Induced universal graphs for families of small graphs

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Abstract

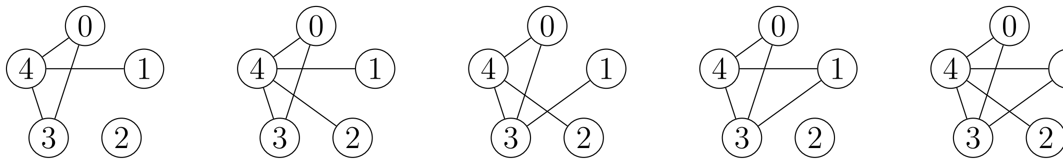
For $0 \leq k \leq 6$, we give the minimum number of vertices $f(k)$ in a graph containing all k-vertex graphs as induced subgraphs, and show that $16 \leq f(7) \leq 18$. For $0 \leq k \leq 5$, we also give the counts of such graphs, as generated by brute-force computer search. We give additional results for small graphs containing all trees on k vertices.

1 Introduction

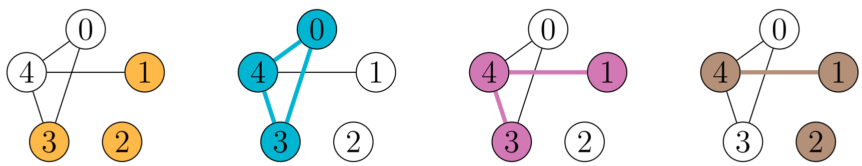
Given a collection $\mathcal{F}$ of graphs, graph G is *induced universal* for $\mathcal{F}$ if and only if each element of $\mathcal{F}$ is an induced subgraph of G. Graph G is a *minimal* induced universal graph for $\mathcal{F}$ if it has as few vertices as possible. The problem of finding a minimal induced universal graph of a family of graphs generalises the minimum common supergraph problem [2].

We write $\mathcal{F}(k)$ to denote the family of all graphs on k vertices, and we write $f(k)$ to denote the order (that is, the number of vertices) of a minimal induced universal graph for $\mathcal{F}(k)$. We write $F(k)$ to denote the number of non-isomorphic graphs of order $f(k)$ that are induced universal for $f(k)$. To give an example, we have $f(3) = 5$ and $F(3) = 5$. All five of the minimal induced universal graphs for $\mathcal{F}(3)$ are shown in Figure 1(a). Figure 1(b) shows the four graphs in $\mathcal{F}(3)$ as induced subgraphs of a single induced universal graph.

Moon showed that $f(k) \leq 2^{(k-1)/2}$ [5], and Alon showed that $f(k) = (1 + o(1))2^{(k-1)/2}$ [1]. There is an extensive literature on bounds on the order of minimal induced universal subgraphs for many families of graphs; see the references in [1]. However, to my knowledge the only existing systematic attempt to find *exact* values for families of small graphs is an answer by James Preen on the Mathematics Stack Exchange website that presents results for families of small connected graphs obtained by brute-force search with the Maple programming language [6].



(a) The five induced universal graphs of order 5 for $\mathcal{F}(3)$ (that is, for the family of all 3-vertex graphs)



(b) A demonstration that the first graph in Figure 1(a) is induced universal for $\mathcal{F}(3)$. For each graph G in $\mathcal{F}(3)$ (I_3, K_3, P_3, and a graph with a single edge), an induced subgraph isomorphic to G is shown in a single color.

Figure 1: Induced universal graphs for the family of all graphs on 3 vertices

This paper uses a brute-force approach similar to that of Preen to find minimal induced universal graphs for families of small *general* graphs. Section 2 describes our computer search method, including an optimisation to reduce run times. Sections 3 presents exact values of $f(k)$ for $k \leq 5$ along with the corresponding counts of minimal induced universal subgraphs. Section 4 gives the value of $f(6)$ and and Section 5 gives bounds on $f(7)$; in each of these sections, the lower bound is proven and a graph obtained by heuristic search demonstrating the upper bound is given. Finally, Section 6 gives results for families of trees.

