Hochschild Homology of Completed Group Rings

James Cooper

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Contents

Chapter 1

Introduction

My favourite book is [7], or perhaps [13].

1.1 Methods

In this survey I give the key tools in an attempt to understand K_1 of the completed group rings of a two dimension p-adic Lie group.

Following, [3], in non-commutative Iwasawa theory, K_1 of the (Iwasawa) completed group rings of the Galois group G of p-adic Lie extensions of number fields (the projective limit of the group rings $\mathbb{Z}_p[G/U]$ for open normal subgroups U of G), and K_1 of localizations of these rings play key roles, and so it becomes important to study the structure of these K_1-groups.

For any odd prime p, I aim to understand the K_1-group, $K_1(\mathbb{Z}_p[[G]])$ where G is the p-adic Lie group, arising from the "False Tate Curve",

$$G = \begin{pmatrix} \mathbb{Z}_p^* & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

This semi-direct product of two copies of $\mathbb{Z}_p$ is the simplest non-commutative example, with no commutative subgroup of finite index. Although $K_1(\mathbb{Z}_p[[G]])$ is explicitly calculated in [11], the aim is to understand some of the group substructure.

The strategy of this survey is to establish the machinery to define a map, the Dennis Trace Map, from the K-groups of group rings to a simpler set of groups, the Hochschild homologies. I then concentrate on interpreting the first Hochschild homology in terms of more familiar objects, the usual group homologies.

I attempt to pass to completions of the Group ring, and hope to see their Hochschild homology lying in a completion of the Hochschild homology of the Group ring. Combining this with the Dennis Trace Map would give information on K_1 of the completion of the Group ring.

I explain how a useful theory may be constructed for Novikov rings where decomposition over conjugacy classes recovers elements in Hochschild Homology of finite group rings, but that the Iwasawa Completion does not Decompose Over Conjugacy Classes of a profinite group. However I show this group exists as an inverse limit of direct sums over conjugacy classes and begin to understand the connecting maps, and in particular how conjugacy classes of a group may fuse when passing to a quotient group.

I then investigate the Hochschild Homologies of localizations of these Iwasawa algebras and explain conditions for vanishing limiting the usefulness of this generalised Dennis Trace Map to study characteristic elements.

1.2 Notation

Let R be a ring with a unit element, and R^* denote the group of units of R, and $M_n(R)$ the ring of n x n matrices with entries in R.

Let G be a pro-p, p-adic, compact Lie group containing no element of order p. Define the Iwawawa algebra, $\Lambda(G)$ to be

$$\Lambda(G) = \varprojlim_{H <_o G} \mathbb{Z}_p[G/H]$$

Moreover, unless otherwise stated assume in addition that G has a closed normal subgroup H such that $G/H = \Gamma$ is isomorphic to $\mathbb{Z}_p$, thus fixing a topological generator of Γ and identifying this with $1+T$ we have the correspondence $\Lambda(\Gamma) \cong \mathbb{Z}_p[[T]]$.

Throughout p represents an odd prime, and G is an open subgroup of the p-adic Lie group

$$L = \begin{pmatrix} \mathbb{Z}_p^* & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

For $n \geq 0$, let $U^{(n)} = ker(\mathbb{Z}_p^* \to (\mathbb{Z}/p^n\mathbb{Z})^*) = 1 + P^n\mathbb{Z}_p$.

Write

$$t(a,b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \text{ (so } G = L = t(\mathbb{Z}_p^*, \mathbb{Z}_p))$$

With

$$\epsilon = t(1,1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in L$$

$$< u >= t(u,0) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \in L \text{ for } u \in \mathbb{Z}_p^*$$

I write out theorems for $\mathbb{Q}_p$ and it's valuation ring under p-adic norm, $\mathbb{Z}_p$, but the sam holds for all K, a complete discrete valuation field of characterisitc 0 with perfect residue field k of characteristic p such that p is a prime element of the valuation ring O_K of K.

Write the fraction field of $\Lambda(\Gamma)$ as $Q(\Gamma)$, and denote $\Lambda_O(\Gamma) = O[[T]]$ for the O-Iwasawa algebra of Γ, and $Q_O(\Gamma)$ for its quotient field.

Finally, following [3] we define for such a G a set S in $\Lambda(G)$ which turns out to be an Ore set and hence we can localize $\Lambda(G)$ with respect to S.

1.2.1 Definition (Canonical Ore Set):

Let S be the set of all f in $\Lambda(G)$ such that $\Lambda(G)/\Lambda(G)f$ is a finitely generated $\Lambda(H)$-module.

1.2.2 Remarks (Properties of Canonical Ore Set):

1. For any open subgroup H, M is finitely generated over $\Lambda(H)$ if and only if for any open subgroup H' of H, M is finitely generated over $\Lambda(H')$. So for such a pro-p $H' = J$ (when $\Lambda(J)$ is a local ring, the maximal ideal is the kernal of the augmentation map from $\Lambda(J)$ to $\mathbb{F}_p$). Moreover, we may assume it to be normal in G (since set of G conjugates of such a J is finite). Then we may consider the natural surjections $\phi_J : \Lambda(G) \to \Lambda(G/J)$ and $\psi_J : \Lambda(G) \to \Omega(G/J)$ (or equivalently with right modules).
   - S is the set of $f \in \Lambda(G)$ such that $\Omega(G/J)/\Omega(G/J)\psi_J$ is finite.
   - S is the set of $f \in \Lambda(G)$ such that $\Omega(G/J)/\Omega(G/J)\phi_J$ is a finitely generated $\mathbb{Z}_p$-module.
   - S is the set of $f \in \Lambda(G)$ such that right multipication by $\psi_J(f)$ on $\Omega(G/J)$ is injective (we may also consider right actions throughout).
2. For M a left $\Lambda(G)$-module, M is S-torsion if for each $x \in M$ there exists an $s \in S$ such that $s.x = 0$. For M a finitely generated left $\Lambda(G)$-module, *M is finitely generated over $\Lambda(H)$ if and only if M is S-torsion.*
3. The set S is multiplicatively closed and is a left and right Ore set in $\Lambda(G)$. The elements of S are non-zero divisors in $\Lambda(G)$

Also define $S^* = \bigcup_{n\geq 0} p^n S$. As p lies in the centre of $\Lambda(G)$, S^* is again a multiplicatively closed left Ore set in $\Lambda(G)$ all of whose elements are non-zero divisors. Writing $M(p)$ for the submodule of M consisting of elements of finite order. M is S^*-torsion if and only if $M/M(p)$ is finitely generated over $\Lambda(H)$. Write $\mathfrak{M}_H(G)$ for the category of all finitely generated $\Lambda(G)$-modules which are S^*-torsion (so $M/M(p)$ is finitely generated over $\Lambda(H)$), and within this let $\Lambda(G) - \text{mod}^H$ be full subcategory of $\Lambda(G) - \text{mod}$ consisting of finitely generated $\Lambda(G)$-modules which are finitely generated over $\Lambda(H)$.

We write $\Lambda(G)_S$, $\Lambda(G)_{S^*}$ for localizations and observe

$$\Lambda(G)_{S^*} = \Lambda(G)_S[\frac{1}{p}]$$

1.3 Classical Iwasawa Theory

Let G be a pro-p, p-adic, compact Lie group containing no element of order p. Define the Iwawawa algebra, $\Lambda(G)$ to be

$$\Lambda(G) = \varprojlim_{H <_o G} \mathbb{Z}_p[G/H]$$

Euler Characteristics can be used to give information on the structure of modules which are finitely generated over $\Lambda(G)$. We see in 1.3.5 below that the rank of such Iwasawa modules is interesting, and Euler Characteristics can be used to calculate these:

1.3.1 Theorem (Iwasawa Ranks and Homology - [9], 1.1):

Assume G is a pro-p, p-adic Lie group which contains no element of order p. Let M be a finitely generated $\Lambda(G)$-module. Then the $\Lambda(G)$-rank of M is given by the following "Euler characteristic" formula:

$$rank_{\Lambda(G)}(M) = \sum_{i\geq 0} (-1)^i rank_{\mathbb{Z}_p}(H_i(G, M))$$

The aim of this introduction is to explain how twisting gives rise to Generalised Euler Characterisitcs and how these fit into a larger programme in number theory connecting the algebraic and analytic theory of elliptic curves.

I now describe one of the main results of Classical Iwasawa Theory, before introducing the Selmer group of an Elliptic Curve as a way to study the Mordell-Weil group. I describe important invariants associated with field extensions and show the clever algebraic to analytic interplay between them. Then I introduce Euler Characterisitics and Generalised Euler Characterisitics as clever invariants which classify modules (up to pseudo-isomorphism). Finally I show how these calculations fit into a much larger programme of deducing information on the rank of the Mordell-Weil group from analytic L-functions.

Consider the cyclotomic extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$. This is formed by considering $\mathbb{Q}(\mu_{p^\infty})$ - adjoining all p power roots - there exists a homomorphism from the Galois group of this extension to $\mathbb{Z}_p^*$ and hence there exists a unique subextension whose Galois group over $\mathbb{Q}$ is $\Gamma = \mathbb{Z}_p$.

We then define subfields $\mathbb{Q}_n$ as invariants of the normal subgroups ie.

$$\begin{array}{cc|c|cl}
 & & \mathbb{Q}_\infty & | & \\
 & & | & | & \Gamma_n = p^n\mathbb{Z}_p \\
\Gamma = \mathbb{Z}_p & & \mathbb{Q}_n & | & \\
 & & | & & \\
 & & \mathbb{Q} & &
\end{array}$$

Let p^{e_n} be the highest power of p dividing the class number of $\mathbb{Q}_n$. Then Iwasawa's amazingly simple control theorem on e_n states:

1.3.2 Theorem (Growth of Class Numbers - [12], 5.3.17):

There exist integers λ, μ, ν such that

$$\mathbf{e_n} = \lambda\mathbf{n} + \mu\mathbf{p^n} + \nu$$

for all sufficiently large n.

This can be proved using the general philosophy of Iwasawa to study the Galois group Γ acting on it's normal subgroups (called X) by inner automorphism.

Serre made this method more transparent by viewing X as a module over the ring $\Lambda = \mathbb{Z}_p[[T]]$, where for γ some topological generator of Γ, T acts on the $\mathbb{Z}_p$-module X as $\gamma - 1$. This gives that the action of T on X is "Topologically Nilpotent" - T^nX gets arbitrarily small with n - so the power series action is well defined.

Key Point to this new approach is that there exists a simple structure theorem for such Λ-modules, see [12], 5.3.8:

1.3.3 Theorem (Structure Theorem of Cohen):

Let X be a finitely generated Iwasawa module. Then there exist irreducible Weierstrass polynomials F_j, and natural numbers r, m_i, n_j and a homomorphim with finite kernel and cokernel:

$$X \sim \Lambda^r \oplus \bigoplus_{i=1}^{s} \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^{t} \Lambda/F_j^{n_j}$$

where r, m_i, n_j and the prime ideals $F_j\Lambda$ are unique.

We may then associate an invariant, the Λ-rank to X, $r(X) = \mathrm{rank}_\Lambda(M) = r$. We may specialise this structure theorem to the torsion case:

1.3.4 Theorem (Torsion Structure Theorem):

Let X be a finitely generated, torsion, Λ-module, then there exists a pseudo-isomorphism (a Λ-module homomorphism with finite kernel and cokernel):

$$X \longrightarrow \bigoplus_{i=1}^{t} \Lambda/(f_i(T)^{a_i})$$

where the f_i are irreducible elements of Λ and the constants are defined up to reordering.

Given this Structure Theorem we can now define associated invariants:

Definition (Characteristic Polynomial):

The Characteristic Polynomial of X, is defined to be

$$f_X(T) = \prod_{i=1}^{t} f_i(T)^{a_i}$$

which is determined up to a unit in $\Lambda(G)$. Let,

1. $\lambda = \deg(f_X(T))$
2. $\mu =$ largest μ such that p^μ divides f_X in the Iwasawa algebra Λ.

1.3.5 Selmer Groups

In this section I will justify the suggestive notation above.

Suppose K is an algebraic extension of $\mathbb{Q}$.

By considering purely local conditions we define a subgroup $Sel_E(K)$ of $H^1(G_K, E(\overline{\mathbb{Q}})_{tors})$ which fits into the exact sequence:

$$0 \longrightarrow E(K) \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z}) \longrightarrow Sel_E(K) \longrightarrow \underline{|||}_E(K) \longrightarrow 0$$

where the group $\underline{|||}_E(K)$, a subgroup of $H^1(K, E(\overline{K}))$ is conjectured to be finite.

Studying the image of $E(K)/nE(K)$ under the Kummer mapping we may deduce the Mordell-Weil Theorem which gives the structure for $E(K)$, points lying on the curve over a field K having form

$$E(K) \cong \mathbb{Z}^r \text{ X } T$$

where T is a finite group and r, the rank of the elliptic curve, a non-negative integer.

Although it is possible to calculate the rank of an elliptic curve in some cases by considering dual isogenies and descent, there is no algorithm for this, and in general it is difficult to prove results about it directly.

Given our construction, notice

$$\text{rank}(Sel_E(K)) \geq r$$

with equality when $\underline{|||}_E(K)$ is finite.

So it is natural to study the rank through the Selmer group.

Since $Sel_E(\mathbb{Q}_\infty)_p$ is a subset of a Galois cohomology group, we may study the Pontajagin dual of it's p-part, $X = \widehat{Sel_E(\mathbb{Q}_\infty)}_p$ as an Iwasawa module. A theorem of Kato gives that the Selmer group is cotorsion - it's dual is torsion, and so the Strurcture Theorem applies. Iwasawa's construction gives that the invariants λ, μ controlling class group growth may be recovered explicitly as the structural invariants, λ, μ of X.

This elegant idea is only part of the programme as it is conjectured that the whole characterisitc polynomial f_X can also be obtained by purely analytic means from the elliptic curve.

1.3.6 Euler Characteristics

From the definitions we have injections

$$\begin{array}{ccc} Sel(E/\mathbb{Q}_\infty)_p & \hookrightarrow & H^1(\mathbb{Q}_\infty, E_{p^\infty}) \\ & & \uparrow \\ Sel(E/\mathbb{Q}_n)_p & \hookrightarrow & H^1(\mathbb{Q}_n, E_{p^\infty}) \end{array} \quad \text{induced from RESTRICTING elements of galois group}$$

The Hochschild-Serre spectral sequence gives that the image of the induced map actually lies in $H^0(\Gamma_n, H^1(\mathbb{Q}_n, E_{p^\infty}))$ - the Γ_n-invariants.

We may translate back to induce a map $Sel(E/\mathbb{Q}_n)_p \to {Sel(E/\mathbb{Q}_\infty)_p}^{\Gamma_n}$. Mazur's Control Theorem (see **[Ge1]** Chapter4) gives that this map has "nice" (finite) kernel and cokernel, moreover the orders of these groups is bounded independently of n.

ie.

$$\begin{array}{ccc} \text{finite} & & \\ \uparrow & & \\ Sel(E/\mathbb{Q}_\infty)_p & \hookrightarrow & H^1(\mathbb{Q}_\infty, E_{p^\infty}) \\ & & \uparrow \\ Sel(E/\mathbb{Q}_n)_p & \hookrightarrow & H^1(\mathbb{Q}_n, E_{p^\infty}) \\ \uparrow & & \\ \text{finite} & & \end{array} \quad \text{induced from RESTRICTING elements of galois group}$$

But instead of studying $Sel(E/\mathbb{Q}_\infty{}^{\Gamma_n})$ we consider it's dual, the torsion group $\widehat{Sel(E/\mathbb{Q}_\infty)}_{\Gamma_n}$. The coinvariants behave well with respect to Iwasawa algebra: $\Lambda_{\Gamma_n} = \mathbb{Z}_p[\Gamma/\Gamma_n]$, a cyclic group with p^n elements, hence $\Lambda_{\Gamma_n} = \mathbb{Z}_p^{p^n}$, so we can recover structure of original module, leading to information on $Sel(E/\mathbb{Q}_n)_p$.

We must be careful studying invariants, Z^G, or equivalently $H^0(G,Z)$, since it is not an exact functor, and is not well defined up to pseudo-Isomorphism, since the G invariants of a finite group are not necessarily Zero.

Instead we study the structure indirectly using Euler Characteristics, these are well defined up to pseudo isomorphism (ie. $\chi(G, \text{finite}) = 1$). Explicitly, in the 1-dimensional $\Gamma = \mathbb{Z}_p$ case:

Let γ be a topological generator for Γ. Let A be finite, then we have the exact sequence:

$$0 \longrightarrow A^\Gamma = \text{ “invariants” } = H^0(\Gamma, A) \longrightarrow A \xrightarrow{\gamma - 1} A \longrightarrow A_\Gamma = \text{ “coinvariants” } = H^1(\Gamma, A) \longrightarrow 0$$

(Higher cohomolgy groups vanish since, cohomological dimension = topological dimension = 1) Rank-Nullity gives:

$$\frac{|A^\Gamma|}{|A|} \cdot \frac{|A|}{|A_\Gamma|} = 1 \quad \Longrightarrow \quad \chi(\Gamma, \textit{ finite }) = 1$$

So Euler Characterisitc is well defined up to pseudo isomorphism.

Denoting the characteristic polynomial of a module X by $z(X)$, we have for a short exact sequence of Λ-torsion modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow B \longrightarrow C \longrightarrow 0$$

that

$$< z(B) > = < z(A) \text{ x } z(C) >$$

Hence we can build a characteristic polynomial from it's factors. Thus to show that *the Euler Characteristic defines modules up to pseudo isomorphism*, it remains to show this is true for modules of the form $M = \Lambda/(f)$ (and then extend using the structure theorem).

The following result from **[Ho3]** gives a connection between the Euler characteristic $\chi(G, \Lambda/(f))$ and the evaluation of the annihilator (a polynomial over the Iwasawa algebra) at 0.

1.3.7 Theorem (Euler Characteristics and Characteristic Elements - [9]):

$$\mathrm{ord}_{\mathbf{p}}(\chi(\mathbf{G}, \mathbf{\Lambda}/(\mathbf{f}))) = \mathrm{ord}_{\mathbf{p}}(\mathbf{f_M}(\mathbf{0}))$$

We can get more information on the annihilator twisting by a character and working with “Generalised Euler Characteristics”. I describe the procedure below:

1. Let V be a finite dimensional vector space on which G acts via ρ:

$$\rho : G \longrightarrow GL(V)$$

2. Suppose T is a $\mathbb{Z}_p$-lattice of V (ie. a $\mathbb{Z}_p$ module of the same dimension as the space) fixed by G, then T is a right $\Gamma = \mathbb{Z}_p[G]$ module.

3. For a finitely generated torsion Λ-module M, and for V as above, choosing a lattice T, define the generalised Euler characteristic, twisiting by ρ to be

$$\chi(G, V, M) = \prod_{i \geq 0} \#(Ext_i^\Gamma(T, M))^{(-1)^i}$$

provided this is defined - all Ext groups are finite, and zero for n sufficiently large.

4. **Claim: (see [10] Lemma 3.1)**

   $\chi(G, V, M)$ is independent of the choice of T, provided M is Λ-torsion.

   Hence Kato's work gives $\chi(G, V, M)$ is well defined independent of T for $M = X$

This construction leads to the more generalised result connecting E.C.s with an infinite number of valuations of the annihilator:

1.3.8 Theorem (see [10] Proposition 3.2)

$$\mathrm{ord}_p(\chi(G, V, M)) = \mathrm{ord}_p(\det\rho(f_M))$$

Recall the construction of the Iwasawa algebra, for a topological generator γ of the Galois group Γ:

$$\begin{array}{cc} \mathbb{Z}_p[[\Gamma]] & \gamma - 1 \\ \downarrow & \downarrow \\ \mathbb{Z}_p[[T]] & T \end{array}$$

Thus the first case, Theorem 1 corresponds to γ acting trivially on $\mathbb{Z}_p$: $(\gamma - 1)(x) = 0 \, \forall x \in \mathbb{Z}_p$.

Then, considering these twisted Euler characterisitcs corresponds to representations $\phi : \Gamma \to \mathbb{Z}_p^*$.

Thus we can identify the zeros of f_M, which determines f_M up to a unit, and hence module Λ/f_M is well defined just by understanding these twisted (generalised) Euler Characteristics.

1.3.9 p-adic L-functions

Let p be a prime of good reduction of the elliptic curve E.

Dualling a module swaps it's zeroth and first cohomology groups:

For a Γ_n-module D:

$$\widehat{H^0(\Gamma_n, D)} = \widehat{D^{\Gamma_n}} = \widehat{D}_{\Gamma_n} = H^1(\Gamma_n, \widehat{D})$$

So for Γ_n of cohomological dimension 1,

$$\chi(\Gamma_n, D) = \chi(\Gamma_n, \widehat{D})^{-1}$$

Hence calculating the Euler characteristic of it's module and it's dual are equivalent.

But we can approach these calculations from an entirely different angle, using the p-adic L-function constructed by Mazur and Swinnerton-Dyer.

Let E be a modular elliptic curve over $\mathbb{Q}$, fo ρ a Dirichlet character let $L(E/\mathbb{Q}, \rho, s)$ be the Hasse-Weil L-series for E twisted by the character ρ. Fix an embeddding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}_p}$.

They can be constructed in many different ways, but all give an interpolation of this twisted L-function to $\mathbb{Q}_p$.

Explicitly there exists an element, $\mathbf{L}(E/\mathbb{Q}, \overline{T}) \in \Lambda \otimes \mathbb{Q}_p$. Using the Weierstrass Preparation Theorem we may factorise this as,

$$\mathbf{L}(E/\mathbb{Q}, \overline{T}) = p^{\mu_E^{anal.}}.u(T).f(T)$$

where $U(T)$ is an invertible power series and $f(T)$ a distinguished polynomial.

Define, $\mathbf{f(T) = f_E^{anal.}(T)}$ **the Analytic Characteristic Polynomial.**

We now have the bridge between the algebraic and analytic sides, the Iwasawa Main Conjecture (I.M.C.):

1.4 Conjecture (Analytic and Algebraic Characteristic Elements - Mazur):

$$\mathbf{f_{anal}} = \mathbf{f_{alg.}}$$

If true, this gives an analytic way to compute the growth invariants λ, μ.

However, Kato has shown that in this classical setting, $f_{alg}|f_{anal}$ in $\mathbb{Q}_p[T]$. Hence, if the analytic invariants $\lambda_{anal}, \mu_{anal}$ agree with the algebraic structual invariants, λ_{alg}, μ_{alg}, then $f_{anal} = f_{alg.}$ and the main conjecture is verified in this case.

1.5 Non-Commutative Iwasawa Theory

What if the Galois group of the extension is no longer $\mathbb{Z}_p$ but a more general G? Then the best strucuture theory (for G p-valued) says there exists an exact sequence:

$$0 \to \bigoplus_{i=1}^{r} \Lambda(G)/L_i \to M/M_0 \to D \to 0$$

for L_i non-zero reflexive ideals of $\Lambda(G)$, M_0 the max. pseudo-null submodule of M, and D some pseudo-null $\Lambda(G)$-module.

Unfortunately the method then collapses - reflexive ideas need not be principal, and it is no longer the case that the Euer characterisitc $\chi(G, D)$ is finite implies $\chi(G, D) = 1$ for D pseudo-null.

QUESTION: How can the Euler Characteristic be modified to give a handle on the non-commutative case? An appropriate definition would overcome problems with pseudo-null modules not vanishing, be consistent with $G = \mathbb{Z}_p$ to recover the classical case, behave well under exact sequences, and a result of the form of 1.3.7 would suggest a definition for a suitable characteristic element.

1.5.1 Motivation

Following [4] we now use the well understood structure theory in the classical case 1.3.4), where characteristic elements are defined up to units in $\Lambda(\Gamma)$ to associate an element to $M \in \mathfrak{M}_H(G)$.

The long exact sequence of H-homology gives that since M is a finitely generated over $\Lambda(G)$, and $M/M(p)$ is finitely generated over $\Lambda(H)$ that $H_i(H, M)$ is a torsion $\Lambda(H)$-module for $i \geq 0$. Hence these homology groups, viewed as $\Lambda(H)$-modules have an associated characteristic element: $g_i \in \Lambda(H)$ say.

1.5.2 Definition (Akashi Series):

For $M \in \mathfrak{M}_H(G)$ let,

$$Ak(M) = \prod_{i \geq 0} g_i^{(-1)^i} \; (mod\ \Lambda(H)^*) \in Q(H)$$

I quote a few properties showing how this invariant behaves well - killing p-power order, and is multiplicative in exact sequences.

Lemma (Akashi Multiplicative over Exact Sequences - [4] 4.1):

- If $M(p)$ is the submodule of M in $\mathfrak{M}_H(G)$ consisting of all elements of p-power order then $Ak(M) = Ak(M/M(p))\,(\text{mod }\Lambda(H)^*)$.
- If we have an exact sequence of modules in $\mathfrak{M}_H(G)$, $0 \to M_1 \to M_2 \to M_3 \to 0$, then $Ak(M_2) = Ak(M_1).Ak(M_3)\,(\text{mod }\Lambda(H)^*)$.

Lemma (Euler Characteristics and Akashi Series - [4] 4.2):

Assume that $M \in \mathfrak{M}_H(G)$ has finite G-Euler Characteristic. Then $[Ak(M)](0)$ is defined and non-zero and we have

$$\chi(G, M) = |[Ak(M)](0)|_p^{-1}$$

This follows from the Hochschild-serre spectral sequence giving the exact sequence:

$$0 \to H_0(H, H_i(H, M)) \to H_i(G, M) \to H_1(H, H_{i-1}(H, M)) \to 0$$

Let π_H denote the natural projection $\Lambda(G) \to \Lambda(H)$, a ring homomorphism.

Lemma (- see [4] 4.3):

Let g be a non-zero element of $\Lambda(G)$ such that $N = \Lambda(G)/\Lambda(G)g \in \mathfrak{M}_H(G)$ - N is principal. Then $H_i(H, N) = 0 \,\forall\, i > 0$. Moreover $Ak(N) = \pi_H(g) \in Q(H)$ is in $\Lambda(H)$ (easy to calculate), and $[Ak(N)](0) \neq 0$ if and only if f has finite G-Euler Characterisitic (consistent with above).

This Akashi series also gives a good test for pseudo-null elements:

Lemma (Akashi Series of Pseudo-null modules - see [4] 4.4):

Let M be a module in $\mathfrak{M}_H(G)$. If $G \cong \mathbb{Z}_p^r$ for some $r \geq 1$ then $Ak(M) = 1\,(mod\; \Lambda(H)^*)$ if and only if M is pseudo-null as a $\Lambda(G)$-module.

An example is given in [4] to show despite this result suggesting that the invariant $Ak(M)$ is a unit for all pseudo-null modules $M \in \mathfrak{M}_H(G)$ it is not the case.

Thinking of the Akashi series as a virtual object - a difference of 2 sums of characteristic elements triggers the connection with Grothendieck Groups, and thus with K-theory. In [22], Venjakob describes the details connecting these ideas to K-theory, and uses the standard theory in this area to build a characteristic element.

For $M \in \mathfrak{M}_H(G)$, the Aksahi series : $Ak_G(H, M) = \prod_{i \geq 0} \mathrm{char}_H(H_i(H, M))^{(-1)^i}$, thought of as an element in $Q(H)^*/\Lambda(H)^*$ induces a map of K-groups:

$$Ak_G(H, -) : K_0(\Lambda(G) - \mathrm{mod}^H) \to K_0(\Lambda(H), Q(H)) \cong Q(H)^*/\Lambda(H)^*,$$

the Swan relative K-group of ring homomorphism $\Lambda(H) \to Q(H)$. K-theory then gives a long exact sequence of Localisation for Ore set S^*:

$$\cdots \to K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\delta_G} K_0(\mathfrak{M}_H(G)) \to K_0(\Lambda(G)) \to K_0(\Lambda(G)_{S^*}) \to 0$$

1.5.3 Definition (Characteristic Element):

It is known that δ_G is surjective, and we now define for each $M \in \mathfrak{M}_H(G)$, a characteristic element of M is any ξ_M in $K_1(\Lambda(G)_{S^*})$ such that,

$$\delta_G(\xi_M) = [M]$$

Proposition (Akashi Series and Characterisitc Element - see [22] 8.1):

Let $M \in \mathfrak{M}_H(G)$. Then the following holds:

$$Ak_G(H, M) = \pi_H(char_G(M)) = \pi_H(F_M)(mod\; \Lambda(G)^*)$$

This connection with Akashi series now yields a good understanding of the Euler Characterisitics. Since twisting preserves exact sequences we may generalise results of the form 1.5.2 to cover all twists of a module.

Theorem (Non-commutative Characteristic Elements and Euler Characteristics - see [3] 3.6):

For a continuous homomorphism $\rho : G \to GL_n(O)$, where $m_\rho = [L : \mathbb{Q}_p]$, L the quotient field of O, and $\hat{\rho}$ the contragradient representation of G: $\hat{\rho}(g) = \rho(g^{-1})^t$. Then, assuming G has no element of order p. For $M \in \mathfrak{M}_H(G)$ let ξ_M be a characteristic element of M. Then, if $\chi(G, tw_{\hat{\rho}}(M))$ is finite we have

- $\xi_M(0) \neq 0, \infty$
- $\chi(G, tw_{\hat{\rho}}(M)) = |\xi(0)|_\rho^{-m_\rho}$

This Theorem shows how all G-Euler characterisitcs may be recovered from the characteristic element, and following 1.3.7 suggests it is a good theory to study.

1.5.4 Machinery

I end this section by giving results from [1] on how one goes about calculating these Characteristic elements. By restricting to p-torsion modules we may write a closed form for the characteristic element.

Let $\mathfrak{D}$ denote the category of finitely generated p-torsion Λ_G-modules, then $T = \{1, p, p^2, \dots\}$ plays the role of the central Ore set giving the exact sequence:

$$\dots \to K_1(\Lambda(G)) \to K_1((\Lambda(G))_T) \xrightarrow{\delta_G} K_0(\mathfrak{D}) \to K_0(\Lambda(G)) \to K_0((\Lambda(G))_T) \to 0$$

Then δ_G is again surjective, and a characteristic element ξ_M is such that $\delta_G(\xi_M) = [M] \in K_0(\mathfrak{D})$.

T is always contained in S^* so there exists a natural commutative diagram of K-groups:

$$\begin{array}{ccc} K_1((\Lambda_G)_T) & \xrightarrow{\delta_G} & K_0(\mathfrak{D}) \\ \downarrow & & \downarrow \\ K_1((\Lambda_G)_{S^*}) & \xrightarrow{\delta_G} & K_0(\mathfrak{M}_H(G)) \end{array}$$

This shows compatibility of characteristic elements, and allows us to study $K_1((\Lambda_G)_{S^*})$ through $K_1((\Lambda_G)_T)$.

We may define the i-th twisted μ-invariant of M for $i = 1, \dots, s$ (running through the s (finite) simple modules $V_1, \dots, V_s$ of $\Lambda(G)$ up to Isomorphism):

$$\mu_i(M) = \frac{\log_p(G, (gr_p M) \otimes_{\mathbb{F}_p} V_i^*)}{\dim_{\mathbb{F}_p} \mathrm{End}_{\Omega(G)}(V_i)},$$

where the grading is with respect to the p-adic filtration on M, giving a finitely generated $\Omega(G)$-module.

It is shown that this number is an integer and moreover we can build up the characteristic element from these:

1.5.5 Theorem (Closed Form for Characteristic Element - see [1] 1.5):

Let $\theta : (\Lambda(G))_T^* \to K_1((\Lambda(G)_T)$ be the canonical homomorphism, and let M be a finitely generated p-torsion $\Lambda(G)$-module. Then,

$$\xi_M = \theta(\prod_{i=1}^{s} f_i^{\mu_i(M)}),$$

where $f_i = 1 + (p-1)e_i$ for e_i an idempotent in $\Lambda(G)$ such that V_i is the unique simple quotient module of $e_i\Lambda(G)$.

We may write out each map in the Localisation Sequence:

For R a ring, and T an Ore Set in R consisting of regular elements we know a localisation R_T exists. The canonical map $\phi : R \to R_T$ gives rise to the exact sequence:

$$K_1(R) \to K_1(R_T) \to K_0(R, \phi) \to K_0(R) \to K_0(R_T)$$

If the ring R is also Noetherian and Regular (meaning every finitely generated R-module has finite projective dimension). This is true for rings of finite global dimension. Then it is known $K_0(R,\phi)$ may be identified with the Grothendieck group $K_0(\mathfrak{C})$, $\mathfrak{C}$ the category of all finitely generated S-torsion R-modules, by the invertible map m given below.

Moreover, when R is regular Noetherian, there exists an isomorphism $\gamma : G_0(R) \to K_o(R)$ giving the exact sequence,

$$\begin{array}{ccccccc} K_1(R) \to & K_1(R_S) & \xrightarrow{\delta} & K_0(\mathfrak{C}) & \xrightarrow{\alpha} & K_0(R) \xrightarrow{\beta} K_0(R_S) \to 0 \\ & \theta\uparrow & l\searrow & m\downarrow || & \nearrow n & \\ & R_S^* & & K_0(R,\phi) & & \end{array}$$

Then we have the following descriptions, denoting by θ the canonical embedding of the units, $R_S^* \to K_1(R_S)$.

- $\alpha([M]) = \gamma([M])$ for all $M \in \mathfrak{C}$.
- $\beta([M]) = [M \otimes_R R_S]$.
- $\gamma([M]) = \sum_{j=0}^{n}(-1)^j[X_j]$ for any choice of finite Projective Resolution of the R-module M,

$$0 \to X_n \to \cdots \to X_0 \to M \to 0$$

.

- $\delta(\theta(x)) = [R/xR] \in \mathfrak{C}$, for all $x \in R \cap R_S^*$. This means that $\theta(x) \in K_1(R_S)$ is a characteristic element for the principal module R/xR when $x \in R \cap R_S^*$.
- The Relative K-group, $K_0(R,\phi)$ is identified with triples (M,N,f), where M and N are Λ_G-modules and f gives an Isomorphism between the induced modules $f : M \uparrow_{\Lambda_G}^{(\Lambda_G)_S} \cong N \uparrow_{\Lambda_G}^{(\Lambda_G)_S}$ modulo obvious commuting squares.
- For M a Λ_G-module, $m(M) \in K_0(R,\phi)$ is defined as follows. Let $\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$ be a finite projective resolution of M by Λ_G-modules. Projectivity allows us to choose a section at each stage $F_{2m} \to F_{2m+1}$, and gluing together a map between even and odd sums, ψ:

$$F^+ = \bigoplus_{i \text{ even}}, \quad F^- = \bigoplus_{i \text{ odd}}$$

  Take $m(M) = [F^+, F^-, \psi]$. This is independent of the choices of resolution and sections, and is additive on short exact sequences.

- For an element of $K_1[(\Lambda_G)_T]$ represented by $g \in GL_n(\Lambda_G)$, thought of as an invertible linear map $g : [(\Lambda_G)_T]^n \to [(\Lambda_G)_T]^n$. Let,

$$l(g) = ((\Lambda_G)^n, (\Lambda_G)^n, g) \in K_0(R,\phi)$$

  . When g is already an isomorphism before inducing, the triple is trivial and we see the inclusion of $K_1(R)$ in $K_1(R_T)$ followed by $l : K_1(R_T) \to K_0(R,\phi)$ is trivial (necessary for Exact Sequence).

- F0r $(M,N,f) \in K_0(R,\phi)$,

$$n(M,N,f) = [M] - [N] \in K_0(R)$$

  Then it is easily seen $n \circ l(g) = [(\Lambda_G)^n] - [(\Lambda_G)^n] = 0$.

1.5.6 Example (False Tate Curve):

The case we will look at to illustrate these techniques comes from picking a field k (assumed to contain the group μ_p of p-th power roots of unity) by adjoining to k_{cyc} the p-power roots of an element in k^* which is not just a root of unity itself. Their union k_∞ is by Kummer theory a Galois extension with Gaois group, $G \cong \mathbb{Z}_p(1) \rtimes \mathbb{Z}_p$ - a semi-direct product where the action is given by the cyclotomic character.

Chapter 2

Background

2.1 Uniform Groups

2.1.1 Profinite Groups and Pro-p Groups

pro-p group special type of profinite group

2.1.2 Powerful p-groups

The key to understanding the structure of analytic pro-p groups lies in the properties of a special class of finite groups

2.1 Definition

1. *A finite p-group G is powerful if p is odd and G/G^p is abelian.*
2. *A subgroup N of a finite p-group G is powerfully embedded in G, written N p.e. G if p is odd and $[N,G] \leq N^p$.*

Think of G powerful as G almost abelian.

2.3 Proposition

Let G be a finite p-group and $N \leq G$. If N p.e. in G then N^p p.e G

Define lower central series for G a finite p-group:

$$P_i(G) = G,\ P_{i+1} = p_i(G)^p[P_i(G), G] \text{ for } \geq 1$$

Notation

$$G_i = P_i(G)$$

2.4 Lemma

Let G be a powerful p-group.

1. *For each i, G_i p.e. G and $G_{i+1} = G_i^p = \Psi(G_i)$*
2. *For each i, the map $x \mapsto x^p$ induces a homomorphism from G_i/g_{i+1} onto G_{i+1}/G_{i+2}*

2.5 Lemma

If $G =< a_1, \ldots, a_d >$ *is a powerful p-group, then* $G^p =< a_1^p, \ldots, a_d^p >$

2.6 Proposition

If G is a powerful p-group then every element of G^p *is a p-th power in G*

2.8 Corollary

If $G =< a_1, \ldots, a_d >$ *is a powerful p-group, then* $G =< a_1 > \cdots < a_d >$ *i.e. G is the product of its cyclic subgroups* $< a_i >$*.*

2.9 Theorem

If G is a powerful p-group and $H \leq G$ *then* $d(H) \leq d(G)$*.*

2.13 Theorem

Let G be a finite p-group of rank r. Then G has a powerful characteristic subgroup of index at most $p^{r\lambda(r)}$ *if p is odd.*

2.1.3 Pro-p groups of finite rank

Characterise the pro-p groups of finite rank as exactly those which contain a finitely generated powerful open subgroup

3.1 Definition

Let G be a pro-p group

1. G is powerful if p is odd and $G/\overline{G^p}$ is abelian.
2. Let $N \leq oG$. Then N is powerfully embedded in G if p is odd and $[N, G] \leq \overline{N^p}$.

3.2 Proposition

Let G be a pro-p group and $N \leq_o G$. Then N p.e G if and only if NK/K p.e in G/K for every K open normal in G.

3.3 Corollary

A topological group G is a powerful pro-p group if and only if G is the inverse limit of an inverse system of powerful finite p-groups in which all the maps are surjective.

Now carry over result from previous chapters, but need that the lower central series is well behaved - consists of open subgroups. True in case of finitely generated pro-p groups.

3.4 Lemma

Let G be a powerful finitely generated pro-p group. Then every element of G^p is a p-th power in G, and $G^p = \Psi(G)$ is open in G.

3.5 Corollary

As above, for each i we have

$$G^{pi} = (G^{pi-1})^p = \{x^{pi} | x \in G\} \text{ p.e. } G^{pi-1}$$

3.6 Theorem

Let $G = \overline{< a_1, \ldots, a_n >}$ be a finitely generated powerful pro-p group, and put $G_i = P_i(G)$ for each i.

- G_i p.e. G
- $G_{i+k} = P_{k+1}(G_i) = G_i^{pk}$ for each $k \geq o$, and in particular $G_{i+1} = \Psi(G_i)$
- $G_i = G^{pi-1} = \overline{< a_1^{p^{i-1}} \ldots a_d^{p^{i-1}} >}$
- the map $x \mapsto x^{pk}$ induces a homomorphism from G_i/g_{i+1} onto G_{i+1}/G_{i+2}

3.7 Proposition

If $G = \overline{< a_1, \ldots, a_d >}$ is a powerful pro-p group, then $G = \overline{< a_1 >} \ldots \overline{< a_d >}$ i.e. G is the product of its procyclic subgroups $\overline{< a_i >}$.

For any topological subgroup G, $d(G)$ denotes the minimal cardinality of a topological generating set for G. If G is a finitely generated pro-p group, we thus have

$$d(G) = dim_{\mathbb{F}_p}(G/\Psi(G))$$

Thus

3.8 Theorem

Let G be a powerful finitely generated pro-p group and H a closed subgroup. Then $d(H) \leq d(G)$.

3.13 Theorem

Let G be a pro-p group. Then G has finite rank if and only if G is finitely generated and G has a powerful open subgroup; in that case, G has a powerful open characteristic subgroup.

3.17 Theorem

Let G be a pro-p group. Then the following are equivalent:

1. G is the product of finitely many procyclic subgroups;
2. G is the product of finitely many closed subgroups of finite rank;
3. G has finite rank;
4. G is finitely generated as a $\mathbb{Z}_p$-powered group. i.e. G has a finite subset X such that every element of G is equal to a product of the form $x_1^{\lambda_1} \ldots x_s^{\lambda_s}$ with $x_j \in X$ and $\lambda_j \in \mathbb{Z}_p$.
5. G is countably generated as a "$\mathbb{Z}_p$-powered group".

2.1.4 Uniformly powerful groups

Shown every pro-p group of finite rank has an open normal subgroup which is powerful. show can satisfy a stronger condition - uniformaly powerful. plus in these groups the group operation can be smoothed out to give a new, abelian group structure, and that this new group is naturally a finitely generated free $\mathbb{Z}_p$-module.

4.1 Definition

A pro-p group G is uniformaly powerful if

1. G is finitely generated
2. G is powerful, and
3. for all i,$|P_i(G) : P_{i+1}(G)| = |G : P_2(G)|$

Usually abbreviate uniformally powerful to uniform.
Then homo $x \mapsto x^p$ is actually an isomorphism $P_i/P_{i+1} \to P_{i+1}/P_{i+2}$.

4.2 Theorem

Let G be a finitely generated powerful pro-p group. Then $P_k(G)$ is uniform for all sufficiently large k.

