SUTD 2019 10.001 Advanced Math 1 - 2D Project Report

James Raphael Tiovalen, Sharmayne Lim Zhiyu, Phoebe Leam Xin Ni, Diya Bhujanga Kotian
Singapore University of Technology and Design
8 Somapah Rd, Singapore 487372

1. We have the original error equation as

$$e(K_M) = \sum_{i=1}^{n} \left(\frac{K_M}{V_{max}} \frac{1}{x_i} + \frac{1}{V_{max}} - \frac{1}{v_i} \right)^2 . \tag{1}$$

By differentiating the error equation with respect to K_M once, we will get

$$e'(K_M) = \sum_{i=1}^{n} 2 \left(\frac{K_M}{V_{max}} \frac{1}{x_i} + \frac{1}{V_{max}} - \frac{1}{v_i} \right) \left(\frac{1}{V_{max} x_i} \right) . \tag{2}$$

By equating $e'(K_M) = 0$ and since V_{max} is a constant, we get

$$\sum_{i=1}^{n} \left(\frac{K_M}{V_{max}} \frac{1}{{x_i}^2} + \frac{1}{V_{max} x_i} - \frac{1}{v_i x_i} \right) = 0 \tag{3}$$

$$K_M \sum_{i=1}^{n} \left(\frac{1}{x_i} \right)^2 + \sum_{i=1}^{n} \left(\frac{v_i}{v_i x_i} - \frac{V_{max}}{v_i x_i} \right) = 0. \tag{4}$$

Therefore,

$$K_M = \frac{\sum_{i=1}^{n} \left(\frac{V_{max} - v_i}{v_i x_i} \right)}{\sum_{i=1}^{n} \left(\frac{1}{x_i} \right)^2} . \tag{5}$$

Using Excel, with $n = 21$ and $V_{max} = 2.17$ mg/min, we obtain $K_M \approx 2.03$ mg/mL.
By differentiating the error equation with respect to K_M twice, we will get

$$e''(K_M) = 2 \sum_{i=1}^{n} \left(\frac{1}{V_{max} x_i} \right)^2 . \tag{6}$$

Thus, as $\frac{1}{V_{max} x_i} > 0 \ \forall \ i \in \mathbb{N}_{\neq 0}$, $e''(K_M) > 0$. Hence, $e(K_M)$ is a global minimum when $K_M \approx 2.03$ mg/mL.

2. 2.1. (a) By integrating both sides of the equation with respect to t, we get

$$\int \frac{1}{g - \frac{\beta}{m} v} \frac{\mathrm{d}v}{\mathrm{d}t} \mathrm{d}t = \int 1 \, \mathrm{d}t. \tag{7}$$

Solving the equation, we will get

$$-\frac{m}{\beta} \ln \left| g - \frac{\beta}{m} v \right| = t + C \tag{8}$$

$$g - \frac{\beta}{m} v = (e^{-\frac{\beta}{m} t})(\pm e^D), \tag{9}$$

where C is a constant and $D = -\frac{\beta}{m} C$.
As $v(0) = 0$, we get $\pm e^D = g$, and thus

$$v(t) = \frac{m}{\beta} g \left(1 - e^{-\frac{\beta}{m} t} \right) . \tag{10}$$

(b) By integrating both sides of Equation (10) with respect to t, we get

$$\int v \,\mathrm{d}t = \int \frac{m}{\beta} g \left(1 - e^{-\frac{\beta}{m}t}\right) \mathrm{d}t. \tag{11}$$

Solving the equation, we will get

$$s = \frac{m}{\beta} g \left(t + \frac{m}{\beta} e^{-\frac{\beta}{m}t}\right) + E, \tag{12}$$

where E is another constant.
As $s(0) = 0$, we get $E = -\frac{m^2}{\beta^2} g$, and thus

$$s(t) = \frac{m}{\beta} g \left[t + \frac{m}{\beta}\left(e^{-\frac{\beta}{m}t} - 1\right)\right]. \tag{13}$$

