SUTD 10.004 Advanced Mathematics II 2019 Midterm Exam Practice Worksheet Answers

1. For $f(x,y) = x^3 + 3x^2y - 2y^2$, $f_x(1,2) = 3(1)^2 + 6(1)(2) = 15$ and $f_y(1,2) = 3(1)^2 - 4(2) = -5$.

2. Directional derivative of f at (x_0, y_0) in the direction of the unit vector $\vec{u}$ is the rate of change of f in the direction of $\vec{u}$ at (x_0, y_0), which is equal to

$$\lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

, provided that the limit exists.

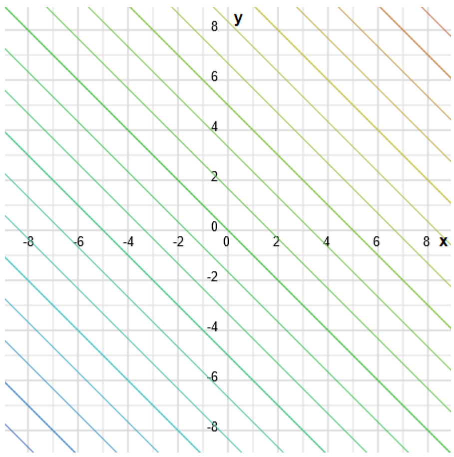
3. For $f(x,y) = x^2y$ and $\vec{v} = [4, -3]$, $D_{\vec{v}}f(2,6) = f_x(2,6)v_1 + f_y(2,6)v_2 = (2)(2)(6)(4) + (2)^2(-3) = 84$.

4. The graph $f(x,y) = 2$ looks like a **horizontal plane**.

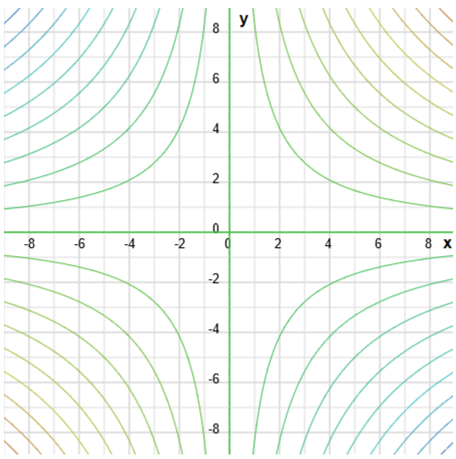
5. The level curves of $z = e^{\sin(x^2+y^2)}$ will be **circles**.

6. **False**. A possible counterexample would be the parabolic cylinder with equation $f(x,y) = y^2$.

7. (a) The contours are equally-spaced parallel straight lines described by the equation $y = -x + C$, where C is a real-valued constant.



(b) The contours are curves described by the equation $y = \frac{D}{x}$, where D is a real-valued constant, except at $f(x,y) = 0$, where the contour lines are $x = 0$ and $y = 0$. As $|f(x,y)|$ increases, the contour lines become more closely spaced.



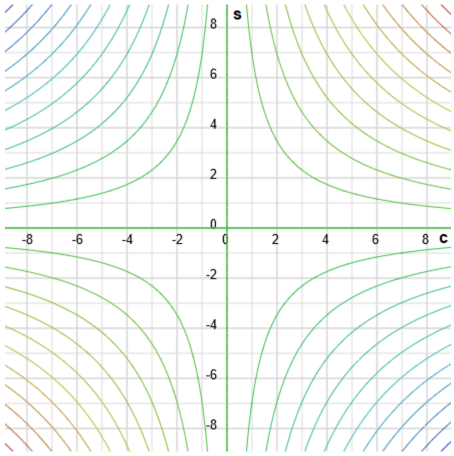
8. Rearranging $2(x-z) = 3(x+y)$, we would get $x + 3y + 2z = 0$. Thus, a normal vector to this plane would be $[k, 3k, 2k]$, for any real value of k.