4.3 Corollary

A pro-p group of finite rank has a characteristic open uniform subgroup.
G is uniform if and only if $d(G_i/G_{i+1}) = d(G_1/G_2) = d$ for all i.

4.4 Proposition

Let G be a powerful finitely generated pro-p group. TFAE:

1. G is uniform;
2. $d(P_i(G)) = d(G)$ for all $i \geq 1$;
3. $d(H) = d(G)$ for every powerful open subgroup H of G.

Simplest characterisation of uniform groups:

4.5 Theorem

A powerfully finitely generated pro-p group is uniform if and only if it is **torsion free**.

4.6 Lemma

If A and B are open uniform subgroups of some pro-p group then $d(A) = d(B)$

4.7 Definition

Let G be a pro-p group of finite rank. The dimension of G is

$$dim(G) = d(H)$$

where H is any open uniform subgroup of G (unambiguous).

Later show dim(G) dimension of G as a p-adic analytic group. Here set up homos between uniform group and ${\mathbb{Z}_p}^d$. Definition makes sense:

4.8 Theorem

Let G be a pro-p group of finite rank and N a closed normal subgroup of G. Then

$$dim(G) = dim(N) + dim(G/N)$$

Recall $G = \overline{<a_1>} \dots \overline{<a_d>}$ thus for each a in G,

$$a = a_1^{\lambda_1} \dots a_d^{\lambda_d}$$

with $\lambda_1 \dots \lambda_d \in \mathbb{Z}_p$, moreover:

4.9 Theorem

Let G be a uniform pro-p group and $\{a_1, \ldots, a_d\}$ a topological generating set for G, where $d = d(G)$. Then the mapping

$$(\lambda_1, \ldots, \lambda_d) \rightarrow a_1^{\lambda_1} \ldots a_d^{\lambda_d}$$

from ${\mathbb{Z}_p}^d$ to G is a **homeomorphism**.

The following Lemmas set up an additive structure, using IM coming from uiniformity.

4.10 Lemma

Let $n \in \mathbb{N}$. The mapping $x \mapsto x^{p^n}$ is a homeomorphism from G onto G_{n+1}. For each k and m, it restricts to a bijection $G_k \rightarrow G_{k+n}$ and induces a bijection $G_k/G_{k+m} \rightarrow G_{n+k}/G_{n+k+m}$

4.10 shows that each element $x \in G_{n+1}$ has a unique p^n-th root in G, denoted $x^{P^{-n}}$. Use this bijection to TRANSFER THE GROUP OPERATION FROM G_{n+1} to G, to define a new structure on G:

For $x, y \in G$ we define

$$x +_n y = (x^{p^n} y^{p^n})^{p^{-n}}$$

The map $x \mapsto x^{p^{-n}}$ is an IM from G_{n+1} onto the group $(G, +_n)$.

4.11 Lemma

For $n > 1$, $x, y \in G$, $u, v \in G_n$

$$xu +_n yv \cong x +_n y \cong x +_{n-1} y \ (mod\, G_{n+1})$$

and for all $m > n$

$$x +_m y \cong x +_n y \ (mod\, G_{n+1})$$

So this mean that for a given pair (x, y) the sequence $(x +_n y)$ is a CAUCHY SEQUENCE, define:

4.12 Definition

For $x, y \in G$,

$$x + y = Lim_{n \rightarrow \infty} x +_n y$$

4.13 Proposition

The set G with the operation + is an abelain group, with identity element 1 and inversion given by $x \rightarrow x^{-1}$.

Using additive notation we have.

4.14 Lemma

1. If $xy = yx$ then $x + y = xy$
2. For each integer m, $mx = x^m$
3. For each $n \geq 1$, $p^{n-1}G = G_n$.
4. If $x, y \in G_n$ then $x + y \cong xy \ (mod\, G_{n+1})$

4.15 Corollary

For each n, G_n is an additive subgroup of G; the additive cosets of G_n in G are the same as the multiplicative cosets of G_n in G. Also the identity map $G_n/G_{n+1} \rightarrow G_n/G_{n+1}$ is an isomorphism of the additive group G_n/G_{n+1} onto the multiplicative group G_n/G_{n+1}, and the index of G_n in the additive group $(G, +)$ is equal to $|G : G_n|$.

4.16 Proposition

With the origional topology of G, $(G,+)$ is a uniform pro-p group of dimension $d = d(G)$. Moreover, any set of topological generators for G is a set of topological generators for $(G,+)$.

As $(G,+)$ is a pro-p group, it admits a natural action by $\mathbb{Z}_p$. Since $(G,+)$ is abelain we make it into a $\mathbb{Z}_p$-module. Structure of module givenby

4.17 Theorem

Let G be a uniform pro-p group of dimension d, and let $\{a_1,$
$dotsa_d\}$ be a topological generating set for G. Then, with the operations defined above, $(G,+)$ is a free $\mathbb{Z}_p$-module on the basis $\{a_1,$
$dotsa_d\}$.

2.1.4.1 4.18 Corollary

Let G be a uniform pro-p group of dimension d. Then the action of $Aut(G)$ on G is $\mathbb{Z}_p$-linear with respect to the $\mathbb{Z}_p$-module structure on $(G,+)$. Hence, $Aut(G)$ may be identified with a subgroup of $GL_d(\mathbb{Z}_p)$.

There is a nice structure theorem for such groups:

4.22 Corollary

Let G be a finitely generated powerful pro-p group of dimension d. Then $Aut(G)$ is isomorphic to a subgroup of $GL_d(\mathbb{Z}_p)\,X\,F$ for some finite group F. In particular $Aut(G)$ is isomorphic to a linear group over $\mathbb{Z}_p$.

Passing from the uniform pro-p group G to the $\mathbb{Z}_p$-module $(G,+)$ described above means forgetting a lot of information about the structure of G, since all free $\mathbb{Z}_p$-modules of a given rank are IM!!!

Save information by using lie operation:

Definition of lie bracket

For $x, y \in G$ and $n \in \mathbb{N}$

$$(x,y)_n = [x^{p^n}, y^{p^n}]^{p^{-2n}}$$

2.1.4.2 4.28 Lemma

If $n > 1$, $x, y \in G$ and $u, v \in G_n$, then

$$(xu, yv)_n \cong (x,y)_n \cong (x,y)_{n-1}\,(mod\,G_{n+1})$$

and for all $m > n$ we have

$$(x,y)_m \cong (x,y)_n\,(mod\,G_{n+2})$$

Thus for given x and y, $((x,y)_n)$ is a Cauchy sequence and we may define:

2.1.4.3 4.29 Definition

For $x, y \in G$,

$$(x,y) = lim_{n\to\infty}(x,y)_n$$

2.1.4.4 4.30 Theorem

With the operation $(-,-)$ the $\mathbb{Z}_p$-module $(G,+)$ becomes a Lie algebra over $\mathbb{Z}_p$.

log is equivalent to passing to the Lie algebra, and the C-H formula shows that xy can be recovered from the Lie algebra structure of $(G,+)$ so don't loose as much information.

Uniformally powerful pro-p groups is a new idea, and loosely corresponds to Lazard's class of "groupes p-saturables" - he views the situation in terms of filtrations.

2.1.5 Automorphism Groups

Here we show that the automorphism group of a pro-p group of finite rank is itself virtuallya pro-p group of finite rank.

Special case for $\mathbb{Z}_p^d$ when we study the Aut group $\Gamma = GL_d(\mathbb{Z}_p)$.

A base for the neighbourhoods of 1 in Γ is given by the congruence subgroups:

$$\Gamma_i = \{\gamma \in \Gamma | \gamma \cong 1_d \, (mod \, p^i)\}$$

It follows that Γ is profinite and that Γ_1 is a pro-p group.

Γ is a compact p-adic analytic group; a fundamental property of such groups is that they contain an open powerful finitely generated pro-p subgroujp, and verify directly:

5.1 Lemma

If p is odd and $n \geq 2$ then every element of Γ_n is the p-th power of an element of Γ_{n-1}.

In general $Aut(G)$ will not itself be a profinite group, however,

5.3 Theorem

If G is a finitely generated profinite group then $Aut(G)$ is a profinite group.

2.1.5.1 5.5 Proposition

Let G be a finitely generated pro-p group. Then $\Gamma(\Phi(G))$ is a pro-p group.

A profinite group G is said to have a property **v**irtually if G has an open normal subgroup H such that H has the property.

2.1.5.2 5.6 Theorem

Let G be a finitely generated profinite group. If G is virtually a pro-p group then $Aut(G)$ is also a virtually pro-p group.

2.1.5.3 5.7 Theorem

Let G be a profinite group. If G is virtually a pro-p group of finite rank, then so is $Aut(G)$.

2.1.6 Normed algebras

Work towards the Campbell-Hausdorff formula.

6.1 Definition

A norm on a ring R is a function $||-|| : R \to \mathbb{R}$ such that for all $a, b \in R$

1. $||a|| \geq 0$; $||a|| = 0$ if and only if $a = 0$.
2. $||1_R|| = 1$ and $||ab|| \leq ||a||.||b||$;
3. $||a \pm b|| \leq max\{||a||, ||b||\}$.

If these hold then R is said to be a normed ring.

6.2 Definition

- The normed ring $(R, ||-||)$ is complete if every Cauchy sequence in R converges to an element in R.
- A normed ring $(\hat{R}, ||-||)$ is called a completion of R if R is a dense subring of $\hat{R}$, and the norm on $\hat{R}$ extends the norm on R, and $\hat{R}$ is complete.
- **For any normed ring such a completion exists and is unique up to norm preserving IM which restricts to the identity on R**

This can also be approached via filtrations:

6.5 Lemma

Let R be a ring and

$$R = R_0 \supset R_1 \supset \cdots \supset R_i \supset \ldots$$

a chain of ideals such that

- $\bigcap_{i \in \mathbb{N}} R_i = 0$
- for all $i, j \in \mathbb{N}$, $R_i R_j \subset R_{i+j}$

Fix a real number $c > 1$ and define $||-|| : R \to \mathbb{R}$ by

$$||0|| = 0; \quad ||a|| = c^{-k} \text{ if } a \in R_k - R_{k+1}$$

Then $(R, ||-||)$ is a normed ring.

6.6 Definition

Let A be a $\mathbb{Q}_p$-algebra. Then $(A, ||-||)$ is a normed $\mathbb{Q}_p$-algebra if $||-||$ is a norm on the ring A and the following holds:

$$||\lambda a|| = |\lambda|.||a|| \text{ for all } a \in A \text{ and } \lambda \in \mathbb{Q}_p$$

Introduce general notion of convergence

6.8 Definition

Let T be a countably infinite set and let $n \to a_n$ be a map of T into R. Let $a, s \in R$.

1. The family $(a_n)_{n \in T}$ converges to a, if, for each $\epsilon > 0$ there exists a finite subset T' of T such that $||a - a_n|| < \epsilon$ for all $n \in T - T'$.
2. The series $\sum_{n \in T} a_n$ converges with sum s if for each $\epsilon > 0$ there exists a finite subset T' of T such that for all finite sets T'' for which $T' \subset T'' \subset T$ we have $||s - \sum_{n \in T} a_n|| < \epsilon$

6.9 Proposition

Let T be a ccountably infinte set and let $n \mapsto a_n$ be a map from T into R. Let $i \mapsto n(i)$ be a bijection from $\mathbb{N}$ to T.

1. $lim_{n \in T} a_n = a$ if and only if $lim_{i \to \infty} a_{n(i)} = a$.
2. The series $\sum_{n \in T} a_n$ converges in R if and only if $lim_{n \in T} a_n = 0$.
3. $\sum_{n \in T} a_n = s$ if and only if $\sum_{i=0}^{\infty} a_{n(i)} = s$.
4. If $\sum_{n \in T} a_n = s$ then $||s|| \leq sup\{||a_n|| \, |n \in T\}$
5. If $\sum_{n \in T} a_n = s$ and for some $m \in T$, $||a_m|| > |a_n||$ for all $n \in T - \{m\}$, then $||s|| = ||a_m||$.

6.10 Proposition

Let T be the disjoint union of a countable family $\{T_\lambda | \lambda \in \Lambda\}$ of countable sets T_λ. Suppose that $\sum_{n\in T} a_n$ is a convergent series in R, with sum s. Then each of the series $\sum_{n\in T_\lambda} a_n$ converges in R with sum s_λ say, and $\sum_{\lambda\in\Lambda} s_\lambda = s$.

6.11 Corollary - DOUBLE SERIES

Let S_1 and S_2 be countable sets. Suppose that for each $(m,n) \in S_1\, x\, S_2$, a_{mn} is an element of R, and that $lim_{(m,n)\in S_1\, x\, S_2} a_{mn} = 0$. Then the double series $\sum_{m\in S_1}(\sum_{n\in S_2} a_m n)$ and $\sum_{n\in S_2}(\sum_{m\in S_1} a_m n)$ both converge and their common sum equals $\sum_{(m,n)\in S_1\, xS_2} a_{mn}$.

6.12 Corollary - CAUCHY MULTIPLICATION OF SERIES

Suppose that $(T,*)$ is a countable set with a binary operation $*$, and that $\sum_{n\in T} a_n$ and $\sum_{n\in T} b_n$ are convergent series in R. Then, for each $n \in T$, the series

$$\sum_{(r,s)\; s.t.\; r*s-n} a_r b_s$$

converges with sum c_n say, and the series $\sum_{n\in T} c_n$ converges with

$$\sum_{n\in T} c_n = (\sum_{n\in T} a_n)(\sum_{n\in T} b_n)$$

Result on uniqueness of power series:

6.13 Proposition

Let A be a complete normed $\mathbb{Q}_p$-algebra and let a_n $(n \in \mathbb{N})$ be elements of A. Suppose there exists a neighbourhood V of 0 in $\mathbb{Q}_p$ such that

$$\sum_{n\in\mathbb{N}} \lambda^n a_n = 0 \text{ for all } \lambda \in V$$

Then $a_n = 0$ for all $n \in \mathbb{N}$

For $(A, ||-||)$ a complete normed $\mathbb{Q}_p$-algebra, we define non-commutative power series, considering elements of $W(X_1,\ldots,x_n)$ as WORDS. Concateate mutliples, and recall empty word, of degree 0 is the identity.

6.14 Definition

The ring of formal power series in the non-comm variables $X_1,\ldots,X_n$ denoted

$$\mathbb{Q}_p << X_1,\ldots,X_n >>$$

is the set of all formal sums

$$F(\mathbf{X}) = \sum_{w\in Words} a_w w \,(a_w \in \mathbb{Q}_p \text{ for all } w)$$

made into a $\mathbb{Q}_p$-algebra with componentwise addition and scalar multiplication:

$$\sum_{w\in W} a_w w \sum_{w\in W} b_w w = \sum_{w\in W} c_w w$$

where $c_w = \sum_{uv=w} a_u b_v$.

$\mathbb{Q}_p << \mathbf{X} >>$ is indeed a $\mathbb{Q}_p$-algebra.

The set of all power series where the power series converges upon substituting the coordinate $\mathbf{X}$ is denoted $E_{\mathbf{X}}$.

6.16 Lemma

Let $x = (x_1, \ldots, x_n) \in A^n$

1. The subset E_X is a subalgebra of $\mathbb{Q}_p << X >>$.
2. The mapping $F(X) \to F(x)$ of $E(X)$ into A is a $\mathbb{Q}_p$-algebra homomorphism.

MAIN DEFINITION - A^n given the product topology. Write $w(||x||) = w(||x_1||, \ldots, ||x_n||)$

Definition

Let $f : D \to A$ be a mapping, where D is a non-empty open subset of A^n. Then f is STRICTLY ANALYTIC on D if there exists $f(X) = \sum_{w \in W} a_w w \in \mathbb{Q}_p << X >>$ such that, for each $x = (x_1, \ldots, x_n) \in D$,

1. $lim_{w \in W} |a_w| w(||x||) = 0$, and
2. $f(x) = F(x)$
3. first condition is a kind of absolute convergence since $||a_w w(x)|| \leq |a_w| w(||x||)$.

Now show how coefficients in power series must be well behaved if it is to have a nice representation:

2.1.6.1 6.18 Lemma

Suppose that f is strictl analytic on a non-empty open set $D \subset A^n$ and that f is represented by $F(X) = \sum_{w \in W} a_w w \in \mathbb{Q}_p << X >>$. Then there exists $k \in \mathbb{N}$ such that $p^{k\, deg\, w} a_w \in \mathbb{Z}_p$ for all $w \in W - \{1\}$

6.19 Proposition

Let D be a non-empty open subset of A^n. If f is a strictly analytic function on D then f is continuous on D.

Now we define exponential and logorithm functions:

6.20 Lemma

For each positive integer n,

$$v(n!) \leq (n-1)/(p-1)$$

6.21 Definition

Power series in $\mathbb{Q}_p << X >>$:

$$\mathbb{E}(X) = \sum_{n=0}^{\infty} \frac{1}{n!} X^n$$

$$\mathbb{L}(X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} X^n$$

Let $A_0 = \{x \in A | \, ||x|| \leq p^{-1}\}$ if $p \neq 2$.

6.22 Proposition

There exist strictly analytic functions,

$$exp : A_0 \to 1 + A_0$$
$$log : 1 + A_0 \to A_o$$

such that

$$exp(x) = \mathbb{E}(x)$$
$$log(x) = \mathbb{L}(x)$$

Reference, $||\frac{(-1)^{n+1}x^n}{n}|| \leq ||\frac{x^n}{n!}|| \leq ||x||$.
We now explore composition of power series.

6.23 Definition

Let

$$G(Y) = \sum_{v \in W(Y)} b_v w \in \mathbb{Q}_p << Y >>$$

$$F_i(X) = \sum_{w \in W(X)} a_{iw} w \in \mathbb{Q}_p << X >> fori = 1, \ldots, m$$

where $X = (X_1 \ldots x_n)$ and $Y = (Y_1, \ldots Y_n)$. Assume that for $i = 1, \ldots, m$ the constant term a_{i1} is equal to zero. For $v(Y) = Y_{i_1} \ldots Y_{i_d} \in W(Y)$, define the coefficients $c_{vw} \in \mathbb{Q}_p$ by

$$v(F_1(X), \ldots, F_m(X)) = F_{i_1}(X) \ldots F_{i+d}(X) = \sum_{w \in W(X)} c_{vw} w(X)$$

The composite of G and $F = (F_1, \ldots F_m)$ is defined to be the formal power series

$$(G \circ F)(X) = \sum_{w \in W(X)} .(\sum_{v \in W(Y)} b_v c_{vw}) w(X)$$

Now show that under certain conditions the operations of composition and evaluation commute.

6.24 Theorem

Let $(A, || - ||)$ be a complete normed $]QP$-algebra, and suppose that $F_1(X), \ldots, F_m(X)$ and $G(Y)$ are formal power series satisfying the conditions above. Suppose that $F_1(X), \ldots F_m(X)$ can all be evaluated at some point $x \in A^n$. For each i put $\tau_i = sup\{||a_{iw} w(x)|| \, | w \in W(X)\}$, and suppose that

$$lim_{v \in W(Y)} |b_v| v(\tau_1, \ldots, \tau_m) = 0$$

Then $G(F_1(x), \ldots, F_m(X))$ and $(G \circ F)(x)$ both exist and are equal.

6.25 Corollary

Let $x \in A_0$. Then

1. $log(exp(x)) = x$;
2. $exp(log(1 + x)) = 1 + x$;
3. $log((1 + x)^n) = n \, log(1 + x)$ for each $n \in \mathbb{Z}$;
4. $\exp(nx) = (exp(x))^n$ for each $n \in \mathbb{Z}$

now introduce campbell-hausdorff series - provides a link between an analytic pro-p group and its associated lie algebra.

6.26 Definition

Let

$$P(X,Y) \quad = \mathbb{E}(X)\mathbb{E}(Y) - 1 \in \mathbb{Q}_p << X,Y >> C(X,Y) \quad = \mathbb{E}(-X)\mathbb{E}(-Y)\mathbb{E}(X)\mathbb{E}(Y) - 1 \in \mathbb{Q}_p << X,Y >>$$

The C-H formula is defined by

$$\Phi(X,Y) = (\mathbb{L} \circ P)(X,Y)$$

and the commutator series by

$$\Psi(X,Y) = (\mathbb{L} \circ C)(X,Y)$$

Moreover it can be shown they can both be evaluated at $x, y \in A_0$:

$$\phi(x,y) = log(exp(x).exp(y))$$

$$\psi(x,y) = log(exp(-x).exp(-y).exp(x).exp(y)$$

Remarkably C-H may be expressed as a sum of Lie elements!!!
Notation: for $e = (e_1, \dots e_n)$ of positive integers we write

$$< e >= e_1 + \cdots + e_n$$

$$(X,Y)_e = (X, Y, Y, ..., Y, X, , ...X, ...)$$

e_1 times X, e_2 times Y ....

C-H formula 6.28 Theorem

Let $\phi(X,Y) = \sum_{n \in \mathbb{N}} u_n(X,Y)$ where $u_n(X,Y)$ is the sum of terms of degree n. Then

$$u_0(X,Y) = 0$$

$$u_1(X,Y) = X + Y$$

$$u_2(X,Y) = \frac{1}{2}(XY - YX)$$

and for each n,

$$u_n(X,Y) = \sum q_e(X,Y)_e$$

Similarly,

$$\psi(X,Y) = XY - YX + \text{ higher degree terms}$$

Now associativity gives a neat composition:
Write

$$H_i(X_1, X_2, X_3) = X_i$$
$$H_{ij}(X_1, X_2, X_3) = (X_1, X_2)$$

Then

$$\boldsymbol{\Phi \circ (H_1, \Phi \circ H_{23}) = \Phi \circ (\Phi \circ H_{12}, H_3)}$$

The following is an important consequence of this!!!!

2.1.6.2 6.38 Corollary

Let $L \cong {\mathbb{Z}_p}^d$ be a Lie algebra over $\mathbb{Z}_p$, and suppose that $(L, L) \subset p^{\epsilon} L$. Let $x, y \in L$, and for $n \in \mathbb{N}$ define the element $u_n(x, y) \in \mathbb{Q}_p L$ by $u_n(x, y) = \sum_{<e>=n-1} q_e(x, y)_e$. Then

1. $u_n(x, Y) \in L$ for all $n \in \mathbb{N}$;
2. the series
$$\overline{\Phi}(x, y) = \sum_{n \in \mathbb{N}} u_n(x, y)$$
converges in L
3. $\overline{\phi}(x, y) - (x + y) \in pL$

So in some sense this $\overline{\Phi}$ picks out the unit part of $x + y$.

2.1.7 The Group Algebra

We saw how to endow the underlying set of a uniform pro-p group with an additive structure.. G made into a lie algebra over $\mathbb{Z}_p$. Make a new lie algebra as subablgebra of commutation lie algebra on an associative algebra which we get by COMPLETING THE GROUP ALGEBRA $\mathbb{Q}_p[G]$ wrt to a norm which is hard to set up!!! Then show how log gives a map between the group and it's associated lie algebra.

Definition of norm

- G finitely generated pro-p group
- for $M \leq N$, M and N normal subgroups of G get map $G/M \to G/N$ induces epi:
$$\mathbb{Z}_p[G/M] \to \mathbb{Z}_p[G/N]$$
- inverse limit of $\mathbb{Z}_p$ algebras:
$$\mathbb{Z}_p[[G]] = \varprojlim_{N \text{ open normal } G} (\mathbb{Z}_p[G/N])$$
known as COMPLETED GROUP ALGEBRA.
- will view CGA as completion of GA wrt some norm.
- for G uniform can even extend this to a norm on the larger group algebra $\mathbb{Q}_p[G]$.

Let's write R for the Group algebra, $\mathbb{Z}_p[G]$, and G_k for it's lower central series (which of course takes a wwonderfully simple form for G uniform), and take I_k to be the kernel of the natural epimorphism.

$$\begin{aligned} R &= \mathbb{Z}_p[G] \\ G_k &= P_k(G) \\ I_k &= (G_k - 1)R = ker(R \to \mathbb{Z}_p[G/G_k]) \end{aligned}$$

Definition of cofinal

of finite index so define same inverse limit

Lemma

Since the family (G_k) is cofinal with the family of all open normal subgroups in G, identify $\mathbb{Z}_p[[G]]$ with

$$\varprojlim_{k \in \mathbb{N}} (\mathbb{Z}/p^k\mathbb{Z})[G/G_k] \cong \varprojlim (R/(I_k + p^k R))$$

Introduce chain of ideals

Cofinal with chain $(I_k + p^k R)$ but better for norms: POWERS OF IDEAL,

$$J = I_1 + pR$$

since $I_1 = (G-1)R = \sum_{x \in G}(x-1)R$ augmentation ideal. So J is kernel of epi

$$R \to \mathbb{F}_p$$

sending all to 1.

2.1.7.1 7.1 Proves (J^k) and $(I_k + p^k R)$ are cofinal

Let $k \geq 1$. Then

1. $J^k \supset I_k + p^k R$
2. *for each* $j \geq 1$, $I_k + p^j R \supset J^{m(k,j)}$ *for* $m(k,j) = j.|G/G_k|$

7.2 Corollary

$$\bigcap_{l=1}^{\infty} J^l = 0$$

7.3 Definition

Since clearly $J^i J^j = J^{i+j}$ invoke Lemma 6.5 to give a norm, $||-||$ on $\mathbb{Z}_p[G]$ defomed by

$$\begin{aligned} ||c|| &= p^{-k} \text{ if } c \in J^k - J^{k+1} \\ ||0|| &= 0 \end{aligned}$$

From Lemma 7.1 the topology on R given by this norm induces on G the origional topology of G.
Now complete R, written,
$\hat{(R)}$ = completion of $(R, ||-||)$

Useful Theorem

In this setup, $\hat{R}$ identified with $\varprojlim_k R/J^k$ thus combining gives,

$$\hat{R} \cong \varprojlim_k (\mathbb{Z}/p^k\mathbb{Z})[G/G_k] \cong \mathbb{Z}_p[[G]]$$

IN UNIFORM CASE TRY TO EXTEND THIS COOL NORM ON GROUP ALGEBRA TO NORM ON THE G.A. $\mathbb{Q}_p[G]$

Notation

- $\{a_1, \ldots, a_d\}$ is a top gen set for G
- $d = d(G)$
- set $b_i = a_i - 1$ for $i = 1, \ldots d$.
- For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$, and d-tuple vector $v = (v_1, \ldots, v_d)$, we write

$$\begin{aligned} <\alpha> &= \alpha_1 + \cdots + \alpha_d \\ v^{\alpha} &= v_1^{\alpha_1} \ldots v_d^{\alpha_d} \end{aligned}$$

STATEMENT OF MAIN RESULTS:

7.4 Theorem

1. If G is POWERFUL then every element of $\mathbb{Z}_p[G]$ is equal to the sum of a convergent series, with $\lambda_\alpha \in \mathbb{Z}_p$ for each α.
$$\sum_{\alpha\in\mathbb{N}^d} \lambda_\alpha b^\alpha$$
2. If G is UNIFORM then the series above is uniquely determined by its sum!!!

2.2 Iwasawa Algebras

Iwasawa Algebras are a kind of Completed Group Rings. Let G be a finitely generated pro-p group. For $M \leq N$ open normal subgroups of G the natural map

$$G/M \to G/N,$$

induces an epimorphism of group algebras

$$\mathbb{Z}_p[G/M] \to \mathbb{Z}_p[G/N],$$

giving an inverse system of $\mathbb{Z}_p$-algebras, whose inverse limit is denoted

$$\mathbb{Z}_p[[G]] = \varprojlim_{N\lhd_o G}(\mathbb{Z}_p[G/N])$$

and is called the Iwasawa Algebra of G. This can also be defined as a completion of the group ring and is known as the completed group algebra of G.

Writing $R = \mathbb{Z}_p[G]$, $G_k = P_k(G)$ (lower p-series) take I_k to be the kernel of the natural epimorphism:

$$I_k = (G_k - 1)R = \ker(R \to \mathbb{Z}_p[G/G_k]),$$

I now recall a result giving the tight correspondence between the algebraic and analytic structure of pro-p groups:

2.2.0.2 Lemma (Proposition 1.16 [6])

Let G be a finitely generated pro-p group, then $P_i(G)$ is open in G for each i, and the set $\{P_i(g)|i \geq 1\}$ is a base for neighbourhoods of 1 in G.

Hence the family (G_k) is cofinal with the family of all open normal subgroups of G, and we may identify $\mathbb{Z}_p[[G]]$ with

$$\varprojlim_{k\in\mathbb{N}}(\mathbb{Z}/p^k\mathbb{Z})[G/G_k] \cong \varprojlim(R/(I_k + p^kR))$$

There is a chain of ideals more suited to defining a norm than (I_k), the cofinal series consisting of powers of the following ideal:

$$J = I_1 + pR = \mathrm{Ker}(R \to \mathbb{F}_p)$$

Cofinality follows from

2.2.0.3 Lemma (Lemma 7.1 [6])

Let $k \geq 1$. Then

- $J_k \supset I_k + p^kR$;
- *for each $j \geq 1$, $I_k + p^jR \supset J^{m(k,j)}$ where $m(k,j) = j|G/G_k|$.*

These definitions give $\bigcap_{l=1}^{\infty} J^l = 0$, and $J^iJ^j = J^{i+j}$ and we may now define a norm with respect to this chain of ideals:

2.2.0.4 Definition

The norm $\| - \|$ *on* $\mathbb{Z}_p[G]$ *is defined by*

$$\begin{aligned} \| c \| &= p^{-k} \text{ if } c \in J^k - J^{k+1} \\ \| 0 \| &= 0 \end{aligned}$$

Writing $\hat{R}$ for the completion of the group algebra with respect this norm, $\hat{R}$ may be thought of as $\varprojlim_k R/J^k$, and Lemma 2.2.0.3 gives

$$\hat{R} \cong \varprojlim_k (\mathbb{Z}/P^k\mathbb{Z})[G/G_k] \cong \mathbb{Z}_p[[G]],$$

This justifies the name "Completed Group Algebra of G" for the Iwasawa Algebra, $\mathbb{Z}_p[[G]]$.

Observe, since $g - 1 \in J = \ker\colon \mathbb{Z}_p[G] \to \mathbb{F}_p$, each element $g \in G$ satisfies $\| g - 1 \| \leq p^{-1}$.

2.2.1 Modules over Iwasawa Algebras

In this section I introduce the topological Nakayama lemma to deduce properties of Iwasawa modules. The structural results we have for projective modules will be important when it comes to constructing projective resolutions in order to calculate Hochschild homology.

Let $\mathfrak{m} = p\mathbb{Z}_p$. Recall that $\mathbb{Z}_p/\mathfrak{m}$ is a finite field (of characteristic p). Therefore $\mathbb{Z}_p[[G]]$ is an inverse limit of finite discrete (hence Artinian) rings, and denoting the kernel of the natural epimorphism $\mathbb{Z}_p[[G]] \twoheadrightarrow \mathbb{Z}_p[[G/U]]$ by $I(U)$, we have that the ideals

$$\mathfrak{m}^n\mathbb{Z}_p[[G]] + I(U), \ n \in \mathbb{N}, U \subset G \text{ open normal}$$

are a fundamental system of neighbourhoods of $0 \in \mathbb{Z}_p[[G]]$.

We denote by $Rad_G \subset \mathbb{Z}_p[[G]]$ the radical of $\mathbb{Z}_p[[G]]$, i.e. the inverse limit of the radicals of $\mathbb{Z}_p/\mathfrak{m}^n[G/U]$ - the intersection of all left maximal ideals. Then Rad_G is a closed two-sided ideal which is the intersection of all open left maximal ideals. The powers $(Rad_G)^n$, $n \geq 1$, define a topology on $\mathbb{Z}_p[[G]]$, which we call the **R-topology**.

2.2.1.1 Proposition (Different Topologies & Properties of Iwasawa Algebras - [12], 5.2.16):

1. *The R-topology is finer than the canonical topology on* $\mathbb{Z}_p[[G]]$*, in particular it is Hausdorff.*
2. *The following assertions are equivalent*
   - *The R-topology coincides with the canonical topology on* $\mathbb{Z}_p[[G]]$
   - $Rad_G \subset \mathbb{Z}_p[[G]]$ *is open.*
   - $\mathbb{Z}_p[[G]]$ *is a semi-local ring.*
   - $(G : G_p) < \infty$*, where* G_p *is a p-Sylow subgroup in* G*.*
3. $\mathbb{Z}_p[[G]]$ *is a local ring if and only if* G *is a pro-P-group. In this case, the maximal ideal of* $\mathbb{Z}_p[[G]]$ *is equal to* $\mathfrak{m}\mathbb{Z}_p[[G]] + I_G$*, the kernel of the composition of augmentation with reduction modulo p:* $\mathbb{Z}_p[[G]] \xrightarrow{g=1} \mathbb{Z}_p \xrightarrow{mod\ p} \mathbb{F}_p$.
4. *More generally, for* $\mathfrak{O}$ *a commutative ring, complete in its* $\mathfrak{m}$*-adic topology, for* $\mathfrak{m}$ *a maximal ideal, and* $\mathfrak{k} = \mathfrak{O}/\mathfrak{m}$ *a finite field of characteristic p -* $\mathfrak{O}$ *compact.* $\mathfrak{O}[[G]]$ *is a semi-local ring if and only if* G *is pro-p. The global dimension,* $gl\mathfrak{O}[[G]]$ *equals* $cd_pG + gl\mathfrak{O}$*, where* cd_p *denotes p-cohomological dimension. By a result of Serre,* cd_pG *is finite if and only if* G *does not contain an element of order p.*

5. *Assume that $\mathfrak{O}$ is finitely generated as a $\mathbb{Z}_p$-module and let G be a compact p-adic Lie group. Then $\mathfrak{O}[[G]]$ is Noetherian.*

Now assume that M is a compact $\mathbb{Z}_p[[G]]$-module. Then in addition to the given topology there are two other topologies on M:

1. The topology given by the sequence of submodules $\{\mathfrak{m}^n M + I(U)M\}_{n,U}$ where $n \in \mathbb{N}$, and U runs through the open normaal subgroups of G. We call this topology the $(\mathfrak{m}, I)$**-topology**.
2. The R-topology, which is given by the sequence of submodules $\{(Rad_G)^n M\}_{n\in\mathbb{N}}$.

From above, the R-topology is obviously finer than the $(\mathfrak{m}, I)$-topology, and both coincide if G is a pro-P-group.

2.2.1.2 Proposition

1. *The $(\mathfrak{m}, I)$-topology is finer than the original toplogy on M. In particular, the $(\mathfrak{m}, I)$- and the R-topology are Hausdorff.*
2. *If M is finitely generated, then the $(\mathfrak{m}, I)$-topology coincides with the original topology on M.*

Thus the topologies are equivalent in the case of interest here, of the False Tate Curve.

2.2.1.3 Corollary (Nakayama's Lemma):

- *If $M \in \mathfrak{C}$ and $Rad_G M = M$, then $M = 0$.*
- *Assume that G is a pro-p-group, hence $\mathbb{Z}_p[[G]]$ **is local with maximal ideal** M. Then M is generated by $x_1, \ldots, x_r$ if and only if $x_i + \mathfrak{M}M$, $i = 1, \ldots, r$, generate $M/\mathfrak{M}M$ as an $\mathbb{Z}_p[[G]]/\mathfrak{M}$ - vector space.*

2.2.1.4 Corollary

Let G be a pro-p-group and let $P \in \mathfrak{C}$ be finitely generated. Then P is a free $\mathbb{Z}_p[[G]]$-module if and only if P is projective.

This will be an essential tool when we manipulate projective resolutions to calculate Hochschild Homologies of the Iwawawa algebra in **??**.

2.2.2 Iwasawa Algebras formed by Power Series Expansion

By refining the filtration $\{J^i\}$, [6] produce a complete, seperated and exhaustive (increasing) filtration $F_\bullet\Lambda$ of $\mathfrak{O}[[G]]$ (for G p-valuable) such that the induced grading,

$$\begin{aligned}\mathrm{gr}\ \mathfrak{O}[[G]] &= \mathfrak{k}[X_0, X_1, \ldots, X_d] \text{ if } \mathfrak{O} \text{ is a DVR} \\ &= \mathfrak{k}[X_1, \ldots, X_d] \text{ if } \mathfrak{O} \text{ is a finite field}\end{aligned}$$

This is key to producing a criterian for any Linear Combination of group elements to be in the Iwasawa Algebra in terms of convergence of a power series (see [22] 2.1), and I then quote the norm of such an element as a function of power series coefficients. I recall these formula from [6] section 7:

2.2.2.1 Notation (Generating Set):

Let $\{a_1, \dots a_d\}$ be a topological generating set for G, where the dimension is $d = d(G)$. Define,

$$b_i = a_i - 1 \text{ for } i = 1, \dots, d$$

For $\alpha = (\alpha_1 \dots, \alpha_d) \in \mathbb{N}^d$ and $\mathbf{v} = (v_1, \dots, v_d)$ any d-tuple, denote,

$$< \alpha >= \alpha_1 + \dots \alpha_d$$
$$\mathbf{v}^\alpha = v_1^{\alpha_1} \dots v_d^{\alpha_d}$$

The structure of the Iwasawa Algebra is given by

2.2.2.2 Theorem (Power Series Expansion of Iwasawa Algebra):

1. *If G is powerful then each element of $\mathbb{Z}_p[[G]]$ is equal to the sum of a convergent series*

$$\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \mathbf{b}^\alpha$$

with $\lambda_\alpha \in \mathbb{Z}_p$ for each α.

2. *If G is uniform then the above series is uniquely determined by its sum*

Given such a power series expansion we can calculate its norm just by looking at coefficients:

2.2.2.3 Theorem (Norm of Power Series):

Assume that G is uniform.If $c = \sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \mathbf{b}^\alpha \in \mathbb{Z}_p[G]$, where $\lambda_\alpha \in \mathbb{Z}_p$ for each α, then

$$\| c \| = sup\ _{\alpha \in \mathbb{N}^d} p^{-<\alpha>} |\lambda_\alpha|.$$

2.2.2.4 Claim (Topological Generation):

$$a_1^{\lambda_1} \dots a_d^{\lambda_d} = \sum_{\alpha \in \mathbb{N}^d} \left(\begin{array}{c} \lambda_1 \\ \alpha_1 \end{array} \right) \dots \left(\begin{array}{c} \lambda_d \\ \alpha_d \end{array} \right) \mathbf{b}^\alpha$$

where, for $\lambda \in \mathbb{Z}_p$, and $1 \leq r \in \mathbb{N}$,

$$\left(\begin{array}{c} \lambda \\ r \end{array} \right) = \frac{\lambda(\lambda - 1) \dots (\lambda - r + 1)}{r!} \text{ belongs to } \mathbb{Z}_p$$

Thus any element in the group ring $\mathbb{Z}_p[G]$ converges to a power series, of form 2.2.2.2, and we verify it lies in the Iwasawa Algebra $\mathbb{Z}_p[[G]]$.

2.2.3 False Tate Curve

I now specialise to a specific example arising from number theory and show how this gives a counter example to any attempt to give a general construct for the completion of the Hochschild Homology of a Group Algbera as the Hochschild Homology of an Iwasawa Algebra.

For $\alpha \in k^*$ a unit of k which is not a root of unity, we define a Galois extension of a number field k which arises from the p-adic Galois representation coming from the Tate Elliptic Curve. Assuming k already contains the p-th roots of unity, μ_p, denote it's cyclotomic $\mathbb{Z}_p$-extension, $k_{cycl} = k(\mu_{p^\infty})$, and k_∞ as the field arising from adjoining the p-power roots of α, and thus all p-power roots of unity, $k_\infty = k(\mu_{p^\infty}, p^{p^{-n}})$. By Kummer Theory, the Galois Group, $G = G(k_\infty/k_{cycl})$ is isomorphic to the (non-abelian) semi-direct product of 2 copies F, H of $(\mathbb{Z}_p, +)$.

The arithmetic of k_∞/k depends on the choice of α, [22] describes the case of $\alpha = p$:

2.2.3.1 Lemma (Ramification Properties):

Let $k = \mathbb{Q}(\mu_p)$ and $k_\infty = k(\mu_{p^\infty}, p^{p^{-\infty}})$. Then the extension k_∞/k

1. *totally ramifies at the unique place over p, in particular there is just one prime of k_∞ above p,*
2. *unramified outside p.*

I explicitly construct the semi-direct product, and simplify the power series expansion for elements in $\mathbb{Z}_p[[G]]$. Let $F, H = (\mathbb{Z}_p, +)$, where ,

$$\begin{array}{rcl} \rho : H & \to & \text{Aut } G = (\mathbb{Z}_p^*, *) \\ 1 & \to & 1+p \\ n & \to & (1+p)^n \end{array}$$

2.2.3.2 Notation (FTC):

$$\begin{array}{rcl} G & = & F \rtimes H \\ & = & \{(f,h) | (f,h).(f',h') = (f + \rho(h).f', h+h')\} \end{array}$$

Thus $(f,h)^{-1} = (-\rho(-h).f, -h)$, and we denote the ***topological generators of*** *G* ***as*** *$f = (1,0)$* ***and*** *$h = (0,1)$*

We can already see that the group is powerful - that G/G^p is abelian, or equivalently that $[G,G] \subset G^p$. Observe $(p.n, 0) = (n,0)^p \in G^p \ \forall n \in \mathbb{Z}_p$. Thus, a general element in the commutator, (for details of calculation see calculation of conjugacy classes below),

$$\begin{array}{rcl} (a,b)^{(g,h)}(a,b)^{-1} & = & (\rho(-h)(a + (\rho(b)-1)g), b)(-\rho(-b).a, -b) \\ & = & ((\rho(-h)-1)a + \rho(-h)(\rho(b)-1)g, 0) \\ \text{Since } p|(\rho(n)-1)\,\forall n \in \mathbb{Z}_p & & \\ & = & (m.p, 0), \text{ some } m \in \mathbb{Z}_p \\ & \in & G^p \end{array}$$

Hence G is powerful - a non-commutative pro-finite group with associated Lie Algebra F X H, the commutative Cartesian product. This follows from the good behaviour under taking powers (if $(g,h)^n = (a,b)$, then $h = b/n$, and $g = a.\frac{1-(1+p)^{b/n}}{1-(1+p)^b}$).