2 Generating all induced universal graphs

Given k and n, I used a simple brute-force approach to find the set of n-vertex graphs that are induced universal for the family $\mathcal{F}(k)$ of all k-vertex graphs. Each possible n-vertex graph G from Brendan McKay's lists of graphs[1] was tested in turn. I used a Python implementation of the McSplit algorithm [4] to test whether each k-vertex graph is an induced subgraph of G. Although the McSplit algorithm was designed for the more general maximum common induced subgraph problem, it performs well on

[1] `https://users.cecs.anu.edu.au/~bdm/data/graphs.html`

the subgraph isomorphism problem if both input graphs are small. (Although I have described the application of this method to families of all graphs of a given order, the method could easily be applied to any family $\mathcal{F}$ of graphs, and indeed Section 6 gives results from the application of this method to families of trees.)

For those graphs G that do not contain a copy of each element of $\mathcal{F}(k)$, it improves the speed of our algorithm if we can iterate over $\mathcal{F}(k)$ in an order that allows us to quickly reject G by finding a graph that is not an induced subgraph of G. The approach I used was to sort $\mathcal{F}(k)$ in descending order of number of isomorphisms. In particular, this means that the clique K_k and the independent set I_k are the first two graphs tested.

The full set of experiments described in this paper, run sequentially, took less than an hour to complete on a laptop with an Intel Core i5-6200U CPU and 8 GB RAM.[2]

3 Results for $k \leq 5$

Table 1 shows, for $0 \leq k \leq 5$, the value of $f(k)$. The table also shows $F(k)$, the number of minimal induced universal graphs for the family of all k-vertex graphs. To my knowledge, the value of $f(5)$ and the values of $F(k)$ have not been published previously.

The values $f(1)$, $f(2)$ and $f(3)$ are equal to to the simple lower bound $2k-1$ given in a question by "Chain Markov" on Mathematics Stack Exchange [3]. The values $f(4)$ and $f(5)$ are equal to a lower bound given in a comment by "bof" on the same question: $f(k) \geq 2k$ if $k \geq 4$. (For $k < 10$, this bound improves upon Moon's lower bound $f(k) \leq 2^{(k-1)/2}$.) To briefly summarise the proof, if $f(k) \leq 2k$ then G must be a split graph (that is, a graph whose vertices can be partitioned into a clique and an independent set); therefore, G cannot contain the cycle C_4 as an induced subgraph. An example of an 8-vertex induced universal graph for the family of 4-vertex graphs was given by "Chain Markov" as a comment on the same question.

Figure 2 shows examples of minimal induced universal graphs for the families of all graphs with three, four and five vertices respectively.

4 $f(6) = 14$

This section shows that $f(6) = 14$. We begin with the lower bound. For $k \geq 6$, we can increase by 2 the lower bound by "bof". In particular, this means that $f(6) \geq 14$.

Proposition 1. *$f(k) \geq 2k + 2$ for all $k \geq 6$.*

[2] The code and results from this paper, including lists of minimal induced universal graphs in graph6 format, are available from `https://github.com/jamestrimble/small-universal-graphs`

k	$f(k)$	$F(k)$
0	0	1
1	1	1
2	3	2
3	5	5
4	8	438
5	10	22

Table 1: For each k, $f(k)$ is the minimum order of a graph containing all k-vertex graphs as induced subgraphs, and $F(k)$ is the number of distinct $f(k)$-vertex graphs that contain all k-vertex graphs as induced subgraphs.