9. The gradient of a function is the direction of the **greatest rate of increase**.

10. $\nabla f(x, y) = \left[\frac{15}{2}x^4, -\frac{24}{7}y^5\right]$.

11. (a) $\frac{\partial p}{\partial c}$ represents the instantaneous rate of change of a person's blood pressure with respect to a change in the rate of volume of blood flowing through a person's heart with respect to time.

(b) i. The level curves represent the possible values of c and s at each value of p.



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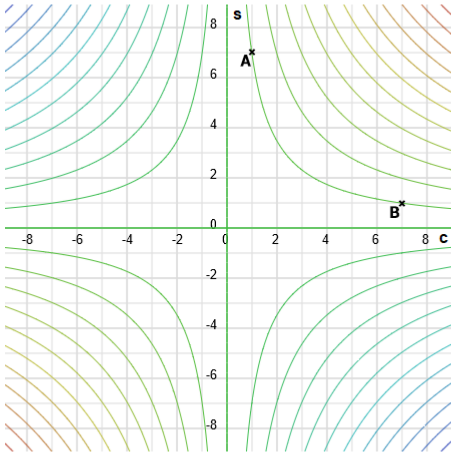

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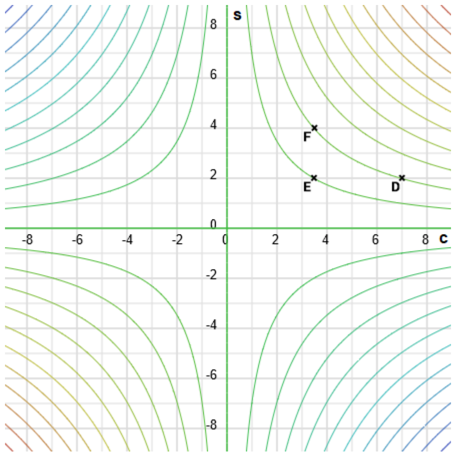


12. (a) $a = k$.

    (b) Coordinates of P: (x_2, y_2).

    (c) M is a minimum.

    (d) $m < 0$ and $n > 0$.

13. The z-component of the normal vector is -1, not the z-coordinate (z_0). Furthermore, the original equation $f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$ can be re-arranged to form $f_x(x_0, y_0)x + f_y(x_0, y_0)y - z = f_x(x_0, y_0)x_0 + f_y(x_0, y_0)y_0 - z_0$. In other words,

$$\begin{bmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f_x(x_0, y_0) \\ f_y(x_0, y_0) \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$

    Thus, as seen from this equation, it is trivial that this relationship is applicable for any coordinate (x_0, y_0, z_0) on a surface defined by any function $f(x, y)$. This is similar to the trick question of dividing both sides of the equation $0 \cdot 5 = 0 \cdot 2$ by 0.

14. Equation of the tangent plane: $x - y - \frac{1}{e}z = -1$.

15. (a) The point $(0, 0)$ is a local maxima.

    (b) The value of f along the x-axis and y-axis is always less than 10. To maintain continuity of the function, $(0, 0)$ must be at least a local maxima. $(0, 0)$ is not guaranteed to be a global maxima, since there could be a global maxima not along the x- and y-axes within the region $[-1, 1] \times [-1, 1]$.

16. $\frac{\partial z}{\partial u} = \frac{1}{uv} \cos \frac{\ln u}{v}$ and $\frac{\partial z}{\partial v} = -\frac{\ln u}{v^2} \cos \frac{lnu}{v}$.