Let $\mathbf{b}_1 = f - 1 \in \mathbb{Z}_p[G]$, and $\mathbf{b}_2 = g - 1 \in \mathbb{Z}_p[G]$, then 2.2.2.2 gives

2.2.3.3 Theorem (Power Series Expansion of FTC):

For $G = F \rtimes H$ each element of $\mathbb{Z}_p[G]$ is equal to the sum of a uniquely determined convergent series

$$\sum_{(\alpha_1,\alpha_2)\in\mathbb{N}^2} \lambda_{(\alpha_1,\alpha_2)} \mathbf{b}_1^{\alpha_1} \mathbf{b}_2^{\alpha_2}$$

with $\lambda_{(\alpha_1,\alpha_2)} \in \mathbb{Z}_p$.

We will now see how this is a special case of a well understood theory:

2.2.3.4 Definition (Skew Power Series Rings):

Let R be a ring, $\sigma : R \to R$ a ring endomorphism and $\delta : R \to R$ a σ-derivation of R - a group homomorphism satisfying

$$\delta(rs) = \delta(r)s + \sigma(r)\delta(s) \text{ for all} r, s \in R.$$

The the (formal) skew power series ring

$$R[[X; \sigma, \delta]]$$

is defined to be the ring whose underlying set consists of the usual power series $\sum_{n=0}^{\infty} r_n X^n$, with $r_n \in R$. Where the multiplication of two such power series is defined by the formula

$$Xr = \sigma(r)X + \delta(r)$$

Hence, all products are convergent, and if R is complete with respect to the I-adic topology, I some σ-invariant ideal such that

$$\delta(R) \subset I, \;\; \delta(I) \subset I^2$$

then multiplication is uniquely defined.

2.2.3.5 Proposition (FTC as Skew Power Series):

Writing, $X = \mathbf{b}_1$, and $Y = \mathbf{b}_2$, we have

$$\mathbb{Z}_p[[G]] \cong \mathbb{Z}_p[[X]][[Y; \sigma, \delta]]$$

the skew power series ring, where $R = \mathbb{Z}_p[[X]]$, and for $\epsilon = \rho(h)$ the ring automorphism σ is induced by $X \to (X+1)^\epsilon$, and $\delta = \sigma -$ id.

2.2.3.6 Corollary (FTC as commutator):

Proof

R is complete with respect to the topology induced by it's maximal ideal $\mathfrak{m}$, generated by X and p. σ is just choosing another generator of H, so $\sigma(\mathfrak{m}) = \mathfrak{m}$. Convergence follows if we can verify $\delta(\mathfrak{m}^k) \subset \mathfrak{m}^{k+1}$ for $k = 0, 1$. Since δ and σ are $\mathbb{Z}_p(\mathbb{F}_p)$-linear, we need only show for $r = X$. But, following [22],

$$\begin{aligned} \delta X &= \sigma(X) - X \\ &= (X+1)^\epsilon - 1 - X \\ &= \sum_{i\geq 1} \binom{\epsilon}{i} X^i - X \\ &= (\epsilon - 1)X^k + \text{ terms of higher degrees} \end{aligned}$$

Since $p|(\epsilon - 1)$ it follows $\delta(X) \subset \mathfrak{m}^2$, as required.

To complete the proof we must verify that σ and δ describe how coefficients fail to commute with variables, that $YX = \sigma(X)Y + \delta(X)$, by linearity it is sufficient to prove:

$$YX = \sigma(X)Y + \delta(X)$$

$$\begin{aligned}
\text{LHS} &= ((0,1)-1)((1,0)-1) \\
&= (0+(1+p).1,1)-(0,1)-(1,0)+(0,1) \\
&= (1+p,1)-(1,0)-(0,1)+(0,0) \\
\text{RHS} &= ((1+p,0)-1)((0,1)-1)+(1+p,0)-(1,0) \\
&= (1+p+1.0,1)-(0,1)-(1+p,0)+1+(1+p,0)-(1,0) \\
&= \text{LHS, as required}
\end{aligned}$$

I now investigate conjugation in this semi-direct product, let $(a,b),(g,h) \in G$ then

$$\begin{aligned}
(a,b)^{(g,h)} &= (g,h)^{-1}(a,b)(g,h) \\
&= (\rho(-h)g,-h)(a,b)(g,h) \\
&= (\rho(-h)g,-h)(a+\rho)b)g,b+h) \\
&= (\rho(-h)g+\rho(-h).(a+\rho(b)g),b) \\
&= (\rho(-h)(a+(\rho(b)-1)g),b)
\end{aligned}$$

Thus $(1,0) = f \sim (\mathbb{Z}_p^*,0)$, since $\rho = (h \to \rho(-h)(1+0)) : \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}_p^*$. And conjugating by the set $(0,\mathbb{N}) \subset G$, we see that **the infinte sequence** $f^1, f^{(1+p)}, f^{(1+p)^2}, f^{(1+p)^3}, \ldots$ **all lie in the same conjugacy class.**

2.2.3.7 FTC as Skew Power Series Recap

$$\begin{aligned}
\mathfrak{O}_K[[L]] &= \mathfrak{O}_K[[\epsilon-1]] << \mathbb{Z}_p >> \quad (2.1)\\
&= \mathfrak{O}_K[[X]] << Y,\sigma >> \quad (2.2)
\end{aligned}$$

Where Y corresponds to $\gamma - 1$ for γ a topological generator of $\mathbb{Z}_p$.

To understand multiplication of $\sum r_n X^n$ it is sufficient to understand

$$\begin{aligned}
Yr &= \sigma(r).Y + \delta(r) = \sigma(r).Y + \sigma(r) - r \text{ so,} \quad (2.3)\\
(1+Y).r &= \sigma(r).(1+Y) \quad (2.4)
\end{aligned}$$

where $\sigma : (X+1) \to (X+1)^{\chi(\gamma^{-1})} - 1 = (X+1)^{1+p}$ WLOG.

i.e

$$\Lambda(G) \cong \mathfrak{O}_K[[X]] << Y,\sigma >> \cong \{\sum r_n Y^n \,|\, (1+Y).r = \sigma(r).(1+Y) \text{ and } \sigma : (X+1) = (X+1)^{1+p}\}$$

2.2.3.8 Proposition (Construction of Completion):

We recall the construction of a completion of Hochschild homology for Novikov rings. Describing elements in $\mathbb{Z}_p[[G]]$ as formal linear combinations of elements, an n-chain used to calculate $HH_n(\mathbb{Z}_p[[G]])$ has form:

$$\sum_{g_1 \in G} n_{g_1} g_{n+1} \otimes \cdots \otimes \sum_{g_{n+1} \in G} n_{g_{n+1}} g_{n+1}$$

For finite elements, in $\mathbb{Z}_p[G]$, this may be simplified to an element of $C_n(\mathbb{Z}G,\mathbb{Z}G)$, a finite sum:

$$(\star) \qquad \sum_{g_1,\ldots,g_{n+1}\in G} n_{g_1}\ldots n_{g_{n+1}} g_1 \otimes \cdots \otimes g_{n+1}$$

Although a general element taken from $\widehat{\mathbb{Z}G}_\chi$ would be an infinite sum of tensors which does not give a well defined element of $C_n(\mathbb{Z}_pG, \mathbb{Z}_pG)$, the homology of the iwasawa algebra lies in a completion of $HH_*(\mathbb{Z}_pG)$ iff the following holds:

"Given a conjugacy class $\gamma \in \Gamma$ there are only finitely many nonzero summands in $(\star)$ with marker in γ: such that $g_1 \ldots g_{n+1} \in \gamma$."

Notice that since $\mathbb{Z}_p[[G]]$ is a ring, if this property holds for calculation of the zeroth homology - in other words that an element of $\lambda \in \mathbb{Z}_p[[G]]$ can only finite support in any one conjugacy class, then the same holds for the formal product $\lambda_1 ... \lambda_{n+1}$ since $\lambda_1 .. \lambda_{n+1} \in \mathbb{Z}_p[[G]]$.

2.2.3.9 Proposition (Iwasawa Completion does not Decompose Over Conjugacy Classes):

The Hochschild homology of a general Iwasawa algebra as a completion of the Hochschild homology of the group ring is not contained in the direct product over conjugacy classes ($HH(\widehat{(\mathbb{Z}G)}) \subset \prod_{\gamma, conj.\ class} HH(\mathbb{Z}G)_\gamma$). Indeed, for G the non-commutative semi-direct product of two copies of $\mathbb{Z}_p$ arising from the False Tate Curve, there exists an element of the Iwasawa Algebra with infinte support in a single conjugacy class.

From above, this is equivalent to demonstrating an element in an Iwasawa Algebra $\lambda \in \mathbb{Z}_p[[G]]$ with a conjugacy class having infinite support.

Consider the False Tate Curve, $G = F \rtimes H$, by 2.2.3.5, for $X = f - 1$,

$$\begin{aligned} \lambda &= \sum_{n\geq 0} \lambda_n X^{(1+p)^n} \\ \text{where, } \lambda_n &= p^n \\ \text{thus, } \lambda &= 1.(f-1) + p.(f-1)^{(1+p)} + p^2.(f-1)^{(1+p)^2} + \ldots \end{aligned}$$

Then the series is convergent, thus $\lambda \in \mathbb{Z}_p[[G]]$. Where

$$\begin{aligned} \text{coeff. of } f^1 &= 1 + p.\mu_{1,1} + p^2.\mu_{1,2} + p^3.\mu_{1,3} + ... \in \mathbb{Z}_p^* \\ \text{coeff. of } f^{(1+p)} &= p + p^2.\mu_{2,1} + p^3.\mu_{2,2} + p^4.\mu_{2,3} + ... \in p\mathbb{Z}_p^* \\ \text{coeff. of } f^{(1+p)^2} &= p^2 + p^3.\mu_{3,1} + p^4.\mu_{3,2} + p^5.\mu_{3,3} + ... \in p^2\mathbb{Z}_p^* \\ &\vdots \end{aligned}$$

Where $\mu_{i,j} \in \mathbb{Z}_p$, thus coeffs. of $g^1, g^{(1+p)^2}.g^{(1+p)^3}, \ldots$ are all nonzero (contained in $p^n\mathbb{Z}_p^*$), and hence **there exists an element of the Iwasawa Algebra with infinte support in a conjugacy class**, as a completion over conjugacy classes described as a direct sum requires completion of the group algebra itself in each coordinate, and does not lie in the direct product over conjugacy classes of the usual group algebra.

However, when G is infinte it is no longer true that $HH_1(G, k) = G^{ab}$, and so although we have shown, $HH(\widehat{(\mathbb{Z}G)}) \subset \prod_{\gamma,\text{conj. class}} HH(\mathbb{Z}G)_\gamma$, does does not imply $HH_1(\widehat{(\mathbb{Z}G)}) \neq \prod_{\gamma,\text{conj. class}} Z(\gamma)^{ab}$. This question will be resolved below.

2.3 Logarithms

2.3.1 Marcus Background

2.3.2 Definition of Logarithm

2.3.3 Dwork-Dieudonne argument in Commutative Case

R any ring, then for $f(X) \in 1 + R[[X]]$, $f(X)$ is uniquely expressible as

$$f(X) = \prod_{n=1}^{\infty} (1 - a_n X^n)$$

with $a_n \in R$.

2.3.4 Abelian Group Law

2.3.5 Completions of Universal Enveloping Algebra following Ardakov

2.4 Integral Logarithms

⋆ **Finite G** ⋆

2.4.1 Introduction of Integral Logarithm following Kakde

The Integral Logarithm is the composition of the usual p-adic logarithm with a linear endomorphism to make it integer valued. Taylor has used these to ger additive descriptions of K_1 of group rings of finite groups, as well as dealing with Class groups, and in particular the Frohlich Conjectures - [15].

2.4.2 Motivation in the Commutative Case

When G is abelian, the p-th power map Φ is a ring endomorphism and so we may re-arrange,

$$\begin{array}{rcl} \Lambda : K_1(\mathbb{Z}_p[G]) & \to & \mathbb{Z}_p[\mathrm{ccl}(G)] \\ [u] & \to & log(u) - \dfrac{1}{p}.log(\Phi(u)) = \dfrac{1}{p}.log(u^p/\Phi(u)) \end{array}$$

converges for $u \in 1 + J(\mathbb{Z}_p[G])$ where J denotes the Jacobson radical: $J(\mathbb{Z}_p[G]) = ker(\mathbb{Z}_p[G] \xrightarrow{\text{aug}} \mathbb{Z}_p \to \mathbb{F}_p)$. Observe that

$$\begin{array}{rcl} u^p & \cong & \Phi(u)(\text{mod } p\mathbb{Z}_p[G]), \text{ or in other words,} \\ u^p/\Phi(u) & \cong & 1(\text{mod } p\mathbb{Z}_p[G]), \text{ thus,} \\ u^p/\Phi(u) & \in & 1 + p\mathbb{Z}_p[G], \text{ so it's image,} \\ log(u^p/\Phi(u)) & \in & p\mathbb{Z}_p[G], \\ \Gamma(u) & \in & \mathbb{Z}_p[G]. \end{array}$$

2.4.3 Re-interpretation in terms of determinant following Taylor, using Adams operator

2.4.4 Tactic of Kakde/Kato to calculate $K_1(\mathbb{Z}_p[G])$

2.4.5 Calculation of Kernel and Cokernel

The following calculation is taken from Oliver's book, [14]. He uses the Isomorphism, $H_0(G;\mathbb{Z}_p[G]) \cong \mathbb{Z}_p[\mathrm{ccl}(G)]$, where action of G on $\mathbb{Z}_p[G]$ is by conjugation.

Lemma (p-th powers of Group Elements - [14] 6.3):

For any group G and any element $g \in G$,

$$(1-g)^p \cong (1-g^p) - p(1-g) \,(\mathrm{mod}\; p(1-g)^2\mathbb{Z}_p[G]).$$

The main technique used to study the image of Γ is to work inductively and compare $K_1(\mathbb{Z}_p[G])$ and $[K_1(\mathbb{Z}_p[G]/z)$, where $z \in Z(G)$ is central of order p, in particular when z itself is a commutator, $z = [g,h]$.

Theorem (Image of Integral Logarithm):

Let p be an odd prime. Define

$$\begin{array}{rcl} \omega : \mathbb{Z}_p[ccl(G)] & \rightarrow & G^{ab} \\ \sum a_i[g_i] & \rightarrow & \prod \overline{[g_i]}^{a_i} \end{array}$$

Then the sequence

$$1 \rightarrow K_1(\mathbb{Z}_p[G])/torsion \xrightarrow{\Gamma} \mathbb{Z}_p[ccl(G)] \xrightarrow{\omega} G^{ab} \rightarrow 1$$

is exact

Proof:

The mapping ω is well defined since each coset $a[G,G]$ is a normal subgroup of G, so is made up as a union of conjugacy classes.

Consider first $G = 1$. Since torsion (roots of unity) vanish -

$$log(\mathbb{Z}_p^*) = log(1+p\mathbb{Z}_p) = p\mathbb{Z}_p = p.\ker\,\omega$$

Since $log(1+p\mathbb{Z}_p)$ is ϕ-invariant,

$$\Gamma(\mathbb{Z}_p^*) = (1-\frac{1}{p}.\phi)(log(\mathbb{Z}_p^*)) = \frac{p-1}{p}log(\mathbb{Z}_p^*)$$

So $Im(\Gamma) = \ker\,(\omega)$ in this case.

For G now a nontrivial p-group it is enough, by naturality to consider G abelian.

Let I denote the augmentation ideal, $I = \{\sum r_i g_i \in \mathbb{Z}_p[G] : \sum r_i = 0\}$. Then for any $u = 1+\sum r_i(1-a_i)g_i \in 1+I$, $r_i \in \mathbb{Z}_p$, we have

$$\begin{array}{rcl} u^p & \cong & 1+p\sum r_i(1-a_i)g_i + \sum r_i^p(1-a_i)^p g_i^p \;(\mathrm{mod}\; pI^2) \\ & \cong & 1+p\sum r_i(1-a_i)g_i + \sum r_i[(1-a_i^p) - p(1-a_i)]g_i^p \text{ (using above Lemma)} \\ & \cong & \Phi(u) + p\sum r_i(1-a_i)(g_i - g_i^p) \cong \Phi(u) \end{array}$$

This shows that $u^p/\Phi(u) \in 1+pI^2$, and hence that

$$\Gamma(u) = \frac{1}{p}.log(u^p/\Phi(u)) \in I^2$$

For $r \in \mathbb{Z}_p$ and $a, b, g \in G$ we have

$$\omega(r(1-a)(1-b)g) = (g)^r(as)^{-r}(bg)^{-r}(abg)^r = 1 \in G^{\text{ab}}$$

Thus $\Gamma(1+I) \subset I^2 \subset \ker\,\omega$ and so

$$\Gamma(K_1(\mathbb{Z}_p[G])) = \Gamma(\mathbb{Z}_p^* * (1+I)) =< \Gamma(\mathbb{Z}_p^*), \Gamma(1+I) >\subset \ker\ (\omega)$$

We now have $Im(\Gamma) \subset Ker(\omega)$ but what about the other inclusion?

Fix a central element $z \in Z(G)$ of order p. If the group is non-abelian we may take z to be a commutator (see [14] Lemma 6.5). Set $G' = G/z$, and assume inductively that the theorem holds for G', the base case $G = 1$ was discussed above. Let $\alpha : G \to G'$ be the projection, then we may induce maps, and note that $K(\alpha) : K_1(\mathbb{Z}_p[G])/\text{tors} \to K_1(\mathbb{Z}_p[G'])/\text{tors}$ is onto. For I an ideal in the ring R, define $K_1(R, I) = GL(R, I)/E(R, I)$, where $GL(R, I)$ is the group of invertible matrices which are congruent to the identity modulo I, and elementary matrices $E(R, I) = [GL(R), GL(R, I)]$. We have the exact sequence,

$$K_2(R) \to K_2(R/I) \to K_1(R, I) \to K_1(R) \to K_1(R/I)$$

This gives the following commutative diagram.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & K_1(\mathbb{Z}_p[G], (1-z)\mathbb{Z}_p[G])/\text{tors} \arrow[r, "\Gamma_0"] \arrow[d] & H_0(G; (1-z)\mathbb{Z}_p[G]) \arrow[r, "\omega_0"] \arrow[d] & \text{Ker}(\alpha^{ab}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & K_1(\mathbb{Z}_p[G])/\text{tors} \arrow[r, "\Gamma_G"] \arrow[d, "K(\alpha)"'] & H_0(G; \mathbb{Z}_p[G]) \arrow[r, "\omega_G"] \arrow[d, "H(\alpha)"'] & G^{ab} \arrow[r] \arrow[d, "\alpha^{ab}"'] & 1 \\
1 \arrow[r] & K_1(\mathbb{Z}_p[G'])/\text{tors} \arrow[r, "\Gamma_{G'}"] \arrow[d] & H_0(G'; \mathbb{Z}_p[G']) \arrow[r, "\omega_{G'}"] \arrow[d] & G'^{ab} \arrow[r] \arrow[d] & 1 \\
 & 1 & 1 & 1 &
\end{tikzcd}$$

Since $K(\alpha)$ is onto the columns are all exact. By the inductive hypothesis the bottom row is exact. Also, ω_0 is clearly onto and so top row is exact. Now, since $\omega_G \circ \Gamma_G = 1$, the middle row is exact by the $3*3$ Lemma.

⋆ **Infinite G** ⋆

2.4.6 Schneider/Venjakob Approach

The following is taken from Schneider's ICMS talk Summer 2009.

Let p be a prime different from 2, and G a pro-p p-adic Lie Group.

The integral logarithm of Oliver and Taylor, [21], gives a $\mathbb{Z}_p$-homomorphism,

$$\begin{aligned}
\Gamma : K_1(\Lambda(G)) = (\Lambda(G)^*)^{\text{ab}} &\to \Lambda(G)^{\text{ab}} \\
&= \Lambda(G)/(\text{additive commutators } xy - yx) \\
&= \mathbb{Z}_p[[\text{ccl}(G)]] \\
\lambda &\to log(\lambda) - \frac{1}{p}\Phi(log(\lambda))
\end{aligned}$$

Where $\phi : \Lambda(G)^{\mathrm{ab}} \to \Lambda(G)^{\mathrm{ab}}$, induced by

$$\begin{array}{rcl} \phi : G & \to & G^p \\ g & \to & g^p \end{array}$$

is a well defined homomorphism.

We now make three additional hypotheses:

Hypothesis (Φ):

The map $\phi : G \to G$ is injective, and $\phi^n(G)$ is open in G for any $n \geq 1$.

Hypothesis (P):

The image $\phi(G)$ is a (normal) subgroup of G, take $p^d = |G : \phi(G)|$.

Hypothesis (SK):

For any U in some Fundamental System of open normal subgroups of G, the map

$$K_1(\mathbb{Z}_p[G/U]) \to K_1(\mathbb{Q}_p[G/U])$$

is injective.

Notice, for G any uniform group, Hypotheses (Φ) and (P) are immediately satisfied.

Then assuming (Φ) and (SK) we establish the following commutative diagrma using results from Oliver's book, [14]:

$$\begin{tikzcd}
 & & 1 \arrow[d] & 1 \arrow[d] & \\
 & & \mu_{p-1} * G^{ab} \arrow[r, equal] \arrow[d] & \mu_{p-1} * G^{ab} \arrow[d] & \\
0 \arrow[r] & \Lambda(G)^{ab} \arrow[r, "\exp(p.-)"] \arrow[d, equal] & K_1(\Lambda(G)) \arrow[r, "\mathrm{Proj}"] \arrow[d] & K_1(\Omega(G)) \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \Lambda(G)^{ab} \arrow[r, "p-\Phi"] & \Lambda(G)^{ab} \arrow[r, "(\star)"] \arrow[d] & \Lambda(G)^{ab}/(p-\Phi) \arrow[r] \arrow[d] & 0 \\
 & & G^{ab} \arrow[r, equal] \arrow[d] & G^{ab} \arrow[d] & \\
 & & 1 & 1 &
\end{tikzcd}$$

Where $(\star)$ is the map quotienting out by image of $\Lambda(G)^{\mathrm{ab}}$ under $p - \Phi$. Clearly, any pre-image under $(\star)$ may have all it's p-th powers removed, giving that the conjugacy classes of non-p-th powers, $\mathbb{Z}_p[[\mathrm{ccl}(G - \phi(G))]] \subset \Lambda(G)^{\mathrm{ab}}$ is a section of $(\star)$.

Then the following theorem allows us to split up the complicated group $K_1(\Lambda(G))$ by pulling back the above section.

Theorem (Schneider [17])

Assuming (Φ) *and (SK) we have the section*

$$K_1^{\Phi}(\Lambda(G)) := \Gamma^{-1}(\mathbb{Z}_p[ccl(G-\phi(G))]) \cong K_1(\Omega(G))$$

under the map

$$\Gamma^{-1}(\mathbb{Z}_p[ccl(G-\phi(G))]) \subset K_1(\Lambda(G)) \xrightarrow{Proj} K_1(\Omega(G))$$

$$\begin{tikzcd}
1 \arrow[d] & & & & & & \\
\mu_{p-1} * G^{ab} \arrow[d] & & & & & & \\
K_1(\Lambda(G)) \arrow[r, "\log(-)"] \arrow[d, "\log(-)"'] & \Sigma(G) \arrow[r, "\Theta"] & \Lambda(G) \arrow[r, "\mathrm{fuse}"] & \mathbb{Z}_p[[\mathrm{Ccl}\ G]] \arrow[r] & HH_1(\Lambda(G)) \arrow[r] & 1 \\
\Sigma(G) \arrow[d, "\mathrm{id}-p/\Phi"'] & & & & & & \\
\Lambda(G) \arrow[d, "\mathrm{fuse}"'] & & & & & & \\
\mathbb{Z}_p[[\mathrm{Ccl}\ G]] \arrow[r, equal] \arrow[d] & HH_0(\Lambda(G)) & & & & & \\
G^{ab} \arrow[d] & & & & & & \\
1 & & & & & &
\end{tikzcd}$$

2.4.7 Example from Ritter and Weiss

2.4.8 Tactic of Schneider/Venjakob to calculate $K_1(\Lambda_G)$

2.5 Filtrations and Completions

Chapter 3

Homology

3.1 Hochschild Homology of Group Rings

I begin this section by constructing a map from the Grothendieck Group of a ring, R to the abelianisation of R, $Tr : (K_0(R) \rightarrow R/[R,R])$ which gives the motivation, and language for the Dennis Trace map for higher order K-groups.

I give the definition of Hochschild Homology in 3.1.2, this is the homology of Bimodules written $HH_n(R,M)$, and specialise to the case $n = 1$ before giving some explicit examples.

I then introduce the Generalised Trace Map, tr, and explain how this gives the Morita Equivalence of the Hochschild Homology.

In the next section, 3.1.5, I give a decomposition of Hochschild Homology of a group algebra as a direct sum of more familiar objects - Group Homology of centralisers in the group, where the sum is taken over conjugacy class representatives.

Section 3.1.6 considers how the interpretation of $HH_1(kG)$ as both a direct sum, and as Kahler Differentials are equivalent.

After giving the Homotopy definition for K-Groups, $K_i(R)$, $i \geq 0$, I am then in a position to define the Dennis Trace Map, $\delta : (K_i(R) \rightarrow HH_i(R)) \, \forall i \geq 0$, and give explicit matrix examples.

3.1.1 Grothendieck Groups

In this report we are mainly interested in the Dennis Trace Map from the first K-group, $\delta : (K_1(R) \rightarrow HH_1(R))$. This section defines the Tace Map, $Tr : (K_0(R) \rightarrow R/[R,R])$, induced from the standard trace map on square matrices. We will see later that $R/[R,R]$ is just the zeroth Hochschild Homology Group of R, and indeed $Tr = \delta_0 : (K_0(R) \rightarrow HH_0(R))$.

Firstly, recall relevant definitions:

3.1.1.1 Definition (Grothendieck Group):

The Grothendieck Group (G.G.) of a semigroup S is an abelian group given in terms of generators and relations:

- *Generators* $\{[x] : x \in S\}$
- *Relations* $x + y = z$ *in* $S \iff (\star)$ $[x] + [y] = [z]$ *in G.G.*

This operation is easily seen to be functorial, with the maps between semi-groups giving rise to associated maps between Grothendieck Groups, and this makes it an interesting object to study.

3.1.1.2 Definition ($K_0(R)$):

Let R be a ring with unit, the zeroth K-group is defined to be:

$$K_0(R) = G.G. \text{ of semigroup } Proj\, R$$

Where Proj R is the semigroup of Isomorphism classes of finitely generated projective modules over R. In this case ($\star$) splits to simplify the relations: $[x] + [y] = [x \oplus y]$.

The most natural setting for understanding the Trace Map comes from a modification of this into idempotents.

Given a ring R, $P \in \text{Proj } R$ we may associate an element of $GL(R)$ to P. Since P is a direct summand of the free group, $P \oplus Q = R^n$, say. Consider the map given on components,

$$\theta : (P \oplus Q \to P \oplus Q) = (id|_P \oplus 0|_Q)$$

Then $\theta : (R^n \to R^n)$ is an idempotent, $\theta^2 = \theta$, and may be considered an element of $M_n(R) \subset M(R)$

It is easily seen that different idempotents give rise to the same projective modules. A natural question to ask, is to what extent does the module uniquely determine the idempotent θ? The answer is given as an equivalence relation on idempotents ensuring Isomorphism of the corresponding modules.

3.1.1.3 Proposition (Idempotent Equivalnce)

For any ring R, Proj R may be identified with the set of conjugacy orbits of $GL(R)$ on $Idem(R)$. Where the Semigroup operation is induced by $(p, q) \to \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$.

Since $K_0(R)$ is the G.G. of this semigroup we may consider mapping from class of idempotents (defined up to conjugation). The "trace map" is now given by a choice of matrix representing P, and taking it's trace (which is well defined):

$$\begin{aligned} Tr \quad &: \quad K_0(R) \to R/[R, R] \\ &: \quad [P] \to [\text{Tr } P] \text{ class of idempotent matrix} \end{aligned}$$

3.1.2 Hochschild Homology of a Ring

For k a commutative ring, and R a k-algebra (with unit), I consider an $R - R$ bimodule M (where $(r_1 m)r_2 = r_1(m r_2)$), and introduce an homology where both left and right actions are considered - the Hochschild Homology of M over R.

3.1.2.1 Definition (Hochschild Homology as Tor-group):

The n-th Hochschild Homology of R with coefficients in the bimodule M is $HH_n(R, M) = Tor_n^{R \otimes R^{op}}(R, M)$.

There is an explicit complex which may be used for computation:

3.1.3 Proposition (HH as Homology of a given Complex):

Hochschild Homology is the homology of Hochschild Chain Complex, $\{C_*(R, M), d\}$ consisting of

$$C_n(R, M) = k^{\otimes n} \otimes_k M$$

and boundary maps

$$d(r_1 \otimes \cdots \otimes r_n \otimes m) = r_2 \otimes \cdots \otimes r_n \otimes m r_1 + \sum_{i=1}^{n-1} (-1)^i r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n \otimes m + (-1)^n r_1 \otimes \cdots \otimes r_{n-1} \otimes r_n m$$

We will be mainly concerned with $HH_0(R, M)$ and $HH_1(R, M)$ which are computed from,

$$\begin{array}{rcl} \to R \otimes R \otimes M & \xrightarrow{d} & R \otimes M \\ r_1 \otimes r_2 \otimes m & \to & r_2 \otimes m r_1 - r_1 r_2 \otimes m + r_1 \otimes r_2 m \end{array}$$

$$\begin{array}{rcl} R \otimes M & \xrightarrow{d} & M \\ r \otimes m & \to & mr - rm \end{array}$$

3.1.3.1 Notation (Ring acting on Itself):

If $M = R$ with $R - R$ bimodule structure coming from left/right multiplication of elements in ring, we write $HH_n(R)$ for $HH_n(R, M)$.

I now calculate the first few terms of the Hochschild homology sequence of a ring explicitly.

3.1.3.2 Corollary (Calculation of Terms of Low Degree):

For a k-algebra R,

- $HH_0(R) = R/[R, R]$
- *For R commutative, $HH_1(R) \cong \Omega^1(R)$, the space of Kahler differentials*

Proof

From the chain complex above, $HH_0(R) = R/d(R^{\otimes 2})$, and $d(r_1 \otimes r_2) = r_1 r_2 - r_2 r_1 = [r_1, r_2]$. Hence,

$$HH_0(R) = R/[R, R] \, (= R \text{ if } R \text{ is commutative})$$

For R commutative, $d(R^{\otimes 2}) = 0$, hence

$$\begin{array}{rcl} HH_1(R) & = & R \otimes_k R / \operatorname{Im} d(R^{\otimes 3}) \\ & = & R \otimes_k R / < \{r_1 r_2 \otimes r_3 - r_1 \otimes r_2 r_3 + r_3 r_1 \otimes r_2\} > \end{array}$$

Writing, $a_1 \mathfrak{d} a_2$ for the image of $a_1 \otimes a_2$ in this quotient, then $\mathfrak{d} : a_2 \to \mathfrak{d} a_2$ is a k-linear derivation.
I now give an example to show how this complex may be used for calculations:

3.1.3.3 Example (Polynomial Ring):

Consider the polynomial ring in one variable over k, $R = k[t]$ (the same arguments extend to case of Laurent Polynomials). R is free over k, taking as basis the set of monomials $\{t^i : i \geq 0\}$, hence is k-Projective. Since R is commutative, $R = R^{\mathrm{op}}$, and $R \otimes R^{\mathrm{op}} \cong k[t, s]$. Thus if we consider setting t and s to be equal, as a $k[t, s]$-module, $R \cong k[t, s]/(t - s)$.

Hence, $k[t, s] \xrightarrow{(t-s)} k[t, s] \twoheadrightarrow R$ is exact, and thus is a $R \otimes R^{\mathrm{op}}$ resolution of R, giving:

$$HH_1(R) \cong HH_0(R) \cong R$$
$$HH_i(R) = 0 \quad \forall \, i > 1$$

3.1.4 Generalised Trace Map

This section extends the trace map from a map on matrices to the sum of diagonal entries to a map from tensor products of matrix spaces, and will be a key tool in the definition of the Dennis Trace Map.

3.1.4.1 Definition (Generalised Trace Map):

Let $N_1, \ldots, N_n$ be R bi-modules. Then for all r, define generalised trace map,

$$tr: \quad M_r(N_1) \otimes \cdots \otimes M_r(N_n) \to N_1 \otimes \cdots \otimes N_n$$

as the unique k-linear extension of the map defined on elementary matrices,

$$tr: \quad E_{i_0 j_0}(a_0) \otimes E_{i_1 j_1}(a_1) \otimes \cdots \otimes E_{i_n j_n}(a_n) \to \left(\begin{array}{cc} a_0 \otimes a_1 \otimes \cdots \otimes a_m, & \textit{if } j_0 = i_1, j_1 = i_2, \ldots, j_m = i_0 \\ 0, & \textit{otherwise} \end{array} \right.$$

Notice that for $n = 1$, tr recovers the usual trace map as a sum of diagonal elements ($j_0 = i_0$). We now see how this leads to the Hochschild Homology being unchanged when we pass from working over a ring R to the new ring of r x r matrices over R:

3.1.4.2 Proposition (Morita Equivalence):

The Trace map (rewritten as a sum over all $n+1$-tuples) acting on elements of Hochschild Complex:

$$\begin{array}{rcl} Tr: M_r(R)^{\otimes n} \otimes M_r(M) & \to & R^{\otimes n} \otimes M \\ : \alpha \otimes \beta \otimes \cdots \otimes \eta & \to & \displaystyle\sum_{(i_0, i_1, \ldots, i_n)} \alpha_{i_0 i_1} \otimes \beta_{i_1 i_2} \otimes \cdots \otimes \eta_{i_n i_0} \\ \textit{induces a Natural Isomorphism,} & & \\ HH_n(M_n(R)) & \to & HH_n(R) \end{array}$$

Hence Hochschild Homology is Morita Invariant.

Since the functor K_1 is Morita Invariant (again proved using G.T.M.), for a map of functors $K_1(R) \to F(R)$ to be interesting, $F(R)$ must also be invariant. 3.1.4.2 gives hope that such a map may exist between $K_n(R)$ and $HH_n(R)$

3.1.5 Hochschild Homology of Group Ring as a Direct Sum over Conjugacy Classes

The aim of this section is to relate Hochschild Homology of a group ring back to a set of Group Homologies corresponding to centralisers of elements in the group. The key result is:

3.1.5.1 Propositon (Direct Sum Decomposition):

For G a finite group the Hochschild homology of the group algebra breaks up as the direct sum over conjugacy classes of the group homology of centrailsers of representatives of the conjugacy classes with trivial coefficients:

$$HH_*(kG) = \bigoplus_{g_c \in C_G} H_*(Z(g_c))$$

Proof of this in [2] is done using Shapiro's Lemma and the Mackey Decomposition of a space into a sum over double cosets. I give a proof which isolates in 3.1.5.2, and 3.1.5.3 what Benson needs from the above results.

3.1.5.2 Lemma (Centralisers and Conjugacy Classes):

Choosing representatives of conjugacy classes, $g_c \in C$, there exists an isomorphism of sets,

$$C \cong G/Z(g_c)$$

where centraliser of g_c, $Z(g_c) = \{h \in G | h = ghg^{-1}\}$

This follows from the First Isomorphism Theorem, applied to,

$$\begin{array}{ccl} G & \twoheadrightarrow & C \\ g & \rightarrow & gg_cg^{-1}, \text{ whose kernel } \cong Z(g_c) \end{array}$$

3.1.5.3 Lemma (Induction of Centralisers):

$$H_*(G, k(G/Z(g_c))) \cong H_*(Z(g_c), k)$$

Shapiro's Lemma for $H \leq G$, and an H-module M follows by explicitly manipulating resolutions and gives a relation between homology of H with coefficients in M, and the homology of G with coefficients in the induced module $M \uparrow_H^G = M \otimes_{kH} kG$, and states

$$H_*(H, M) = H_*(G, M \uparrow_H^G)$$

The Lemma 3.1.5.3 follows immediately from Shapiro noting

- $Z(g_c) \leq G$
- $k(G/Z(g_C)) \cong k \otimes_{kZ(g_c)} kG$

Using these Lemmas I now prove 3.1.5.1 by showing how Hochschild complex splits up over conjugacy classes.

Proof of 3.1.5.1

Partition G into the union of conjugacy classes, $kG = \bigoplus_{C \in C_G} kC$. In Hochschild Chain Complex, each generating chain $c = g_1 \otimes \cdots \otimes g_n \otimes m$ can be thought of (canonically) as $g_1 \otimes \cdots \otimes g_n \otimes g_n^{-1} \ldots g_1^{-1} g$ where the product of elements, $g = g_1 \ldots g_n m$ is the "marker" of generating chain.

Claim (boundary map preserves markers):

All generating chains occuring in the sum giving boundary, $d(c)$ have "markers" in $C(g)$ (in the same conjugacy class).

This can be seen by considering each summand of the boundary in turn, for example, the first may be thought of as $g_2 \otimes \cdots \otimes g_n \otimes g_n^{-1} \ldots g_2^{-1}(g_1^{-1} g g_1)$ with "marker" $g_1^{-1} g g_1 \in C(g)$.

Thus, we can separate the chains with markers in a particular conjugacy class. For $C \in C_G$, a particular conjugacy class, denote by $C_*(kG, kG)_C$ the subset of $C_*(kG, kG)$ generated by chains with markers in C. This gives the decomposition: $C_*(kG, kG) \cong \bigoplus_{C \in C_G} C_*(kG, kG)_C$ inducing the sum over homology of C-components:

$$HH_*(kG, kG) \cong \bigoplus_{C \in C_G} HH_*(kG, kG)_C$$

To complete the proof we understand how Hochschild homology, may be reduced to group homology where group action on a left module incorporates both left/right action of bimodule.

For N a $kG - kG$ bimodule, let $\overline{N}$ be the left kG-module with underlying abelian group N, and left action given by $g \circ m = (g \circ_l m) \circ_r g^{-1}$. Then isomorphims at resolution level - between 3.1.3, and homogeneous bar complex (used to calculate group homology, see [23]) or Shapiro's Lemma 3.1.5.3 give,

$$HH_*(kG, N) = H_*(G, \overline{N})$$

As a direct sum of left modules, $kG = \bigoplus_{C \in C_G} kC$, hence restricting action to conjugacy classes:

$H_*(G, \overline{kG}) = \bigoplus_{C \in C_G} H_*(G, kC)$. 3.1.5.2, and 3.1.5.3 give Hochschild Homology of Group Algebra in terms of conjugacy class representatoves $\{g_C\}$

$$\begin{aligned} HH_*(kG, kG) &= \bigoplus_{C \in C_G} HH_*(kG, kG)_C \\ &= \bigoplus_{C \in C_G} H_*(G, kG) \\ &= \bigoplus_{\{g_C\}} H_*(G, k(G/Z(g_C))) \\ &= \bigoplus_{\{g_C\}} H_*(Z(g_C)) \end{aligned}$$

where $\underline{HH_*(kG, kG)_C \text{ corresponds to } H_*(Z(g_C))}$. This completes the proof of 3.1.5.1.

These ideas can be extended using the same proof to the case of semiconjugacy - for a group homomorphism ϕ, define an equivalence relation $g_1 \sim g_2$ when $g_1 = g g_2 \phi(g^{-1})$ for some $g \in G$. Then the decomposition theory of "markers" passes unchanged, and denoting the set of "semiconjugacy classes" by G_{C_ϕ},

$$HH_*(kG) = \bigoplus_{g_c \in C_{G_\phi}} H_*(k(g_c))_\phi$$

3.1.5.4 Corollary (Zeroth Hochschild Homology):

$HH_0(kG, kG) \cong kG$ the free abelain group generated by conjugacy classes.

3.1.5.5 Corollary (First Hochschild Homology):

Hochschild Homology splits as a sum of abelianisations:

$$HH_1(kG, kG) \cong \bigoplus_{C \in C_G} H_1(Z(g_C)) \cong \bigoplus_{C \in C_G} (Z(g_C))^{ab}$$

These corollories explain the general strategy of the Introduction to get information on first K-groups, since abelianisations are easy to calculate, the image of the Dennis Trace Map is well understood. Thus, if we understand the map itself we can get information on the domain space, $K_1(kG)$.

3.1.6 Equivalence of Definitions of First Hochschild Homology of a Group Ring, $HH_1(kG)$

We have 2 different interpretations of the first Hochschild Homology of a group ring, as Kahler differentials (3.1.3.2), and also lying in a direct sum (3.1.5.1). I give an isomorphism Θ connecting the approaches. For ease of notation I consider the case of G commutative, when each element forms it's own conjugacy class, whose centraliser is the whole group $G = G^{\mathrm{ab}}$.

$$\begin{tikzcd}
& HH_1(kG) = <\{g\otimes h\}>/(\sim) \arrow[dl] \arrow[dr] & \\
<\{g\mathfrak{d}h\}> \arrow[rr, "\Theta"] & & \bigoplus_{g\in G} G
\end{tikzcd}$$

The concept of "markers" introduced in the proof of 3.1.5.1 gives which conjugacy class each term lies in, and thinking of $g\mathfrak{d}h$ as $g\mathfrak{d}g^{-1}(gh)$, the mapping into direct sum is (unique) linear extension to group algebras of

$$g\mathfrak{d}g^{-1}(gh) \xrightarrow{\Theta} (1,\dots,1,g,1,\dots,1),$$

with the identity in each component except for g lying in gh-position

The derivative property of $\mathfrak{d}$ follows immediately from the following commutative diagram, since both $g\mathfrak{d}hk$ and $h\mathfrak{d}gk$ have the same marker, ghk.

$$\begin{array}{ccc}
(\dots,1,gh,1,\dots) & = & (\dots,1,g,1,\dots).(\dots,1,h,1,\dots) \\
\downarrow\Theta & \circlearrowright & \downarrow\Theta \\
gh\mathfrak{d}k & = & g\mathfrak{d}hk + h\mathfrak{d}kg
\end{array}$$

I complete this section by giving another, perhaps more familiar interpretation of Kahler Differentials from that given in 3.1.3.2.