(c) We have the error function as

$$e(\beta) := \sum_{i=1}^{n} \left(s(t_i) - s_i\right)^2 \tag{14}$$

$$e(\beta) = \sum_{i=1}^{n} \left(\frac{m}{\beta} g t_i + \frac{m^2}{\beta^2} g e^{-\frac{\beta}{m}t_i} - \frac{m^2}{\beta^2} g - s_i\right)^2. \tag{15}$$

By differentiating $e(\beta)$ with respect to β using Chain Rule, we will get

$$e'(\beta) = \sum_{i=1}^{n} 2\left(-\frac{m^2}{\beta^5} g^2\right)\left(\beta t_i + m e^{-\frac{\beta}{m}t_i} - m - \frac{s_i}{mg}\beta^2\right)\left(\beta t_i + 2m e^{-\frac{\beta}{m}t_i} + \beta t_i e^{-\frac{\beta}{m}t_i} - 2m\right). \tag{16}$$

By differentiating $e'(\beta)$ with respect to β using Product Rule, we will get

$$e''(\beta) = \sum_{i=1}^{n} 2\left[AB\left(\frac{5m^2}{\beta^6} g^2\right) - \frac{m^2}{\beta^5} g^2 B\left(t_i - t_i X - \frac{2s_i}{mg}\beta\right) - \frac{m^2}{\beta^5} g^2 t_i A\left(1 - 2X + X\left(1 - \frac{\beta}{m} t_i\right)\right)\right], \tag{17}$$

where

$$A = \beta t_i + m e^{-\frac{\beta}{m}t_i} - m - \frac{s_i}{mg}\beta^2, \tag{18}$$

$$B = \beta t_i + 2m e^{-\frac{\beta}{m}t_i} + \beta t_i e^{-\frac{\beta}{m}t_i} - 2m, \tag{19}$$

$$X = e^{-\frac{\beta}{m}t_i}. \tag{20}$$

To get the minimum $e(\beta)$, we will need to find the value of β where $e'(\beta) = 0$. As it is difficult to find the root(s) of $e'(\beta)$ analytically, we will solve it numerically by using Newton's Method, where

$$\beta_{k+1} = \beta_k - \frac{e'(\beta_k)}{e''(\beta_k)}. \tag{21}$$

Using Excel, with $n = 10$, $g = 9.80665$ m/s^2 and $m = 0.31983$ kg, we obtain $\beta \approx 0.0818$ kg/s.

From Excel, when $\beta \approx 0.0818$, $e''(\beta) \approx 248.5 > 0$. The graph of $e(\beta)$ against β also shows that there is only one minimum point of $e(\beta)$ $\forall\ \beta \in \mathbb{R}_{>0}$. Thus, $e(\beta)$ is a global minimum when $\beta \approx 0.0818$ kg/s.

2.2. (a) We approximate the original differential equation as

$$T(t_n) - \mu_0 mgcos\theta - mgcos\theta - \beta v_n = m\frac{v_{n+1} - v_n}{t_{n+1} - t_n}, \tag{22}$$

with $v_0 = 0$, $t_0 = 0$, $T(t_0) = 0$, $\mu_0 = 0.342$, $m = 0.048$ kg, $g = 9.80665$ m/s^2, $\theta = 4.8°$, $\beta \approx 0.0818$ kg/s, $\Delta t = t_{n+1} - t_n = 0.2$ s and $0 \leq n \leq 33$, where $n \in \mathbb{N}$.

Rearranging the equation, we obtain

$$v_{n+1} = \frac{T(t_n) - \mu_0 mgcos\theta - mgcos\theta - (\beta - 5m)\, v_n}{5m}. \tag{23}$$

We use this equation, where v_{n+1} is the subject, in Excel to solve the recurrent sequence.