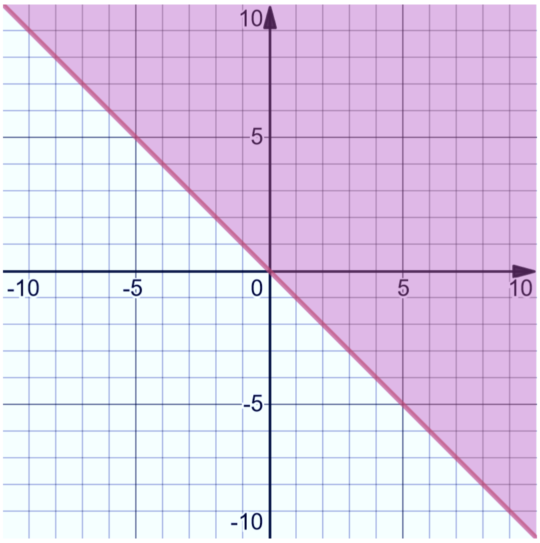
17. (a) $\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x} + f'(t) \left[\left(\frac{\partial g}{\partial y} \right) + 2f(t) \left(\frac{\partial g}{\partial z} \right) \right]$.

    (b) $f'(t) = -1$.

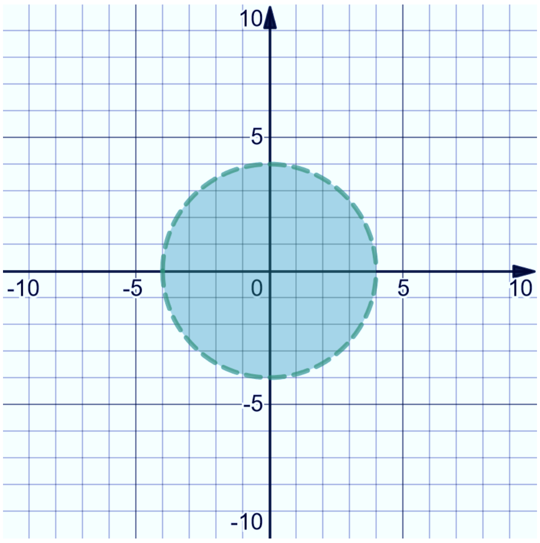
18.

$$\begin{aligned} \left(\frac{\partial z}{\partial r} \right)^2 + \frac{1}{r^2} \left(\frac{\partial z}{\partial \theta} \right)^2 &= \left[\left(\frac{\partial z}{\partial x} \right) (\cos \theta) + \left(\frac{\partial z}{\partial y} \right) (\sin \theta) \right]^2 + \frac{1}{r^2} \left[\left(\frac{\partial z}{\partial x} \right) (-r \sin \theta) + \left(\frac{\partial z}{\partial y} \right) (r \cos \theta) \right]^2 \\ &= \left(\frac{\partial z}{\partial x} \right)^2 (\sin^2 \theta + \cos^2 \theta) + \left(\frac{\partial z}{\partial y} \right)^2 (\sin^2 \theta + \cos^2 \theta) \\ &+ \left(\frac{\partial z}{\partial x} \right) \left(\frac{\partial z}{\partial y} \right) \sin 2\theta - \left(\frac{\partial z}{\partial x} \right) \left(\frac{\partial z}{\partial y} \right) \sin 2\theta \\ &= \left(\frac{\partial z}{\partial x} \right)^2 + \left(\frac{\partial z}{\partial y} \right)^2 \end{aligned}$$

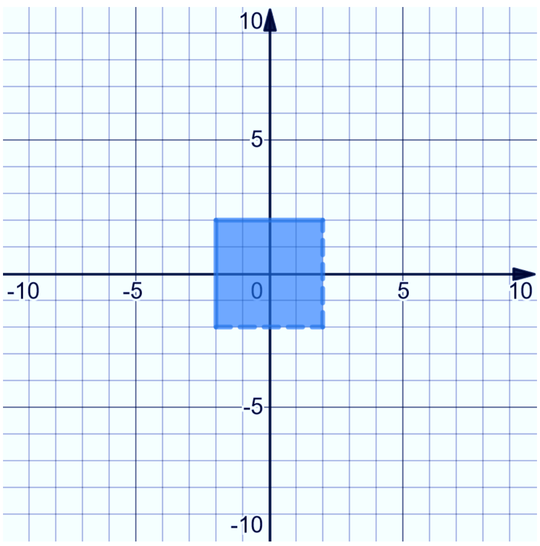
19. (a) $\{(x, y) \in \mathbb{R}^2 : x + y \geqslant 0\}$