For a ring R, let I be the kernel of the augmentation map $R\otimes R \to R$, induced from $a\otimes b \to ab$. Then there is an isomorphism from Kahler differentials to the quotient of ideals I/I^2:

$$\begin{array}{rcl}
<\{a\mathfrak{d}b\}>/(\sim) & \cong & I/I^2 \\
a\mathfrak{d}b & \xrightarrow{\psi} & ab\otimes 1 - a\otimes b
\end{array}$$

This is clear except for it being a map on the quotient, or equivalently that $\mathfrak{d}$ is a derivation:

Claim (ψ respect derivation):

$$\begin{array}{rcl}
\psi((a\mathfrak{d}bc)-(ab\mathfrak{d}c+ac\mathfrak{d}b)) & \equiv & (abc\otimes 1 - a\otimes bc)-(abc\otimes 1 + ab\otimes c + abc\otimes 1 - ac\otimes b) \\
& \equiv & (abc\otimes 1 - a\otimes bc + ab\otimes c - ac\otimes b) \\
& \equiv & (a\otimes 1)(b\otimes 1 - 1\otimes b)(c\otimes 1 - 1\otimes c) \in I^2 \\
& \equiv & 0
\end{array}$$

Hence, $\psi(a\mathfrak{d}bc) \equiv \psi(ab\mathfrak{d}c + ac\mathfrak{d}b)$, as required.

3.2 Novikov Rings

I consider 2 different completions of the group ring kG.

The first, forming the Novikov ring is used in the study of Dynamical Systems, see [19], where the Dennis Trace Map is used to define a noncommutative zeta function for a closed 1-form on a manifold M, this lies in the first Hochschild Homology of a Novikov ring, and gives information on the orbit structure of gradient flows. Secondly, I try to mimic this construction to give the Hochschild Homology of an Iwasawa algebra as a completion of the Hochschild Homology of group algebras.

3.2.1 Definition of Novikov Rings

Let G be a group and $\chi : G \to \mathbb{R}$ be a homomorphism.

Let $\widehat{\widehat{\mathbb{Z}G}}$ denote the abelian group of all functions $G \to \mathbb{Z}$. Given $\lambda \in \widehat{\widehat{\mathbb{Z}G}}$ pick out the nonvanishing, interesting elements - those in the support of λ, writing supp $\lambda = \{g \in G | \lambda(g) \neq 0\}$.

Elements in Novikov ring (for the homomorphism χ) are defined by a finiteness condition on the support:

$$\widehat{\mathbb{Z}G}_\chi = \{\lambda \in \widehat{\widehat{\mathbb{Z}G}} | \, \forall r \in \mathbb{R} \;\; \# \text{ supp } \lambda \cap \chi^{-1}([r, \infty)) < \infty\}$$

where ring multiplication for $\lambda_1, \lambda_2 \in \widehat{\mathbb{Z}G}_\chi$ gives a well defined element in $\widehat{\mathbb{Z}G}_\chi$:

$$(\lambda_1.\lambda_2)(g) = \sum_{h_1, h_2 \in G | h_1 h_2 = g} \lambda_1(h_1)\lambda_2(h_2)$$

The usual group ring is contained in the Novikov ring as finitely supported maps, and $\mathbb{Z}G = \widehat{\mathbb{Z}G}_\chi$ if and only if χ is the zero homomorphism.

The Novikov ring arises as the completion of the group ring with respect to the following non-Archimedean norm:

3.2.1.1 Definition (Novikov Norm):

The norm of $\lambda \in \mathbb{Z}G_\chi$ *is defined to be*

$$\| \lambda \|_\chi = inf\{t \in (0, \infty) | supp \ \lambda \subset \chi^{-1}((-\infty, log \ t])\},$$

We may avoid use of "log" by giving an equivalent norm arising from a filtration of ideals. Let $\mathbb{Z}G_{\chi \geq 0}$ be the linear span on the elements g such that $\chi(g) \geq 0$, similiarly for all $n \in \mathbb{Z}$, take

$$J_n = \mathbb{Z}G_{\chi \geq n} = < \{g | \chi(g) \geq n\} >$$

This leads to the filtration

$$\cdots \geq J_{-1} \geq F_0 \geq J_1 \geq \cdots \geq J_n \geq J_{n+1} \geq \dots$$

corresponding to

$$\mathbb{Z}G \geq \cdots \geq \mathbb{Z}G_{\chi \geq -1} \cdots \geq \mathbb{Z}G_{\chi \geq 0} \cdots \geq \mathbb{Z}G_{\chi \geq 1} \geq \dots$$

Defining a norm $\| \lambda \|$ on $\mathbb{Z}_p[G]$ by

$$\begin{aligned} \| \lambda \| &= p^{-k} \text{ if } \lambda \in J^k - J^{k+1} \\ \| 0 \| &= 0 \end{aligned}$$

we recover the Novikov ring by completing the group ring with respect to $\| \lambda \|$.

3.2.1.2 Example (Laurent Polynomials):

Let G be the infinte cyclic group, $G = < g >$, identifying g with the indeterminant T the group algebra becomes the Laurent polynomials: $\mathbb{Z}G = \mathbb{Z}[T, T^{-1}]$.

Define the homomorphism to the integers,

$$\begin{aligned} \chi : G &\to \mathbb{Z} \\ \chi : T &\to 1 \\ (\chi : T^{-1} &\to -1) \end{aligned}$$

Then $J_n = \mathbb{Z}G_{\chi \geq n} = T^n \mathbb{Z}[T]$, and the filtration becomes

$$\cdots \geq T^{-1}\mathbb{Z}[T] \geq \mathbb{Z}[T] \geq T\mathbb{Z}[T] \geq T^2\mathbb{Z}[T] \geq \dots$$

The completion with respect to the norm coming from this filtration are the power series in T, T^{-1} with a finite number of terms of positive index: $\overline{\mathbb{Z}G}_\chi = \bigcup_n T^n \mathbb{Z}[T^{-1}]$.

3.2.2 Decomposition of Novikov Complex in Calculation of Hochschild Homology

The Hochschild homology of the finite group ring, 3.1.5.1 is well understood. In this section, following [19], section 4.1, I show how the Hochschild homology of a Novikov ring lies in the completion of the direct product over conjugacy classes of group homology.

Describing the elements in $\widehat{\mathbb{Z}G}_\chi$ as formal linear combinations of elements, an n-chain used to calculate $HH_n(\widehat{\mathbb{Z}G}_\chi)$ has the form:

$$\sum_{g_1 \in G} n_{g_1} g_1 \otimes \cdots \otimes \sum_{g_{n+1} \in G} n_{g_{n+1}} g_{n+1}$$

For finite elements, in $\mathbb{Z}G$, this may be simplified to an element of $C_n(\mathbb{Z}G, \mathbb{Z}G)$, a finite sum:

$$(\star) \qquad \sum_{g_1, \ldots, g_{n+1} \in G} n_{g_1} \ldots n_{g_{n+1}} g_1 \otimes \cdots \otimes g_{n+1}$$

Although a general element taken from $\widehat{\mathbb{Z}G}_\chi$ would be an infinite sum of tensors which does not give a well defined element of $C_n(\mathbb{Z}G, \mathbb{Z}G)$ (the process of breaking down an n-chain in $C_n(\widehat{\mathbb{Z}G}, \widehat{\mathbb{Z}G})$ into form $\star$ may not give a well defined n-chain in $C_n(\mathbb{Z}G, \mathbb{Z}G)$). The following restrictive fact allows us to proceed:

3.2.2.1 Claim (Finiteness of Conjugacy Classes):

Given a conjugacy class $\gamma \in \Gamma$ there are only finitely many nonzero summands in $(\star)$ with marker in γ: such that $g_1 \ldots g_{n+1} \in \gamma$.

Proof

The constructive proof given in [19] relies on χ being well defined on a conjugacy class. This allows us to write $\chi(\gamma) = \chi(g_1) + \ldots \chi(g_{n+1})$, and we may interpret the Novikov completion as allowing elements to only support a finite number of "large" group members, where "large" means $\chi(g) > r$ for any given $r \in \mathbb{R}$.

Thus given the $n+1$ elements of Hochschild Novikov chain, $\sum_{g_1 \in G} n_{g_1} g_1, \ldots, \sum_{g_{n+1} \in G} n_{g_{n+1}} g_{n+1}$, for the union of nonzero terms, $S = \bigcup_1^{n+1} \text{Supp}\left(\sum_{g_k \in G} n_{g_k} g_k\right)$, $\chi(S)$ is bounded above, by M say, giving $\sum_{k \neq i} g_k < n.M$, hence if $\chi(\gamma) = \chi(g_1) + \ldots \chi(g_{n+1})$, then for all i, $\chi(g_i) > \chi(\gamma) - n.M$ giving a finite number of combinations.

Moreover, Schutz gives an explicit map giving which terms in an element of the Novikov ring may be involved in a product giving an element in γ, and then decomposing into conjugacy classes as in the proof of 3.1.5.1 gives a chain homomorphism $\theta_\gamma : C_n(\widehat{\mathbb{Z}G}, \widehat{\mathbb{Z}G}) \to C_n(\mathbb{Z}G, \mathbb{Z}G)_\gamma$, and joining them gives,

$$\theta_* : C_n(\widehat{\mathbb{Z}G}, \widehat{\mathbb{Z}G}) \to C_n(\mathbb{Z}G, \mathbb{Z}G)$$

Notice, that to only prove the finiteness of markers in conjugacy classes, it is sufficient to show this holds for a single element $\lambda \in \widehat{\mathbb{Z}G}_\chi$ (that the "markers" - single elements in this case - have only finite support in a conjugacy class), and (n+1)-degree product, giving markers, follows from considering the formal product $\prod \lambda_i$ which is also in the Novikov Ring. But this is immediate because for a conjugacy class $\gamma \in \Gamma$, $\chi(\gamma) = r$ is well defined, and by definition of the Novikov ring there is only finite support for λ in $\chi^{-1}([r, \infty))$. The same argument is used to consider the case of Iwasawa Algebras in 2.2.3.8.

Defining,

$$C_*(\mathbb{Z}G)_\chi = \{(c_\gamma) \in \prod_{\gamma \in \Gamma} C_*(\mathbb{Z}G, \mathbb{Z}G)_\gamma \,|\, \forall r \in \mathbb{R}, \;\; \#\{c_\gamma \neq 0 | \chi(\gamma) \geq r\} < \infty\}$$

and $\widehat{HH}_*(\mathbb{Z}G)_\chi = H_*(C_*(\mathbb{Z}G)_\chi)$. Then,

$$\bigoplus_{\gamma \in \Gamma} H_*(C_*(\mathbb{Z}G, \mathbb{Z}G)_\gamma) \subset \widehat{HH}_*(\mathbb{Z}G)_\chi \subset \prod_{\gamma \in \Gamma} H_*(C_*(\mathbb{Z}G, \mathbb{Z}G)_\gamma)$$

Then, θ_* factors through $\widehat{HH}_*(\mathbb{Z}G)_\chi$, and is onto, hence $\widehat{HH}_*(\mathbb{Z}G)_\chi$ a completion of the hochschild homology of a group algebra is the hochschild homology of the Novikov ring. Writing θ for the restriction of $HH_n(\widehat{\mathbb{Z}G}, \widehat{\mathbb{Z}G})$ to $\widehat{HH}_*(\mathbb{Z}G)_\chi$:

$$\begin{array}{ccc} HH_*(\mathbb{Z}G) & \stackrel{\cong}{\to} & \bigoplus_{\gamma\in\Gamma} H_*(C_*(\mathbb{Z}G, \mathbb{Z}G)_\gamma) \\ \downarrow & & \downarrow \\ HH_*(\widehat{\mathbb{Z}G}_\chi) & \stackrel{\theta}{\to} & \widehat{HH}_*(\mathbb{Z}G)_\chi \end{array}$$

3.3 Completion of Hochschild Homology

I begin by recalling a result from 3.1.5 giving the Hochschild homology of a group ring in terms of Group homology of related rings. I then mirror this proof in the Iwasawa case.

We transform right-modules into left ones by introducing the opposite action. Where $kG \cong kG^{\mathrm{op}}$ via $g \to g^{-1}$, thus $k(G\mathrm{x}G) \cong kG \otimes kG^{\mathrm{op}}$. Moreover, kG as a $KG \otimes kG^{\mathrm{op}}$-module (or equivalently a G-G-bimodule) is isomorphic to $k \uparrow_{\Delta(G)}^{G\mathrm{x}G}$ where

$$\Delta(G) = \{(g, g) \in G\mathrm{x}G,\ g \in G\}$$

is a diagonal subgroup of $G\mathrm{x}G$.

Thus,

$$\begin{aligned} HH_*(kG, M) &\cong Tor_*^{kG\otimes kG^{\mathrm{op}}}(kG, M) \text{ definition} \\ &\cong Tor_*^{k(G\mathrm{x}G)}(kG, M) \\ &\cong Tor_*^{k(G\mathrm{x}G)}(k \uparrow_{\Delta(G)}^{G\mathrm{x}G}, M) \end{aligned}$$

Working with resolutions, $k(G\mathrm{x}G)$ is a free, hence projective $\Delta(G)$-module, meaning $- \otimes_{k\Delta(G)} k(G\mathrm{x}G)$ is exact:

$$HH_*(kG, M) \cong Tor_*^{k\Delta(G)}(k, M_{\Delta(G)})$$

where $M_{\Delta(G)}$ denotes M considered as a $\Delta(G)$-module - this conjugation action by elements of G may be written $\overline{M}$:

$$\begin{aligned} HH_*(kG, M) &\cong Tor_*^{kG}(k, \overline{M}) \\ &\cong H_*(G, \overline{M}) \end{aligned}$$

Iwasawa Completion of Hochschild Homology

I now recreate this construction replacing tensor products by completed tensor products (to give "Tor" groups over Iwasawa algebras) and introducing inverse limits where necessary. I establish two preliminary lemmas:

Lemma (Λ_G as an induced module):

Completed induction (via completed tensor product) gives,

$$\mathbb{Z}_p \uparrow_{\Lambda_{\Delta(G)}}^{\Lambda_G \widehat{\otimes} \Lambda_G^{op}} \cong \Lambda_G$$

Proof

In other words I need to show $[\mathbb{Z}_p]\widehat{\otimes}_{\Lambda_{\Delta(G)}}[\Lambda_G \otimes_{\mathbb{Z}_p} \Lambda_G^{op}] \cong \Lambda_G$ as a $\Lambda_G \otimes_{\mathbb{Z}_p} \Lambda_G$-module.

Define the bihomorphism α as follows:

$$\begin{array}{rcl} \alpha : [\mathbb{Z}_p] \text{ x } [\Lambda_G \otimes_{\mathbb{Z}_p} \Lambda_G^{op}] & \rightarrow & \Lambda_G \\ [\mu] \text{ x } [a \otimes b^{-1}] & \rightarrow & \mu(ab^{-1}) \end{array}$$

α is indeed a bihomorphism, and we see that for any other bihomomorphism, $\phi : [\mathbb{Z}_p] \text{ x } [\Lambda_G \otimes_{\mathbb{Z}_p} \Lambda_G^{op}] \rightarrow R$, the trivial action of $\Lambda_G \otimes_{\mathbb{Z}_p} \Lambda_G^{op}$ on $\mathbb{Z}_p$ gives

$$\begin{array}{rcl} \phi([\mu] \text{ x } [a \otimes b^{-1}]) & = & \phi([\mu] \text{ x } [ga \otimes (gb^{-1})] \, \forall g \in G \\ & = & \phi([\mu] \text{ x } [i \otimes (ab^{-1})] \\ & = & \phi([i] \text{ x } [i \otimes \mu(ab^{-1})] \end{array}$$

Referring to definition B.5, the completed tensor product is defined by universality with respect to bihomomorphisms, and we may uniquely define $\mathbb{Z}_p$-homomorphism, $g : \Lambda_G \rightarrow R$ by taking $g(\mu(ab^{-1})) = \phi([\mu] \text{ X } [a \otimes b^{-1}])$, then $\phi = g \circ \alpha$ and the following commutes:

$$\begin{array}{ccc} [\mathbb{Z}_p] \text{ x } [\Lambda_G \otimes_{\mathbb{Z}_p} \Lambda_G^{op}] & \stackrel{\alpha}{\rightarrow} & \Lambda_G \\ & \phi \searrow & \downarrow g \\ \circlearrowleft & & R \end{array}$$

Lemma (Exact Tensors):

As a functor $-\widehat{\otimes}_{\Lambda_{\Delta(G)}}(\Lambda_G \widehat{\otimes} \Lambda_G^{op})$ *is exact. Thus,* $\Lambda_G \widehat{\otimes} \Lambda_G^{op}$ *is a "projective"* $\Lambda_{\Delta(G)}$*-module.*

Proof

$(\Lambda_G \widehat{\otimes} \Lambda_G^{op})$ is a free, hence projective, hence flat $\Lambda_{\Delta(G)}$-module. Alternatively, the completed tensor product functor is formed from usual ones, each of which is an exact functor.

Definition

The n-th completed Hochschild homology of an Iwasawa Algebra Λ_G *with coefficient in the bimodule* M *is:*

$$\begin{array}{rcl} HH_n(\Lambda_G, M) & = & Tor_n^{\Lambda_G \widehat{\otimes} \Lambda_G^{op}}(\Lambda_G, M) \text{definition} \\ & = & Tor_n^{\Lambda_{\Delta(G)}}(\mathbb{Z}_p, \overline{M}) \text{using Lemma 1 and Lemma 2} \\ & = & Tor_n^{\Delta(G)}(\mathbb{Z}_p, \overline{M}) \text{from B.6.1} \\ & = & H_n(G, \overline{M}) \end{array}$$

Writing $G = \varprojlim(G/U)$ and $M = \varprojlim \overline{\mathbb{Z}_p[G/U]} = \varprojlim A_i$ we see that the conjugation actions are compatible in the sense of B.4.3:

For $U \leq V$ where $U, V < G$ are open normal subgroups, then U is normal in V, and moreover, $G/V = (G/U)/(V/U)$, thus to show general compatibility it is sufficient to consider the case $H = G/U \leq G$ for some $U < G$ normal.

We need that conjugation is complatible with

$$\begin{array}{ccc} G & \stackrel{\phi}{\rightarrow} & G/U \\ g & \rightarrow & gU \end{array}$$

Hence we require $\forall m \in kG$, $g \in G$,

$$g \circ m \xrightarrow{\phi} \phi(g) \circ \phi(m)$$

i.e.

$$gmg^{-1}U = (gU).(mU).(g^{-1}U)$$

But this is immediate since U is normal in G.

Invoking, B.4.5, we may take out a single inverse limit:

$$HH_*(\Lambda_G, \Lambda_G) = \varprojlim H_*(G/U, \mathbb{Z}_p[G/U])$$

Hence,

$$\begin{aligned}
HH_*(\Lambda_G) &= HH_*(\varprojlim \mathbb{Z}_p[G/U]) \\
&= \varprojlim HH_*(\mathbb{Z}_p[G/U]) \\
&= \varprojlim \bigoplus_{\gamma \in C_{G/U}} HH_*(\mathbb{Z}_p[G/U])_\gamma \\
HH_1(\Lambda_G) &= \varprojlim \bigoplus_{\gamma \in C_{G/U}} (Z(g_\gamma))^{ab}
\end{aligned}$$

where for each term in inverse limit, we have a sum over conjugacy classes. In $n = 1$ case this reduces to a sum of centralisers, $Z(g_\gamma) \leq G/U$

3.3.1 What is the topology underlying this inverse limit?

We may describe a topology which recovers this inverse limit using the standard concepts of product and quotient topology.

If we interpret $HH_n(\Lambda_G)$ as some quotient of the tensor product $(\Lambda_G)^{\otimes n}$, then $HH_n(\Lambda_G)$ is the quotient of some Cartesian product $(\Lambda_G)^n$, and if powers of $J = \text{Ker } (\mathbb{Z}_p[[G]] \to \mathbb{F}_p)$ give a topology on Λ_G taking $||\lambda|| = p^-k,$ where $x \in J^k - J^{k+1}$.

In the case of interest to us here, $HH_1(\Lambda_G) \cong [\Lambda_G \otimes \Lambda_G]/\sim$, and for $A \otimes B \in [J^i \otimes J^j] - [J^{i+1} \otimes J^j] - [J^i \otimes J^{j+1}]$, then $||A \otimes B|| = p^{-(i+j)}$ is well defined and displays $HH_1(\Lambda_G)$ as the inverse limit $\varprojlim HH_1(\mathbb{Z}_p[G/U])$.

3.3.2 Induced Maps

Where an argument using markers shows for a finite group G acting on kG by conjugation -

$$\bigoplus_{C \in ccl(G)} HH_*(kG, kG) = \bigoplus_{C \in ccl(G)} H_*(G, kC) = \bigoplus_{g_c \in ccl(G)} H_*(G, k(G/Z(g_c))) = \bigoplus_{g_c \in ccl(G)} H_*(Z(g_c), k)$$

Thus it is sufficient to understand how the quotient map on finite groups, $G \to G/U$ induces to give a map

$$\bigoplus_{g_c \in ccl(G)} (Z(g_c))^{ab} = HH_*(kG) \to HH_*(kG/U) = \bigoplus_{f_c \in ccl(G/U)} (Z(f_c))^{ab}$$

The augmentation map gives rise to a $\mathbb{Z}G$-free resolution of k in terms of augmentation ideal:

$$0 \to IG \hookrightarrow kG \to k \to 0$$

This allows us to calculate H_1 explicitly, and perform the induction giving rise to centralisers explicitly. Since

$$0 \to H_1(G, B) \to B \otimes IG \to B \otimes \mathbb{Z}G \to H_0(G, B) \to 0$$

We have

$$\begin{array}{rcl} H_1(G,B) = \ker(B \otimes_G IG \to B) = B \otimes IG/(IG)^2 & \cong & B \otimes G/G' \\ \text{where the isomorphism is given on generators by} & & \\ b \otimes (x-1) & \to & b \otimes x \end{array}$$

Thus, we may view $H_1(G,k) \cong G/G' = G^{ab}$ on generators in IG, $g-1 \to g$.

The key calculation is done in the following table, recalling $k(G/Z(g_c)) = k \uparrow_{kZ(g_c)}^{kG}$, on RHS I follow a representative element which generates whole group.

$$\begin{array}{ccccc} H_1(G,k(G/Z(g_c))) & = & \ker(IG \otimes_{kG} k(G/Z(g_c)) \to kG \otimes_{kG} k(G/Z(g_c)) & & \\ & = & \ker(IG \otimes_{kG} [kG \otimes_{kZ(g_c)} k] \to kG \otimes_{kG} [kG \otimes_{kZ(g_c)} k]) & & g-1 \otimes kG \otimes k \\ ||\wr & & & & \downarrow \\ H_1(Z(g_C),k) & = & \ker(IG \otimes_{kZ(g_c} k \to kG \otimes_{kZ(g_c)} k) & & g-1 \otimes k \\ ||\wr & & & & \downarrow \\ (Z(g_c))^{ab} & & & & g \end{array}$$

3.3.2.1 Conjugacy Classes

Consider

$$\begin{array}{rcl} G & \to & G/U \\ g & \to & gU \end{array}$$

Then if f and h are conjugate, meaning there exists a g such that $f = g^{-1}hg$, then certainly $fU = g^{-1}hgU$, and since U is normal in G, this gives $fU = g^{-1}U.hU.gU$, and so fU is conjugate to hU in G/U.

$$f, h \text{ conj in } G \Longrightarrow fU, hU \text{ conj in } G/U$$

But if elements with representatives fU, and hU are conjugate, we can only deduce there exists a g such that $fU = (gU)^{-1}(hU)(gU)$, thus $U = f^{-1}g^{-1}hgU$ which gives $f^{-1}g^{-1}hg \in U$, but is not necessarily the identity. i.e.

$$fU, hU \text{ conj in } G/U \not\Rightarrow f, h \text{ conj in } G$$

meaning that the *conjugacy classes in G may fuse when we pass to G/U.*

We may now combine these ideas with the explicit maps between Hochschild homology groups and direct sums over conjugacy classes of abelianisations of conjugacy classes given above.

It is sufficient to consider what happens to each summand over conjugacy classes, so wlog consider the representative (g_1, g) where $g \in (Z(g_1))$, meaning $g_1 g g_1^{-1} = g$. Certainly then $g_1 U g U g_1^{-1} U = gU$ and thus $gU \in (Z(g_1 U))$ giving meaning to the chain:

$$\begin{array}{cc} \bigoplus (Z(g_*))^{ab} & (g_1, g) \\ ||\wr & \downarrow \\ \bigoplus H_1(Z(g_*),k) & (g_1, g-1 \otimes k) \\ ||\wr & \downarrow \\ \bigoplus H_1(G,k(G/Z(g_*))) & (g_1, g-1 \otimes kG \otimes k) \\ \downarrow & \downarrow \\ \bigoplus H_1(G/U,k((G/U)/Z(g_*U))) & (g_1U, gU-U \otimes kG/U \otimes k) \\ ||\wr & \downarrow \\ \bigoplus H_1(Z(g_*U),k) & (g_1U, gU-U \otimes k) \\ ||\wr & \downarrow \\ \bigoplus (Z(g_*U))^{ab} & (g_1U, gU) \end{array}$$

We are actually dealing with elements in the abelianization here, so we need to check $(g_1, g) \to (g_1U, gU)$ is well defined up to multiplication by commutators. *By $g \in [Z(g_1)]^{ab}$ we are referring to a class gV where*

$$V = \{aba^{-1}b^{-1}|a, b \in Z(g_1)\} = \{aba^{-1}b^{-1}|ag_1 = g_1a \,\&\, bg_1 = g_1b\} = \{aba^{-1}b^{-1}|[a, g_1] = i \,\&\, [b, g_1] = i\}$$

Then $gU \in [Z(g_1U)]^{ab}$ we are referring to a class gUW where

$$W = \{aba^{-1}b^{-1}U|aUg_1U = g_1UaU \,\&\, bUg_1U = g_1UbU\} = \{aba^{-1}b^{-1}U|[a, g_1] \in U \,\&\, [b, g_1] \in U\} \supset V$$

hence $gVU \subset gW$ and so the map is well defined.

The fusion mentioned above means there might be more than one contribution to each summand in the quotient, suppose $g_1, \ldots, g_n$ are representatives of distinct conjugacy classes in G, but are all conjugate to each other (and to $f = g_1U$ say) in G/U, we can think of the summands lying over each other in the following sense:

$$\underbrace{[Z(g_1)]^{ab} \bigoplus \cdots \bigoplus [Z(g_n)]^{ab}}_{[Z(f)]^{ab}} \quad \bigoplus \cdots \qquad \bigoplus \cdots$$

3.3.2.2 Hochschild Homology under Quotient Map - Special Conjugacy Classes

The lack of canonical representatives for conjugacy classes makes the choices in the representation arbitrary and thus only defined up to conjugacy.

Take representatives of the conjugacy classes of G/U:

$$\{f_1 \in C_1, \ldots f_n \in C_n\}$$

Moreover choose representatives of the conjugacy classes of G:

$$\{g_{(1,1)} \in C_{(1,1)}, \ldots g_{(1,m_1)} \in C_{(1,m_1)}, g_{(2,1)} \in C_{(2,1)}, \ldots g_{(n,1)} \in C_{(n,1)}, \ldots g_{(n,m_n)} \in C_{(n,m_n)}\}$$

Where the projections, $\{g_{(1,1)}U, \ldots, g_{(1,m_1)}U\}$ are not only conjugate to each other, lying in C_1, conjugate to f_1, but in fact map to f_1U under projection (otherwise we might have to conjugate elements to get them lying in correct centraliser).

Similarly, for all $i = 1, 2, \ldots, n$, $\{g_{(i,1)}U, \ldots, g_{(i,m_i)}U\}$ are all conjugate to each other, lying in C_i, projecting to f_i.

Then, for $h_{(i,j)} \in Z(g_{(i,j)})$ we have the map:

$$\begin{array}{ccccc}
HH_1(kG) & \left(\underbrace{h_{(1,1)}, \ldots, h_{(1,m_1)}}\right) & ,\ldots, & \left(\underbrace{h_{(n,1)}, \ldots, h_{(n,m_n)}}\right) \\
\downarrow & & \downarrow & \\
HH_1(kG/U) & \left(\underbrace{(\prod h_{(1,1)} \ldots h_{(1,m_1)}U)}_{\text{Conj. Class } C_1}\right) & ,\ldots, & \left(\underbrace{(\prod h_{(n,1)} \ldots h_{(n,m_n)}U)}_{\text{Conj. Class } C_n}\right)
\end{array}$$

3.3.2.3 Hochschild Homology under Quotient Map - General Conjugacy Classes

From 3.3.2.1, if gU and hU are conjugate in G/U, then g is conjugate to a product of h with an element of U, in G. Conversely, if g and h are conjugate then gU and $huU = hU$ are conjugate for all $u \in U$. Thus elements of G which become conjugate to g when passing to the quotient is $X_g = \{g_iU|g_i \text{ is conjugate to } g \text{ in } G\}$.

Understanding how elements fuse requires us to consider how many elements of X_g are already conjugate (to avoid repetition in counting).

In the commutative case, when each element is it's own conjugacy class, $|U|$ conjugacy classes (elements of G) lie over each conjugacy class (elements of G/U).

Since U is normal it can be written as a sum of conjugacy classes of G: $U = \bigcup_{i=1}^{n} C_i$. I begin by observing that elements mapping to $\overline{id}$ in G/U are infact just elements of $U \subset G$, a union of (n) conjugacy classes. The same is true for other conjugacy classes - $X_g = \cup\{g_i U\}$ is a sum of conjugacy classes. This is seem by considering the conjugate by $h \in G$ os a general element $g_i u \in X_g$:

$$\begin{aligned} h^{-1}(g_i u)h &= (h^{-1}g_i h)(h^{-1}uh) \\ &= g_j.u' \;\; \text{(since } U \text{ is normal)} \\ &\in X_g \end{aligned}$$

We may visualize conjugacy class fusion for projection G to G/U, where $U = \bigcup_{i=1}^{n} C_i$.

$$\begin{array}{ccccccc} id \in C_1 & \dots & C_n & g \in C_{n+1} & \dots & C_m & \dots \\ \searrow & \downarrow & \swarrow & \searrow & \downarrow & \swarrow & \\ & B_1 & & & B_2 & & \\ & id.U & & & g.U & & \end{array}$$

Let us denote by $\begin{array}{c} h \\ | \\ g \end{array}$ the component of the direct sum lying over conjugacy class of g, where $h \in Z(g)$.

Now, to understand the inverse limit I ask if the connecting map,

$$\bigoplus_{g_c \in ccl(G)} (Z(g_c))^{ab} \to \bigoplus_{f_c \in ccl(G/U)} (Z(f_c))^{ab}$$

need be onto under extension of map $\begin{array}{c} h \\ | \\ g \end{array} \to \begin{array}{c} hU \\ | \\ gU \end{array}$.

We observe, $\begin{array}{c} hU \\ | \\ gU \end{array}$ is equivalent to

$$\begin{aligned} hU.gU.h^{-1}U &= gU \\ hgh^{-1} &= gu \text{ for some } u \in U \end{aligned}$$

Whereas, $\begin{array}{c} h \\ | \\ g \end{array}$ is equivalent to

$$hgh^{-1} = g \tag{3.1}$$

giving image under projection $\begin{array}{c} hU \\ | \\ gU \end{array}$ with

$$hgh^{-1} = g.id$$

Thus the possibility of hitting different values for commutator $[g, h^{-1}]$ could only arise as the image of a representative in a different conjugacy class which only becomes conjugate after projection. Without loss of generality we may reduce to the case of the contribution from n where $g = nu$ is the representative we want:

For $\begin{array}{c} m \\ | \\ n \end{array}$, where $[m,n] = m^{-1}n^{-1}mn = 1$ we get a contribution, $\begin{array}{c} mU \\ | \\ nuU \end{array}$, where $m^{-1}(nu)^{-1}.m.nu = m^{-1}u^{-1}n^{-1}mnu = m^{-1}u^{-1}mu = 1$ if and only if $u \in Z(m)$. This has reduced the question to how the decomposition of G into left cosets of $Z(m)$ interplays with conjugacy class structure.

This gives that the number of values, u' attainable as a commutator under projection of $\begin{array}{c} m \\ | \\ n \end{array}$ is less than or equal to the number of conjugacy classes lying above gU which is less than or equal to $|U|$ with equality only if G is abelain (and then we see that this is indeed sufficient for the connecting map to be onto).

I now give a non-commutative example to show this map is not commutative in general:

3.3.2.4 Dihedral Example

Take $G = D_8 = \{i, \alpha, \alpha^2, \alpha^3, \beta, \alpha\beta, \alpha^2\beta, \alpha^3\beta | \alpha^4 = id = \beta^2, \beta\alpha = \alpha^3\beta\}$

Then this has conjugacy classes,

$$\begin{array}{rcl} C_1 & = & \{id\} \\ C_2 & = & \{\alpha, \alpha^3\} \\ C_3 & = & \{\alpha^2\} \\ C_4 & = & \{\beta, \alpha^2\beta\} \\ C_5 & = & \{\alpha\beta, \alpha^3\beta\} \end{array}$$

moreover, $A = C_1 \cup C_3 = \{id, \alpha^2\}$ is a normal subgroup. We see that the map $G \to G/A \cong V$ (non-cyclc, commutative group of order 4).

$$\begin{array}{ccccccc} G & & C_1 & C_3 & C_2 & C_4 & C_5 \\ \downarrow & & \downarrow & \swarrow & \downarrow & \downarrow & \downarrow \\ G/A & & \overline{id} \equiv \overline{\alpha^2} & & \overline{\alpha} \equiv \overline{\alpha^3} & \overline{\beta} \equiv \overline{\alpha^2\beta} & \overline{\alpha\beta} \equiv \overline{\alpha^3\beta} \end{array}$$

Hence, $Z(\alpha\beta) = (id, \alpha\beta, \alpha^3\beta, \alpha^2) \to (\overline{id}, \overline{\alpha\beta}) \in Z(\overline{\alpha\beta})$.

But the Vier group is Abelian, thus for all $v \in V$, $Z(v) = V$. Therefore, $Z(\overline{\alpha\beta}) = (\overline{id}, \overline{\alpha}, \overline{\beta}, \overline{\alpha\beta})$.

So there is no pre-image of $\begin{array}{c} \overline{\alpha} \\ | \\ \overline{\alpha\beta} \end{array}$ showing that connecting maps in the inverse limit defining the Hochschild homology of Iwasawa completed group algebras need not be surjective.

3.3.3 Does $HH_1(\Lambda_G)$ Reduce to a Direct Product over Conjugacy Classes?

In 2.2.3.9 I showed that the Hochschild homology of an Iwasawa algebra is not contained in the direct product over conjugacy classes,

$$\varprojlim HH_1 \mathbb{Z}_p[G/U] = HH_1(\widehat{(\mathbb{Z}_p G)}) \subset \prod_{\gamma, \text{conj. class}} HH_1(\mathbb{Z}_p G)_\gamma = \prod_{\gamma, \text{conj. class}} (Z(\gamma))^{ab}$$

When G is infinite the first homology group is no longer just the abelianisation. The universal defining property of $\varprojlim$ gives an injective map,

$$\prod_{\gamma, \text{conj. class}} (Z(\gamma))^{ab} \hookrightarrow \varprojlim (HH_1(\mathbb{Z}_p[G/U])) = HH_1(\widehat{(\mathbb{Z}_p G)})$$

I now ask if this is an Isomorphism:

Claim (Completion of Homology):

$$\prod_{\gamma, conj.\ class} (Z(\gamma))^{ab} \cong \varprojlim (HH_1(\mathbb{Z}_p[G/U])) = HH_1(\widehat{(\mathbb{Z}_p G)})$$

Consider the Classical $\mathbb{Z}_p$-extension - $G \cong \mathbb{Z}_p$ then everything is abelian and connecting maps are just projections modulo p^n (see section ?? for justification of this), so we are reduced to giving an element in $\varprojlim (HH_1(\mathbb{Z}_p[\mathbb{Z}/p^n\mathbb{Z}])) = \varprojlim \bigoplus_{\mathbb{Z}/p^n\mathbb{Z}} (\mathbb{Z}/p^n\mathbb{Z})$ not in $\prod_{\mathbb{Z}_p}(\mathbb{Z}_p)$. I will use the notation $\begin{matrix}(\dots, a, \dots)\\ | \\ (\dots, b, \dots)\end{matrix}$ to denote the element a lying in coordinate b. For p odd, $\sum_{i=0}^{p^n-1}(-1)^i = 1$, giving consistency of following elements to build up an element x in $\varprojlim \bigoplus_{\mathbb{Z}/p^n\mathbb{Z}} (\mathbb{Z}/p^n\mathbb{Z})$.

$$\begin{array}{cc}
(\mathbb{Z}/p^0\mathbb{Z}): & \begin{matrix}(1)\\ | \\ (0)\end{matrix} \\
\uparrow & \\
(\mathbb{Z}/p^1\mathbb{Z}): & \begin{matrix}(\overbrace{1,-1,1,\dots,-1,1})\\ | \\ (0, 0+1.p^0, 0+2.p^0, \dots, 0+(p-1).p^0)\end{matrix} \\
\uparrow & \\
(\mathbb{Z}/p^2\mathbb{Z}): & \begin{matrix}(\overbrace{1,-1,1,\dots,1},\overbrace{1,-1,1,\dots,1},\dots,\overbrace{1,-1,1,\dots,1})\\ | \\ (0, 0+1.p^1, \dots, 0+(p-1).p^1, 1, 1+1.p^1, \dots, 1+(p-1).p^1, \dots, \\ p-1, p-1+1.p^1, \dots, p-1+(p-1).p^1,)\end{matrix} \\
\uparrow & \\
\vdots &
\end{array}$$

We now study if $x \in \prod_{\mathbb{Z}_p}(\mathbb{Z}_p)$. If it was, then element lying over $a_0+a_1p+\dots+a_np^n$ is $sgn(a_0+a_1+\dots+a_n) = (-1)^{a_0+a_1+\dots+a_n}$. By Fermat's Little Theorem this is just $(-1)^{a_0+a_1p+\dots+a_np^n}$. Since $a^x = exp(ln\, a.x)$, whether this continues to be defined for all exponentials in $\mathbb{Z}_p$, not necessarily finite, is equivalent to whether $ln\,(-1)$ is defined. But $ln: 1+A_0 \to A_0$ where $A_0 = \{x \text{ such that } ||x|| \leq p\} = p\mathbb{Z}_p$. But $-1 \in \mathbb{Z}_p - p\mathbb{Z}_p$ and so the logarithm, and hence the exponential are not defined. Hence, $x \in \varprojlim \bigoplus_{\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{Z}/p^n\mathbb{Z})$ does not come from an element of $x \in \prod_{\mathbb{Z}_p}(\mathbb{Z}_p)$, as required.

3.3.4 Products of Finite Groups

Given a family of groups, there are many different ways of combining these to get another group.

Let $\{G_\lambda | \lambda \in \Lambda\}$ be a given set of groups over a (not necessarily finite) indexing set Λ.

3.3.4.1 Catesian Product

The **cartesian** (or unrestricted direct) product,

$$C = \mathrm{Cr}_{\lambda \in \Lambda} G_\lambda$$

is the group with underlying set the product of the G_λ as sets - vectors whose λ-component lies in G_λ, and whose group operation is defined by multiplication of components: $(g_\lambda)(h_\lambda) = (g_\lambda h_\lambda)$ for $g_\lambda, h_\lambda \in G_\lambda$.

The cartesian product may also be given a universal construction as follows:

Define the projections $\pi_\lambda : G \to G_\lambda$ by taking $\pi\lambda(x)$ to be the λ-th component of x. π_λ is a homomorphism for each λ.

Given a family of homomorphisms $\phi_\lambda : H \to G_\lambda$ for some group H, there exists a unique homomorphism $\phi : H \to G$ such that $\phi \circ \pi_\lambda = \phi_\lambda$ for all λ.

The existence of the map ϕ gives the following commutative diagram:

$$\begin{tikzcd} H \arrow[d, "\phi"] \arrow[dr, "\phi_\lambda"] & \\ G \arrow[r, "\pi_\lambda"] & G_\lambda \end{tikzcd}$$

3.3.4.2 Direct Product

The subset of the (g_λ) such that $g_\lambda = a_\lambda$ for almost all λ, so sequence is trivial with finitely many exceptions, is called the **external direct** product,

$$D = \mathrm{Dr}_{\lambda\in\Lambda} G_\lambda$$

The G_λ are referred to as the direct factors. D is a normal subgroup of C, and equal for a finite indexing set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The products are then written,

$$D = G_{\lambda_1}\, X\, G_{\lambda_2}\, X\, \ldots\, X\, G_{\lambda_n}$$

And should the groups be abelian, and written additively,

$$D = G_{\lambda_1} \oplus G_{\lambda_2} \oplus \ldots \oplus G_{\lambda_n}$$

3.3.4.3 Free Products

A free product of the family G_λ is a group G together with a collection of homomorphisms $l_\lambda : G_\lambda \to G$ with universal property that for another such group H and set of homomorphisms $l_\lambda : G_\lambda \to H$, there is a unique homomorphism of groups $\phi : G \to H$ such that $l_\lambda \circ \phi = \phi_\lambda$, and the following diagram commutes:

$$\begin{tikzcd} G_\lambda \arrow[r, "l_\lambda"] \arrow[d, "\phi_\lambda"] & G \arrow[dl, "\phi"] \\ H & \end{tikzcd}$$

The free product is sometimes denoted $F = \mathrm{Fr}_{\lambda\in\Lambda} G_\lambda$.