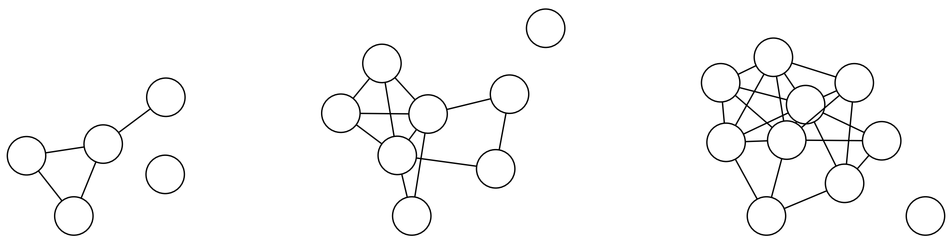


Figure 2: Minimal induced universal graphs for the families of all graphs with 3, 4, and 5 vertices

Proof. Suppose that G is an induced universal graph for the family of all graphs on k vertices, and that G has no more than $2k+1$ vertices. Graph G must have K_k and I_k as induced sugraphs. This clique and independent set may overlap by no more than one vertex, so their union must contain at least $2k-1$ vertices. Therefore it is possible to partition the vertices of G into three sets: a clique S_1, an independent set S_2, and a third set S_3 containing at most 2 vertices.

We will show that G cannot contain as induced subgraphs both of the graphs in Figure 3. These graphs are complements of each other, and we refer to them as H and H' respectively.

Let G_1 be an induced subgraph of G that is isomorphic to H. Since there are no edges between the two three-vertex cliques in G_1, it must be the case that the vertex set of at least one of these cliques does not intersect S_1. Since S_2 is an independent set in G, this clique must have exactly one vertex in S_2 and two vertices in S_3. We can deduce, then, that S_3 contains exactly two vertices, and that these vertices are adjacent in G.

Now consider graph H'. Since H' is an induced subgraph of G, it follows by taking complements of H' and G that H is an induced subgraph of G' (the complement of G). We can repeat the argument of the previous paragraph with the roles of S_1 and

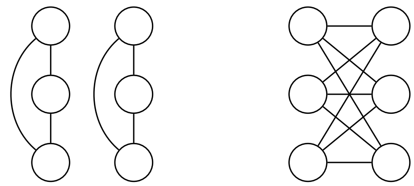


Figure 3: If $k > 5$ and G is a graph on $2k + 1$ or fewer vertices, then G cannot have as induced subgraphs all four of the following graphs: the clique K_k, the independent set I_k, and the two graphs shown. See the the proof of Proposition 1

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S_2 reversed to show that the two vertices in S_3 must be adjacent in the complement of G, and therefore must not be adjacent in G. Since we previously showed that these vertices are adjacent in G, we have a contradiction. $\square$

To give an upper bound of 14 on $f(6)$, Figure 4 shows the adjacency matrix of a 14-vertex graph that is induced universal for the family of all graphs on six vertices. This was generated using a simple local search algorithm. We begin by generating a random graph on 14 vertices as follows. We number the vertices from zero; the generator makes the k vertices numbered 0 to $k - 1$ a clique, and the k vertices numbered $k - 1$ to $2k - 2$ an independent set.[3] Each possible edge that is not involved in either the clique or the independent set is added with probability 1/2. We then repeatedly "flip" the status of a random edge from present to absent or vice versa, but always leave the large clique and independent set intact. After each flip, we count the number of 156 graphs on 6 vertices that are isomorphic to a subgraph of our 14-vertex graph. If the most recent flip decreased this number, we revert it. After each 1000 flips, we restart the algorithm with a new random graph.

5 Bounds on $f(7)$

By Proposition 1, we have $f(7) \geq 16$. Figure 5 shows the adjacency matrix of an 18-vertex induced universal graph for the family of all seven-vertex graphs. This was generated with the heuristic described in Section 4, with two modifications. First, 10000 rather than 1000 edge-flips were permitted before each restart, as this was found to be more effective in a preliminary run of the experiment. Second, the overlapping six-vertex clique and independent set were replaced with a clique on seven vertices and an independent set on seven vertices. Again, these had one vertex in common.

[3]The clique and the independent set thus have one vertex in common. The proof of Proposition 1 can be modified straightforwardly to show that there is no 14-vertex induced universal graph for this family of graphs that contains a 6-vertex clique and a 6-vertex independent set as induced subgraphs with disjoint vertex sets.

