

Induced universal graphs for families of small graphs

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Abstract

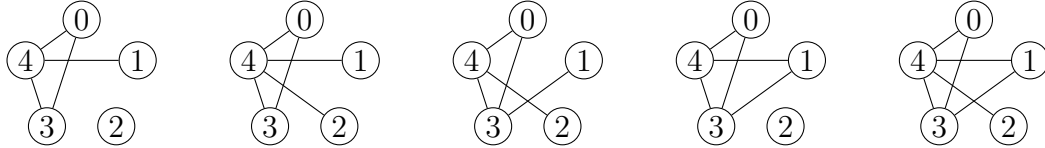
We present exact and heuristic algorithms that find, for a given family of graphs, a graph that contains each member of the family as an induced subgraph. For $0 \leq k \leq 6$, we give the minimum number of vertices $f(k)$ in a graph containing all k -vertex graphs as induced subgraphs, and show that $16 \leq f(7) \leq 18$. For $0 \leq k \leq 5$, we also give the counts of such graphs, as generated by brute-force computer search. We give additional results for small graphs containing all trees on k vertices.

1 Introduction

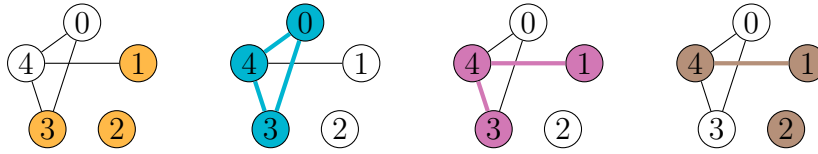
Given a collection \mathcal{F} of graphs, graph G is *induced universal* for \mathcal{F} if and only if each element of \mathcal{F} is an induced subgraph of G . Graph G is a *minimal* induced universal graph for \mathcal{F} if it has as few vertices as possible. The problem of finding a minimal induced universal graph of a family of graphs generalises the minimum common supergraph problem [2], which applies only to families containing exactly two graphs.

We write $\mathcal{F}(k)$ to denote the family of all graphs on k vertices, and we write $f(k)$ to denote the order (that is, the number of vertices) of a minimal induced universal graph for $\mathcal{F}(k)$. We write $F(k)$ to denote the number of non-isomorphic graphs of order $f(k)$ that are induced universal for $\mathcal{F}(k)$. To give an example, we have $f(3) = 5$ and $F(3) = 5$; all five of the minimal induced universal graphs for $\mathcal{F}(3)$ are shown in Figure 1(a). Figure 1(b) shows the four graphs in $\mathcal{F}(3)$ as induced subgraphs of a single induced universal graph.

Moon showed that $f(k) \leq 2^{(k-1)/2}$ [12], and Alon showed that $f(k) = (1 + o(1))2^{(k-1)/2}$ [1]. There is an extensive literature on bounds on the order of minimal induced universal subgraphs for many families of graphs; see the references in [1]. However, to my knowledge the only existing systematic attempt to find *exact* values for families of small graphs is an answer by James Preen on the Mathematics Stack Exchange website that presents results for families of small connected graphs obtained by brute-force search with the Maple programming language [13].



(a) The five induced universal graphs of order 5 for $\mathcal{F}(3)$ (that is, for the family of all 3-vertex graphs)



(b) A demonstration that the first graph in Figure 1(a) is induced universal for $\mathcal{F}(3)$. For each graph G in $\mathcal{F}(3)$ (I_3 , K_3 , P_3 , and a graph with a single edge), an induced subgraph isomorphic to G is shown in a single color.

Figure 1: Minimal induced universal graphs for the family of all graphs on 3 vertices

This paper uses a brute-force approach similar to that of Preen to find minimal induced universal graphs for families of small graphs. Our contributions are (1) an ordering strategy to reduce the number of subgraph isomorphism calls required by the brute force algorithm; (2) exact values for $f(k)$ ($5 \leq k \leq 6$) and $F(k)$ ($1 \leq k \leq 5$); (3) an upper bound of 18 for $f(7)$; (4) a hill-climbing search algorithm to find small (but not necessarily optimal) induced universal graphs (5) exact values of the order of a minimal induced universal graph for the families of all trees on k vertices, for $1 \leq k \leq 8$.

The sequel is structured as follows. Section 2 describes the basic brute force method. Section 3 compares four methods for sorting the graphs in \mathcal{F} , and compares their effect on the number of subgraph isomorphism calls made by the brute-force algorithm. Section 4 presents exact values of $f(k)$ and $F(k)$ for $k \leq 5$. Section 5 gives the value of $f(6)$ and Section 6 gives bounds on $f(7)$; in each of these sections, the lower bound is proven and a graph obtained by heuristic search demonstrating

the upper bound is given. Finally, Section 7 describes a specialised algorithm for families of trees, and gives results for these families.