From the category-theoretic viewpoint this free product is a coproduct in the category of groups (the product being the cartesian product defined above).

For each λ, taking $H = G_\lambda$, and maps $\phi_\lambda = \mathrm{id}$, other ϕ_μ trivial, we see that $\phi \circ l_\lambda = \mathrm{id}|_{G_\lambda}$ and so each l_λ is injective.

Uniqueness of this construction is clear from the universal property. Existence can be shown using an explicit description on words, where letters are taken from the disjoint union of the G_λ (we are only working up to Isomorphism of G_λ, so may assume they do not intersect), products are by juxtapostion, and the only relations are contracting/ expanding letters lying in the same group G_λ, and absorbing/inserting identity elements.

If Λ is finte, $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, it is usual to write product as

$$G_{\lambda_1} \star G_{\lambda_2} \star \ldots \star G_{\lambda_n}$$

The free product has the following properties:

1. For $\{G_\lambda | \lambda \in \Lambda\}$ there is a natural epimorphism of groups from $\mathrm{Fr}_{\lambda\in\Lambda} G_\lambda$ to $\mathrm{Dr}_{\lambda\in\Lambda} G_\lambda$, defined by ************, and with kernel *****************.

2. $(\mathrm{Fr}_{\lambda\in\Lambda} G_\lambda)^{\mathrm{ab}} \cong \mathrm{Dr}_{\lambda\in\Lambda} (G_\lambda)^{\mathrm{ab}}$.

3. So for abelian families G_λ, we have $(\mathrm{Fr}_{\lambda\in\Lambda}G_\lambda)^{\mathrm{ab}} \cong \mathrm{Dr}_{\lambda\in\Lambda}G_\lambda$. For a finite index we have:

$$(\bigstar_{\lambda\in\Lambda}G_\lambda)^{\mathrm{ab}} \cong \bigoplus_{\lambda\in\Lambda} G_\lambda$$

3.3.5 Products of Profinite Groups

Following Melnikov I explain how the topology of profinte groups and spaces can be used to control what is happening in the inverse limits of such products, and how such products may themselves be represented as products.

I will discuss p-groups and pro-p groups in this section although the same ideas could be applied to and pro- groups for a full class of finite groups (meaning it is closed under subgroups, homomorphic images and group extensions).

3.3.5.1 When the indexing familiy is discrete

3.3.5.2 When the indexing family varies continuously over a profinite space

3.3.5.3 Inverse limits and free products

3.3.5.4 Reformulation of Completed Hochschild Homology

We begin by explaining why Inverse Limits and Abelianization Commute. We can of course identify $H_1(G,\mathbb{Z}_p)$ with G^{ab} for any group G, and this gives,

$$\begin{aligned}
\varprojlim_{U\leq_o G} (G/U)^{ab} &= \varprojlim_{U\leq_o G} H_1(G/U,\mathbb{Z}_p) \\
&= [\varinjlim_{U\leq_o G} H^1(G/U,\mathbb{Z}_p)]^* \\
&= [\varinjlim_{U\leq_o G} H^1(G/U,\mathbb{Z}_p^{G/U})]^* \\
&= [H^1(G,\mathbb{Z}_p)]^* \\
&= H_1(G,\mathbb{Z}_p) \\
&= G^{ab}
\end{aligned}$$

$$\begin{aligned}
\varprojlim_{U\leq_o G} \bigoplus_{g_U\in\mathrm{ccl}(G/U)} Z(g_U)^{ab} &= \varprojlim_{U\leq_o G} [\bigstar_{g_U\in\mathrm{ccl}(G/U)} Z(g_U)]^{ab} \\
&= [\varprojlim_{U\leq_o G} \bigstar_{g_U\in\mathrm{ccl}(G/U)} Z(g_U)]^{ab} \\
&= [\bigstar_{g\in\mathrm{ccl}(G)} \varprojlim_{U\leq_o G} Z(gU)]^{ab} \\
&= [\bigstar_{g\in\mathrm{ccl}(G)} Z(g)]^{ab}
\end{aligned}$$

Chapter 4

Trace Maps

4.1 K-theory and semi-local Rings

R is semilocal if it has a finite number of maximal ideals, and such that for J the Jacobson radical, R/J is a simple Artinian Ring (so certainly true when R is local - R/J field). By Wedderburn's Theorem, a semi-simple Artinian ring id a direct sum of matrix rings, so R/J is a full matrix ring. We will see that in this case, $K_1(R)$ is "small".

4.1.1 Idempotents

e and $1-e$ always orthog for idempotent e.

Idempotent is primative if it does not split as sum of 2 idempotents. '

4.1.2 Blocks

For a ring A,

$$A = B_1 \oplus \cdots \oplus B_r$$

decomposition into into 2 sided ideals - whenever A is NOETHERIAN.

B_is are the blocks of A, generated by central idempotent, decomposition gives:

$$1 = e_1 + \ldots e_r$$

$B_i = e_1 A$ is itself a ring, with identity e_i

4.1.3 Grothendieck Groups

$$K_0(A) = K_0(\mathbb{P}(A))$$
$$K_0(A) = K_0(\mathbb{M}(A))$$

4.1.4 Semisimple Rings

4.1.4.1 Lemma

Let A be a semisimple ring and let $V_1, \ldots, V_s$ be a complete list of representatives for the isomorphism classes of simple A-modules.

- If P, Q are finitely generated A-modules, then

$$P \cong Q \text{ if and only if } [P] = [Q] \text{ in } K_0(A)$$

- $$K_0(A) = \oplus_{i=1}^{s} \mathbb{Z}[V_i]$$

- $$b(A) = rk\, K_0(A)$$

4.1.5 Semilocal Rings

Notation $J(A)$ for jacobson radical, and $\overline{A} = A/J(A)$. A is

- semilocal if $\overline{A}$ is Artinian
- local if $\overline{A}$ is simple Artinian
- scalr local if $\overline{A}$ is a division ring

A is a complete semilocal ring if A is semilocal and complete wrt $J(A)$-adic filtration.

4.1.6 Idempotent Lifting

For A a complete semilocal ring... with simple modules as above.

Since $\overline{A}$ is semisimple we can find primative orthogonal idempotents, $a_1, \dots a_s$ in $\overline{A}$ such that $V_i \cong a_i\overline{A}$ as an $\overline{A}$-module.

A is $J(A)$-adically complete so can lift a_i to primative orthogonal idempotents of A

$$a_i = \overline{e}_i$$

4.1.7 Quillen's Construction of the Higher K-groups

First, I give a general construction of K-groups which will allow me, in 4.2.1 to simultaneously define a whole set of Dennis Trace Maps, $\delta : K_n(R) \to HH_n(R),\ n \geq 0$, and then I focus on the $n = 1$ case, when $HH_n(kG)$ has an especially simple form as a direct sum of commutative groups, see 3.1.5.5. It is first necessary to recall the maps which form the background to Quillen's construction of the higher K-groups.

1. *Hurewicz Map, $*$*

   "$*$" is a map from the n-th Homotopy Group of a based space (X, x_0) to it's n-th Homology Group. A loop $f \in \Pi(X, x_0)$ is a map from the n-sphere based at s_0 to the based space $f : (S^n, s_0) \to (X, x_0)$. Taking homologies of the spaces, f induces a homomorphism $f_* : H_n(S, s_0) \to H_n(X, x_0)$. But $H_n(S, s_0) = \mathbb{Z}$, and so the homomorphism is completely determined by specifying the image of 1, where $f_* : 1 \to f_*(1) \in H_n(X, x_0)$ is associated to the loop f giving the Hurewicz map:

$$\begin{aligned} " * " : \Pi_n(X, x_0) &\to H_n(X, x_0) \\ f &\to f_*(1) \end{aligned}$$

2. *Fusion Map, ϕ*

   For a ring, R, let $\phi' : kGL_r(R) \to M_r(R)$ be componentwise evaluation of the formal sum. This induces $\phi'' : HH_n(kGL_r(R) \to HH_n(M_r(R))$, and composing with the Generalised Trace Map (defined in 3.1.4) construct,

$$\phi = Tr \circ \phi'' : HH_n(GL_r(R)) \to HH_n(R)$$

3. *Plus Construction and Milnor's Base Space, + and B*

   The Base map "B" is constructed to utilise geometry to solve algebraic question by associating to a given group a space, and " + " is a map between spaces, turning homology equivalences into homotopy equivalences, and preserving homology - $H_n(BG^+, k) \cong H_n(G, k)$. They are defined in such a way that for n=0,1,2 the groups $\Pi_n(B(GL(R))^+) = K_n(R)$, the classically defined K-groups. Defining the higher K-groups as $K_n(R) = \Pi_n(B(GL(R))^+)$, $n \geq 0$, the long exact sequence of such "K-groups" which immediately arises from the homotopy suggests this is the correct definition. A full account is given in [2], section 2.10.

4.1.8 Dieudonne Determinant

Let A be a K-algebra, $a \in A$ let char.pol.$_{A/K}(a)$ be the characteristic polynomial of the K-linear map $x \to ax$, $x \in A$. Then trace, $T_{A/K}$ and norm maps, $N_{A/K}$ are defined as follows:

$$\text{char.pol.}_{A/K} a = X^m - (T_{A/K} a) X^{m-1} + \cdots + (-1)^m N_{A/K} a$$

However, if we take $A = M_n(K)$, so each $\mathbf{a} \in A$ is a matrix over K.

$$\text{char.pol.}_{A/K} \mathbf{a} = \{\text{char.pol.matrix}_{A/K} \mathbf{a}\}^n$$

giving $T_{A/K}\mathbf{a} = n.$(trace of matrix $\mathbf{a}$) and $N_{A/K}\mathbf{a} = (\det\ \mathbf{a})^n$, suggesting the trace and determinant are not fine enough for calculations. We introduced a "reduced" theory to overcome this.

Let A be an arbitrary separable K-algebra and E a splitting field for A over K, so there is an isomorphism of E-algebras

$$E \otimes_K A \cong \bigsqcup_{i=1}^{s} M_{n_i}(E)$$

For each $a \in A$, let

$$h(1 \otimes a) = \bigsqcup \phi_i(a), \text{ where } \phi_i(a) \in M_{n_1}(E), 1 \leq i \leq s$$

Take reduced characterisitc polynomial to be,

$$\text{red.char.pol.}_{A/K} a = \prod_{i=1}^{s} \text{char.pol.}\phi_i(a), \forall a \in A$$

This polynomial turns out to be independent of the choice of E, and moreover its coefficients lie in the ground field K. Taking reduced norm and trace, $nr_{A/K}$ and $tr_{A/K}$ as coefficients we overcome the problem discused above and still have nice multiplicative/ additive properties.

Suppose $A = M_n(D)$ with D a skewfield with centre K. Write $K^\bullet = K - \{0\}$. Then

$$nr_{A/K} : GL_n(D) \to K^\bullet$$

is a multiplicative homomorphism which may be thought of as a "determinant map" in some sense.

Let $D^\cdot = D - \{0\}$, $D^\# = D^\bullet/[D^\bullet, D^\bullet]$, then $nr_{D/K} : D^\bullet \to K^\bullet$ is a homo and induces

$$nr_{D/K} : D^\# \to K^\bullet$$

There is also a homomorphism, $det : GL_n(D) \to D^\#$, the **Dieudonne Determinant** such that,

$$nr_{D/K}\det\ a = nr_{A/K} a \text{ for all } a \in GL_n(D)$$

Let $\mathbf{E} \in GL_n(D)$ be an elementary matrix. Then for $\mathbf{X} \in GL_n(D)$, $\mathbf{EX}$ is obtained from $\mathbf{X}$ by increasing some row of $\mathbf{X}$ by a left multiple of another row, and similarly for $\mathbf{XE}$ and columns.

Given any such $\mathbf{X}$ we can find products $\mathbf{P}$ and $\mathbf{Q}$ of elementary matrices such that,

$$\mathbf{PXQ} = \mathrm{diag}(a_1, \ldots, a_n),\ a_i \in D$$

By, 4.1.15 $\mathrm{diag}(a, a^{-1})$ is itself a product of elem. matrices we may improve matrices to give:

$$PXQ = \mathrm{diag}(1, \ldots, 1, a) \text{ where } a = \prod_1^n a_i \in D^\bullet$$

The image of a in $D^\#$ is the Dieudonne Determinant of $\mathbf{X}$, det $\mathbf{X}$.

If D is a field then $D^\# = D^\bullet$, and det$\mathbf{X}$ is the usual determinant of $\mathbf{X}$. If D is not a field then the element $a \in D^\bullet$ is not uniquely determined by $\mathbf{X}$, but it's image in $D^\#$ is.

- Every elementary matrix has determinant 1.
- $\begin{pmatrix} \mathbf{X} & * \\ 0 & \mathbf{Y} \end{pmatrix} = (\det\mathbf{X})(\det\mathbf{Y})$ for $\mathbf{X} \in GL_n(D)$, $\mathbf{Y} \in GL_m(D)$
- $\det(\mathbf{XY})) = (\det\mathbf{X})(\det\mathbf{Y})$ for $\mathbf{X}, \mathbf{Y} \in GL_n(D)$

I now work towards the key theorem connecting this determinant with the Whitehead Goups. Recall,

4.1.9 Lemma (Grothendieck Groups of Semisimple Rings (see [1] 2.4))

Let A be a semisimple ring and let $V_1, \ldots, V_s$ be a complete list of representatives for the isomorphism classes of simple A-modules.

1. *If P, Q are finitely generated A-modules, then*

$$P \cong Q \textit{ if and only if } |P| = |Q| \textit{ in } K_0(A)$$

2. *$K_0(A) = \bigoplus_{i=1}^s \mathbb{Z}[V_i]$ is free of rank s.*
3. *The number of blocks of A, giving a direct sum decomposition into 2-sided ideals, $b(A) = rkK_0(A)$*

For a left semisimple ring A, the number of homogeneous components $\{A_i\}$ of the left regular module ${}_AA$ is finite, and A is their direct sum: $A = A_1 \bigoplus \cdots \bigoplus A_i$ These A_i are called the Wedderburn components of A, each is a 2-sided ideal in A, with $A_iA_{i'} = 0$ if $i \neq i'$. Moreover, each A_i is a simple Artinian ring. Each projective A-module $M \in \mathfrak{P}(A)$ decomposes into a direct sum $M = \bigoplus M_i$, where each $M_i \in \mathfrak{P}(A_i)$ is a projective A_i-module, and each $\mu \in \mathrm{Aut}\ M$ has a corresponding decomposition. Then,

$$K_1(A) \cong \bigsqcup_i K_1(A_i)$$

and the problem is reduced to the case of simple Artinian rings.

Let A be a simple Artinian Ring, and thus is Morita Equivalent to a division ring D, and as K_1 is unchanged under this equivalence,

$$K_1(D) \cong K_1(A)$$

Then $A = \mathrm{End}_D V$ where V is a finite dimension vector space over D. Then the isomorphism between projective D-modules and projective A-modules is given by mapping $M \in \mathfrak{P}(D)$ onto $V \otimes_D M \in \mathfrak{P}(A)$. The isomorphism follows from $\mathrm{Aut}_D M \cong \mathrm{Aut}_A(V \otimes_D M)$ by $[M, \mu] \to [V \otimes_D M, 1 \otimes \mu]$ for $M \in \mathfrak{P}(D)$, $\mu \in \mathrm{Aut}_D M$ and interpreting K_1 as maps from a free space to itself.

We now calculate $K_1(D)$, where D is a skewfield, making essential use of the Dieudonne Determinant. Let $D^\bullet = D - \{0\}$, the $D^\# = D^\bullet/[D^\bullet, D^\bullet]$ is an abelian multiplicative group.

4.1.10 Theorem: (K_1 and Dieudonne Determinant ([5] 38.32))

The Dieudonne determinant gives an isomorphism of groups

$$K_1(D) \cong D^{\#}$$

for every skewfield D.

Moreover, the proof will show that infact every element of $K_1(D)$ has the form $[D, d_r]$ for some $d \in D^{\bullet}$, where d_r represents right multiplication by d. Also, when R is a local ring (not necessarily commutative) Klingenberg has shown there is a well defined isomorphism det : $K_1(R) \cong (R^*)^{ab}$.

Proof:

For each ordered pair (M, μ), where $M \in P(D)$ and $\mu \in \mathrm{Aut}_D M$. Each $M \in \mathfrak{P}(D)$ has a finite D-basis, relative to which μ may be represented by a matrix $\mathbf{X} \in GL_n(D)$, where $n = \dim_D M$. Of course change of basis just replaces $\mathbf{X}$ by $\mathbf{TXT^{-1}}$ for some $\mathbf{T} \in GL_n(D)$. The Dieudonne determinant

$$\det\mathbf{X} = \det\mathbf{TXT^{-1}} \in D^{\#}$$

Setting det $\mu = \det\mathbf{X} \in D^{\#}$ this is well defined.

Consider an exact sequence of pairs (with morphisms coming from commuting square), with L, M, N being D-spaces.

$$0 \to (L, \lambda) \to (M, \mu) \to (N, \nu) \to 0$$

We may choose a D-basis of M so that the matrix $\mathbf{X}$ representing μ has the form

$$\mathbf{X} = \begin{pmatrix} \mathbf{X}_1 & * \\ 0 & \mathbf{X}_2 \end{pmatrix}$$

with $\mathbf{X}_1$ the matrix of λ, and $\mathbf{X}_2$ that of ν. Clearly, $\det\mathbf{X} = (\det\mathbf{X}_1)(\det\mathbf{X}_2)$, giving

$$\det\mu = (\det\lambda).(\det\mu)$$

Moreover, $\det\mu\mu' = (\det\mu).(\det\mu')$ for $\mu, \mu' \in \mathrm{Aut}\ M$. This gives a well-defined homomorphism

$$\det : K_1(D) \to D^{\#}$$

which maps $[M, \mu]$ onto $\det\mu$.

Surjectivity is clear. For injectivity suppose $\det\mu = 1$, with μ represented by some matrix $\mathbf{X}$. The we can write down products $\mathbf{P}, \mathbf{Q}$ of elementary matrices over D such that

$$\mathbf{PXQ} = \mathrm{diag}(1, \ldots, 1, d)$$

where $d \in [D^{\bullet}, D^{\bullet}]$. We now observe that every elementary matrx $\mathbf{E}$ represents the zero element of $K_1(D)$ as it may always be written in the form

$$\begin{pmatrix} \mathbf{I} & * \\ 0 & \mathbf{I} \end{pmatrix} \text{ or } \begin{pmatrix} \mathbf{I} & 0 \\ * & \mathbf{I} \end{pmatrix}$$

And we may think of it lying in the s.e.s.:

$$0 \to (X_1, 1) \to (X_2, *) \to (X_3, *) \to 0$$

with $[X_1, 1] = 0 = [X_2, 1]$ in $K_1(D)$. Finally, $\mathrm{diag}(1, \ldots, 1, d)$ represents the zero element in $K_1(D)$ since $d \in [D^{\bullet}, D^{\bullet}$ and $K_1(D)$ is abelian. Hence, $\det\mu = 1$ ensures $[M, \mu] = 0$ in $K_1(D)$ hence map is injective.

4.1.11 $K_1(R)$ for Non-Commutative Semilocal Rings

When R is a commutative ring, the existence of a simple determinant function and good control of the commutator gives a nice structure for $K_1(R)$, where the determinant map gives an obvious splitting

$$K_1(R) \cong SK_1(R) \text{ x } R^*$$

And since the commutators all occur as elementary matrices ([5], 40.25)

$$SK_1(R) \cong SL(R)/E(R)$$

However, there is a strong result due to Vaserstein which gives a good understanding of K_1 for R semilocal: when $R/J(R)$ is semisimple Artinian, where $J(R)$denotes the Jacobson Radical of R. We write $\overline{R}$ for the image, $R/J(R)$. Moreover, we say that R is *complete semilocal ring* if R is semilocal and complete with respect to the $J(R)$-adic filtration.

We first recall some background Lemmas:

4.1.12 Lemma: (Units in Semisimple Rings - [5] ex 40.1)

*Let A be a **s**emisimple ring, and suppose that $A = Ax + m$ for some left ideal M of A and some $x \in A$. Show that $x + m \in A^*$ for some $m \in M$.*

4.1.13 Lemma: (Units in Ring - [5] ex 5.2)

*An element x of a ring A is a **u**nit if and only if the image of x in $A/rad\,A$ is a **u**nit*

4.1.14 Lemma: ([5] 40.8)

For $\mathbf{X}, \mathbf{Y} \in GL(\Lambda)$, we have

$$\begin{pmatrix} \mathbf{X} & * \\ \mathbf{0} & \mathbf{Y} \end{pmatrix} = \begin{pmatrix} \mathbf{X} & \mathbf{0} \\ * & \mathbf{Y} \end{pmatrix} = \mathbf{XY} \textit{ in } K_1(\Lambda)$$

4.1.15 Lemma: (Elementary Matrices - [5] 40.22)

*Let $E_n(R)$ be an elementary matrix in $GL_n(R)$ is it is obtained from the identity by changeing one off-diagonal entry. Let $\mathbf{E}_{ij}(A) = \mathbf{I} + a\mathbf{e}_{ij} \in M_n(R)$ for $1 \leq i, j \leq n, i \neq j, a \in R$, where $\mathbf{e}_{ij}$ is the matrix unit with 1 at position (i, j) and zeroes elsewhere. Calculation gives $\{\mathbf{E}_{ij}(a)\}$ satisfy the **Steinberg Relations** for $1 \leq i, j, k, l \leq n$:*

$$\mathbf{E}_{ij}(a)\mathbf{E}_{ij}(b) = \mathbf{E}_{ij}(a + b) \text{ if } i \neq j$$

$$[\mathbf{E}_{ij}(a), \mathbf{E}_{kl}(b)] = 1 \text{ if } i \neq j, k \neq l, j \neq k, i \neq l$$

$$[\mathbf{E}_{ij}(a), \mathbf{E}_{jk}(b)] = \mathbf{E}_{ik}(ab) \text{ if } i, j, k \text{ distinct}$$

The third relation gives $[E_n(R), E_n(R)] = E_n(R)$ if $n \geq 3$, andso the direct limit, $E(R)$ is a perfect group (coincides with it's commutator subgroup). Whitehead's Lemma now states $E(R) = [GL(R), GL(R)]$ and is deduced from the following identity:

$$(*) \begin{pmatrix} \phi & 0 \\ 0 & \phi^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \phi^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \phi^{-1} - 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\phi & 1 \end{pmatrix}$$

which holds for each $\phi \in GL_n(R)$, hence $[GL_n(R), GL_n(R)] \subset E_{2n}(R)$ for all $n \geq 1$.

4.1.16 Theorem (Whitehead Group of Semilocal Rings ([5], 40.31)

Let R be a semilocal ring. Then the natural map $R^* \to K_1(R)$ is surjective, that is, each element of $K_1(R)$ is of the form (u) for some $u \in R^*$. Moreover, the kernel of this surjection is the subgroup of R^* generated by all expressions $(1+xy)(1+yx)^{-1}$ in R^* (for $x, y \in R$).

Proof:

Suppose that $R = L + Rb$ for some left ideal L of R and some $b \in R$. If bars denote reduction mod $J(R)$, then $\overline{R}$ is a semisimple ring, and $\overline{R} = \overline{L} + \overline{Rb}$.

By 4.1.12 there exists an element $y \in \overline{L}$ with $y + \overline{b} \in \overline{R}^*$. Therefore $x + b \in R^*$ for some $x \in L$, by 4.1.13. We shall use these facts below.

An n-tuple $\mathbf{a} = (a_1, \ldots, a_n)^t$ of elements of R is called *unimodular* if $\sum_{i=1}^n Ra_i = R$. If $\mathbf{X} \in GL_n(R)$, then from $\mathbf{X}^{-1}\mathbf{X} = \mathbf{I}$ it follows that every column of $\mathbf{X}$ is unimodular. If $\mathbf{x} = (x_1, \ldots, x_n)^t$ is the first column of $\mathbf{X}$, and $\mathbf{E}_{ij}(a)$ is as in 4.1.15, the first column of $\mathbf{E}_{ij}(a)\mathbf{X}$ is

$$(x_1, \ldots, x_{i-1}, x_i + ax_j, x_{i+1}, \ldots, x_n)^t$$

We write $\mathbf{x} \sim \mathbf{x}'$ if the vector $\mathbf{x}'$ can be obtained from $\mathbf{x}$ by a finite number of such elementary (row) operations. If $n \geq 2$ and $x_1 \in R^*$, it is clear from identity in 4.1.15 that

$$(x_1, \ldots, x_n)^t \sim (1, 0, \ldots, 0)^t$$

We use these ideas to show that when R is semilocal, every unimodular $\mathbf{x} =^t (x_1, \ldots, x_n)$ is equivalent to ${}^t(1, 0, \ldots, 0)$ if $n \geq 2$. By hypotheses, $Rx_1 + \cdots + Rx_n = R$; by the frst step in the proof we obtain

$$a_1x_1 + \cdots + a_{n-1}x_{n-1} + x_n \in R^*$$

for some elements $a_i \in R$. Then,

$$\mathbf{x} \sim (x_1, \ldots, x_{n-1}, a_1x_1 + \cdots + a_{n-1}x_{n-1} + x_n)^t \sim (1, 0, \ldots, 0)^t$$

as desired.

In matrix form, let $\mathbf{X} \in GL_n(R)$ with $n \geq 2$. By the above there exists a product $\mathbf{E}$ of elementary matrices such that

$$\mathbf{EX} = \begin{pmatrix} 1 & * \\ 0 & \mathbf{X}_1 \end{pmatrix} \text{ for some } \mathbf{X}_1 \in GL_{n-1}(R).$$

But then $\mathbf{X}$ and $\mathbf{X}_1$ represent the same element of $K_1(R)$, since from $(*)$, for $\mathbf{X}, \mathbf{Y} \in GL(R)$, we have

$$\begin{pmatrix} \mathbf{X} & * \\ 0 & \mathbf{Y} \end{pmatrix} = \begin{pmatrix} \mathbf{Y} & 0 \\ * & \mathbf{X} \end{pmatrix} = \mathbf{XY} \text{ in } K_1(R)$$

Continuing this procedure with $\mathbf{X}_1$ (if $n \geq 3$), we eventually obtain an element $\mathbf{X} = (u)$ in $K_1(R)$ for some $u \in R^*$. This completes the proof.

If R is semilocal, the surjection $R^* \to K_1(R)$ induces a surjection

$$R^*/[R^*, R^*] \to K_1(R).$$

If R is a skewfield, or more generally a local ring, this surjection is an isomorphism with inverse the Dieudonne determinant (rmks end 4.1.8).

4.2 Dennis Trace Map

4.2.1 Homotopy Definition of Dennis Trace Map

Writing $G = GL(R)$ I now define the D.T.M., $\delta : K_n(R) \to HH_n(R)$ as the composition of maps defined in 4.1.7, and $\chi : H_n(G,k) \hookrightarrow HH_n(kG)$ which is inclusion as a direct summand by 3.1.5.1.

$$\begin{array}{ccc} K_n(R) & \xrightarrow{\delta} & HH_n(R) \\ \| \text{ def.} & & \uparrow \phi \\ \Pi_n(BG^+) & & HH_n(kG) \\ \downarrow \text{"} * \text{"} & & \uparrow \chi \\ H_n(BG^+,k) & \xrightarrow{\cong} & H_n(G,k) \end{array}$$

In the case $n = 1$, we observe $K_1(R) = H_1(G,k)$, and indeed the composition is trivial, and so the D.T.M., δ is reduced to understanding inclusion in the direct sum, χ, and the fusion map, ϕ:

$$\delta = \phi \circ \chi : K_1(R) = H_1(G,k) \to HH_1(R)$$

This is all rather abstract so I now give an example as illustratation.

4.2.1.1 Theorem (Example of Laurent Polynomials):

Let k be a commutative ring, and $R = k[v, v^{-1}]$ the Laurent polynomial ring in one variable. From 3.1.3.3,

- $HH_n(R) = 0 \ \forall n > 1$
- $HH_1(R) =$ *free R-module on generator $\mathfrak{d}v$.*

Then the Dennis Trace map sends the class of $v \in R^X \subset K_1(R)$ (embedded as a 1 X 1-matrix to the Kahler differential $v^{-1}\mathfrak{d}v \in HH_1(R)$, the logarithmic derivative.

$$\begin{array}{rcl} \delta : R^X \subset K_1(R) & \to & HH_1(R) \\ v & \to & v^{-1}\mathfrak{d}v \end{array}$$

Proof

Let G be the infinite cyclic group with generator v, $G =< v >$. Then $G \subset R^{\mathrm{X}}$, and the group algebra $kG = R$, thus the fusion map, $\phi = i$, and to compute the D.T.M. on the class of $[v]$ we need only look at the assembly map χ.

In Hochschild Homology, $[v]$ is sent to the class of the cycle corresponding to v in the chain complex $C_1(G)$, the first term in the Hochschild Chain Complex, which is just $v^{-1} \otimes v$ where we are thinking of $HH_*(kG)$ as $H_*(G, kG)$ where G acts by conjugation - see claim in the proof of 3.1.5.1. Identification of $HH_1(R)$ with Kahler differentials gives: $[v] \to v^{-1}\mathfrak{d}v$.

The case of Laurent polynomials is infact a universal example for the whole of the commutative case, giving a simple result on the direct summands $R^{\mathrm{X}} \subset GL_1(R) \subset K_1(R)$, and SK_1, the kernel of the induced determinant map splitting $K_1(R) = R^{\mathrm{X}} \oplus SK_1(R)$. The proof uses functoriality of K-groups and Homology, and Morita Equivalence, together with the method of calculating the inverse of a matrix using adjoints, see [16] section 10.3.

4.2.1.2 Corollary (Commutative Case):

For R a commutative k-algebra, the D.T.M. vanishes on $SK_1(R)$, and sends any $g \in R^X \subset K_1(R)$ to it's logarithmic derivative, $g^{-1}\mathfrak{d}g \in HH_1(R)$

Putting these results together, and scaling a given matrix, g, to one of determinant 1, $\overline{g}$:

4.2.1.3 Corollary (D.T.M. as a Determinant):

For R commutative, splitting an element of $K_1(R)$, considered as $g \in GL(R)$ into the pair $(det\ g, \overline{g})$, the D.T.M., $\delta : g \to (det\ g)^{-1}\mathfrak{d}(det\ g)$.

In the non-commutative case, the same arguments give $\delta : K_1(kG) \to HH_1(kG) : M \to Tr(M^{-1} \otimes M)$. This tensor image still has the same derivative property, but the pair must now commute:

$$HH_1(R) = \{a \otimes b | ab = ba\}/(\sim)$$

This is indeed well defined since for N a commutator, $Tr(N^{-1} \otimes N) \equiv 0$.

The direct sum decomposition, 3.1.5.1 still holds, but we must be careful, even in the commutative case to separate out the additive structure of $\mathfrak{d}$ from the multiplicative structure of $\bigoplus_{g \in G} G$: $\Theta : (\det\ M)^{-1}\mathfrak{d}(\det\ M) \nrightarrow ((\det\ M)^{-1}, 1, \dots)$ as we must multiply out all pairs before applying Θ. This is best illustrated by an example.

Take $G = C_5 =< h >$, the cyclic group of order 5 generated by h. Then we may calculate image of element $g = \begin{pmatrix} i & h + h^2 \\ h & i + h^2 + h^4 \end{pmatrix}$, using that for $\lambda \in k$ such that $2.\lambda = 1$, we have $\lambda.(i+h).(i-h+h^2-h^3+h^4) = \lambda.2.i = \lambda$

$$\begin{aligned} \delta : K_1(kG) &\to HH_1(kG) \cong \bigoplus_{g \in G} G \\ g &\to (\det\ g)^{-1}\mathfrak{d}(\det\ g) = \lambda(i+h)\mathfrak{d}(i - h + h^2 - h^3 + h^4) \\ &\cong \lambda(i, h^{-1}, h^2, h^{-3}, h^4).(h^4, i, h^{-1}, h^2, h^{-3}) \\ &\cong \lambda(h^4, h^4, h, h^4, h) \\ &\cong (h^2, h^2, h^3, h^2, h^3) \in \bigoplus_{g \in G} G \end{aligned}$$

In the way the determinant function appears in the commutative case, the Dieudonne Determinant, with image consisting of cosets arises from the D.T.M. in the non-commutative case, so we are interested in elements occuring as determinants which are units in the group ring.

In general, identifying the units in a group ring is difficult, and only isolated theories exist. The simplest being:

4.2.1.4 Proposition (Splitting of Group Rings):

The units in the group ring, $(\mathbb{Z}G)^*$ splits up as a direct sum with the group G itself as one of the summands, using the following maps:

$$\begin{aligned} (\mathbb{Z}G)^* &\cong G \oplus \dots \\ 1.g &\leftarrow g \\ \sum n_g g &\to \prod g^{n_g} \end{aligned}$$

4.3 Zeta Functions for Novikov Rings

In [19], Schutz uses the one-parameter fixed point theory of Geoghegan and Nicas to get information about the closed orbit structure of transverse gradient flows of closed 1-forms on a closed manifold M. He defines a noncommutative zeta function in an object related to the first Hochschild homology group of the Novikov ring associated to the 1-form, essentially an image under the Dennis Trace Map and relates this to the torsion of a natural chain homotopy equivalence between the Novikov complex and a completed simplicial complex of $\overline{M}$, the universal cover of M.

Recall, from 4.2.1, for any ring R there is a Dennis trace homomorphism $DT : K_1(R) \to HH_1(R)$ from K-theory to Hochschild homology (in all dimensions, but only dimension 1 is used). They use a variant of DT,, the Dennis Trace map, denoted $\mathfrak{DT} : \overline{W} \to \widehat{HH_1(\mathbb{Z}G)}_\xi$. Here $\overline{W}$ is a subgroup of $K_1(\overline{\mathbb{Z}G}_\xi)$ containing the torsion of $\phi(v)$, and $\widehat{HH_1(\mathbb{Z}G)}_\xi$ is a completion of $HH_1(\mathbb{Z}G)$, see **??** which is related to the Hochschild homology of the Novikov ring by a natural homomorphism $\theta : HH_1(\widehat{\mathbb{Z}G_\xi}) \to \widehat{HH_1(\mathbb{Z}G_\xi)}$.

The main theorem says that if one applies this modified Dennis trace homomorphism to the torsion of the equivalence $\phi(v)$, one gets the (topologically important part of the) closed orbit structure of the flow induced by v in a recognizable form - that of a (noncommutative) zeta function. In other words the Dennis trace of the torsion equals the zeta function.

Theorem (DTM, Torsion Equivalence and Zeta Functions - [19] 1.1):

Let ω be a Morse form on a smooth connected closed manifold M^n. Let $\xi : G \to \mathbb{R}$ be induced by ω and let $C_*^\Delta(\overline{M})$ be the simplicial $\mathbb{Z}G$ complex coming from a smooth triangulation of M. For every transverse ω-gradient, ν there is a natural chain homotopy equivalence $\phi(v) : \widehat{\mathbb{Z}G_\xi} \otimes_{\mathbb{Z}G} C_*^\Delta(\overline{M}) \to C_*(\omega,\nu)$ whose torsion $t(\phi(v))$ lies in $\overline{W}$ and satisfies

$$\mathfrak{DT}(t(\phi(v))) = \zeta(-v)$$

I now ask if the same in true in the Iwasawa completed case - Does the completed Dennis Trace Map from **??**, performed on the characteristic elements in K-theory of [3] carry important information?

4.4 Dennis Trace Map on $K_1(\Lambda_G)$

Abelianisation and taking of inverse limits does not commute, hence considering ring $\Lambda_G = \varprojlim_{U \le G} \mathbb{Z}_p[G/U]$, $K_1(\Lambda_G) = K_1(\varprojlim \mathbb{Z}_p[G/U]) \neq \varprojlim K_1(\mathbb{Z}_p[G/U])$, but the universal defining property of $\varprojlim$ gives a map,

$$K_1(\Lambda_G) = K_1(\varprojlim \mathbb{Z}_p[G/U]) \to \varprojlim K_1(\mathbb{Z}_p[G/U])$$

It was shown in 3.3 that

$$HH_1(\Lambda_G) = HH_1(\varprojlim \mathbb{Z}_p[G/U]) = \varprojlim HH_1(\mathbb{Z}_p[G/U])$$

Moreover from 3.3.3, there exists a map $\prod_{\gamma,\text{conj. class}}(Z(\gamma))^{ab} \to \varprojlim HH_1(\mathbb{Z}_p[G/U]) = HH_1(\Lambda_G)$.
I now use these maps to build up a generalised trace map on K_1 of completed group algebras.
From 4.2.1 there exists a map $\delta_U : K_1(\mathbb{Z}_p[G/U]) \to HH_1(\mathbb{Z}_p[G/U])$ for these to induce a map

$$\delta : \varprojlim K_1(\mathbb{Z}_p[G/U]) \to \varprojlim HH_1(\mathbb{Z}_p[G/U])$$

We need to show the following square commutes for $U \le V$ giving projections $G/U \xrightarrow{\pi} (G/U)/(V/U) = G/V$:

$$\begin{array}{ccc} K_1(\mathbb{Z}_p[G/U]) & \xrightarrow{\delta_U} & HH_1(\mathbb{Z}_p[G/U]) \\ \downarrow \pi & & \downarrow \pi \\ K_1(\mathbb{Z}_p[G/V]) & \xrightarrow{\delta_V} & HH_1(\mathbb{Z}_p[G/V]) \end{array}$$

Equivalently, without loss of generality it is enough to show this holds when passing to a quotient group for finite G, $G \xrightarrow{\pi} G/U : g \to \overline{g}$:

$$\begin{array}{ccc} K_1(\mathbb{Z}_p[G]) & \xrightarrow{\delta} & HH_1(\mathbb{Z}_p[G]) \\ \downarrow \pi & & \downarrow \pi \\ K_1(\mathbb{Z}_p[G/U]) & \xrightarrow{\delta_U} & HH_1(\mathbb{Z}_p[G/U]) \end{array}$$

For $M \in M_n(\mathbb{Z}_p[G])$,

$$\begin{aligned}\pi \circ \delta(M) &= \pi(Tr(M^{-1}\mathfrak{d}M)) \\ &= Tr(\overline{M}^{-1}\mathfrak{d}\overline{M}) \text{ since Projection commutes with Trace and Inverses} \\ &= \delta_U \circ \pi(M)\end{aligned}$$

Thus the maps δ_U are consistent and we may define their inverse limit:

$$\delta = \varprojlim \delta_U : \varprojlim K_1(\mathbb{Z}_p[G/U]) \to \varprojlim HH_1(\mathbb{Z}_p[G/U])$$

This gives the definition of a Dennis Trace Map, $\Delta : K_1(\Lambda_G) \to \varprojlim HH_1(\Lambda_G)$ given by the composition:

$$\Delta : K_1(\Lambda_G) \to \varprojlim K_1(\mathbb{Z}_p[G/U]) \to \varprojlim HH_1(\mathbb{Z}_p[G/U]) = HH_1(\Lambda_G)$$

Thus, the amount of useful information this Dennis Trace Map can carry is limited by how much is lost in passing from $K_1(\Lambda_G)$ to $\varprojlim K_1(\mathbb{Z}_p[G/U])$.

We hope to define a characteristic element in this image - $\varprojlim HH_1(\mathbb{Z}_p[G/U])$, the inverse limits of direct sums. However, the following commutative diagram shows we are infact just dealing with a direct product:

Claim (DTM Commutes with Direct Products):

For Λ_G semi-local the following diagram commutes, considering appropriate sums formally. Let element of $K_1(\Lambda_G)$ be represented by $x \in (\Lambda_G)^*$.

$$\begin{array}{ccc} x \in \Lambda_G^* & \to & x^{-1}\mathfrak{d}x \\ \downarrow & & \downarrow \Theta \\ \varprojlim \overline{x_U} \in \varprojlim K_1([G/U]) & & \prod y \in \prod_{\gamma, conj.\ class\ in\ G} (Z_{(G)}(\gamma))^{ab} \\ \downarrow \delta_U & & \downarrow \\ \overline{x_U}^{-1}\mathfrak{d}\overline{x_U} & \xrightarrow{\Theta} & \varprojlim \bigoplus \overline{y} \in \varprojlim \bigoplus_{\gamma, conj.\ class\ in\ G/U} (Z_{(G/U)}(\gamma))^{ab} = HH_1(\Lambda_G) \end{array}$$

Thus to calculate image under Dennis Trace Map in the Semi-Local case it is enough to understand the image in the formal object $\prod_{\gamma,\text{conj. class in } G} (Z_{(G)}(\gamma))^{ab}$.

For this Trace Map to be useful we need it to be well defined on characteristic elements - elements of $K_1(\Lambda_G) \hookrightarrow K_1((\Lambda_G)_T)$ must vanish. I investigate this in 6.1.4, and show that infact the Hochschild homology id too corse, for the number theoretical situations we are interested in: localisation kills the whole group and not just elements induced from un-localised $HH_1(R) \to HH_1(R_S)$.

4.5 Factoring Dennis Trace Map using adjusted Logarithms

4.6 Classical Case and calculations of Bentzen-Madsen

In this section we consider the extension for $G = \mathbb{Z}_p$, look at the Hochschild homology, and following methods of [19] we calculate the image of representatives of $K_1(\Lambda_G)$ under the Dennis Trace Map, and ask if this carries significant information.

