Small induced universal graphs

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August 24, 2021

1 Introduction

Given a collection $\mathcal{F}$ of graphs, graph G is *induced universal* for $\mathcal{F}$ if and only if each element of $\mathcal{F}$ is an induced subgraph of G. Graph G is a *minimal* induced universal graph for $\mathcal{F}$ is it has as few vertices as possible. We write $f(k)$ to denote the order (number of vertices) of a minimal induced universal graph for the family of all graphs on k vertices. Moon showed that $f(k) \leq 2^{(k-1)/2}$ [4], and Alon showed that $f(k) = (1 + o(1))2^{(k-1)/2}$ [1]. There is an extensive literature on bounds on the order of minimal induced universal subgraphs for many families of graph; see the references in [1]. However, I am not aware of any existing systematic attempts to find exact values for families of small graphs. This paper presents exact values of $f(k)$ for $k \leq 5$, obtained by brute-force computational search. The corresponding counts of minimal induced universal subgraphs, and results for families of trees, are also presented.

2 Method

A simple brute-force approach was used to find the set of n-vertex graphs that are induced universal for the family $\mathcal{F}$ of all k-vertex graphs. Each possible n-vertex graph G from Brendan McKay's lists of graphs[1] was tested in turn. We used a Python implementation of the McSplit algorithm [3] to test whether each k-vertex graph is an induced subgraph of G. Although the McSplit algorithm was designed for the more general maximum common induced subgraph problem, it performs well on the subgraph isomorphism problem if both input graphs are small.

[1] `https://users.cecs.anu.edu.au/~bdm/data/graphs.html`

For those graphs G that do not contain a copy of each element of $\mathcal{F}$, it is desirable to iterate over $\mathcal{F}$ in an order that allows us to quickly reject G by finding a graph that is not an induced subgraph of G. The approach we used was to sort $\mathcal{F}$ in descending order of number of isomorphisms. In particular, this means that the clique K_k and the independent set I_k are the first two graphs tested.

The full set of experiments described in this paper, run sequentially, required around 14 million calls to the subgraph isomorphism algorithm and took around 12 minutes to complete on a laptop.

3 Results

Table 1 shows, for $0 \leq k \leq 5$, the value of $f(k)$. The table also shows $F(k)$, the number of minimal induced universal graphs for the family of all k-vertex graphs. To my knowledge, the value of $f(10)$ and the values of $F(k)$ have not been published previously.

The values $f(4)$ and $f(5)$ are equal to a simple lower bound given in a comment by "bof" on Mathematics Stack Exchange [2]: $f(k) \geq 2k$ if $k \geq 4$. (For $k < 10$, this bound improves upon Moon's lower bound.) To briefly summarise the proof, if $k \leq 2k$ then G must be a split graph (that is, a graph whose vertices can be partitioned into a clique and an independent set); therefore, G cannot contain the cycle C_4 as an induced subgraph. An example of an 8-vertex induced universal graph for the family of 4-vertex graphs was given by "Chain Markov" as a comment on the same question.

k	$f(k)$	$F(k)$
0	0	1
1	1	1
2	3	2
3	5	5
4	8	438
5	10	22

Table 1: For each k, $f(k)$ is the minimum order of a graph containing all k-vertex graphs as induced subgraphs, and $F(k)$ is the number of distinct $f(k)$-vertex graphs that contain all k-vertex graphs as induced subgraphs.

Figure 3 shows three examples of minimal induced universal graphs. These are for the families of all graphs with three, four and five vertices respectively.

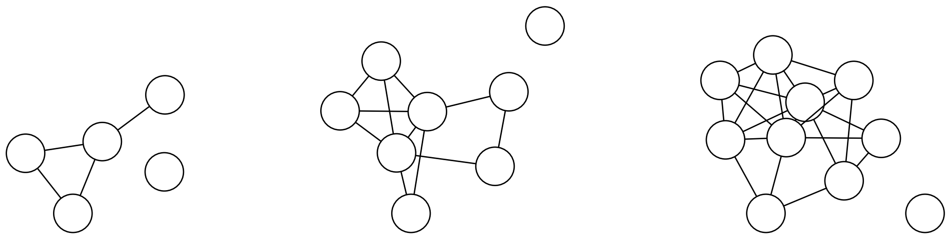

Figure 1: Induced universal graphs for the families of all graphs with 3, 4, and 5 vertices

4 Trees

Table 2 gives the order $t(k)$ of a minimal induced universal graph for the family of k-vertex trees, and the number $T(k)$ of such graphs. Figure 4 shows one of the 66 minimal induced universal graphs for the family of 6-vertex trees.

k	$t(k)$	$T(k)$
0	0	1
1	1	1
2	2	1
3	3	1
4	5	2
5	7	18
6	9	66

Table 2: For each k, $t(k)$ is the minimum order of a graph containing all k-vertex trees as induced subgraphs, and $T(k)$ is the number of distinct $t(k)$-vertex graphs that contain all k-vertex trees as induced subgraphs.

References

[1] N. Alon. Asymptotically optimal induced universal graphs. *Geometric and Functional Analysis*, 27(1):1–32, 2017.

[2] Chain Markov. What is the minimal possible size of an n-universal graph? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3246874 (version: 2019-06-01).

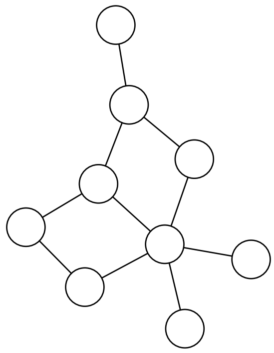

Figure 2: A universal graph for the family of all trees with 6 vertices

[3] C. McCreesh, P. Prosser, and J. Trimble. A partitioning algorithm for maximum common subgraph problems. In C. Sierra, editor, *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017*, pages 712–719. ijcai.org, 2017.

[4] J. W. Moon. On minimal n-universal graphs. *Proceedings of the Glasgow Mathematical Association*, 7(1):32–33, 1965.