0	1	1	1	1	1	0	1	1	0	0	0	1	0
1	0	1	1	1	1	1	1	1	1	1	0	0	1
1	1	0	1	1	1	0	1	1	0	1	0	1	0
1	1	1	0	1	1	1	0	0	0	1	1	0	0
1	1	1	1	0	1	0	1	0	0	0	0	0	0
1	1	1	1	1	0	0	0	0	0	0	1	0	1
0	1	0	1	0	0	0	0	0	0	0	1	1	1
1	1	1	0	1	0	0	0	0	0	0	1	1	0
1	1	1	0	0	0	0	0	0	0	0	0	1	0
0	1	0	0	0	0	0	0	0	0	0	1	0	1
0	1	1	1	0	0	0	0	0	0	0	1	1	0
0	0	0	1	0	1	1	1	0	1	1	0	0	1
1	0	1	0	0	0	1	1	1	0	1	0	0	0
0	1	0	0	0	1	1	0	0	1	0	1	0	0

Figure 4: The adjacency matrix of a 14-vertex induced universal graph for the family of all six-vertex graphs

(We also tried making the clique and independent set vertex-disjoint, but did not find an 18-vertex solution in four hours with this approach.)

Thus we have $16 \leq f(7) \leq 18$.

6 Trees

Table 2 gives the order $t(k)$ of a minimal induced universal graph for the family of k-vertex trees, and the number $T(k)$ of such graphs. Figure 6 shows one of the 66 minimal induced universal graphs for the family of 6-vertex trees.

7 Acknowledgements

I would like to thank Brendan McKay for helpful feedback on this paper, and Persi Diaconis for introducing me to induced universal graphs and for interesting email discussions on related topics.

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0 1 1 1 1 1 1 1 1 1 0 1 1 1 1 1 0 0
1 0 1 1 1 1 1 0 1 1 1 0 1 1 0 1 0 0
1 1 0 1 1 1 1 1 0 1 0 1 0 0 0 0 0 1
1 1 1 0 1 1 1 0 1 0 0 1 0 0 0 0 0 1
1 1 1 1 0 1 1 1 1 0 0 0 0 0 1 0 0 0
1 1 1 1 1 0 1 0 1 0 0 0 0 1 1 1 1 1
1 1 1 1 1 1 0 0 0 0 0 0 0 1 0 0 0 1
1 0 1 0 1 0 0 0 0 0 0 0 0 0 1 1 1 0
1 1 0 1 1 1 0 0 0 0 0 0 0 1 1 1 0 1
1 1 1 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1
0 1 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1 0
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1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1
1 1 0 0 0 1 1 0 1 0 1 0 0 0 0 0 0 1
1 0 0 0 1 1 0 1 1 1 0 1 0 0 0 0 0 1
1 1 0 0 0 1 0 1 1 1 0 1 1 0 0 0 1 1
0 0 0 0 0 1 0 1 0 1 1 1 1 0 0 1 0 1
0 0 1 1 0 1 1 0 1 1 0 0 1 1 1 1 1 0
```

Figure 5: The adjacency matrix of an 18-vertex induced universal graph for the family of all seven-vertex graphs

k	$t(k)$	$T(k)$
1	1	1
2	2	1
3	3	1
4	5	2
5	7	18
6	9	66

Table 2: For each k, $t(k)$ is the minimum order of a graph containing all k-vertex trees as induced subgraphs, and $T(k)$ is the number of distinct $t(k)$-vertex graphs that contain all k-vertex trees as induced subgraphs.

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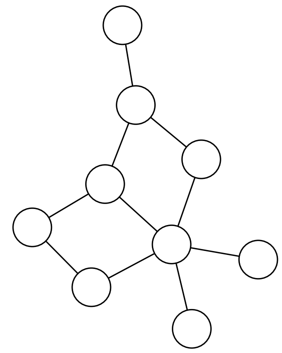

Figure 6: A universal graph for the family of all trees with 6 vertices

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