4.6.1 Calculation of Image of D.T.M. in $HH_1(\Lambda_G)$

Any Iwasawa module may be written (up to pseudo-isomorphism) as $\Lambda^r \oplus \bigoplus_{i=1}^s \Lambda/p^{m_i} \oplus \bigoplus_{j=1}^t \Lambda/F_j^{n_j}$, see [12], the corresponding characteristic element from [3] is just image under the natural inclusion

$$\begin{aligned}(\Lambda(G))^* &\hookrightarrow K_1(\Lambda(G)) \\ u &\to \begin{pmatrix} u & & \\ & 1 & \\ & & \ddots \end{pmatrix}\end{aligned}$$

1.3.8 tells us that $\Lambda_G \cong \mathbb{Z}_p[[T]]$, a power series in 1-varibale, and hence units,

$$(\Lambda_G)^* \cong \{a_0 + a_1 T + \dots | a_0 \in \mathbb{Z}_p^*, a_i \in \mathbb{Z}_p \ \forall i \geq 1\}$$

Since G is pro-p, Λ_G is local, hence semi-local, and the above map is onto. So to understand the image of $K_1(\Lambda_G)$ in $HH_1(\Lambda_G)$ under the Dennis Trace Map homomorphism, it is sufficient to understand where a power series, $F = a_0 + a_1 T + \dots | a_0 \in \mathbb{Z}_p^*$, gets mapped. I will freely interchange between the 2-different interpretations of Kahler differentials, see 3.1.6.
From 4.2.1, $\delta : K_1(kG) \to HH_1(kG) : M \to Tr(M^{-1} \otimes M)$, and after the natural inculsion above, the D.T.M. reduces to

$$F \to F^{-1}\mathfrak{d}F$$

Denoting the inverse of F in the power series ring, $\mathbb{Z}_p[[T]]$ by $G = b_0 + b_1 T + b_2 T^2 + \dots = a_0^{-1} - a_1 a_0^{-2}.T + \dots$ we have $F \xrightarrow{\delta} G\mathfrak{d}F$. We now simplify this using the derivative property:

$$\begin{aligned} T^2\mathfrak{d}(-) &= T\mathfrak{d}T(-) + T\mathfrak{d}(-)T \\ &= T\mathfrak{d}2T(-) \ (\mathbb{Z}_p \text{ commutative}) \end{aligned}$$

More generally $T^n\mathfrak{d}(-) = nT\mathfrak{d}T^{n-1}(-)$
We see,

$$\delta F = (\sum a_i T^i)\mathfrak{d}(\sum b_j T^j) = \sum_{j\geq 0, i \geq 1} T\mathfrak{d}(ia_i)b_j T^{i-1}T^j + a_0\mathfrak{d}\sum b_j T^j = \sum_{j \geq 0, i\geq 1} T\mathfrak{d}(ia_i b_j)T^{i+j-1}$$

Recall, $T = \gamma - 1$ for γ a generator of $G = \mathbb{Z}_p$.

$$\begin{aligned} \delta F &= (\gamma - 1)\mathfrak{d} \sum_{i+j-1=n, i\geq 0, j \geq 0} (ia_i b_j)(\gamma - 1)^n \\ &= (\gamma)\mathfrak{d} \sum_{i+j-1=n, i \geq 0, j\geq 0} (ia_i b_j) \sum_{k=0}^{n} (-1)^k \gamma^k \begin{pmatrix} n \\ k \end{pmatrix} \end{aligned}$$

The coefficient of γ^m on RHS, corresponding to coordinate of γ^{m+1} (once we have included contribution of γ from LHS to marker):

$$\begin{aligned} &= a_1 + (b_m - \begin{pmatrix} m+1 \\ m \end{pmatrix} b_{m+1} + \begin{pmatrix} m+2 \\ m \end{pmatrix} b_{m+2} - \begin{pmatrix} m+3 \\ m \end{pmatrix} b_{m+3} + \dots) + 2a_2 \dots \\ &= 1/(m!).a_1[(m).(m-1)\dots 1.b_m - (m+1).(m)\dots 2.b_{m+1} + (m+2).(m+1)\dots 3.b_{m+2} + \dots] + \dots \\ &= 1/(m!)[a_1.[F]^{(m)}|_{-1} + 2a_2.[F]^{(m)}|_{-1} + \dots] \\ &= 1/(m!)(a_1.[F^{(m)}]|_{-1} + 2a_2[T.F^{(m)}]|_{-1} + 3a_3[T^2.F^{(m)}]|_{-1} + \dots) \\ &= 1/(m!)[(a_1 + 2a_2 T + 3a_3 T^2 + \dots)F]^{(m)}|_{-1} \\ &= 1/(m!)[G'.F]^{(m)}|_{-1} \\ &= 1/(m!)[F'/F.F]^{(m)}|_{-1} \\ &= 1/(m!)[(\ln F)']^{(m)}|_{-1} \\ &= 1/(m!)[\ln F]^{(m+1)}|_{-1} \end{aligned} \tag{4.1}$$

So the coordinate of γ^{m+1} is simply given by $[\ln F]^{(m+1)}|_{-1}$, where $(-)^{(m)}$ denotes the m-th derivative with respect to T.

We now may rewrite the image, taking places corresponding to $0, 1, m \in \mathbb{Z}_p$ as:

$$\delta F = (0, (F(-1)/F'(G(-1)), \dots, [\ln F]^{(m+1)}|_{-1}, \dots) \in \prod_{\gamma \in \mathbb{Z}_p} (\mathbb{Z}_p)$$

It is now natural to ask about the kernel and image of this map as a way to deduce information on the structure of $K_1(\Lambda_G)$.

What is the Kernel of Dennis Trace Map, $\delta : K_1(\Lambda_G) \to HH_1(\Lambda_G)$ in the Classical Case?

Suppose $G'F^{(m)} = 0$ for all m, then analyticity gives $G'F =$ const. $= G'F(0) = -a_1 a_0^{-2} a_0 = -a_0^{-1} a_1$
Hence, $G' = -a_0^{-1} a_1 / F = -a_0^{-1} a_1 .G$, and so $(\ln G)' = -a_0^{-1} a_1$. Thus, for constants K, K',

$$\begin{aligned} G &= K'.e^{-a_0^{-1} a_1 T} \\ F &= K'.e^{a_0^{-1} a_1 T} \\ \text{Considering} \quad T &= 0, \\ F &= a_0 e^{a_0^{-1} a_1 T} \\ &= a_0 + a_1 T + \cdots + a_0.(a_1 a_0^{-1})^n/(n!).T^n + \dots \end{aligned}$$

So we have free choice over a_0, a_1 but then all other coefficients (and hence the power series itself) are determined. Thus kernel is isomorphic to $\mathbb{Z}_p^*$ x $\mathbb{Z}_p$ as sets.

"ln" surjects $1 + T\mathbb{Z}_p[[T]]$ onto $T\mathbb{Z}_p[[T]]$, and differentiating, $(\ln F)'$ surjects onto $\mathbb{Z}_p[[T]]$ (we may factor out $a_0 \in \mathbb{Z}_p^*$ constant term from each coefficient, since it cancels in F'/F and it is absorbed into the kernel, see above). So δ surjects onto $\prod_{\gamma \neq 0}(\mathbb{Z}_p)$. This recovers the structure of the units in Λ_G, or equivalently of $K_1(\Lambda_G)$ in this commutative case as $\mathbb{Z}_p^*$ x $\mathbb{Z}_p$ x $\mathbb{Z}_p$ x ... where the group of units corresponds to the constant term of the power series representing elements in Iwasawa Algebra being invertible.

Chapter 5

Connections

5.1 Dennis Trace Map and Integral Logarithms

5.2 Kernel and Cokernel of Dennis Trace Map

Chapter 6

Extensions

6.1 Localizations

[3] explains how characteristic elements lying in the Whitehead group of a Localized Iwasawa algebra code interesting information for non-commutative Iwasawa Theory.

In this section I consider whether the generalized Dennis Trace Maps discussed in **??** behaves well with respect to localization by the canonical Ore set, the image of the characteristic element under this map lying in the first Hochschild homology of a localized Iwasawa algebra. Hence, the study of how K_1 and HH_1 behave with respect to localization of the ring is relevant.

6.1.1 Commutative Localization

For R a commutative ring, let S be a subset of R which is multiplicatevly closed. The idea is to invert elements of S to give a ring R_S whose units are precisely the elements of S.

If S contains zero divisors, wjen we invert S, the elements of R which kill an element of S must be zero in R_S. Define the assasinator of S as follows:

$$\text{ass}(S) = \{x \in R : xs = 0 \text{ for some } s \in S\} = \bigcup_{s \in S} \text{ann}(s)$$

Then a localisation of R at S is a ring R_S with a homomorphism $\phi : R \to R_S$ (and if R_S exists it is unique up to isomorphism given these conditions)

1. $\phi(s)$ is a unit in R_S for all $s \in S$.
2. Every element of R_S can be written in the form $\phi(r)\phi(s)^{-1}$ for soome $r \in R$ and $s \in S$.
3. $\ker\phi = \text{ass}(S)$, so ϕ is not in general injective

The localisation may be thought of as a ring of fractions $r/s <-> (r, s)$ modulo the equivalence relation:

$$(r, s) \sim (u, v) \text{if and only if} rvt = uvt \text{ for some } t \in S$$

R-modules may be localised to $M \otimes_R R_S$ which we may think of as the set of equivalence classes $\{m/s : m \in M, s \in S\}$ in M x S modulo the equivalence:

$$(m, s) \sim (n, t) \text{ if and only if } mtu = nsu \text{ for some } u \in S$$

One of the most important properties of localization is that it is *exact*, meaning it takes exact sequences to exact sequences. Given $f : M \to N$ a map of R-modules, we define a map $f_S : M_S \to N_S$ by setting

$$f_S(m/s) = f(m)/s \text{ for all } m \in M, s \in S$$

6.1.2 Non-Commutative Localization

There are much tighter conditions on S for a localization to exist. Suppose a right localisation (considering right fractions rs^{-1}) exists, then for all $r \in R$, $s \in S$ the element $\phi(s)^{-1}\phi(R)$ must lie in R_S, and so is expressable as

$$\phi(s)^{-1}\phi(r) = \phi(a)\phi(b)^{-1}$$

Hence $\phi(rb - sa) = 0$ so there exists a $t \in S$ such that $r(bt) - s(at)$, and $bt \in S$ since S is mult. closed.

Definition (Ore Set)

A mult. closed set S is a right Ore set, if and only if for all $r \in R, s \in S$ there exist $r' \in R, s' \in S$ such that $rs' = sr'$.

This is automatically satisfied in Commutative rings.

I have already explained that if the right localisation exists, then S must be an Ore set, but conversely if S is an Ore set then $\text{ass}(S)$ is a two-sided ideal of R, and provided $\overline{S}$ consists of regular elements in $\overline{R} = R/\text{ass}(S)$ then R_S exists.

6.1.3 Localization of K_1-groups

We showed in 2.2.1.1 that the Iwasawa algebra is local (with unique maximal ideal the kernel of the composition of augmentation and reduction modulo p: $\Lambda_G \to \mathbb{Z}_p \to \mathbb{F}_p$), hence semilocal and the elements of K_1 are all realized as units, see 4.1.16.

Using that Localisation is exact, and a technical Lemma it is established in [3], Proposition 4.2, that $(\Lambda_G)_S$ is semi-local allowing us to again apply 4.1.16 so the elements of $K_1[(\Lambda_G)_S]$ are all realized as units.

However we are interested in characteristic elements lying in $K_1[(\Lambda_G)^*_S]$ where $S^* = \bigcup p^n S$. [3] explains that this Ring need no longer be semilocal. In False Tate Curve, $\Lambda_G = \mathbb{Z}_p[[U,V]]$ where $U+1$, $V+1$ correspond to fixed topological generators, and S is the complement of (p, U) in R. From above, R_S is a local ring of dimension 2. So $R_{S^*} = R_S[1/p]$ is a ring of dimension 1. We see for g any irreducible distinguished polynomial in $\mathbb{Z}_p[U]$, gR is a prime ideal, so gR_{S^*} is also prime ($S^* \cap gR$ is empty) giving R_{S^*} has infinitely many prime ideals, so is not semilocal.

However, the statement of Vasserstein's theorem on K_1 still holds.

Theorem (K_1 of Localisation of Completed Group Rings - see [3] 4.4):

Assume that G has no element of order p. Then the natural map,

$$(\Lambda_G)^*_{S^*} \to K_1[(\Lambda_G)_{S^*}] \text{ is surjective}$$

Kato shows, for G an open subgroup of the Galois group corresponding to the False Tate extension with associated finite group Δ (see **??**),

$$K_1(\mathbb{Z}_p[[G]]_{S^*}) \cong K_1(\mathbb{Z}_p[[G]]_S) \oplus \mathbb{Z}^{|\Delta|}$$

Thus the study of $K_1(\mathbb{Z}_p[[G]]_{S^*})$ is reduced to the study of $K_1(\mathbb{Z}_p[[G]]_S)$. Finally, after localising the ideals $A_{G,n}$ of **??** he produces a group $\phi_S(G)$, and a homomorphism ([11], 8.14),

$$\theta_{G,S} : K_1(\mathbb{Z}_p[[G]]_S) \to \phi_S(G)$$

which is conjectured to be bijective, and shows how this depends on the bijectivity of a simpler map.

6.1.4 Localization of HH_1-groups

Theorem (Localisation of Tor-groups - see [23] 8.7.3):

Let S be an Ore Set in ring R, and M and N two R-modules. Then

$$Tor_*^{R_S}(M_S, N_S) \cong Tor_*^R(M_S, N) \cong [Tor_*^R(M, N)]_S$$

Since the Hochschild homology may be considered as group homology with coefficients in group ring where group acts by conjugation we have

6.1.4.1 Corollary (Hochshild Homology of Localization of Group Rings):

Let S be an Ore Set in ring R, a k-algebra. Then

$$HH_*((kG_S)) \cong H_*^{(kG)_S}(\overline{((kG)_S)}) \cong [HH_*(kG)]_S$$

Corollary (Hochshild Homology of Localisation of Completed Group Rings):

$$\varprojlim_{U \leq G} [HH_*(k[G/U])]_S \cong \varprojlim_{U \leq G} HH_*((k[G/U])_S)$$

Thus understanding whether inverse limits commute with localization by S will lead to a connection between the object of interest, $HH_*((\Lambda_G)_S)$, above, and $(\varprojlim HH_*(k[G/U]))_S = [HH(\Lambda_G)]_S$ which is easy to calculate.

Localization and taking Inverse Limits does not Commute

However, Universal Property of Inverse Limit gives a map $S^{-1} \varprojlim R_U \to \varprojlim S^{-1} R_U$ (not necessarily onto).

Take $S = \{1, p, \dots\}$, and $R = \mathbb{Z}_p = \varprojlim_n \mathbb{Z}_p/p^n\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^i\mathbb{Z}$ (more generally, for F a Weierstrass polynomial, the injection $\mathfrak{O}[T] \hookrightarrow \mathfrak{O}[[T]]$ induces an isomorphism $\mathfrak{O}[T]/F\mathfrak{O}[T] \to \mathfrak{O}[[T]]/F\mathfrak{O}[[T]]$, see [12] 5.3.3).

Consider image of $m \in \mathbb{Z}/p^i\mathbb{Z}$ in $S^{-1}\mathbb{Z}/p^i\mathbb{Z}$. Certainly $p^i m = 0$, but $\frac{1}{p^i} \in S^{-1}$ so $\frac{1}{p^i}(p^i m) = 0$ hence $\overline{m} = 0 \in S^{-1}M \, \forall \, m \in \mathbb{Z}/p^i\mathbb{Z}$. This gives the result since

$$\varprojlim_i [S^{-1}(\mathbb{Z}/p^i\mathbb{Z})] = 0 \neq \mathbb{Q}_p = S^{-1} \varprojlim_i (\mathbb{Z}/p^i\mathbb{Z})$$

So we see that the Torsion elements map to zero, and non-torsion elements are not in kernel of Localization.

6.1.4.2 Proposition (Hochschild Homology of Localised Iwasawa Algebras of pro-p Groups):

For G a pro-p group, $HH_1((\Lambda_G)_T)$ is trivial, and thus by commuting diagram of 1.5.4, $HH_1((\Lambda_G)_{S^*})$ is trivial.

Proof

$$\begin{array}{rcl} HH_1[(\Lambda_G)_T] & = & HH_1[(\varprojlim \mathbb{Z}_p[G/U])_T] \\ \text{Localization commutes with the inverse limit} & & \text{(no torsion in group algebra)} \\ & = & HH_1[\varprojlim(\mathbb{Z}_p[G/U])_T] \\ & = & \varprojlim[HH_1(\mathbb{Z}_p[G/U])_T] \text{ see } \mathbf{??} \\ & = & \varprojlim[HH_1(\mathbb{Z}_p[G/U])]_T \text{ see 6.1.4.1} \\ & = & \varprojlim[\bigoplus_{\gamma \in \mathrm{ccl}(G/U)} [Z(\gamma)]^{\mathrm{ab}}]_T \end{array}$$

G is pro-p implies G/U is a p-group, all elements have order some power of p, and thus the subgroups $Z(\gamma)$ also have this property, meaning they are $T = \{1, p, p^2, \dots\}$-torsion and are killed when we localize - $HH_1[(\Lambda_G)_T] = \varprojlim \bigoplus_\gamma \overline{id} = \overline{id}$ in $\bigoplus_{\gamma \in \mathrm{ccl}(G/U)} [Z(\gamma)]^{\mathrm{ab}}$.

Appendix A

Derived Functors and Homology of Groups

A.1 Derived functors and Homology

A.1.1 Introduction to Derived Functors

Derived Functors are a central tool in Homological Algebra.

Given an additive functor $F : \mathbb{U} \mapsto \mathbb{B}$ from the abelian category $\mathbb{U}$ to the abelian category $\mathbb{B}$, we may form its left derived functors. $L_n F : \mathbb{U} \mapsto \mathbb{B}$. This is subject to the technical condition that $\mathbb{U}$ has enough projectives - that every object of $\mathbb{U}$ has a projective presentation.

$L_n F(A)$ is then a function depending on the two variables F and A. The key property is that these give rise to 2 Exact Sequences - arising from varying A and F respectively.

2 cases are of special interest to us:

1. We may study F as the tensor product $F_B : \mathbb{M} \mapsto \mathbb{U}$ where $F_B(A) = A \otimes_\Lambda B$, or equivalently, by symmetry of the tensor product $G_A : \mathbb{M} \mapsto \mathbb{U}$ where $G_A(B) = A \otimes_\Lambda B$. Which leads to $L_n F_B(A) \cong L_n G_A(B)$, which are both equal to the familiar functor Tor_n^Λ which measures by how much the functor $\otimes$ fails to be right exact.

2. Similarly, provided there are enough injectives we can study right derived functors. The functor Hom then leads to a generalisation of the Extension group to Ext_Λ^n. We will see how these groups degenerate to give cohomology groups of a group acting on one module, and we can then view these Ext-groups as a natural 2-module analogue of the standard cohomology groups. It is then natural ro study alternating products of their orders of the type used in Euler Characteristics of 1 module.

I aim to give a survey of the theorems in this area and how they fit together to form the machinery rather than quoting each proof.

A.1.2 Homotopy

A.1.3 Background

Let $\varphi, \psi : \mathbf{C} \mapsto \mathbf{D}$ be 2 chain maps between chain complexes. φ, ψ induce homomorphisms: $H(\mathbf{C}) \mapsto H(\mathbf{D})$ and it is an important question when they are the same.

Homotopy gives a sufficient relation that $\varphi_* = \psi_* : H(\mathbf{C}) \mapsto H(\mathbf{D})$

A.1.4 Definition

A homotopy $\Sigma : \varphi \mapsto \psi$ for chain maps $\varphi, \psi : \mathbf{C} \mapsto \mathbf{D}$ is a morphism of degree +1 of graded modules $\Sigma : \mathbf{C} \mapsto \mathbf{D}$ such that $\psi - \varphi = \delta\Sigma + \Sigma\delta$. ie. such that for all $n \in \mathbb{Z}$

$$\psi_n - \varphi_n = \delta_{n+1}\Sigma_n + \Sigma_{n-1}\delta_n$$

A.1.5 Theorem

If two chain maps $\varphi, \psi : \boldsymbol{C} \mapsto \boldsymbol{D}$ are homotopic then $H(\varphi) = H(\psi) : H(\boldsymbol{C}) \mapsto H(\boldsymbol{D})$.

Proof

Let $z \in Ker\delta_n$ be a cycle in C_n. If $\Sigma : \varphi \mapsto \psi$, then

$$(\psi - \varphi)z = \delta\Sigma z + \Sigma\delta z = \delta\Sigma z$$

since $\delta z = 0$. Hence $\psi(z) - \varphi(z)$ is a boundary in D_n, ie. $\psi(z)$ and $\varphi(z)$ are homologous.

The homotopy relation is very powerful because its additive structure behaves well with respect to chain maps and additive functors: it is an equivalence relation, and book keeping shows that:

1. If $\varphi \cong \phi : \mathbf{C} \mapsto \mathbf{D}$ and $\varphi' \cong \phi' : \mathbf{D} \mapsto \mathbf{E}$, then $\phi'\phi \cong \varphi'\varphi : \mathbf{d} \mapsto \mathbf{E}$
2. Let $F : \mathbb{M}_\Lambda \mapsto \mathbb{M}_{\Lambda'}$ be an additive functor. If $\mathbf{C}$ and $\mathbf{D}$ are chain complexes of Λ-modules and $\phi \cong \varphi : \mathbf{C} \mapsto \mathbf{D}$, then $F\phi \cong F\varphi : F\mathbf{C} \mapsto F\mathbf{D}$.
3. If $\phi \cong \varphi : \mathbf{C} \mapsto \mathbf{D}$ and if F is an additive functor, then $H(F\phi) = H(F\varphi) : H(F\mathbf{C}) \mapsto H(F\mathbf{D})$.

However, this description is only sufficient - I now quote an example from [8] to demonstrate it is not necessary:

A.1.6 Example

Take ring $\Lambda = \mathbb{Z}$.

C chain:

$$0 \to C_1 \to C_0 \to 0$$

$$0 \to (s_1) \to^{\delta : s_1 \mapsto 2s_0} (s_0) \to 0$$

Hence, $H_1(C) = 0$, $H_0(C) = \mathbb{Z}/2$

D chain:

$$0 \to D_1 \to 0$$

$$0 \to (t_1) \to 0$$

Hence $H_1(D) = 0$

Define chain map $\varphi : C \mapsto D : s_1 \mapsto t_1$. Both φ, and the Zero chain map $0 : C \mapsto D$ induce zero maps in homology. But they are not homotopic using the 3rd compatibility condition above. Take the functor to be $- \otimes \mathbb{Z}/2$, we obtain

$$\begin{array}{ccc} H_1(\varphi \otimes \mathbb{Z}/2) & \neq & H_1(0 \otimes \mathbb{Z}/2) \\ || & & || \\ 1 : \mathbb{Z}/2 \mapsto \mathbb{Z}/2 & & 0 : \mathbb{Z}/2 \mapsto \mathbb{Z}/2 \end{array}$$

A.1.7 Projective Resolutions

A.1.8 Definition

The positive chain complex $\mathbf{C} : \cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to 0$ (with $C_n = 0$ for $n < 0$) is **Projective** if C_n is projective for all $n \geq 0$, and is **Acyclic** if $H_n(\mathbf{C}) = 0 \,\forall\, n \geq 0$.

Equivalently, $\mathbf{C}$ is acyclic if and only if $\cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to H_0(\mathbf{C}) \to 0$ is **exact**.

A Projective Acyclic resolution $\mathbf{P}$ with an isomorphism $H_0(\mathbf{P}) \cong A$ is a **Projective Resolution of A**. This is a basic tool in the construction of derived functors. The following lemma shows why they are so commonly used.

A.1.9 Lemma

To every Λ-module A there exists a Projective Resolution.

Proof

Choose a projective presentation $0 \to R_1 \to P_0 \to A \to 0$ of A: then a projective presentation $0 \to R_2 \to P_1 \to R_1 \to 0$ of R_1, etc. Clearly, the complex

$$\mathbf{P} : \cdots \to P_n \to^{\delta_n} P_{n-1} \to \cdots \to P_0$$

where $\delta_n : P_n \to P_{n-1}$ is defined by $0 \to P_n \to R_n \to P_{n-1} \to 0$, is a projective resolution of A. Since it is projective, acyclic, and $H_0(\mathbf{P}) = A$.

This proof is instructive as every projective resolution arises in this way - as we may collapse down at any stage to a projective presentation. Thus, the existence of a projective presentation and projective resolution for all modules are equivalent.

A.1.10 Proposition

Two Projective Resolutions os A are canonically of the same homotopy type.

The proof builds on a result which, given a homomorphism between the zeroth homology of a projective chain complex and an acyclic chain complex builds a chain map, unique up to homotopy. We have both conditions in both chains here and we can apply the result twice in different directions to get uniqueness.

Later on we are using these resolutions in calculations, uniqueness up to homotopy ensures that groups cohomology groups of derived functors are well defined, see A.1.5.

A.1.11 Derived Functors

As mentioned in the introduction derived functors are a generalisation of the much older theories of Tor and Ext,

Let $T : \mathbb{M}_A \mapsto \mathbb{U}b$ be an additive covariant functor(in the case of Tor : $T_B(A) = A \otimes B$, and $Ext : G_B(a) = Hom(A, B)$). I will defines a sequence of functors $L_nT : \mathbb{M}_A \mapsto \mathbb{U}b$, $n = 0, 1, 2, \ldots$, these are the "*Left Derived Functors of T*".

I will build up this definition in stages:

1. For a Λ-module A, $\mathbf{P}$ a projective resolution of A we defined the abelian groups $L_n^{\mathbf{P}}T(A)$, $n = 0, 1, \ldots$ by considering the homology of the following complex:

$$T\mathbf{P} : \cdots \to TP_n \to TP_{n-1} \to \cdots \to TP_0 \to 0$$

and set

$$L_n^{\mathbf{P}}T(A) = H_n(T\mathbf{P}),\ n = 0, 1, \ldots$$

2. For T an additive functor, the homotopy η of any two projective resolutions **P,Q** of A induces an isomorphism $\eta : L_n^{\mathbf{P}}TA \cong L_n^{\mathbf{Q}}TA$, $n = 0, 1, \ldots$.

3. Hence, we are able to drop the superscript P and write L_nTA for $L_n^{\mathbf{P}}TA$ - we have free choice in calculations to use any projective resolution of A.

4. Finally, given $\alpha : A \mapsto A'$, we need to define induce homomorphisms $\alpha_* : L_nTA \mapsto L_nTA'$, $n = 0, 1, \ldots$. To do this take projective resolutions **P,P'** of A, A' and extend α to a chain map (using the same construction as in proving uniqueness of the projective resolution - both complexes as acyclic and projective) α(**P,P'**). This gives directly a map of derived functors,

$$\alpha_* : L_n^{\mathbf{P}}TA \mapsto L_n^{\mathbf{P'}}TA'$$

A.1.12 Examples

I now give two examples straight from the definition.

1. Recall, a covariant functor $T : \mathbb{M}_\Lambda \mathbb{U}b$ is right exact if, for any sequence $A' \to A \to A'' \to 0$ the sequence $TA' \to TA \to TA'' \to 0$ is exact. Considering $A \to A \oplus B \to B \to 0$ we see (using symmetry) that right exact functors are additive. As an example, $B \otimes_\Lambda -$ is additive.

2. If P is a projective Λ-module, then clearly, $\mathbf{P} : \cdots \to 0 \to P \to 0$ is a projective resolution of P, and taking homology of a functor T applied to this resolution gives:

$$L_nTP = 0 \text{ for } n = 1, 2, \ldots \text{ and } L_0TP = TP$$

A.1.13 Long Exact Sequences of Derived Functors

Firstly, in A.1.15 I will vary the object in $\mathbb{M}_A$ whilst keeping the functor fixed to deduce an exact sequence. But i need 2 preliminaries :the snake lemma for keeping track of kernels in commutative diagrams, and a lemma on existence of projective presentations.

A.1.14 Snake Lemma

Let the following diagram have exact rows.

$$\begin{array}{ccccccc} & & A & \xrightarrow{\mu} & B & \xrightarrow{\varepsilon} & C & \longrightarrow & 0 \\ & & \downarrow\alpha & & \downarrow\beta & & \downarrow\gamma & & \\ 0 & \longrightarrow & A' & \xrightarrow{\mu'} & B' & \xrightarrow{\varepsilon'} & C' & & \end{array}$$

Then there is a "connecting homomorphism" $\omega : ker\gamma \mapsto coker\alpha$ such that the following sequence is exact:

$$ker\alpha \xrightarrow{\mu_*} ker\beta \xrightarrow{\varepsilon_*} ker\gamma \xrightarrow{\omega} coker\alpha \xrightarrow{\mu'_*} coker\beta \xrightarrow{\varepsilon'_*} coker\gamma$$

If μ is monomorphic, so is μ_*; if ε' is epimorphic, so is ε'_*.

Let $c \in ker\gamma$, we choose $b \in B$ with $\varepsilon b = c$. Since $\varepsilon'\beta b = \gamma\varepsilon b = \gamma c = 0$ there exists $a' \in A'$ with $\beta b = \mu' a'$. Define $w(c) = [a']$, the coset of a' in $coker\alpha$. This clearly has the required properties and prove uniqueness by contradiction.

A.1.15 Projective Presentation Lemma

To a short exact sequence $A' \rightarrowtail \varphi A \twoheadrightarrow^{\phi} A''$ and to projective presentations $\varepsilon' : P' \twoheadrightarrow A'$ and $\varepsilon'' : P'' \twoheadrightarrow A''$ there exists a projective presentation $\varepsilon : P \twoheadrightarrow A$ and homomorphisms $\iota : P' \to P$ and $\pi : P \to P''$ such that the *following diagram is commutative with exact rows*

$$\begin{array}{ccccc} P' & \rightarrowtail^{\iota} & P & \twoheadrightarrow^{\pi} & P'' \\ \downarrow \varepsilon' & & \downarrow \varepsilon & & \downarrow \varepsilon'' \\ A' & \rightarrowtail^{\varphi} & A & \twoheadrightarrow^{\phi} & A'' \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

The proof is constructive, taking $P = P' \oplus P''$, and uses the canonical injection of P' into P and the canonical projection of P onto P'' as well as the property that P'' is projective to give the existence of a compatible function to construct ε.

A.1.16 Theorem - L.E.S. from varying the object

Let $T : \mathbb{M}_A \mapsto \mathbb{U}b$ be an additive functor and let $A \rightarrowtail^{\alpha'} A \twoheadrightarrow^{\alpha''} A''$ be a short exact sequence. Then there exist connecting homomorphisms

$$\omega_n : L_nTA'' \to L_{n-1}TA',\ n = 1, 2, \ldots$$

such that the following sequence is exact:

$$\cdots \to L_nTA' \to^{\alpha'_*} L_nTA \to^{\alpha''_*} L_nTA'' \to^{\omega_n} L_{n-1}TA' \to \ldots$$

$$\cdots \to L_1TA'' \to^{\omega_1} L_0TA' \to^{\alpha'_*} L_0TA \to^{\alpha''_*} L_0TA'' \to 0.$$

Proof

The Projective presentation Lemma, A.1.15 is precisely what we need to construct the following commutative diagram with exact rows, where P'_0, P_0, and P''_0 are projective:

$$\begin{array}{ccccc} P'_0 & \rightarrowtail & P_0 & \twoheadrightarrow & P''_0 \\ \downarrow \varepsilon' & & \downarrow \varepsilon & & \downarrow \varepsilon'' \\ A' & \rightarrowtail^{\alpha'} & A & \twoheadrightarrow^{\varepsilon''} & A'' \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0 \end{array}$$

$P_0 = P'_0 \oplus P''_0$ (by construction), and using snake lemma A.1.14, the kernel sequence is exact: $ker\varepsilon' \rightarrowtail ker\varepsilon \twoheadrightarrow ker\varepsilon''$. Repeat this procedure replacing S.E.S. of the theorem with S.E.S. of kernels. Induction gives the exact sequence of complexes:

$$\mathbf{P} \rightarrowtail^{\alpha'} \mathbf{P} \twoheadrightarrow^{\alpha''} \mathbf{P}''$$

For each $n \geq 0$, $P_n = P'_n \oplus P''_n$, and T is additive hence we get short exactness of $0 \to T\mathbf{P}' \to T\mathbf{P} \to T\mathbf{P}'' \to 0$. Then the construction of long exact homology sequence of complexes yields maps $\omega_n : H_n(T\mathbf{P}'') \to H_{n-1}(T\mathbf{P}')$ and also the exactness of the sequence. Homotopy considerations give independence from choice of resolution and chain maps.

There is a similar theorem for varying the functor. To quote it precisely we need first a definition.

A.1.17 Definition

A sequence $T' \to^{\varepsilon'} T \to^{\varepsilon''} T''$ of additive functors $T', T, T'' : \mathbb{M}_\Lambda \to \mathbb{U}b$ and natural transformations $\varepsilon'.\varepsilon''$ is called *exact on projectives* if, for every projective Λ-module P, the sequence $0 \to T'P \to^{\varepsilon'_P} TP \to^{\varepsilon''_P} T''P \to 0$ is exact.

A.1.18 Theorem - L.E.S. from varying the functor

Let the sequence of additive functors $T' \to^{\varepsilon'} T \to^{\varepsilon''} T''$ of additive functors $T', T, T'' : \mathbb{M}_\Lambda \to \mathbb{U}b$ be exact on projectives. Then, for every Λ-module A, there are connecting homomorphisms $\omega_n : L_n T'' A \to L_{n-1} T' A$ such that the sequence

$$\begin{aligned} \ldots \quad &\to L_n T' A \to^{\varepsilon'} L_n T A \to^{\varepsilon''} L_n T'' A \to^{\omega_n} L_{n-1} T' A \to \ldots \\ \ldots \quad &\to L_1 T'' A \to^{\omega_1} L_0 T' A \to^{\varepsilon'} L_0 T A \to \varepsilon'' L_0 T'' A \to 0 \end{aligned}$$

is exact.

Proof

Choose a projective resolution $\mathbf{P}$ of A, then exactness on projectives gives exactness of $0 \to T'\mathbf{P} \to^{\varepsilon'_\mathbf{P}} T\mathbf{P} \to^{\varepsilon''_\mathbf{P}} T''\mathbf{P} \to 0$ and applying Long Exact Homology Sequence of Complexes gives the result.

A.1.19 Application: the functor Ext^n_Λ

A.1.20 Definition

The abelain groups $Ext^n_\Lambda(A, B)$ are calculated as follows:

- Choose a projective resolution $\mathbf{P}$ of A.
- Form complex $Hom_\Lambda(\mathbf{P}, B)$
- Take cohomology to find Ext^n_Λ groups

ie. Consider the right derived functors of the additive covariant functor $Hom_\Lambda(-, B)$, and define:

$$Ext^n_\Lambda(-, B) = R^n(Hom_\Lambda(-, B)),\ n = 0, 1, 2, \ldots$$

Recall, the definition of $Ext(A, B)$ of extensions of A by B as being the abelian group which gives exactness of
$\cdots \to Hom_\Lambda(P_0, B) \to Hom_\Lambda(R_1, B) \to Ext_\Lambda(A, B) \to 0$
where $R_1 \rightarrowtail P_0 \twoheadrightarrow A$ is any projective presentation of A.

From the right derived functor analogue of L.E.S. A.1.16 we have
$\cdots \to Ext^0_\Lambda(P_0, B) \to Ext^0_\Lambda(R_1, B) \to Ext^1_\Lambda(A, B) \to 0$

Observing that $Ext^0_\Lambda = Hom_\Lambda(A, B)$ (since the functor Hom is left exact) we get

$$Ext^1_\Lambda(A, B) \cong Ext_\Lambda(A, B)$$

and the notation is justified.

The following characterises projective modules and the functor Ext^n_Λ:

A.1.21 Proposition

The following are equivalent:

1. A is a projective Λ-module.
2. $Hom_\Lambda(A, -)$ is an exact functor.
3. $Ext^i_\Lambda(A, B)$ vanishes for all $i \neq 0$ and all B. In other words, A is $Hom_\Lambda(-, B)$-acyclic for all B.
4. $Ext^1_\Lambda(A, B)$ vanishes for all B.

$1 \implies 4$ Using equivalence of Ext^1 and Ext: $Ext_\Lambda(A,B)$ is in 1-1 correspondence with the set $E(P,B)$, consisting of classes of extensions of the form $B \rightarrowtail E \twoheadrightarrow A$. But from universal property of projective A, short exact sequences of this form split. Hence $E(A,B)$ consists of just one element, the zero element.

$4 \implies 3$ The key here is that the vanishing of the first Ext group applies to all modules B (if it was just a particular one it is conceivable to continue the exact sequence back with a series of isomorphisms). This allows us to use dimension shifting, where we truncate a given projective resolution so that the second cohomology group of the original resolution is calculated by the first of the new one, and so by induction all Ext groups $(n \geq 0)$ are calculated from the first. Let

$$\mathbf{P} : \cdots \to P_2 \to_{a_2} P_1 \to_{a_1} P_0 \twoheadrightarrow_{a_0} B \to 0$$

be a projective resolution of B. For calculations, consider

$$\cdots \to Hom_\Lambda(A,P_2) \to_{a_{2*}} Hom_\Lambda(A,P_1) \to_{a_{1*}} Hom_\Lambda(A,P_0) \to 0$$

hypothesis (4) gives that

$$H_1(\mathbf{P}) = Ext^1_\Lambda(A,B) = \frac{ker\{Hom(A,P_2) \to Hom(A,P_1)\}}{im\{Hom(A,P_1) \to Hom(A,P_0)\}} = \frac{ker\, a_{1*}}{im\, a_{2*}} = 0$$

.

We can amend the projective resolution to one of P_0/B which preserves the same maps:

$$\cdots \to P_2 \to_{a_2} P_1 \twoheadrightarrow_{a_1} P_0/B \to 0$$

Then we can calculate, by hypothesis (4) the first homology of **P**',

$$H_1(\mathbf{P'}) = Ext^1_\Lambda(A, P_0/B) = 0 = \frac{ker\, a_{2*}}{im\, a_{3*}} = Ext^2_\Lambda(A,B)$$

. Similarly by induction, we have $Ext^i_\Lambda(A,B) = 0$ for all $i \geq 0$, hence (3).

$3 \implies 2$ Apply L.E.S. A.1.16 to $M' \rightarrowtail M \twoheadrightarrow M''$. All higher terms vanish by (3), and so L.E.S. collapses to $Hom_\Lambda(A,M') \rightarrowtail Hom_\Lambda(A,M) \twoheadrightarrow Hom_\Lambda(A,M'')$ and hence functor $Hom_\Lambda(A,-)$ is exact.

$2 \implies 1$ Given that $Hom_\Lambda(A,-)$ is exact and that we are given a surjection $g : B \rightarrowtail C$ and a map $\gamma : A \to C$. We can lift $\gamma \in Hom_\Lambda(A,C)$ to $\beta \in Hom_\Lambda(A,B)$ such that $\gamma = g_*\beta = g \circ \beta$ because g_* is onto. Thus A has the universal lifting property, hence A is projective.

A.1.22 Examples

I now give 3 explicit calculations working over the ring $\mathbb{Z}$, and viewing abelian groups as $\mathbb{Z}$-modules.

1. Let $A = \mathbb{Z}/p$ then we have the resolution

   $$0 \to \mathbb{Z} \to^p \mathbb{Z} \to \mathbb{Z}/p \to 0$$

   Since $Hom(\mathbb{Z},B) \cong B$ we see that to calculate $Ext^*(\mathbb{Z}/p,B)$ we need to take the cohomology of $0 \to B \to^p B \to 0$. Hence,

   $$Ext^n_\mathbb{Z} = \begin{cases} pB & \text{if } n = 0 \\ B/pB & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}$$

2. We know, embedding B in an injective abelian group I^0 and taking its quotient I^1 (which is divisible, and hence again exact) we get that
   $\mathbf{Ext^n_\mathbb{Z}(A,B) = o}$ **for** $n \geq 2$ **and all abelian groups A,B**.

   We can calculate the small terms exactly since every finite abelian group may be expressed in the form $B \cong \mathbb{Z}^m \oplus \mathbb{Z}/p_1 \oplus \cdots \oplus \mathbb{Z}/p_n$. And hence we may calculate the Ext groups of any pair of abelian groups.

3. Now take $B = \mathbb{Z}$. We have the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. This gives $Ext^0_\mathbb{Z}(A,\mathbb{Z}) = 0$ and $Ext^1_\mathbb{Z}(A,\mathbb{Z}) = A^* = Hom(A,\mathbb{Q}/\mathbb{Z})$, the Pontrajagin dual.

A.1.23 Ext^n_Λ where $n \geq 1$ viewed as extensions

Recall, from A.1.20 that $Ext_\Lambda(A,B) = Ext^1_\Lambda(A,B)$ can be interpreted as the group of equivalence classes of extensions. I would like to quote Yoneda's ideas on how to expand this idea using "n-extensions".

An n-extension of A by B is an exact sequence $\mathbf{E} : 0 \to B \to E_n \to \cdots \to E_1 \to A \to 0$ of Λ-modules.

So above, an extension is a 1-extension. We now need a notion of equivalence of n-extensions. We first define a non-symmetric relation for $n \geq 2$. We shall say that the n-extensions **E,E'** satisfy the relation $\mathbf{E} \rightsquigarrow E'$ if there is a commutative diagram:

$$\begin{array}{ccccccccccccc} \mathbf{E}: & 0 & \to & B & \to & E_n & \to \cdots \to & E_1 & \to & A & \to & 0 \\ & & & \| & & \downarrow & & \downarrow & & \| & & \\ \mathbf{E}': & 0 & \to & B & \to & E'_n & \to \cdots \to & E'_1 & \to & A & \to & 0 \end{array}$$

We now induce an equivalence relation. We say **E is equivalent to** E':

$$E \sim E' \iff \exists \text{ a chain } E = E_0, E_1, \ldots, E_k = E' \text{ with } E_0 \rightsquigarrow E_1 \leftrightsquigarrow E_2 \rightsquigarrow \cdots \leftrightsquigarrow E_k$$

Denoting [E] for the equivalence class of the n-extension E, and $Yext^n_\Lambda(A,B)$ for the set of such classes of n-extensions it can be shown that $Yext^n_\Lambda(-,-)$ is a bifunctor and that there is a natural equivalence of set valued bifunctors:

$$Yext^n_\Lambda(-,-) \cong Ext^n_\Lambda(-,-)$$

Giving a more tangible interpretation of the higher Ext groups.