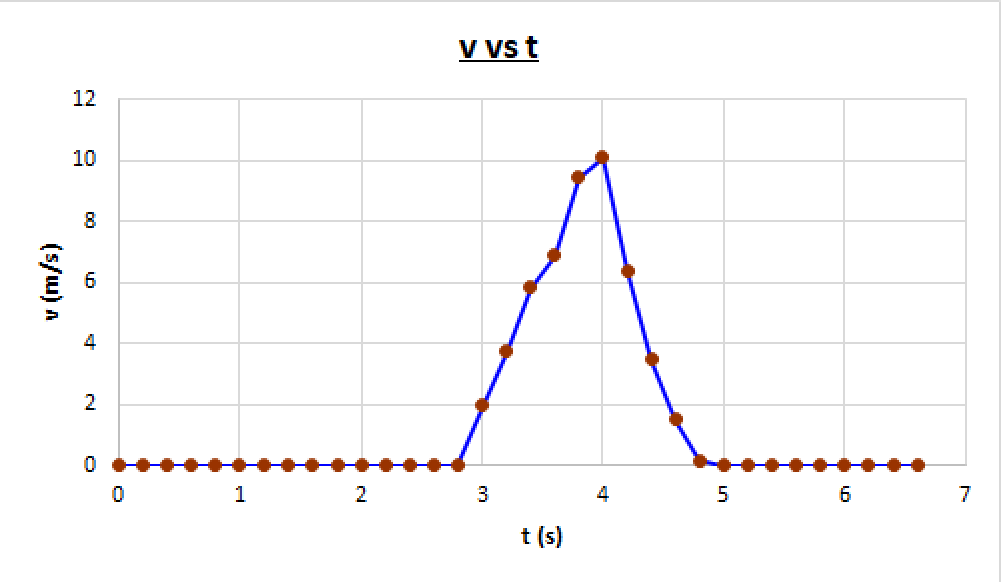

FIG. 1: Graph of v against t.

(b) To estimate $s(t_i)$, we utilise this useful recurrence relation

$$s_{n+1} = s_n + \frac{(v_{n+1} + v_n)\,(t_{n+1} - t_n)}{2}, \tag{24}$$

where $s_0 = 0$ and $\Delta t = t_{n+1} - t_n = 0.2$ s.

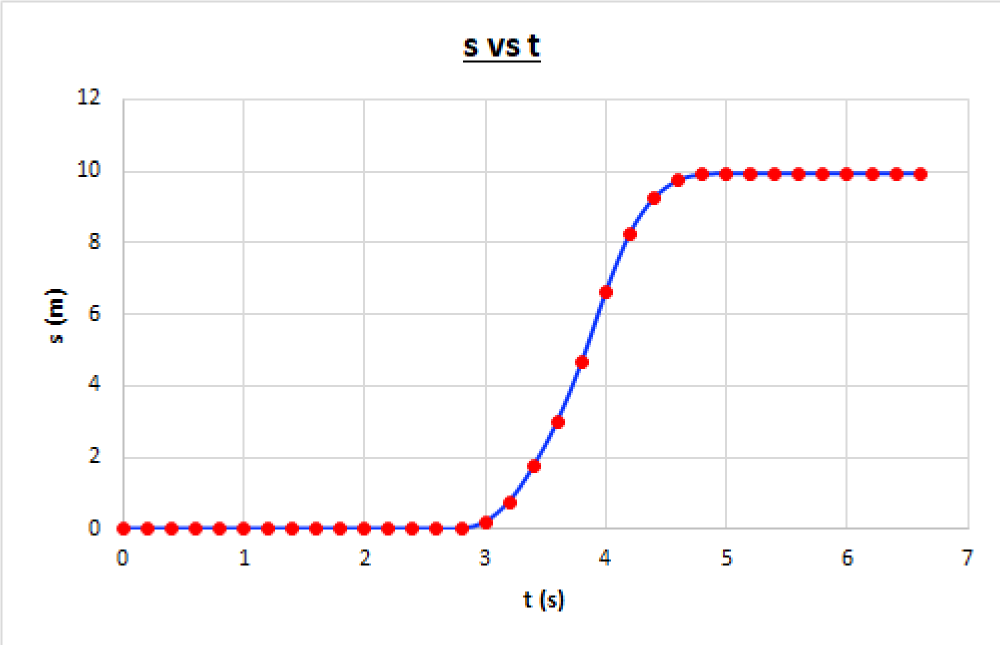

FIG. 2: Graph of s against t.

From Excel, we obtain the total distance travelled of about 9.93 m.

(c) By trial and error in Excel, we can get a total distance travelled of 8 m if the value of $\beta \approx 0.107$ kg/s.