2 Generating all induced universal graphs

Given \mathcal{F} and n , we use the brute-force approach shown in Algorithm 1 — which is essentially the same as the method described by Preen [13] — to find the set of n -vertex graphs that are induced universal for the family \mathcal{F} . The entry point is the function `AllInducedUniversalGraphs()`. Line 14 tests in turn each candidate graph G from the family of n -vertex graphs, and adds to the collection \mathcal{G} those that are induced universal for \mathcal{F} . The function `IsInducedUniversal()` tests whether a graph G is induced universal for \mathcal{F} by checking for each $H \in \mathcal{F}$ that H is isomorphic to an induced subgraph of G .

For the collection $\mathcal{F}(n)$ of candidate graphs on line 14, our implementation uses graphs generated using Brendan McKay’s `geng` program [10].¹ These are read in `graph6` format² from a text file as the program proceeds, and therefore only one candidate graph at a time needs to be stored in memory.

Our implementation is written purely in Python and run using the CPython interpreter. On line 6, the function `InducedSubIso()` calls an algorithm for the induced subgraph isomorphism decision problem; for this, our program uses an implementation by the author of the `McSplit` algorithm [8]. `McSplit` was designed for the more general maximum common induced subgraph problem, but it can be used for induced subgraph isomorphism with a simple modification: we backtrack when the calculated upper bound on the order of a common subgraph is less the order of the smaller of the two input graphs. This method for solving the subgraph isomorphism problem is very fast if both input graphs are small, as the graphs considered in this paper are. While `McSplit` is suitable for our purposes in this paper, we do not claim that it is the fastest induced subgraph isomorphism solver for very small graphs; it would be useful future work to perform an experimental comparison with other subgraph isomorphism solvers.

The full set of experiments described in this paper, run sequentially, took less than a day to complete on a laptop with an Intel Core i5-6200U CPU and 8 GB RAM.³

¹The program `geng` is distributed with Nauty; we used version 27.r3.<https://pallini.di.uniroma1.it/>

²<https://users.cecs.anu.edu.au/~bdm/data/formats.html>

³The code and results from this paper, including lists of minimal induced universal graphs in `graph6` format, are available from <https://github.com/jamestrimble/small-universal-graphs>

Algorithm 1: A brute-force algorithm for finding all order- n induced universal subgraphs of a family \mathcal{F} of graphs. The entry point is $\text{AllInducedUniversalGraphs}(\mathcal{F}, n)$.

```

1 IsInducedUniversal( $\mathcal{F}, G$ )
2 Data: A family of graphs  $\mathcal{F}$  and a graph  $G$ .
3 Result: A boolean value indicating whether  $G$  is induced universal for  $\mathcal{F}$ .
4 begin
5   for  $H \in \mathcal{F}$  do
6     if  $\neg \text{InducedSubIso}(H, G)$  then return false
7   return true

8 AllInducedUniversalGraphs( $\mathcal{F}, n$ )
9 Data: A family of graphs  $\mathcal{F}$  and a natural number  $n$ .
10 Result: The set of all graphs of order  $n$  that are induced universal for  $\mathcal{F}$ .
11 begin
12    $\mathcal{G} \leftarrow \emptyset$ 
13   for  $G \in \mathcal{F}(n)$  do
14     if  $\text{IsInducedUniversal}(\mathcal{F}, G)$  then  $\mathcal{G} \leftarrow \mathcal{G} \cup \{G\}$ 
15   return  $\mathcal{G}$ 

```

3 Iteration order for \mathcal{F}

When determining on line 5 of Algorithm 1 whether a graph G contains every element of \mathcal{F} as an induced subgraph, are some iteration orders of \mathcal{F} better than others? For a graph G that contains a copy of each element of \mathcal{F} , the iteration order is of no importance; every element of \mathcal{F} must be checked before *true* is returned. For a graph G that does *not* contain a copy of each element of \mathcal{F} , however, we would like to iterate over \mathcal{F} in an order such that graphs that are likely to fail the subgraph isomorphism test are checked early; that way, we can quickly return *false* and avoid unnecessary calls to the subgraph isomorphism algorithm.

We begin by considering the case where our family of graphs \mathcal{F} is $\mathcal{F}(k)$ for some k ; that is, \mathcal{F} is the set of all graphs of order k . As Diaconis and Chatterjee note, [4], the complete graph K_n is the ‘hardest’ graph of order n for a random graph to contain. This led me to consider two orderings in which K_k and its complement I_k come first. The first of these approaches is to sort $\mathcal{F}(k)$ in descending order of automorphism count, so that graphs with large automorphism groups appear first in the list. (Intuitively, if a graph has few automorphisms then we can generate many different labelled graphs by permuting the vertex labels. This gives more ‘opportunities’ for

the graph to be an induced subgraph of G .) The second approach considered is to place graphs with unusually high or low edge counts, as measured by the absolute value of $2|E(G)| - \binom{|V(G)|}{2}$, first.