A.2 Group Cohomology

The next two sections study cohomology of groups and algebras. A reference for this is Lazards paper [Lz]. I recall the key points i will need for my calculations.

Following the treatment in [8] I shall define, for an abstract group G (over the integers) the homology groups $H_n(G,B)$ and the cohomology groups $H^n(G,B)$, for $n \geq 0$, where A is a left, and B a right G-module. This follows from the theory of derived functors, taking the group ring $\mathbb{Z}G$ as the "ground ring" Λ".

We begin by giving a precise defintion of the group ring, which we define by giving its underlying abelian group and specifying the composition of 2 elements. Note, in this chapter all groups are written multiplicatively.

A.2.1 Group Ring $\mathbb{Z}G$

A.2.2 Definition

The **Integral Group Ring** $\mathbb{Z}G$ of G has as it's underlying abelian group the free abelain group with basis the set of elements of G, and the product of 2 basis elements is induced from the product in G.

ie.

- Elements are of the form $\sum_{x\in G} m(x)\, x$ (where $m : G \mapsto \mathbb{Z}$ is zero on all but a finite number of elements of G).
- Multiplication is given by

$$\left(\sum_{x\in G} m(x)\, x\right) . \left(\sum_{y\in G} m'(y)\, y\right) = \sum_{x,y\in G} (m(x).m'(y))xy$$

Consider the natural embedding

$$i : G \mapsto \mathbb{Z}G : g \mapsto \sum m(x)\, x \text{ where } m(x) = \begin{cases} 1 & \text{if } x = g \\ 0 & \text{otherwise} \end{cases}$$

I now give a universal property which will be important when we come to generalise group cohomology to Lie algebras.

A.2.3 Proposition: Extension of Nice Multiplicative Functions

Let R be a ring. Given any function $f : G \mapsto R$ which interacts well with the multiplicative structure of the group and ring such that: $f(xy) = f(x).f(y)$ and $f(1_G) = 1_R$ then there exists a unique ring homomorphism $f' : \mathbb{Z}G \mapsto R$ such that $f'i = f$. Clearly

$$f'(\sum_{x \in G} m(x)\, x) = \sum_{x \in G} m(x) f(x)$$

is the only possible choice and this commutes.

This proposition gives an interpretation of a G-module A as a $\mathbb{Z}G$-module (which is a G-module via the embedding i) and so we may use these terms interchangeably.

A.2.4 Augmentation Map

Consider the trivial map from G into integers $\mathbb{Z}$, given by $x \in G \mapsto 1 \in \mathbb{Z} \forall x \in G$. A.2.3 gives a unique ring homomorphism $\varepsilon : \mathbb{Z}G \mapsto \mathbb{Z}$, the augmentation of $\mathbb{Z}G$ defined by:

$$\varepsilon \left(\sum_{x \in G} m(x)\, x \right) = \sum_{x \in G} m(x)$$

Denote the kernel of ε, called the **Augmentation Ideal of G** by IG. It is central to the theory of group cohomology, see for example A.2.10.

A.2.5 Construction of (Co)Homology Groups

For a left G-module A, thinking of $\mathbb{Z}$ as a trivial G-module, we define the **n-th cohomology group of G with coefficients in A** by

$$H^n(G, A) = Ext^n_G(\mathbb{Z}, A)$$

And dually, for a right G-module B, the homology groups are

$$H_n(G, A) = Tor^G_n(B, \mathbb{Z})$$

From now on i will concentrate on calculations relating to cohomology although for each of the results there is an analogous dual result for homology.

A.2.6 Explicit Calculation

- Take a G-progective resolution $\mathbf{P}$ of the trivial (left) module $\mathbb{Z}$.
- Form the complexes $Hom_G(\mathbf{P}, A)$ and $B \otimes_G \mathbf{P}$.
- Compute their homology

As an example i give an explicit description of a resolution of $\mathbb{Z}$ over our given group, known as the **Homogeneous Bar Resolution**. We need to define G-modules to form the projection:

1. Let $\overline{B_n}$, $n \geq 0$ be the free abelian group on all $(n+1)$-tuples $(Y_0, y_1, \ldots, y_n)$, $y_i \in G$. Define the left G-module strucure in $\overline{B_n}$ by

$$y(y_0, y_1, \ldots, y_n) = (yy_0, yy_1, \ldots, yy_n)\, y \in G$$

Hence $\overline{B_n}$ is a free G-module, with base consisting of the $(n+1)$-tuples $(1, y_1, \ldots, y_n)$.

2. Define the differential maps of the sequence

$$\mathbf{\overline{B}} : \cdots \to \overline{B_n} \to^{\rho_n} \overline{B_{n-1}} \to \cdots \to \overline{B_1} \to^{\rho_1} \overline{B_0}$$

by the boundary formula

$$\rho_n(y_0, y_1, \ldots, y_n) = \sum_{i=0}^{n} (-1)^i (y_0, \ldots, \widehat{y_i}, \ldots, y_n)$$

3. It is easily seen that $\rho_{n-1}\rho_n = 0$ for $n \geq 2$ and recalling the augmentation map ε we also have $\varepsilon\rho_1 = 0$.

4. Explicit calculation, or contracting homotpoies from topology give that

$$\cdots \to \overline{B_n} \to^{\rho_n} \overline{B_{n-1}} \to \cdots \to \overline{B_1} \to^{\rho_1} \overline{B_0} \to^{\varepsilon} \mathbb{Z} \to 0$$

is a free G-resolution of $\mathbb{Z}$.

A.2.7 Exact Sequences

From the long exact sequences associated to derived functors, see A.1.16 we have, for a short exact sequence $A' \rightarrowtail A \twoheadrightarrow A''$ of G-modules there exists a long exact cohomology sequence:

$$0 \to H^0(G, A') \to H^0(G, A) \to H^0(G, A'') \to H^1(G, A') \to \ldots$$
$$\cdots \to H^n(G, A') \to H^n(G, A) \to H^n(G, A'') \to H^{n+1}(G, A') \to \ldots$$

The following are other properties of Group Cohomology easily deduced from the theory of derived functors:

A.2.8

If A is injective then $H^n(G, A) = 0$ for all $n \geq 1$.

A.2.9

Hence, if $A \rightarrowtail I \twoheadrightarrow A'$ is an injective presentation of A, then the long exact sequence collapses to give

$$H^{n+1}(G, A) \cong H^n(G, A') \text{ for } n \geq 1$$

A.2.10 Changing the Group Acting

I will show $H^n(G, A) \cong Ext_g^{n-1}(IG, A)$ hence we can equally well work with the group or its augmentation ideal. This is a special case of a more general shifting technique:

Lemma

Let $0 \to K \to P_k \to^{\phi} \cdots \to P_0 \to \mathbb{Z} \to 0$ be an exact sequence of left G-modules with $P_0, \ldots, P_k$ projective. Then the following sequence is exact and specifies the cohomology groups of G:

1. *$Hom_G(P_k, A) \to Hom_G(K, A) \to H^{k+1}(G, A) \to 0$*
2. *$H^n(G, A) \cong Ext_G^{n-k-1}(K, A)$ for $n \geq k + 2$*

Proof

1. This is just the definition of H^{k+1} as the coker of the induced map $Hom(P_k, A) \mapsto Hom(K, A)$.

2. Let $\cdots \to B_1 \to B_0 \to^{\varphi} K \to 0$ be a projective resolution of K used to calculate $Ext_G^{n-k-1}(K, A)$. Then, since $\mathbf{Im}\varphi = \mathbf{K} = \mathbf{ker}\phi$ we can piece together with our original resolution to get a new projective resolution of $\mathbb{Z}$:

$$\ldots B_{n-k-2} \to^{\alpha} B_{n-k-1} \to^{\beta} B_{n-k} \to \cdots \to B_0 \to P_k \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

   Then,

$$Ext_G^{n-k-1}(K, A) = ker\beta/im\alpha = H^n(G, A)$$

   as required.

In particular, using the resolution $0 \to IG \to \mathbb{Z}G \to \mathbb{Z} \to 0$, we have, for $n \geq 2$

$$H^n(G, A) \cong Ext_G^{n-1}(IG, A)$$

Appendix B

Homology of Profinite Groups

B.1 Introduction

In this section we define a cohomology for profinite groups, G, which recovers the usual group cohomology when G is finite and see how this profinite cohomology is built up from finite pieces in B.4.

I then consider completing the ground ring, and then define a cohomology - this is achieved using the completed tensor product in B.5, and i relate back to the cohomology of profinite groups in a surprisingly simple way in B.6.

In B.6.2 I give a duality result for "Tor" and"Ext" groups over completed rings, and together with the usual application of Pontryagin duality the theories of homology and cohomology are dual, and I use them interchangeably.

These techniques give the ground work for **??** where I demonstrate the Hochschild Homology of the completed (Iwasawa) algebras as a completion of the Hochschild homologies of the finite group ring.

B.2 Definition of Cohomology

I outline key constructions without proof - see ([23], 6.11) or ([20]) for details.

For G profinite, an appropriate notion of module is that of "discrete G-module. A discrete G-module is a G-module A such that when A is given the discrete topology, the action of G on A: the multiplication map $G \,\mathrm{x}\, A \to A$ is continuous.

The cohomology groups $H^*(G, A)$ of a profinite group G with coefficients in a discrete G-module A are the right derived functors of the functor $\mathfrak{D} \to \mathrm{Ab}$ sending A to A^G, applied to A.

Thus $H^0(G, A) = A^G$, and so when G is a finite group, dimension shifting arguments give $H^*(G, A)$ agree with usual group cohomology.

Many properties are inherited - $H^*(G, A)$ is contravariant in G via restriction maps, H^* is functorial, and when the map $G \to G/H$ is continuous (when H is a closed normal subgroup of G) we have inflation maps

$$H^*(G/H, A^H) \to H^*(G, A)$$

Cochains and Cocycles

I now give a resolution to calculate these groups.

If A is a discrete G-module, let $C^n(G, A)$ denote the set of continuous maps from G^n to A (defining $C^0(G, A) = A$) - i.e. the maps $\phi : G^n \to A$ which are locally constant - each point in G^n has a neighbourhood on which ϕ is constant.

Endowed with pointwise addition, $C^n(G, A)$ is an abelian group. Moreover, it is easily seen that $C^n(G, A) = \varinjlim_U C^n(G/U, A^U)$ where U runs through all open normal subgroups of G. This gives the key theorem:

Theorem (Definition of Cohomology - [23], 6.11.13):

Let G be a profinite group and A a discrte G-module. Then,

$$\begin{aligned} H^*(G,A) &\cong H^*(C^*(G,A)) \\ &\cong \varinjlim_{U} H^*(G/U, A^U) \end{aligned}$$

B.3 Induction

Recall, a G-module A is called **a**cyclic if $H^n(G,A) = 0$ for all $n > 0$. A is called **cohomologically trivial/ welk/flasque** if

$$H_n(H,A) = 0$$

for all closed subgroups, $H < G$, and all $n > 0$.

For our purposes the most important examples of cohomologically trivial G-modules are those that are **i**nduced, given by

$$\mathrm{Ind}_G(A) = \mathrm{Map}(G,A)$$

where A is any G-module.

The most natural definition is as continuous functions $x : G \to A$, with the discrete topology on A, and where $\sigma \in G$ acts via $(\sigma x)(\tau) = \sigma x(\sigma^{-1}\tau)$. We will use the isomorphism:

$$\mathrm{Ind}_G(A) \cong A \otimes \mathbb{Z}_p[G]$$

given by $x \to \sum_{\sigma \in G} x(\sigma^{-1}) \otimes \sigma$, where $\mathbb{Z}_p[G]$ is the **group ring** of G.

The following proposition from [12], chapter 1 gives the important properties of Induced modules required to handle dimension shifting arguments in homological algebra:

B.3.1 Proposition (Properties of Induction):

1. *The functor $A \to Ind_G(A)$ is exact ($\mathbb{Z}_p[G]$ free, and hence projective as a $\mathbb{Z}$-module)*
2. *An induced G-module A is also an induced H-module for every closed subgroup H of G, and if H is normal, then A^H is an induced G/H-module.*
3. *If one of the G-modules A or B is induced, then so are $A \otimes B$ and $Hom(A,B)$, provided that in the case of $Hom(A,B)$ when A is induced, G is finite.*
4. *If U runs through the open normal subgroups of G, then*
$$Ind_G(A) = \varinjlim_{U} Ind_{G/U}(A^U)$$
5. *The induced G-modules are cohomologically trivial.*

B.4 Structure of Homology Groups

The cohomology groups $H^n(G,A)$ of a profinite group G with coefficients in a G-module A are built up in a simple way from those of the finite factor groups of G.

Let U, V run through the open normal subgroups of G. If $V \subset U$, then the projections

$$G^{n+1} \to (G/V)^{n+1} \to (G/U)^{n+1}$$

induce homomorphisms,

$$C^n(G/U, A^U) \to C^n(G/V, A^V) \to C^n(G, A)$$

which commute with the operators δ^{n+1} and we therefore obtain homomorphisms,

$$H^n(G/U, A^U) \to H^n(G/V, A^V) \to H^n(G, A)$$

The groups $H^n(G/U, A^U)$ thus form a direct system and we have a canonical homomorphism $\varinjlim_U H^n(G/U, A^U) \to H^n(G, A)$, moreover this may be shown to be an isomorphism:

B.4.1 Proposition (Decomposition of Cohomology - [12], 1.2.6):

$$\varinjlim_{\mathbf{U}} \mathbf{H^n}(\mathbf{G}/\mathbf{U}, \mathbf{A^U}) \cong \mathbf{H^n}(\mathbf{G}, \mathbf{A})$$

B.4.2 Corollary (Decomposition of Homology - Dually):

$$\varprojlim_{U} H_n(G/U, A_U) \cong H_n(G, A)$$

B.4.3 Change of the Group G

I investigate how this affects the cohomology groups $H^n(G, A)$. A general situation is for 2 profinite groups, G and G', a G-module A and a G'-module A', and 2 homomorphisms:

$$\phi : G' \to G, \ \ f : A \to A'$$

if such that the homomorphisms are "compatible": $\mathbf{f}(\phi(\sigma')\mathbf{a}) = \sigma'\mathbf{f}(\mathbf{a})$, then we obtain a map of chain complexes which commutes with boundary maps, giving a map of cohomologies:

$$H^n(G, A) \to H^n(G', A')$$

Moreover the map on cohomology groups $H^n(G, A)$ is functorial in both G and A simultaneously.

Let $(G_i)_{i\in I}$ be a projective system of profinite groups and let $(A_i)_{i\in I}$ be a direct system, where each A_i is a G_i-module and the transition maps,

$$G_j \to G_i, A_i \to A_j$$

are compatible in the sense defined above. Then combining the induced homomorphisms $H^n(G_i, A_i) \to H^n(G_j, A_j)$ the cohomology groups $H^n(G_i, A_i)$ form a direct system of abelian groups. We may generalise B.4.1 to

B.4.4 Proposition (Limits of Cohomology Groups - [12], 1.5.1):

If $G = \varprojlim_{i\in I} G_i$ *and* $A = \varinjlim_{i\in I} A_i$*, then*

$$H^n(G, A) \cong \varinjlim_{i\in I} H^n(G_i, A_i)$$

B.4.5 Corollary (Limits of Homology Groups):

If $G = \varprojlim_{i\in I} G_i$ *and* $A = \varprojlim_{i\in I} A_i$*, then*

$$H_n(G, A) \cong \varprojlim_{i\in I} H_n(G_i, A_i)$$

Proof of B.4.4

The compatible pairs of maps give canonical homomorphisms $\kappa : C^n(G_i, A_i) \to C^n(G, A)$. Hence a homomorphism,

$$\kappa : \varinjlim_{i\in I} C^n(G_i, A_i) \to C^n(G, A)$$

which commutes with the boundary homomorphisms. Thus it is sufficient to show that κ is an isomorphism which is achieved by using compactness arguments to reduce to finite quotients.

- SURJECTIVITY:

  Let $y : G^n \to A$ be the inhomogeneous cochain associated to $x \in C^n(G, A)$. Since G^n is compact, A discrete and y continuous, y takes only finitely many values and factors through $\overline{y} : (G/U)^n \to A$ for a suitable open normal subgroup U. The finitely many values are represented by elements of some A_i - $\overline{y}$ is the ccomposite of a function $\overline{y}_i : (G/U)^n \to A_i$ with $A_i \to A$. Also, there exists $j > i$ such that the projection $G \to G/U$ factors through the canonical map $G_j \to G/U$, and this gives the inhomogeneous cochain $y_j : G_j^n \to A_j$ as the composite

  $$G_j^n \to (G/U)^n \xrightarrow{\overline{y}_i} A_i \to A_j$$

  such that the composiite $G^n \to G_j^n \xrightarrow{y_j} A_j \to A$ is y. If $x_j \in C^n(G_j, A_j)$ is the homogeneous cochain associated to y_j then its image in $C^n(G, A)$ is x. This gives the surjectivity of κ

- INJECTIVITY:

  Let $x_i \in C^n(G_i, A_i)$ be a cochain which becomes zero in $C^n(G, A)$, so that the composite

  $$G^{n+1} \to G_i^{n+1} \xrightarrow{x_i} A_i \to A$$

  is zero. Since x_i has only finitely many values, there exists a $j \geq i$ such that the composite

  $$G_j^{n+1} \to G_i^{n+1} \xrightarrow{x_i} A_i \to A_j$$

  is already zero. Thus x_i becomes zero in $C^n(G_j, A_j)$ and hence represents the zero class in $\varinjlim_{i\in I} C^n(G_i, A_i)$. This gives the injectivity of κ.

B.5 Completed Tensor Product

Firstly, recall that every compact $\mathbb{Z}_p[[G]]$-module is the projective limit of finite modules, and the category $\mathfrak{C}$ of compact modules has sufficiently many projectives and exact inverse limits.

Also, every discrete $\mathbb{Z}_p[[G]]$-module is the direct limit of finite modules, and the category $\mathfrak{D}$ of discrete modules has sufficiently many injectives and exact direct limits.

A tensor product for compact $\mathbb{Z}_p[[G]]$-modules is defined by its universal property. Explicitly, let M be a compact right and N be a compact left $\mathbb{Z}_p[[G]]$-module. Then the **complete tensor product** is a compact $\mathbb{Z}_p$-module $M\widehat{\otimes}_{\mathbb{Z}_p[[G]]}N$ coming with an $\mathbb{Z}_p[[G]]$-bihomomorphism α (i.e. α is a continuous $\mathbb{Z}_p$-homomorphism such that $\alpha(m\lambda, n) = \alpha(m, \lambda n)$ for all $m \in M$, $n \in N$ and $\lambda \in \mathbb{Z}_p[[G]]$):

$$\alpha : M \,\mathrm{x}\, N \to M\widehat{\otimes}_{\mathbb{Z}_p[[G]]}N$$

with the following property: given any $\mathbb{Z}_p[[G]]$-bihomomorphism f of $M \,\mathrm{x}\, N$ into a compact $\mathbb{Z}_p$-module R, there is a unique $\mathbb{Z}_p$-module homomorphism $g : M\widehat{\otimes}N \to R$ such that $f = g \circ \alpha$.

The complete tensor product may be constructed in the following way:

$$M\widehat{\otimes}_{\mathbb{Z}_p[[G]]}N = \varprojlim_{U,V} M/U \otimes_{\mathbb{Z}_p[[G]]} N/V$$

where U (respectively V) run through the open $\mathbb{Z}_p[[G]]$-submodules of M (respectively N). In 3.3 we will explicity identify submodules in calculation of Hochschild homology. Since M/U and N/V are both finite, each $M/U \otimes_{\mathbb{Z}_p[[G]]} N/V$ is finite, and thus $M\widehat{\otimes}_{\mathbb{Z}_p[[G]]}N$ is a compact $\mathbb{Z}_p$-module. Moreover, the natural quptient maps $M \,\mathrm{x}\, N \to M/U \otimes N/V$ induce the desired bihomomorphism $\alpha : M \,\mathrm{x}\, N \to M\widehat{\otimes}N$ when passing to the limit. We see from the exact sequence,

$$0 \to \mathrm{im}(M \otimes V + U \otimes N) \to M \otimes N \to M/U \otimes N/V \to 0$$

that infact $M\widehat{\otimes}N$ is a completion of $M \otimes N$ arising from the topology with fundamental system of open neighbourhoods of zero given by $\mathrm{im}(M \otimes V + U \otimes N)$.

B.6 Homology of Completed Group Rings

We are now in a position to define the Tor groups over the Iwasawa algebra: $Tor_\bullet^{\mathbb{Z}_p[[G]]}(-,-)$ is defined as the left derived functors of (the right exact functor) $-\widehat{\otimes}_{\mathbb{Z}_p[[G]]}-$.

Induction takes the form, $M_G \cong \mathbb{Z}_p\widehat{\otimes}_{\mathbb{Z}_p[[G]]}M$ giving the following crucial theorem:

B.6.1 Proposition (Completed "Tor" Groups and Group Homology - [12] chapter V, 5.2.6):

There are canonical isomorphisms for all $i \geq 0$ and all compact modules $M \in \mathfrak{C}$:

$$H_i(G, M) \cong Tor_i^{\mathbb{Z}_p[[G]]}(\mathbb{Z}_p, M)$$

Proof

The above functors agree for $i = 0$. It is sufficient to show that a free $\mathbb{Z}_p[[G]]$-module F has trivial G-homology. Using ********* we reduce to the case $F = \mathbb{Z}_p[[G]]$. We then have a compatible inverse system with the groups G/U acting on the module $\mathbb{Z}_p[G/U]$. By 3.3 we have,

$$H_i(G, \mathbb{Z}_p[[G]]) = \varprojlim_{U \subset G} H_i(G/U, \mathbb{Z}_p[G])$$

where U runs through the open normal subgroups of G. The key observation is that the G/U-module $\mathbb{Z}_p[G] \cong \mathbb{Z}_p \otimes_{\mathbb{Z}} \mathbb{Z}[G/U]$ is induced and hence is cohomologically trivial by B.3.1.

We can similarly define "*Ext*" groups, and the 2 are related using pontryagin duality, a self inverse mapping from an abelian profinite group, A (abelian compact group) to discrete abelian group (discrete abelian torsion group), given by considering a space of maps under compact-open topology: where

$$A^V = \mathrm{Hom}_{cont}(A, \mathbb{Q}_p/\mathbb{Z}_p)$$

where $\mathbb{Q}_p/\mathbb{Z}_p$ is given quotient topology, (see [12], 5.2.9 for details),

B.6.2 Theorem (Relationship Between "Tor" and "Ext" Groups):

For $M \in \mathfrak{D}$ a discrete module and $N \in \mathfrak{C}$ a compact module (guaranteeing both groups defined) there are canonical isomorphisms for all $i \geq 0$:

$$Tor_i^{\mathbb{Z}_p[[G]]}(M, N^V) \cong Ext^i_{\mathbb{Z}_p[[G]]}(M, N)^V$$

where "^{V}" denotes the Pontryagin dual

Thus the theories of homology and cohomology are dual and we may use Pontryagin duality to easily switch between the two.

B.6.3 Profinite Spaces and Profinite Groups

B.6.3.1 Lemma

For a Hausdorff topological space T the following conditions are equivalent:

1. *T is the (topological) inverse limit of finite discrete spaces.*
2. *T is compact and every point of T has a basis of neighbourhoods consisting of subsets which are both open and closed.*
3. *T is compact and totally disconnected.*

The inverse limit of an inverse system of topological groups is just the inverse limit of groups together with the inverse limit topology on the underlying topological space.

B.6.3.2 Definition

*A space T is called a **profinite space** if it satisfies the equivalent conditions above.*

B.6.3.3 Proposition

For a Hausdorff topological group G the following conditiions are equivalent:

1. *G is the (topological) inverse limit of finite discrete groups.*
2. *G is compact aand the unit element has a basis of neighbourhoods consisting of open and closed normal subgroups.*
3. *G is compact and totally disconnected.*

B.6.3.4 Definition

*A Hausdorff Topological Group G is called a **profinite group** if it satisfies the equivalent conditions above.*

Assume homos between profinite groups are continuous and subgroups closed. Since a subgroup is complement of non trivial cosets, open subgroups are closed, and closed subgroups opne iff it is of finite index. Can try to pass finite group theory to profinite group theory:

B.6.3.5 Definition

*Let G be a profinite group. A **topological** G-module M is an abelian Hausdorff topological group M which is endowed with the structure of a G-module such that the action $G\,X\,M \to M$, $(g,m) \to g(m)$, is continuous. The term G-module, without the word topological will always refer to a discrete module - **the topology on M is the discrete topology**.*

B.6.3.6 Lemma

*Let G be a profinite group and let M be a **discrete** G-module, then the following holds:*

1. *For every $m \in M$ the subgroup $G_m = \{g \in G | g(m) = m\}$ is open.*
2. *$M = \bigcup M^U$ where U runs through the open subgroups of G: clear since $m \in M^{G_m}$.*

The groups $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{F}_p$ are always viewed as trivial discrete G-modules, modules with trivial action of G.

B.6.3.7 Definition

We call the group

$$A^V = Hom_{cont}(A, \mathbb{R}/\mathbb{Z})$$

the Pontryagin dual of A.

Given locally compact topological spaces X,Y the set of continuous maps $Map_{cont}(X, YU)$ carries a natural topology, the compact-open topology, with basis the subsets:

$$U_{K,U} = \{f \in Map_{cont}(X, Y_| f(K) \subset U\}$$

where K is compact subset of X and U is open subset of Y.

B.6.3.8 Theorem - Pontryagin Duality

If A is a Hausdorff abelian locally compact topological group, then the same is true for A^V endowed with the compact open topology. There is a canonical homomorphism:

$$A \to (A^V)^V$$

given by

$$a \to \tau_a : A^V \to \mathbb{R}/\mathbb{Z}, \phi \to \phi(a)$$

is an isomorphism of topological groups. Commutes with limits and induces:

$$\text{(abelian compact groups)} \leftrightarrow \text{(discrete abelian groups)}$$

$$\text{(abelian profinite groups)} \leftrightarrow \text{(discrete abelian torsiongroups)}$$

Given family $\{X_i\}_{i \in I}$ of Hausdorff, abelian topological groups let $Y_i \subset X_i$ be given for almost all $i \in I$.

B.6.3.9 Definition

*The **restricted product***

$$\prod_{i \in I} (X_i, Y_i)$$

is the subgroup of $\prod_{i \in I}(X_i)$ such that $x_i \in Y_i$ for almost all i.

The direct product is such an example. The restricted product is again a Hausdorff, abelian topological group.

B.6.3.10 Proposition

If all the X_i are locally compact and almost all Y_i are compact, then again the restricted product is a locally compact group. There is an isomorphism:

$$(\prod_{i\in I}(X_i, Y_i))^V \cong \prod_{i\in I}(X_i^V, (X_i/Y_i)^V)$$

B.6.4 Definition of the Cohomology Groups

Let

$$X^n = X^n(G, A) = Map(G^{n+1}, A)$$

Take

$$d_i : G^{n+1} \to G^n : (\sigma_1, \dots \sigma_n) \to (\sigma_1, \dots \hat{\sigma_i}, \dots \sigma_n)$$

Which in turn induce d_i^* on the $X^{i-1} \to X^i$. Now define,

$$\delta^n = \sum_{i=0}^{n} (-1)^i d_i^*$$

B.6.4.1 Proposition

The sequence

$$0 \to A \to^{\delta^0} X^0 \to^{\delta^1} X^1 \to^{\delta^2} X^2 \to^{\delta^3} X^3 \to \dots$$

is exact

Hence we have a resolution of A by G-modules, applying fixed module functor leads to cohomology groups:

B.6.4.2 Definition

For $n \geq 0$ the factor group

$$H^n(G, A) = Z^n(G, A)/B^n(G, A)$$

is called the n-dimensional cohomology group of G with coefficients in A.

Below trick reduces number of variables in computation by one.

Also often refine defintion of 0th cohom groups in the following sense: For G a finite group, the norm residue group,

$$\hat{H}^0(G, A) = A^G/N_G A$$

where $N_G A$ is the image of the norm residue map

$$N_G : A \to A, \, N_G a = \sum_{\sigma\in G} \sigma a$$

The modified cohomology groups are usual ones except 0th is the norm residue group, equivalent to extending the resolution:

B.6.4.3 Proposition

We have an exact sequence

$$0 \to \hat{H}_0(G, A) \to H_0(G, A) \to^{N_G} H^0(G, A) \to \hat{H}^0(G, A) \to 0$$

Now let G be a profinite group and A a G-module. For every pair of normal subgroups $V \leq U$ of G, we have homos:

$$\hat{H}^0(G/V, A^V) \to \hat{H}^0(G/U, A^U)$$

$$\hat{H}_0(G/V, A^V) \to \hat{H}_0(G/U, A^U)$$

Induced from $id : A^G \to A^G$, respectively $N_{U/V} : A^V \to A^U$. Now define:

$$\hat{H}^0(G, A) = \varprojlim_U H^0(G/U, A^U)$$

Similarly for homology giving relations of the form:

$$0 \to N_G A \to A^G \to H^0(G, A)$$

for $N_G A = \varprojlim N_{G/U} A^U$.

The complete standard resolution of A is defined as the sequence $X^{-1-n} = Hom(X_n, A)$ giving:

$$\dots \to X^{-2} \to X^{-1} \to X^0 \to X^1 \to X^2 \to \dots$$

Now, for every $n \in \mathbb{Z}$ the n-th cohomology group $\hat{H}^n(G, A)$ is defined as the homology group of the complex

$$\hat{C}^\bullet(G, A) = ((X^n)^G)_{n \in \mathbb{Z}}$$

Can interpret small cohomology groups in terms of crossed homomorhisms and pointed sets.

The cohomology groups $H^n(G, A)$ of a profinite group G with coefficients in a G-module A are built up in a simple way from those of the finite factor groups of G.

$$G^{n+1} \to (G/V)^{n+1} \to (G/U)^{n+!}$$

induces maps commuting with differentials:

$$C^n(G//U, A^U) \to C^n(G/V, A^V) \to C^n(G, A)$$

giving homos:

$$H^n(G//U, A^U) \to H^n(G/V, A^V) \to H^n(G, A)$$

therefore the groups form a direct system, with homo:

$$\varprojlim \mathbf{H^n(G/U, A^U) \to H^n(G, A)}$$

which is infact an IOM

B.6.5 The Exact Cohomology Sequence

B.6.5.1 Definition

A G-module A is called acyclic if $H^n(G, A) = 0$ for all $n > 0$, and cohomologically trivial

$$H^n(H, A) = 0$$

for all closed subgroups H of G, and all $n > 0$. INDUCED MODULES ARE COHOM TRIV.

If G is a finite group, $Ind_G(A) = Map(G, A) \cong A \circ \mathbb{Z}[G]$

B.6.5.2 Proposition

1. The functor $A \to Ind_G(A)$ is exact.
2. An induced G-module A is also an induced H-module for every closed subgroup H of G, and for H normal, A^H is an induced g/H-module.
3. If one of A or B is induced so are $a \circ B$ and $Hom(A, B)$
4. For U running through open subgroups of G,

$$Ind_G(A) = \varprojlim Ind_{G/U}(A^U)$$

This leads to

B.6.5.3 Proposition

The induced G-modules are cohomologically trivial, and for G-finite we have moreover the norm residue group, $\hat{H}^n(G, M) = 0$ for all $n \in \mathbb{Z}$.

We can now perform **dimension shifting**:

B.6.6 The Cup-Product

B.6.7 Change of the Group G

B.6.8 Basic Properties

B.6.9 Cohomological Triviality

Appendix C

Lie Groups and Lie Algebras

C.1 Lie Groups

Before going on to define the cohomology of Lie groups and Lie algebras I would like to recall the key properties of these structures and of Lie Theory - the method of passing from a Lie group to its Lie algebra. Later, this report will concentrate on the relation of the cohomology of a Lie group to the cohomolgy of its associated Liealgebra.

C.1.1 Manifolds

I will recall the definition of tangent space which will be needed for Lie Theory later.

C.1.2 Analytic Manifolds

Recall, for X a topological space, a chart c on X is a triple (U, φ, n) such that:

1. $U \subset X$ is open
2. $n \in \mathbb{Z}$ and $n \geq 0$
3. $\varphi : U \mapsto \varphi U \subset k^n$ is open and φ is a homeomorphism.

Charts allow considerations of continuity, analyticity,.. of maps to be passed to questions about the well understood maps $k^n \mapsto k^m$ using homeomorphism φ. Two charts c, c' are compatible if they behave well on their intersection $V = U \cap U'$ - if the maps $\varphi' \circ \varphi^{-1}|_{\varphi(V)}$ and $\varphi \circ \varphi'^{-1}|_{\varphi'(V)}$ are analytic.

An atlas is a collection of compatible charts which "span" X, meaning that $\bigcup U$ cover X. This leads to the notation of compatibility of atlases - where charts are pairwise compatible.

We can now define X as an analytic manifold if it has the extra structure of an equivalence class of compatible atlases.

To understand a morphism f of manifolds we again use charts to pass to simple maps $k^n \mapsto k^m$ and require that f induces a continuous map which is "locally given by analytic functions". So, for a sufficiently refined chart, the maps on coordinates are analytic.

C.1.3 Tangent Spaces

For $x \in X$, define $T_x X = (\mathbf{m}_x / \mathbf{m}_x^2)^*$ = tangent space of X at x, where $\mathbf{m}_x$ denotes functions vanishing at x.

Claim

T_xX is canonically isomorphic to the space $\underline{C_x}$ of "tangent classes of curves at x".

Taking $\underline{F'_x} = \{\text{pairs } (N, \phi) : 0 \in N \subset k \text{ is an open neighborhood, } \phi : N \mapsto X \text{ such that } \phi(0) = x\}$.
We define an equivalence relation on $\underline{F'_x}$. We say (N_1, ϕ_1) is equivalent to (N_2, ϕ_2) if $D(\varphi \circ \phi_1)(0) = D(\varphi \circ \phi_2)(0)$ - this is a valid definition since $\varphi \circ \phi_i$ is defined in the neighborhood $N_i \cup \phi_i^{-1}(U)$ of 0.

Then the tangent space, $\underline{C_x}$ is the set of equivalence classes of $\underline{F'_x}$. The definition of $\underline{C_x}$, and of its induced vector space structure (corresponding to adding linear maps given by differentiating) is independent of the choice of chart.

For $x \in X$, denote $\underline{F_x}$ for the set of pairs (U, φ) where U is an open neighborhood of x, and φ is an analytic function on U.

We define the pairing:

$$\underline{F'_x}\, X\, \underline{F_x} \mapsto k \;:\; (N, \phi)\, X\, (V, f) \mapsto D(f \circ \phi)(0) \in k$$

This induces a pairing $\underline{C_x}\, X\, T^*_xX \mapsto^{\omega} k$ which is bilinear and gives required duality - $\underline{C_x}$ is the dual of T^*_xX.

Of course, the pairing ω is simply differentiation of a function in the direction of the tangent to a curve.

As the simplest example, for a vector space V, and for any point $x \in V$ (it doesn't matter which point we choose by homogeneity), $T_xV = L(k, V) = V$ - the tangent to a curve through V is a vector in V.

Later, we will consider tangent spaces at $x = 1 \in G$, and analytic group when producing its associated Lie algebra $\mathcal{G}$.

C.1.4 Analytic Groups

C.1.5 Definition

For G a topological group and an analytic manifold over k (a field complete with respect to a non-trivial absolute value). G is said to be an **analytic group** or **lie group** if the following hold:

1. The map $G\, X\, G \mapsto G : (x, y) \mapsto xy$ is a morphism.
2. The map $G \mapsto G : x \mapsto x^{-1}$ is a morphism.

These two conditions immediately give structure to the group, and allow us to study any neighborhood as a translation by (1) of a neighborhood of the origin. For example, since taking a chart, G is locally isomorphic to an open subset of k^n (some n), the intersection of neighborhoods of the identity if $\{1\}$, and it is a standard result that this gives G is Hausdorff.

There are two examples which become important later in this report:

1. **General Linear Groups**

   For a finite dimensional algebra R over k, we denote $G_m(R)$ for its group of invertible elements. It is clear that multiplication is a morphism since multiplication in R is bilinear, and has an obvious inverse in the group.

   Familiarly, for $R = End(V)$, the endomorphism ring of a finite dimensional vector space V/k, we call $G_m(R)$ the general linear group of V, $GL(V)$.

   When $V = k^n$, we use the notation $GL(V) = GL(n, k) = GL_n(k)$, whose elements may be represented by invertible matrrices. For a valuation ring A/k, let $GL(n, A) \subset GL(n, k)$ be defined by

   $$GL(n, A) = \{(\alpha_{i,j}) | (\alpha_{i,j}) \in A \text{ is an auto, } det(\alpha_{i,j}) \in \text{ units of } A\}$$

   Then the group $GL(n, A)$ is open and closed in $E(k^n)$ and hence an analytic group. When k is locally compact, $GL(n, A)$ is a compact open subgroup of $GL(n, k)$.

2. **Lie Group $\mathbb{Q}_p$**

   Consider $\mathbb{Q}_p$ as a degenerate Lie group over the field $\mathbb{Q}_p$ with inherited multiplication and inverse. We have the theorem that if k is locally compact then:

C.1.6 Theorem

$GL(n,A)$ is a maximal compact subgroup of $GL(n,k)$ and, if G is a maximal compact subgroup of $GL(n,k)$, then G is a conjugate of $GL(n,A)$.

In this case we can use maximality of $\mathbb{Z}_p$ in $\mathbb{Q}_p$ to reduce the study to that of discrete Lie groups and algebras over $\mathbb{Z}_p$.

C.1.7 Formal Groups

When $R=k$, a complete field, we shall use formal groups to define a functor T:

Analytic Groups → Lie Algebras

Central to this technique will be the 1-1 correspondence between Lie algebras and formal groups which i will now describe.

In a way, Lie Groups are "locally" formal groups(or equivalently, Lie algebras) - which are a kind of "linearisation" of the system.

Definition

Let R be a commutative ring with a unit, and consider the Formal Power Series ring, $R[[X_1,\dots,X_n]]=R[[X]]$ in n-variables, and let $Y=(Y_1,\dots Y_n)$ be a set of a further n variables.

Then, **A Formal Group Law** in n variables is an n-tuple $F=(F_i)$ of formal power series, $F_i\in R[[X,Y]]$, such that:

1. $F(X,0)=X,\ F(0,Y)=Y$
2. $F(U,F(V,W))=F(F(U,V),W)$, a kind of associativity.

This immediately gives tight restrictions on structure - each F_i has the form:

$$F_i(X,Y)=X+Y+\sum_{|\alpha|\geq 1,|\beta|\geq 1}c_{\alpha,\beta}X^\alpha Y^\beta$$

We can think of a Formal Group Law as a bifunctor, which approximates to first order (possibly after shifting the origin) to **addition**.

We could simply take $F(X,Y)=X+Y$.

Alternatively, we could consider multiplication of 2 elements taking origin as the multiplicative identity: $(1+X).(1+Y)=1+X+Y+XY$ gives rise to the Formal Group Law $F(X,Y)=X+Y+XY$.

Formal Groups are very important in the study of the Group Law for rational points on Elliptic Curves:

C.1.8 Group Law for Points on Elliptic Curves

Given a Weierstrass Equation in x,y describing an Elliptic Curve **E**, we make the change of coordinates:

$$z=x/y \text{ and } w=-1/y$$

ie.

$$x=z/w \text{ and } y=-1/w$$

The Weierstrass Equation then becomes:

$$w=z^3+a_1zw+a_2z^2w+a_3w^2+a_4zw^2+a_6w^3$$

We then substitute this equation recursively (and checking convergence) to give:

C.1.9

$$w(z) = z^3(1 + A_1 z + A_2 z^2 + \dots) \in \mathbb{Z}[a_1, \dots, a_6][[z]]$$

where each $A_n \in \mathbb{Z}[a_1, \dots, a_6]$ is homogeneous of weight .

C.1.9 gives rise to Laurent series for x and y:

$$\begin{aligned} x(z) &= \frac{z}{w(z)} = 1/z^2 - a_1/z - a_2 - a_3 z - (a_4 + a_1 a_3) z^2 - \dots \\ y(z) &= \frac{-1}{w(z)} = -1/z^3 + a_1/z^2 + a_2/z + a_3 + (a_4 + a_1 a_3) z + \dots \end{aligned}$$

We now express the additive group law on **E** using these power series in place of x and y.

Let z_1, z_2 be independent indeterminants, define $w_i = w(z_i)$ for $i = 1, 2$. Standard calculations on this new Elliptic Curve (defined by its Weierstrass Equation) gives the z-coordinate of $\ominus(z_1 \oplus z_2)$, called z_3 as

$$\begin{aligned} z_3 &= z_3(z_1, z_2) \\ &= -z_1 - z_2 + \frac{a_1\lambda + a_3\lambda^2 - a_2\nu - 2a_4\lambda\nu - 3a_6\lambda^2\nu}{1 + a_2\lambda + a_4\lambda^2 + a_6\lambda^3} \\ &\in \mathbb{Z}[a_1, \dots, a_6][[z_1, z_2]] \end{aligned}$$

Inverting this point, we get an expression for $z(z_1 \oplus z_2)$:

$$\begin{aligned} F(z_1, z_2) &= i(z_3(z_1, z_2)) \\ &= z_1 + z_2 - a_1 z_1 z_2 - a_2(z_1^2 z_2 + z_1 z_2^2) + \dots \\ &\quad -(2a_3 z_1^3 z_2 - (a_1 a_2 - 3a_3) z_1^2 z_2^2 + 2a_3 z_1 z_2^3) + \dots \\ &\quad \in \mathbb{Z}[a_1, \dots, a_6][[z_1, z_2]] \end{aligned}$$

We define this power series as a formal group law $F(z_1, z_2)$ - all required properties of associativity and commutivity are inherited from the geometric interpretation of the group law.