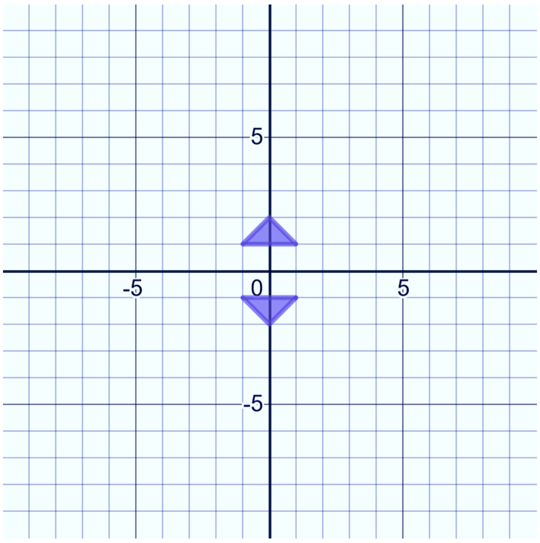
(b) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 16\}$



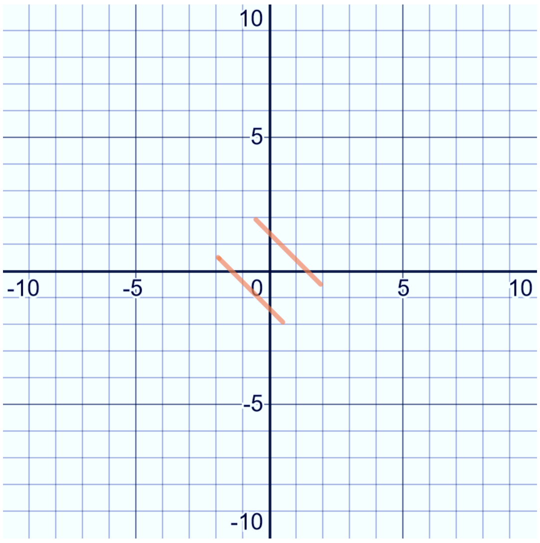
(c) $[-2,2) \times (-2,2]$



20. (a) Bounded and closed.



(b) Bounded and open.



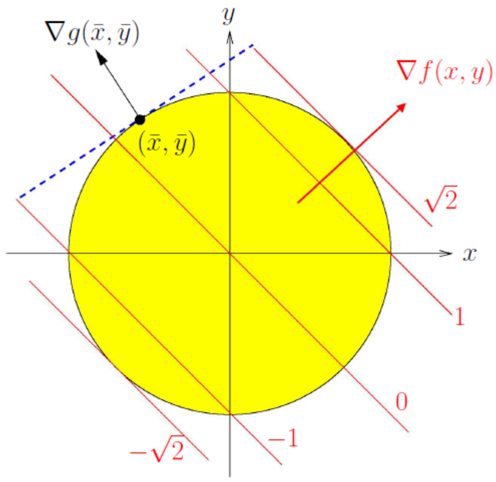
21. Yes, there is a difference.

    A function is closed if for each $\alpha \in \mathbb{R}$, the sublevel set $\{x \in X | f(x) \leq \alpha\}$ is a closed set. A function is bounded if there exists a real number M such that $|f(x)| \leq M$ for all x in X, the function's domain set.

    Meanwhile, a region is closed if the region includes all of its boundary points. A region is bounded if there exists a real number N such that $|x| \leq N$ for all x in the region.

    Extreme Value Theorem is useful when the function's domain region is closed and bounded, not the function itself.

22. Lagrange multipliers are used when we want to maximize (or minimize) the value of $f(x, y)$, subject to some equality constraint(s) $g(x, y) = 0$. By moving along the constraint(s) towards (or away from) the direction of where $\nabla f(x, y)$ has a component, we will eventually reach the point where $\nabla f(x, y)$ is parallel (or anti-parallel) to $\nabla g(x, y)$. At that optimal point, it would be the maximum (or minimum) value of f that we can reach within the specified constraint(s).



