L} = \frac{1}{2}\dot{x}^2 - \omega_0^2 x^2 \tag{5}$$
  - $$\overline{L} = \frac{1}{2}\dot{x}^2 - \omega_0^2 x^2 + \dot{x} = \frac{1}{2}\dot{x}^2 - \omega_0^2 x^2 + \frac{d}{dt}(x) \tag{6}$$
  - $$L = -\ln(\omega_0^2 x^2 + \dot{x}^2) + \frac{2\dot{x}}{\omega_0 x}\tan^{-1}\left[\frac{\dot{x}}{\omega_0 x}\right] \tag{7}$$

  $\tilde{L}$ and L are Nonequivalent, while $\tilde{L}$ and $\overline{L}$ are not
- Note that we have abandoned the $L = T - U$ prescription.

- Lutzky Invariant:

$$\Phi_L = \frac{\frac{\partial^2 \tilde{L}}{\partial \dot{x}^2}}{\frac{\partial^2 L}{\partial \dot{x}^2}} \tag{8}$$

- Example:

$$\begin{aligned} \Phi_L &= \frac{\frac{\partial^2}{\partial \dot{x}^2}\left[\frac{1}{2}\dot{x}^2 - \omega_0^2 x^2\right]}{\frac{\partial^2}{\partial \dot{x}^2}\left[-\ln(\omega_0^2 x^2 + \dot{x}^2) + \frac{2\dot{x}}{\omega_0 x}\tan^{-1}\left[\frac{\dot{x}}{\omega_0 x}\right]\right]} \\ &= \frac{1}{\frac{2}{\omega_0^2 x^2 + \dot{x}^2}} = \frac{1}{2}\omega_0^2 x^2 + \frac{1}{2}\dot{x}^2 = H \end{aligned} \tag{9}$$

- Noether Analysis is limiting
  - Requires strict action invariance
  - Requires continuous symmetries
- NLM Analysis is more flexible
  - Only requires that the equations of motion remain invariant, so can easily deal with discrete symmetries.
  - Allows us to more fully explore invariant space more, because of abandoning action invariance

Results

We now consider the Damped Driven Harmonic Oscillator:

$$\ddot{x} = -\omega_0^2 x - \gamma \dot{x} + f_0 \cos(\omega_0 t) \tag{10}$$

It has Lagrangian:

$$L = e^{\gamma t} \left(\frac{1}{2}\dot{x}^2 - \frac{1}{2}\omega_0^2 x^2 + f_0 x \cos(\omega_0 t) \right) \tag{11}$$

We check that it gives us what we want

$$\begin{aligned} \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{x}}\right] - \frac{\partial L}{\partial x} &= e^{\gamma t}\ddot{x} + \gamma e^{\gamma t}\dot{x} - e^{\gamma t}(-\omega_0^2 x + f_0 \cos(t)) \\ &= \ddot{x} + \omega_0^2 x + \gamma \dot{x} - f_0 \cos(\omega_0 t) = 0 \end{aligned}$$

Results

Now let's see about an invariant. We know the solution to this equation:

$$\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} e^{\rho_+ t} & e^{\rho_- t} \\ \rho_+ e^{\rho_+ t} & \rho_- e^{\rho_- t} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} + \frac{f_0}{\gamma\omega_0} \begin{pmatrix} \sin(\omega_0 t) \\ \omega_0 \cos(\omega_0 t) \end{pmatrix}$$

A and B are constants of integration. Matrix inversion gives us

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{e^{-t(\rho_+ + \rho_-)}}{(\rho_+ - \rho_-)} \begin{pmatrix} \rho_- e^{\rho_- t} & -e^{\rho_- t} \\ -\rho_+ e^{\rho_+ t} & e^{\rho_+ t} \end{pmatrix} \left[\begin{pmatrix} x \\ \dot{x} \end{pmatrix} - \frac{f_0}{\gamma\omega_0} \begin{pmatrix} \sin(\omega_0 t) \\ \omega_0 \cos(\omega_0 t) \end{pmatrix} \right]$$

Results

We use these constants to find an invariant

$$\Phi = \frac{e^{\gamma t}}{\gamma\omega_0(\gamma-4\omega_0^2)}\Big(f_0\omega_0\cos(\omega_0 t)(\gamma x + 2\dot{x}) + f_0\sin(\omega_0 t)(2\omega_0^2 x + \gamma\dot{x}) \\ -\gamma\omega_0(\omega_0 x^2 + \gamma x\dot{x} + \dot{x}^2) - \frac{f_0}{2\gamma}(2\omega_0 + \gamma\sin(2t\omega_0))\Big) \tag{12}$$

Next we need to construct our second Lagrangian using the Lutzky invariant:

$$\Phi_L = \frac{\frac{\partial^2\tilde{L}}{\partial\dot{x}^2}}{\frac{\partial^2 L}{\partial\dot{x}^2}} \qquad \Rightarrow \qquad \frac{\partial^2 L}{\partial\dot{x}^2} = \frac{\frac{\partial^2\tilde{L}}{\partial\dot{x}^2}}{\Phi} = \frac{e^{\gamma t}}{\Phi}$$

Results

And the result is:

$$
\begin{aligned}
L = & \\
& \frac{\sqrt{\gamma^2-4\omega_0^2}}{(\gamma\omega_0 x - f_0\sin(\omega_0 t))}\Bigg(f_0(2\omega_0\cos(\omega_0 t)+\gamma\sin(\omega_0 t))\tanh^{-1}\left[\frac{2f_0\omega_0\cos(\omega_0 t)+f_0\gamma\sin(t\omega_0)-\gamma\omega_0(\gamma x+2\dot{x})}{\sqrt{\gamma^2-4\omega_0^2}(-f_0\sin(\omega_0 t)+\gamma\omega_0 x)}\right] \\
& \qquad +\gamma\omega_0(\gamma x+2\dot{x})\tanh^{-1}\left[\frac{2f_0\omega_0\cos(\omega_0 t)+f_0\gamma\sin(t\omega_0)-\gamma\omega_0(\gamma x+2\dot{x})}{\sqrt{\gamma^2-4\omega_0^2}(f_0\sin(\omega_0 t)+\gamma\omega_0 x)}\right]\Bigg) \\
& + (\gamma^2-4\omega_0^2)\ln\Big[f_0^2(2\omega_0+\gamma\sin(2t\omega_0)) - 2\gamma(f_0\omega_0\cos(\omega_0 t)(\gamma x+2\dot{x}) \\
& \qquad + f_0\sin(\omega_0 t)(2\omega_0^2 x+\gamma\dot{x}) - \gamma\omega_0(\omega_0^2 x^2+\gamma x\dot{x}+\dot{x}^2))\Big]
\end{aligned}
$$

This suggests that the above invariant is not one that is attainable through traditional Noetherian Analysis.

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