The following lemma, whose proof is analogous to the calculation of example C.1.13 is important in handling torsion points.

C.1.10 Lemma

Let $a \in R^*$ and $f(T) \in R[[T]]$ a power series starting

$$f(T) = aT + \dots$$

Then there is a ***unique*** *power series $g(T) \in R[[T]]$ such that $f(g(T)) = T$. It further satisfies $g(f(T)) = T$.*

We can now prove the central position in handling addition in these new coordinates:

C.1.11 Proposition

Let $\mathcal{F}$ be a formal group over R, and let $m \in \mathbb{Z}$.

1. *$[m](T) = mT + (\text{higher order terms})$.*
2. *If $m \in R^*$, then $[m] : \mathcal{F} \mapsto \mathcal{F}$ is an isomorphism.*

Proof

1. For positive m it is immediate from the formal power series F. For negative m observe that the inversion i negates the dominant term.
2. This follows from the above lemma, C.1.10. g provides a two-sided inverse to f, and hence $[m]$ has an inverse and is thus an isomorphism.

The following plays an important role in proving the Mordell-Weil Theorem on the structure of rational points on an Elliptic curve - that, *The Group E(K) is finitely generated.*

C.1.12 Proposition

Let p be the characteristic of k (p=0 being allowed). Then every torsion element of $\mathcal{F}(\mathcal{M})$ has order a power of p.

Proof

We need to only consider torsion elements of order prime to p by multiplying by an arbitrary torsion element of an appropriate value of p. Using this trick, let $m \geq 1$ such that $(m, p) = 1$, $x \in \mathcal{F}(\mathcal{M})$ an element such that $[m](x) = 0$ and we need to show $x = 0$.

Since m is prime to p, $m \notin \mathcal{M}$, hence from C.1.11 $[m]$ is an isomorphism of formal groups and induces $[m] : \mathcal{F}(\mathcal{M}) \cong \mathcal{F}(\mathcal{M})$.

This has trivial kernel, hence $x = 0$, as required.

C.1.13 Formulae

We now study general expressions for the structure of Formal Groups. We use $O(d^0 \geq n)$ to denote a Formal Power Series whose homogeneous parts of degree strictly less than n vanish.

- The structure from the definition may be written:

$$F(X, Y) = X + Y + B(X, Y) + O(d^0 \geq 3)$$

  where B is a bilinear form. We now set

$$[X, Y] = B(X, Y) - B(Y, X)$$

  and I shall refer to this bifunctor as the "Lie Bracket" associated to the Formal Group.

- Consider the construction of the inverse power series φ of Theorem C.1.6 satisfying $F(X, \varphi(X)) = 0 = F(\varphi(X), X)$

  Let $\varphi_i(X)$ be the i-th homogeneous part of $\varphi(X)$.

  Since $F(X, \varphi(X)) = X + \varphi_1(X) + O(d^0 \geq 2) = 0$.

  We have $\varphi(X) = -X + \varphi_2(X) + O(d^0 \geq 2)$.

$$\begin{aligned} \text{Similarly } F(X, \varphi(X)) &= X + (-X + \varphi_2(X) + \dots) + B(X, -X + \dots) + \dots \\ &= \varphi_2(X) - B(X, X) + O(d^0 \geq 3). \end{aligned}$$

  Hence, $\varphi_2(X) = B(x, X)$.

  Combining gives:

$$\varphi(X) = -X + B(X, X) + O(d^0 \geq 3)$$

The following are preliminary to a key theorem on the associativity of the "Lie Bracket" we have constructed.

C.1.14 Lemma

$$XYX^{-1} = Y + [X, Y] + O(d^0 \geq 3)$$

This follows by careful book keeping when expanding the brackets. Similar results hold for Y^{-1}, $X^{-1}Y^{-1}XY$, and together lead to an identity of Hall:

C.1.15 Proposition: Hall

$$(X^Y, (Y, Z)).(Y^Z, (Z, X)).(Z^X, (X, Y)) = 0$$

Finally examining this identity to order 3 recovers Jacobi's identity, justifying my description of $[X, Y]$ as a Lie Bracket:

C.1.16 Identity: Jacobi

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

C.1.17 Formal Groups Giving Rise to Analytic Groups

Let k be a complete Ultrametric field, and let L be the valuation ring of k. Denote the maximal ideal of A by **m**. For $F(X, Y)$ a formal group law over A, we construct an analytic group:

Let $G = \{(x_1, \ldots, x_n) : x_i \in \boldsymbol{m}\}$ and define multiplication on G by the formula $xy = F(x, y)$.

A group G arising from the above construction is known as **standard**.

Definition

Recall that an **analytic group chunk** is a topological space X together with a distinguished element $i \in X$, and an open neighborhood U of i in X, together with a pair of maps $\varphi : U\, X\, U \mapsto X$, and $\phi : U \mapsto U$ such that:

1. For some neighborhood V_1 of e in U, $x \in V_1 \Rightarrow$
$$x = \varphi(x, e) = \varphi(e, x)$$
2. For some neighborhood V_2 of e in U, $x \in V_2 \Rightarrow$
$$e = \varphi(x, \phi x) = \varphi(\phi x, x)$$
3. For some neighborhood V_3 of e in U, $\varphi(V_3\, X\, V_3) \subset U$, and for all $x, y, z \in V_3$:
$$\varphi(x, \varphi(y, z)) = \varphi(\varphi(x, y), z)$$

Clearly φ, ϕ arise locally from a multiplication, and an inverse, and by shrinking U can we can assume this is true on the whole of U.

The following theorem shows how analytic groups naturally have a substructure of a formal group.

C.1.18 Theorem

Any analytic group chunk contains an open subgroup which is standard.

The proof proceeds by shrinking G so that the Formal Power Series Converges, and then the strict unit ball gives the desired open subgroup.

As an immediate corollary we see that in fact any analytic group chunk is equivalent (ie. there exists local homeomorphisms between spaces such that ordered pairs of composition are equivalent to the identity) to an analytic group.

C.2 Lie Theory

C.2.1 Lie Algebra of an Analytic Group Chunk

For a group chunk G/k, define as set isomorphisms $L(G) = \mathcal{G} = T_eG$, the tangent space at the identity, see C.1.3. I now define a Lie Algebra structure on $\mathcal{G}$.

To get a handle on the calculations take a chart $c = (U, \varphi, n)$ of G at e. The group law on G is induced (via φ) from a formal group law F on k^n. Denoting the isomorphism $\cong k^n$ induced by φ, by $\overline{\varphi}$. Define

$$\overline{\varphi}[x, y]_c = [\overline{\varphi}x, \overline{\varphi}y]_F$$

A homomorphism between charts passes to a Lie Algebra homomorphism and hence the induced Lie Bracket is independent of c.

C.2.2 Definition

$\mathcal{G}$ together with it's canonical Lie Algebra structure is the **Lie Algebra of G**.

This construction provides a converse to Theorem C.1.18, since from a given group chunk with given formal group we can construct the associated standard algebra. When we are working with analytic groups, the local functions ϕ, φ of group chunks are global and induce, via charts the formal group F on k^n, this in turn induces the Lie algebra.

Example

Let R be an associative algebra of finite dimension over k. From C.1.5, $G_m(R)$ is an analytic group. $T_1G_m(R) = R$. Multiplication in $G_m(R)$ has the form $(1 + x)(1 + y) = 1 + x + y + xy$. Taking the chart

$$\begin{aligned} \varphi \;&:\; R \mapsto k^n \\ &:\; z \mapsto z - 1 \end{aligned}$$

We have

$$\begin{aligned} \overline{\varphi}[x, y]_c &= [x - 1, y - 1]_F \\ &= 1 + [x, y]_c \\ &= (1 + x)(1 + y) \\ &= 1 + x + y + xy \end{aligned}$$

Hence, $F(x, y) = x + y + xy = x + y + B(x, y)$. Hence, the Lie algebra structure on $T_1G_m(R) = R$ is given by

$$[x, y] = xy - yx$$

Note that in the case where R is an endomorphism ring we recover the usual Lie algebra structure.

C.2.3 Linear Action

Consider a Lie Group G acting on a vector space V. For the continuation, the most important case to us is deducing the action of it's Lie Algebra on V (and hence calculating the Lie Algebra Cohomology with coefficients in V).

Formally, a *Linear Representation of G in V* is an analytic group homomorphism $\sigma : G \mapsto GL(V)$. The, an element $g \in G$ acts on V via σ:

$$g.v = \sigma(g)(v)\, \forall v \in V$$

The induced homomorphism $\overline{\sigma} : L(G) \mapsto E(V)$ gives an induced representation of $L(G) = \mathcal{G}$ on V.

C.2.4 Examples of Linear Representations

I now give two examples of this, there will be more in my own calculations, see chapter 7.

1. **Determinants:**

   Let $G = GL(V)$. Then $det : G \mapsto G_m(k) = k^*$ the determinant map is an analytic homomorphism. Take x as a representative of an element $1 + x \in \mathcal{G} = L(G)$. We have

   $$\begin{aligned} det(1+x) &= 1 + tr(x) + \cdots + det(x) \\ &= 1 + tr(x) + O(d^0 \geq 2) \end{aligned}$$

   Hence, $L(det)(x) = tr(x)$ as Lie Theory picks out the dominant term only (replaces curves by their linear tangents). This is a good example of how the Lie Algebra is a linearisation of the Lie Group, and so is often simpler. This simplicity makes associated calculations of cohomology easier, while the results of Lazard allow us to pass back ti get information on the hard calculations of cohomology of Lie groups.

   Conversely, we may turn this process round, since from [18], p152, there is a theorem of Lie:

C.2.5 Theorem: Third Theorem of Lie

For any Lie Algebra $\mathcal{G}$ there exists a connected and simply connected analytic group G such that $L(G) = \mathcal{G}$. This allows us to use the theory of Lie groups to treat Lie algebras.

2. **Tensoring:**

   Let $V_1, \ldots, V_n$ be vector spaces. Take $V = V_1 \otimes \cdots \otimes V_n$. Then $G = \prod_{i=1}^n GL(V_i)$ acts on V in a natural way via

   $$\begin{aligned} \Theta \quad &: \quad E(V_1)\, X \ldots X\, E(V_n) \mapsto E(V) \\ &: \quad (u_1, \ldots, u_n) \mapsto u_1 \otimes \cdots \otimes u_n \end{aligned}$$

   To work with the Lie Algebra we take a representative in the neighborhood of 1: $\Theta(1 + x_1, \ldots, 1 + x_n) = 1 + \sum_{i=1}^n 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1 + O(d^0 \geq 2)$. Hence, translating the origin:

   $$L\Theta(x_1, \ldots, x_n) = \sum_{i=1}^n 1 \otimes \cdots \otimes x_i \otimes \cdots \otimes 1$$

Appendix D

Lie Algebra Cohomology and Spectral Sequences

D.1 Lie Algebra Cohomology

D.1.1 Definition

Recall, a Lie algebra $\mathcal{G}/K$ is a K vector space, with a bilinear map $[\,,\,] : \mathcal{G}\, X\, \mathcal{G} \mapsto \mathcal{G}$ known as the Lie Bracket. The Lie bracket is anti-commutative, and non-associative the Jacobi identity, see C.1.16 shows how far from being associative it is: $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$, hence

$$[X, [Y, Z]] = [[X, Y], Z] + [[Z, X], Y]$$

The Lie algebra is called abelian if $[x, y] = 0$ for all $x, y \in \mathcal{G}$

Consider the functor $L : \mathcal{G} \mapsto \mathcal{G}_{ab} = \mathcal{G}/[\mathcal{G}, \mathcal{G}]$ which maps to the largest abelian quotient.

For M a K-vector space, we define the n-fold tensor product: $T_n M = M \otimes_k M \otimes_k \cdots \otimes_k M$, and the Tensor algebra (a free K-algebra over M):

$$TM = \oplus_{n=0}^{\infty} T_n M$$

, where the multiplication is induced from product of elements in the a-fold tensor algebra and the b-fold tensor algebra as being the natural gluing to an element in the $a + b$-fold tensor product:

$$(m_1 \otimes \cdots \otimes m_a).(m'_1 \otimes \cdots \otimes m'_b) = (m_1 \otimes \cdots \otimes m_a \otimes m'_1 \otimes \cdots \otimes m'_b)$$

For our Lie algebra, we can now define the **Universal Enveloping Algebra $U\mathcal{G}$ of $\mathcal{G}$**, where

$$U\mathcal{G} = T\mathcal{G}/I \text{ where } I =< x \otimes y - y \otimes x - [x, y] >$$

By forming the quotient of the tensor algebra by the ideal I we are effectively forcing this new algebra to be abelian.

Recall, when defining group cohomology we used the group ring which is characterised by being adjoint to the unit functor: the functor : rings $\mapsto$ groups : $\Lambda \mapsto$ Group of Units of Λ. T has an analogous property - that the universal enveloping algebra functor $\mathcal{G} \mapsto U\mathcal{G}$ is the left adjoint to L, and plays a similar role in Lie algebra cohomology.

For a Lie algebra $\mathcal{G}$ over K, and for a $\mathcal{G}$-module A, we define the n^{th} **cohomology group of $\mathcal{G}$ with coefficients in A**,

$$\mathbf{H^n}(\mathcal{G}, \mathbf{A}) = \mathbf{Ext^n_{U\mathcal{G}}}(\mathbf{K}, \mathbf{A})$$

where we regard K as a trivial $\mathcal{G}$-module.

This leads to similar results as for groups, for example H^0 picks out invariant elements: $H^0(\mathcal{G}, A) = \{a \in A | x \circ a = 0 \,\forall\, x \in \mathcal{G}\}$.

Again, $H^n(\mathcal{G}, A)$ may be computed via any $\mathcal{G}$-projective resolution of K. We could try to mirror the construction of the Bar complex for groups, see A.2.6, but there exists one which is simpler for doing calculations:

D.1.2 The Koszul Complex

As a reference, see for example the paper of Lazard, chapter 5, section 1.3.3.

The n^{th} exterior power of a module M, a universal object on the set of alternating maps is defined as a quotient of the n-fold tensor product:

$$E_n M = T_n M / < x_1 \otimes x_2 \otimes \cdots \otimes x_n - (sgn\,\sigma) x_{\sigma 1} \otimes \cdots \otimes x_{\sigma n} >$$

where $sgn\,\sigma$ is the parity of σ. As examples: $E_0 V \cong K$, and $E_1 V \cong V$.

D.1.3 Alternating Property

Notice $< x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n > \cong < x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n >$, since a transposition has parity -1. Hence, when 2 elements in a bracket are equal, the bracket is its own negation, hence zero. The linearity of $< -, \ldots, - >$ allows us to expand to leave a sum of brackets consisting of basis elements. When $n > dim\,V$ it is inevitable that in each bracket of the expansion basis elements are repeated, hence, $E_n V = 0$.

As when we were forming the tensor algebra, we define the Exterior Algebra, with induced composition by

$$EM = \oplus_{n=0}^{\infty} E_n M$$

D.1.4 Construction of Complex

Set $V =$ underlying vector space of $\mathcal{G}$, where $\mathcal{G}$ is considered as a vector space over K but forgetting the composition given by Lie bracket.

Define C_n as the tensor product of the universal enveloping algebra with the n^{th} exterior power of V:

$$C_n = U\mathcal{G} \otimes_K E_n V = \{u < x_1, \ldots, x_n >\}$$

D.1.5 Lemma

$$\begin{aligned} \text{Let } d_n \quad &: \quad C_n \mapsto C_{n-1} \\ &: \quad < x_1, \ldots, x_n > \mapsto \sum_{i=1}^{n} (-1)^{i+1} x_i < x_1, \ldots, \widehat{x}_i, \ldots, x_n > + \\ &\sum_{1 \le i < j \le n} (-1)^{i+j} < [x_i, x_j], x_1, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots x_n > \end{aligned}$$

and we keep $U\mathcal{G}$ fixed. Then, by mechanical checking we get that the chain $\cdots \to C_n \to^{d_n} C_{n-1} \to \cdots \to C_1 \to C_0 = U\mathcal{G}$ is a $\mathcal{G}$-projective resolution of K, where $C_0 \to K$ is the augmentation map, similar to the construction for groups.

Since, from D.1.3, $E_k V = o$ for $k > dim\,V$, taking homology of above chain we have:

$$H^k(\mathcal{G}, A) = 0 \text{ for } k \ge dim\,V + 1$$

D.1.6 Example

To illustrate the methods discussed above I will now give an example from Totaro's paper. Working with $\mathcal{G}$ a finite dimensional Lie algebra over K. In this case i work with homology groups to ease notation, but of course a dual theory applies for cohomology.

Firstly, recall the definition of the Euler character:

Claim:

Euler Character = *Alt Sum Homologies of Complexes*

= *Alt Sum of dimensions of vector spaces in a complex used to calculate Homology*

Consider calculating complex:

$$\cdots \to^{\epsilon} D \to^{\delta} C \to^{\gamma} B \to^{\beta} A \to^{\alpha} 0$$

Then finiteness allows us to use the rank-nullity identity:

$$\begin{aligned}\text{Alt Sum dimensions} &= dim\,A - dim\,B + dim\,C - dim\,D + \ldots \\ &= (Null\,\alpha) - (Im\,\beta + Null\,\beta) + (Im\,\gamma + Null\,\gamma) - (Im\,\delta + Null\,\delta) + \ldots \\ &= (Null\,\alpha - Im\,\beta) - (Null\,\beta - Im\,\gamma) + \ldots \text{(regrouping)} \\ &= \sum (-1)^i dim|H_*|\end{aligned}$$

We are now ready to prove the following proposition:

Proposition

For $\mathcal{G}$ a finite dimensional Lie algebra (of dimension N), and M a finite dimensional representation of $\mathcal{G}$. The Euler Characteristic,

$$\chi(\mathcal{G}, M) = \sum_{i \geq o} dim_k H_i(\mathcal{G}, M) = \begin{cases} 0 & \mathcal{G} \neq 0 \\ dim\,M & \mathcal{G} = 0 \end{cases}$$

Proof

$\underline{\mathcal{G} = 0}$, $H_0(\mathcal{G}, M) = M$

$\underline{\mathcal{G} \neq 0}$, To calculate homology take Tor groups of the Koszul complex:

$$M \otimes_{U\mathcal{G}} C_n = M \otimes_{U\mathcal{G}} (U\mathcal{G} \otimes_k E_n V) = M \otimes_k E_n V$$

Hence, we are reduced to calculating the homology of the complex of finite dimensional vector spaces,

$$\cdots \to E_2\mathcal{G} \otimes_k M \to \mathcal{G} \otimes_k M \to M \to 0$$

The theory of Exterior Algebras gives $dim E_n\mathcal{G} \otimes_k M = dim\,M.dim\,E_n\mathcal{G} = dim\,M.\begin{pmatrix} dim\mathcal{G} \\ n \end{pmatrix}$, hence invoking the claim gives

$$\chi(\mathcal{G}, M) = \sum_{i=0}^{N} (-1)^i dim\,M.\begin{pmatrix} N \\ i \end{pmatrix} = dim\,M.(1-1)^N = 0 \text{ for } N > 0$$

D.1.7 Structure Theorem

For completeness I include the statement of the Poincare-Birkhoff-Witt Theorem which gives the structure of $U\mathcal{G}$. I will use this theorem to check my calculation of the Universal Enveloping Algebra of my induced Lie Algebra in my calculations section, chapter 7.

Let$\{e_\alpha\}$ be a k-basis of $\mathcal{G}$.

For each sequence $I = (\alpha_1, \ldots, \alpha_p)$ denote $e_{\alpha_1} \ldots e_{\alpha_p} \in U\mathcal{G}$ by e_I. We call a sequence I increasing if $\alpha_1 \leq \cdots \leq \alpha_p$ (and we have a convention that ϕ is increasing, where $e_\phi = 1$).

Theorem: Poincare-Birkhoff-Witt

If $\mathcal{G}$ is a free k-module, then $U\mathcal{G}$ is also a free k-module. Moreover, if $\{e_\alpha\}$ is an ordered basis of $\mathcal{G}$, then the elements e_I with I an an increasing sequence form a basis of $u\mathcal{G}$.

D.1.8 Calculating Ext Groups using Cohomology

Let M and N be left $\mathcal{G}$-modules: k-modules with a k-bilinear product
$\mathcal{G} \otimes_k M \mapsto M$, written $x \otimes m \mapsto xm$, such that:

$$[x, y]m = x(ym) - y(xm)\ \forall\, x, y \in \mathcal{G} \text{ and } m \in M$$

Then $Hom_k(M, N)$ is a $\mathcal{G}$-module by

$$(xf)(m) = xf(m) - f(xm)\,\forall\, x \in \mathcal{G},\ m \in M$$

since

$$\begin{aligned} ([x,y]f)(m) &= [x,y]f(m) - f([x,y]m) \\ &= x(yf(m)) - y(xf(m)) - f(x(ym) - y(xm)) \\ &= xyf(m) - xf(ym) - yf(xm) + f(xym) \\ &\quad -yxf(m) + yf(xm) + xf(ym) - f(yxm) \\ &= \{x(yf)\}(m) - \{y(xf)\}(m) \end{aligned}$$

Hence it is a $\mathcal{G}$-module. We now deduce a crucial isomorphism of $\mathcal{G}$-modules:

D.1.9 Lemma

$$Hom_{\mathcal{G}}(M, N) \cong Hom_k(M, N)^{\mathcal{G}}$$

Proof

Given $f \in Hom_{\mathcal{G}}(M, N)$ we have

1. $f(gm) = gf(m)$ since f is a $\mathcal{G}$ homomorphism.
2. $f(km) = kf(m)$ since $\mathcal{G}$ is a k-module.

We now define $\Theta \in Hom_{\mathcal{G}}(M, N) \cong Hom_k(M, N)$ via

$$[\Theta f](m) = f(m)$$

Then condition 2 gives $\Theta f \in Hom_k(M, N)$, and moreover condition 1 gives invariance under $\mathcal{G}$, hence $\Theta f \in Hom_k(M, N)^{\mathcal{G}}$. Clearly, any such map $Hom_k(M, N)^{\mathcal{G}}$ arises in this way and we may define an inverse morphism by inclusion, giving the required isomorphism.

D.1.10 Corollary

The natural isomorphism above may be extended to a natural isomorphism of functors:

$$Ext^*_{U\mathcal{G}}(M, N) \cong H^*_{Lie}(\mathcal{G}, Hom_k(M, N))$$

Note, by the Global Dimension Theorem, see [W], this implies that the global dimension of $U\mathcal{G}$ equals the Lie algebra cohomological dimension of $\mathcal{G}$.

Proof

Clearly, from definition of cohomology and of Ext groups as a measure of how far from being exact Hom is,

$$\begin{aligned} Ext^0_{U\mathcal{G}}(M,N) &\cong Hom_{\mathcal{G}}(M,N) \\ &\cong Hom_k(M,N)^{\mathcal{G}} \text{ from Lemma above,} \\ &\cong \text{Invariants of } Hom_k(M,N) \text{ under action of } \mathcal{G} \\ &\cong H^0_{Lie}(\mathcal{G}, Hom_k(M,N)) \end{aligned}$$

I now quote [HS], chapter IV, Proposition 5.7:

If T is left-exact, then R^0T is naturally equivalent to T.

Hence, since we know H^* and Ext are both derived functors, the isomorphism may be induced from isomorphism of their values for $n = 0$ which recovers the original additive functors from which the derived functors are constructed.

D.2 Spectral Sequences

Following the treatment in [W] I will introduce the theory of spectral sequences up to the theory behind the Hoschild-Serre spectral sequence as a special case of the Grothendieck spectral sequence. I will avoid reference to the exact couple.

The aim of this chapter is to interpret and apply the formula:

$$E_2^{pq} = Ext^p_R(A, Ext^q_R(S,B)) \Longrightarrow Ext^{p+q}_R(A,B)$$

D.2.1 Spectral Sequences as Approximations of the Total Homology

Recall, a **Double Complex** (bicomplex) in a category $\mathcal{A}$ is a family $\{c_{p,q}\}$ of objects of $\mathcal{A}$, together with maps:

$$d^h : c_{p,q} \to c_{p-1,q} \text{ and } d^v : c_{p,q} \to c_{p,q-1}$$

such that $d^h \circ d^h = d^v \circ d^v = 0 = d^v \circ d^h + d^h \circ d^v$ - an *anticommutative lattice* with chains for rows and columns.

We can now define a total complex $Tot^{\Pi}(C)$ by

$$Tot^{\Pi}(C)_n = \prod_{p+q=n} c_{p,q}$$

and the formula $d = d^h + d^v$ defines maps:

$$d : Tot^{\Pi}(C)_n \to Tot^{\Pi}(C)_{n-1}$$

moreover,

$$d \circ d = (d^h + d^v) \circ (d^h + d^v) = d^h \circ d^h + d^v \circ d^v + (d^h \circ d^v + d^v \circ d^h) = 0$$

ie. anticommutivity property gives that d is a differential.

Suppose double complex E consists of just two columns at p and p-1:

$$\begin{array}{ccc} \vdots & & \vdots \\ \downarrow d^v & & \downarrow d^v \\ E_{p-1,2} & \xleftarrow{d^h} & E_{p,2} \\ \downarrow d^v & & \downarrow d^v \\ E_{p-1,1} & \xleftarrow{d^h} & E_{p,1} \\ \downarrow d^v & & \downarrow d^v \\ E_{p-1,0} & \xleftarrow{d^h} & E_{p,0} \\ \downarrow d^v & & \downarrow d^v \\ \vdots & & \vdots \end{array}$$

For fixed n, set $q = n - p$. Hence, an element of $Tot(E) = T$ is represented by an element $(a, b) \in E_{p-1,q+1} \, X \, E_{pq}$.

Viewing columns as vertical chains we take homologies, with notation:

$$\begin{aligned} E^1_{pq} &= H(E^o_{pq}) \text{ (taken vertically)} \\ &= ker\, d^v_{pq}/Im\, d^v_{pq+1} \end{aligned}$$

The horizontal maps d^h induce maps between these homology groups, since if $x \in ker\, d^v$:

$$d^v(d^h x) = -d^h(d^v x) = -d^h(0) = 0$$

Hence $d^h x \in ker\, d^v$, which induces $d^h : E^1_{p,q} \to E^1_{p-1,q}$

We now take the horizontal homology to form E^2:

$$E_{p-1,q+2} \,\mathrm{x}\, E_{p,q+1} \longrightarrow^{d^v \,\mathrm{x}\, d^h + d^v} E_{p-1,q+} \,\mathrm{x}\, E_{p,q} \longrightarrow^{d^v \,\mathrm{x}\, d^h + d^v} E_{p-1,q} \,\mathrm{x}\, E_{p,q-1}$$

Then,

$$H_{p+q}(T) = \{(x, y) \,|\, d^v x + d^h y = 0 \text{ and } d^v y = 0\}$$

This gives,

$$\begin{aligned} E^2_{pq} &= \{x \in E^1_{pq} \,|\, d^h x = 0\} \\ &= \{d^v x = 0,\, d^h x = 0\} \end{aligned}$$

This gives rise to the canonical injection:

$$\begin{aligned} E^2_{p-1,q+1} = \{x \in E^0_{p-1,q+1} \,|\, d^v x = 0,\, d^h x = 0\} &\hookrightarrow H_{p+q}(T) = \{(x, y) \,|\, d^v x + d^h y = 0,\, d^v y = 0\} \\ x &\hookrightarrow (x, 0) \end{aligned}$$

This is an injection with cokernel $\{y \in E_{pq} \,|\, d^v y = 0 \text{ and } 0 + d^h y = 0\} = E^2_{pq}$. Hence,

$$0 \to E^2_{p-1,q+1} \to H_{p+q}(T) \to E^2_{pq} \to 0$$

Hence, up to extension, E^2 gives homology of $T = Tot(E)$.
Continuing this analysis gives rise to the spectral sequence as an approximation to homology of T.

D.2.2 Definition of Spectral Sequence

Definition

A **homology spectral sequence** (starting with E^a) in an abelian category $\mathcal{A}$ has the following structure:

1. A family $\{E^r_{pq}\}$ of objects of $\mathcal{A}$ defined for all integers p, q, and $r \geq a$.
2. Maps $d^r_{pq} : E^r_{pq} \to E^r_{p-r,q+r-1}$ that are differentials. ie. $d^r \circ d^r = 0$. In other words, lines of slope $-(r+1)/r$ of the lattice E^r_{**} form chain complexes
3. There are isomorphisms between E^{r+1}_{pq} and the homology of E^r_{**} at the pq spot. Hence,

$$E^{r+1}_{pq} \cong ker(d^r_{pq})/image(d^r_{p+r,q-r+1})$$

Immediately we see that E^{r+1}_{pq} is a subquotient of E^r_{pq}. We define the total degree of a term E^r_{pq} as $n = p + q$ (as when defining the total complex), then, viewing the double complex as a lattice, we see the terms of degree n lie on a line of slope -1. Moreover, each differential D^r_{pq} decreases the total degree by 1.

Definition

A homology spectral sequence is said to be **bounded**, if for each n there are only finitely many nonzero terms of total degree n in E^a_{**} (if true for one such a true for all greater ones). Then, for each $p \,\&\, q\, \exists\, r_0$ such that $E^r_{pq} = E^{r+1}_{pq}\, \forall\, r \geq r_0$, and we write E^∞_{pq} for this *STABLE VALUE* of E^r_{pq}.

Definition

A bounded spectral sequence **converges** to H_* if we are given a family of objects H_n, each having a *finite* filtration

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots \subseteq F_t H_n = H_n$$

and there exist isomorphisms $E^\infty_{pq} \cong F_p H_{p+q}/F_{p-1} H_{p+q}$, this situation is denoted

$$E^a_{pq} \Longrightarrow H_{p+q}$$

Definition

A homology spectral sequence **collapses** at E^r $(r \geq 2)$ if there is exactly one nonzero row or column in the lattice $\{E^r_{pq}\}$. If we know a collapsed sequence converges to H_n it is easy to read these values off - H_n is the unique nonzero E^r_{pq} with $p + q = n$. Compare this with the 2 column collapse of D.2.1.

If the spectral sequence does not collapse at some finite level then it becomes very difficult to manipulate.

Definition

A spectral sequence is **regular** if for each pair p, q the differentials d^r_{pq} are zero for all large r. ie if

$$Z^\infty_{pq} = \bigcap_{r=a}^{\infty} Z^r_{pq} = Z^r_{pq}$$

for all large r. As we will see regularity is a very convenient condition ensuring convergence.

D.2.3 Spectral Sequences Arising from a Filtration

To every exhaustive filtration F of a chain complex
$\mathbf{C}$: $\cdots \subseteq F_{p-1}\mathbf{C} \subseteq F_p\mathbf{C} \subseteq \ldots$ (with $\mathbf{C} = \bigcup F_p\mathbf{C}$) we construct an associated spectral sequence (without worrying about its convergence properties).

Theorem

A filtration F of a chain complex $\mathbf{C}$ naturally determines a spectral sequence starting with $E^0_{pq} = F_p C_{p+q}/F_{p-1} C_{p+q}$ and $E^1_{pq} = H_{p+q}(E^0_{p*})$.

D.2.4 Construction

Write $\eta_p : F_p\mathbf{C} \twoheadrightarrow F_p\mathbf{C}/F_{p-1}\mathbf{C} = E^0_p$ - a surjection.

We now define *cycles modulo* $F_{p-r}\mathbf{C}$ - approximate cycles vanishing to a smaller set under the boundary map.

$$A^r_p = \{c \in F_p\mathbf{C} : d(\mathbf{C}) \in F_{p-r}\mathbf{C}\}$$

and their images:

$$\begin{aligned} Z^r_p &= \eta_p(A^r_p) \text{ in } E^0_r \\ B^{r+1}_{p-r} &= \eta_{p-r}(d(A^r_p)) \text{ in } E^0_{p-r} \end{aligned}$$

Taking $Z_p^\infty = \bigcap_{r=1}^\infty$ and $B_p^\infty = \bigcup_{r=1}^\infty B_p^r$ we define a tower of subobjects of E_p^0 -

$$0 = B_p^0 \subseteq B_p^1 \subseteq \cdots \subseteq B_p^r \subseteq \cdots \subseteq B_p^\infty \subseteq Z_p^\infty \subseteq \cdots \subseteq Z_p^r \subseteq \cdots \subseteq Z_p^1 \subseteq Z_p^0 = E_p^0$$

Since $A_p^r \cap F_{p-1}\mathbf{C} = A_{p-1}^{r-1}$, hence $Z_p^r \cong A_p^r / A_{p-1}^{r-1}$ we have

$$E_p^r = \frac{Z_p^r}{B_p^r} \cong \frac{A_p^r + F_{p-1}(\mathbf{C})}{d(A_{p+r-1}^{r-1}) + F_{p-1}(\mathbf{C})} \cong \frac{A_p^r}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}}$$

and the differential of $\mathbf{C}$ induces $d_p^r : E_p^r \to E_{p-r}^r$ and the map d determines isomorphisms $Z_p^r/Z_p^{r+1} \cong B_{p-r}^{r+1}/B_{p-r}^r$ $(\star)$.

The kernel of d_p^r is

$$\frac{\{z \in A_p^r : d(z) \in d(A_{p-1}^{r-1}) + A_{p-r-1}^{r-1}\}}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}} = \frac{A_{p-1}^{r-1} + A_p^{r+1}}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}} \cong \frac{Z_p^{r+1}}{B_p^r}$$

and by $(\star)$ this factors as

$$E_p^r = Z_p^r/B_p^r \to Z_p^r/Z_p^{r+1} \cong B_{p-r}^{r+1}/B_{p-r}^r \hookrightarrow Z_{p-r}^r/B_{p-r}^r = E_{p-r}^r$$

Hence image of d_p^r is B_{p-r}^{r+1}/B_{p-r}^r, and relabelling ($p+r$ instead of p) we have isomorphisms:

$$E_p^{r+1} = Z_p^{r+1}/B_p^{r+1} \cong ker(d_p^r)/im(d_{p+r}^r)$$

and hence get isomorphisms between $E^(r+1)$ and $H_*(E^r)$ which completes the construction.

D.2.5 Spectral Sequences of a Double Complex

Given a Double Complex $\mathbf{C}$ we may collapse either rows or columns to give two different filtrations of $\mathbf{C}$ each giving rise to spectral sequences related to the homology of $Tot(\mathbf{C})$. The tactic here is to compare the two as a way to calculate homology. Let $\mathbf{C} = C_{**}$ be a double complex.

Definition

We first consider **Filtration by Columns**.

Filter the total complex $Tot(C)$ by columns of C:

Set $F_n^I Tot(C)$ as the total complex of the double subcomplex (copying $\mathbf{C}$ for the first n columns and 0 elsewhere)

$$\begin{array}{ccc|ccc} \dots & * & * & 0 & 0 & \dots \\ \dots & * & * & 0 & 0 & \dots \\ \dots & * & * & 0 & 0 & \dots \end{array}$$

This filtration gives a spectral sequence $\{{E_{pq}^r}^I\}$ with $\{{E_{pq}^0}^I\} = C_{pq}$, where d^0 are just the vertical differentials d^v of C -

$${E_{pq}^1}^I = H_q^v(C_{p*})$$

and the horizontal differentials induce $d^1 : H_q^v(C_{p,*}) \to H_q^v(C_{p-1,*})$ giving

$${E_{pq}^2}^I = H_p^h H_q^v(C)$$

If C is a first quadrat double complex. the filtration is bounded, giving convergence of the spectral sequence:

$${E_{pq}^2}^I = H_p^h H_q^v(C) \Longrightarrow H_{p+q}(Tot(C))$$

Secondly consider **Filtration by Rows**.

Let ${F_n}^{II}(Tot(C))$ be he total complex of the subcomplex formed by first n rows, and zero elsewhere:

$$\begin{array}{ccc} \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ - & - & - \\ * & * & * \\ * & * & * \\ \vdots & \vdots & \vdots \end{array}$$

E^0 is calculated as $F_pTot(C)/F_{p-1}Tot(C)$ in the row C_{*p}, ${E_p^0}^{II} = C_{qp}$ and ${E_{pq}^1}^{II} = H_q^h(C_{*p})$, and the vertical differentials d^v induce d^1:

$$ {E_{pq}^2}^{II} = H_p^v H_q^h(C)$$

As above, the spectral sequence converges to $H_*Tot(C)$.

D.2.6 Grothendieck Spectral Sequence

D.2.7 Theorem: Grothendieck

Given $F : \mathcal{U} \mapsto \mathcal{B}$, $G : \mathcal{B} \mapsto \mathcal{C}$, assume that if I is an injective object of $\mathcal{U}$, then $F(I)$ is G-acyclic. Then there is a spectral sequence $\{E_n(A)\}$ corresponding to each object A of $\mathcal{U}$, such that

$$E_1^{p,q} = (R^pG)(R^{q-p}F)(A) \Longrightarrow R^q(GF)(A)$$

which converges finitely to the graded object associated with $\{R^q(GF)(A)\}$, suitably filtered.

To prove this we study first Filtration by Columns as in D.2.4 to obtain $H^q(Tot\mathbf{B}) = R^q(GF)(A)$ where $\mathbf{B}$ is a double chain complex constructed from a base row of a resolution of A, and each column a resolution of elements in the initial resolution. We now explicitly calculate terms using Filtration by Rows. The technical hypotheses give convergence of the 2 spectral sequences to the same value. See [HS] for the details.

D.2.8 Application

Let N be a normal subgroup of K with quotient group Q:

$$N \rightarrowtail^i K \twoheadrightarrow^p Q$$

Let $\mathcal{U}$ be the category of K-modules.

Let $\mathcal{B}$ be the category of Q-modules.

Let $\mathcal{C}$ be the category of Abelian Groups.

Define $F : \mathcal{U} \to B$ where $F(A) = Hom_N(\mathbb{Z}, A) = A^N$ - subgroup of A of elements fixed by N.

Similarly, $G : \mathcal{B} \to C$ where $G(B) = Hom_Q(\mathbb{Z}, B) = B^Q$

Then A^N is a Q-module via $(\star)(px) \circ a = xa$, $x \in K$, $a \in A^N$ making F (and G) additive functors. Also,

$$GF(A) = Hom_K(\mathbb{Z}, A) = A^K$$

Given this structure, as an application of D.2.7 we now prove,

D.2.9 Hochschild-Serre

Let A be a K-module, then there is a natural action of Q on the cohomology groups $H^m(N, A)$. Moreover, there is a spectral sequence $\{E_n(A)\}$ such that

$$E_1^{p,q} = H^p(Q, H^{q-p}(N, A)) \Longrightarrow H^q(K, A)$$

which converges finitely to the graded group associated with $\{H^q(K, A)\}$ suitably filtered.

By $\star$, F preserves monomorphisms, hence injectives. In particular, injectives are mapped to acyclics, and thus the hypotheses of D.2.8 are satisfied.

Since $\mathbb{Z}K$ is a free $\mathbb{Z}N$-module, a K-injective resolution of A is also an N-injective resolution. Given any such K-injective resolution of A, $I_0 \to I_1 \to I_2 \to \dots$ the complex $Hom_N(\mathbb{Z}, A) \to Hom_N(\mathbb{Z}, I_0) \to Hom_N(\mathbb{Z}, I_1) \to \dots$ is a Q-complex and so the groups $H^m(N, A)$ become Q-modules with

$$R^m F(A) = H^m(N, A)$$

Since, $R^m G(B) = H^m(Q, B)$ we have $R^m(GF)(A) = H^m(K, A)$. Recalling D.2.8 completes the proof.

Alternatively, since $N \to G \to G/N$ we can take cohomolgy once more to state the theorem as:

For G a profinite group, N a closed normal subgroup of G,

$$H^p(G/N, H^q(N, A)) \Longrightarrow_p H^q(G, A)$$

D.2.10 Sequences Formed by Terms of Low Degree

Once we have the existence of a spectral sequence $E_2^{pq} \Longrightarrow H^*$ we can look at the first few terms to form short exact sequences.

Notation:

A filtration on A is a family of subgroups $\{A^n\}_{-\infty}^{\infty}$ of A with $A^{n+1} \subseteq A^n$, ”$A^{-\infty}$” $= \cup A^n = A$, ”A^{∞}” $= \cap A^n = \{0\}$. These two objects together form a filtered abelian group.

The definition of $E_\infty^{p,q}$ gives

$$E_\infty^{p,q} = \frac{H^{p+q}(A) \cap H^*(A)^p}{H^{p+q}(A) \cap H^*(A)^{p+1}}$$

Thus, for $p + q = n$ the $E_\infty^{p,q}$ are just the composition factors in the filtration

$$H^n(A) \supseteq H^n(A)^1 \supseteq H^n(A)^2 \supseteq \dots \supseteq H^n(A)^v \supseteq \dots$$

ie. $E_\infty^{p,n-p}$ is the p-th composition factor in $H^n(A)$. Representing $E_r^{p,q}$ as a lattice, $d_r^{p,q}$ is an arrow ”going over r and down $r - 1$”.

This interpretation immediately gives, dually to D.2.1, that when a *spectral sequence degenerates except for columns $j, j+1$ we have only 2 terms $E_2^{j+1,i-1}$ and $E_2^{0,i}$ in the filtration of H^{i+j}*, hence:

Lemma

If the spectral sequence degenerates as above, then:

$$0 \to E_2^{j+1,i-1} \to H^{i+j} \to E_2^{j,i} \to 0$$

Example

Invoking the Hochschild-Serre spectral sequence above, we have if
$E_2^p q = H^p(G/N, H^{q-p}(N, W))$ vanishes for $p \neq 0, 1$ (for example if G/N has cohomological dimension 1) then the Lemma gives:

$$0 \to H^1(G/N, H^{i-1}(N, W)) \to H^i(G, W) \to H^0(G/N, H^i(N, W)) \to 0$$

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