23. The function contains local minimas along the line $y = x$, except the origin (since the function is undefined there), with the value of $f(x, y) = \frac{40}{3}$. The function also contains local maximas along the line $y = -x$, except the origin (since the function is undefined there), with the value of $f(x, y) = 0$.

24. (a) All 4 critical points $(1, 1, -2)$, $(1, -1, -2)$, $(-1, 1, -2)$ and $(-1, -1, -2)$ are saddle points.

    (b) Equation of tangent plane: $24x + 12y - z = 53$.

    (c) Considering that $f(x, 1) = x^4 - 2x^2 - 1 = (x^2 - 1)^2 - 2 \geqslant -2$ for all $x \in \mathbb{R}$ and $f(0, y) = -y^4 \leqslant 0$ for all $y \in \mathbb{R}$, $f(x, y)$ is not bounded above and not bounded below.

25. (a) True.

    (b) False. The minimum and maximum are not constrained to only the x direction or the y direction. As long as at a point (x^*, y^*) the function is a minimum in one direction and a maximum in another direction, it is considered a saddle point.

    (c) False. The global minimum would have a value of $c + \frac{1}{5}$.

26. Minimum point of $f(x, y) = x^2 + y^2$ subject to the constraint $2x + 4y = 15$ is $(\frac{3}{2}, 3, \frac{45}{4})$.

27. Fubini's Theorem states that if $f(x, y)$ is continuous on R, a rectangular region $[a, b] \times [c, d]$ with finite values of a, b, c and d, then

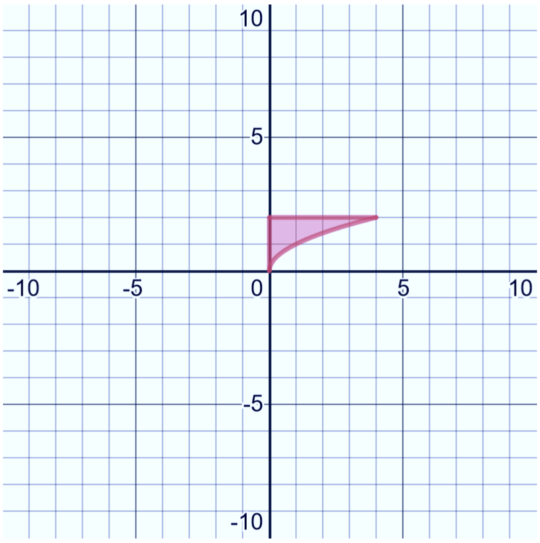
$$\iint_R f(x, y)\, \mathrm{d}A = \int_a^b \left(\int_c^d f(x, y)\, \mathrm{d}y \right) \mathrm{d}x = \int_c^d \left(\int_a^b f(x, y)\, \mathrm{d}x \right) \mathrm{d}y$$

28. **False**. This is because at $(0, 0)$, $\frac{x^2 - y^2}{(x^2 + y^2)^2}$ is undefined and thus it is discontinuous at the origin. As it is discontinuous, it is non-integrable and thus, Fubini's Theorem is not applicable.

29. The region of integration of an integral is the domain over which the integration of a function is conducted.

30. $\int_0^2 \int_0^3 (x^2 + y^2)\, \mathrm{d}y\, \mathrm{d}x = 26$.

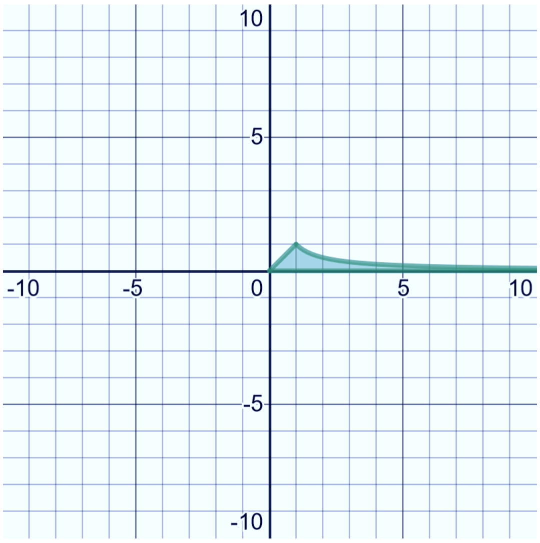
31. Region of integration: $0 \leqslant x \leqslant y^2$ and $0 \leqslant y \leqslant 2$.



32. The region of integration bounded by the lines $x = y + 1$ and $x = \frac{y^2}{2} - 3$ is **horizontally simple**.

33. $\int_0^1 \int_0^1$ min(x, y) dx d$y = \frac{1}{3}$.

34. (a) Region of integration: $y \leqslant x \leqslant \frac{1}{y}$ and $0 \leqslant y \leqslant 1$.



(b) $I = \frac{1}{8}(e - 1)^2$.

35. (a) $2 \int_0^R \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{R^2 - x^2 - y^2}$ dx dy.

(b) $\int_0^R \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} 1$ dz dy dx.

(c) $2 \int_0^{2\pi} \int_0^R r\sqrt{R^2 - r^2}$ dr dθ.

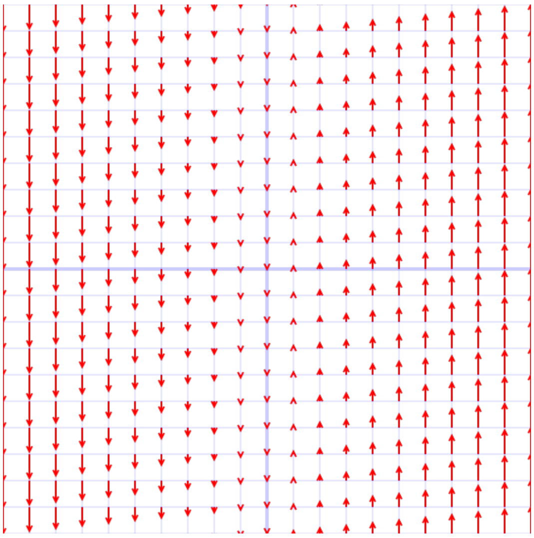
(d) $\int_0^{\frac{\pi}{2}} \int_0^R \int_0^{\sqrt{R^2-r^2}} r$ dz dr dθ.

(e) $\int_0^{\pi} \int_0^{2\pi} \int_0^R \rho^2 \sin\phi$ dρ dθ dϕ.

36. $\iiint f(x,y,z)\ \mathrm{d}V$ could be used to calculate the mass of a three-dimensional object, given that $f(x,y,z)$, which is the density of the object at each point, is not a constant.

37. **(e) None of the above**. The line integral is equal to $\frac{1}{5}\left(e^{\pi}+9\right)^{\frac{5}{2}} - 20\sqrt{10}$.

38. Vector field of $F(x,y) = \begin{pmatrix} 0 \\ 2x \end{pmatrix}$.



39. Gradient Theorem states that if $\vec{\boldsymbol{F}} = \nabla f$ is a conservative vector field from $\mathbb{R}^n$ to $\mathbb{R}^n$ and γ is any (piecewise) differentiable curve in $\mathbb{R}^n$ with initial point $\vec{\boldsymbol{a}}$ and final point $\vec{\boldsymbol{b}}$, then

$$\int_{\gamma} \vec{\boldsymbol{F}}(\vec{\boldsymbol{p}}) \cdot \mathrm{d}\vec{\boldsymbol{p}} = f(\vec{\boldsymbol{b}}) - f(\vec{\boldsymbol{a}}).$$

40. (a) Work done by $\vec{\boldsymbol{F}} = \frac{\pi}{2}$.

(b) With the new path, work done by $\vec{\boldsymbol{F}} = 1$.

(c) The vector field $\vec{\boldsymbol{F}}$ is not conservative, since the work done by $\vec{\boldsymbol{F}}$ is path-dependent.