Figure 2 examines whether graphs that appear early in the orderings of \mathcal{F} generated by each of these two strategies are contained in few graphs with a given number of vertices, as we would hope to be the case. For this experiment, we let $\mathcal{F} = \mathcal{F}(5)$, and we consider the problem of finding induced universal graphs of order 8. Each point on the plots represents one of the 34 graphs of order 5 (fewer than 34 points are visible due to exactly-overlapping points, as we would expect). On the vertical axis, we have the number of graphs in $\mathcal{F}(8)$ containing G ; we would like graphs for which this value is small to appear early in our order. On the horizontal axes we have each of our two measures: size of automorphism group in the first plot and ‘extremeness’ of edge count in the second plot. As we can see from the first plot, the automorphism measure has a strong correlation with the number of graphs containing G as an induced subgraph, suggesting that sorting by this measure could be a good strategy for reducing the number of subgraph isomorphism calls in Algorithm 1.

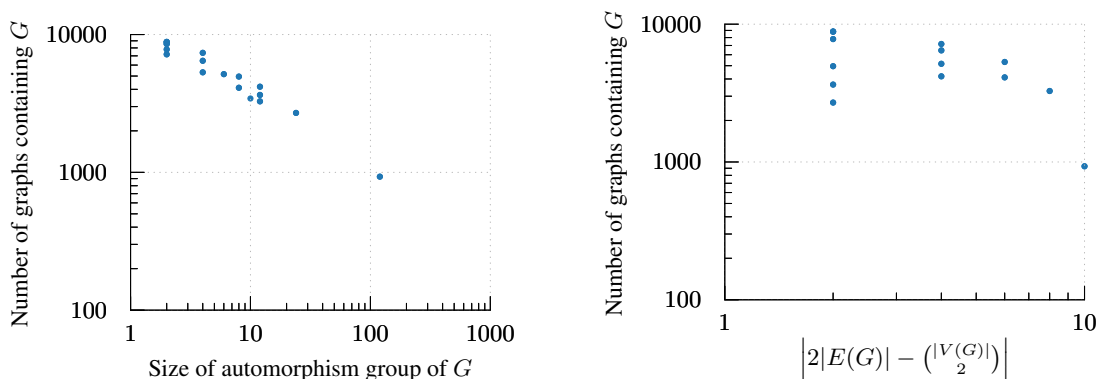


Figure 2: For each graph $G \in \mathcal{F}(5)$, we plot on the vertical axis the number of graphs in $\mathcal{F}(8)$ that contain G as an induced subgraph. This value has a strong negative correlation with the size of the automorphism group of G (left plot), but a weaker correlation with a measure of how extreme the edge count of G is (right plot). All axis scales are logarithmic.

3.1 Testing the strategies: $\mathcal{F} = \mathcal{F}_5$

We now test whether each of our ordering strategies results, as hoped, in a reduced number of calls to the subgraph isomorphism function in Algorithm 1. As our first test case, we consider `AllInducedUniversalGraphs($\mathcal{F}(5), 9$)` — finding all induced

universal graphs of order 9 for the family of all graphs of order 5. We consider four strategies.

The first two of these are the automorphism-count and ‘edge-count extremeness’ measures described above. The fourth strategy is a random order. Finally, we used one additional strategy that we call “almost random”. Under this strategy, I_5 and K_5 are placed at the start of the list, and the remaining 32 graphs are in random order. This strategy was not intended to be useful in practice, but to give insight into whether the success of the first two strategies was purely due to having I_5 and K_5 at the start of the list of graphs. Table 1 summarises these ordering strategies.

Name	Description of strategy
Automorphisms	Sort in descending order of automorphism group size
Edges	Sort in descending order of $ 2 E(G) - \binom{ V(G) }{2} $ (thus, give priority to graphs that are very sparse or very dense)
Almost-random	Shuffle the list of graphs in \mathcal{F} , then move K_k and I_k to the start if they are present
Random	Shuffle the list of graphs in \mathcal{F}

Table 1: Ordering strategies tested for the brute-force algorithm

For each of these strategies, we ran the program 50 times.

The results are shown in Figure 3. Over 50 runs, the automorphisms strategy required a mean of 304,746 calls to the subgraph isomorphism solver, while the edges strategy required marginally fewer calls: a mean of 304,701. Surprisingly, the almost-random strategy had the lowest average number of calls of all the strategies, at 304,387. The random strategy fared poorly, requiring over 800,000 calls on average. The variance in number of calls was small for all but the random strategy, with between 304,000 and 305,000 calls being required for each run.

We note, first, that edges strategy appears to be at least as effective as the automorphisms strategy for this instance, despite the lower correlation observed in Figure 2. Perhaps most surprising is that the almost-random strategy performed better than either of these two strategies; our expectation had been that this simple strategy would give some but not all of the benefit of the first two strategies. We conjecture that the almost-random strategy is effective because it quickly checks a diverse range of graphs. Consider, for example, Table 2, which shows the graphs of order 5 listed in descending order of automorphism count. This is the order in which they are checked under the automorphisms strategy (with the order within each row chosen randomly).

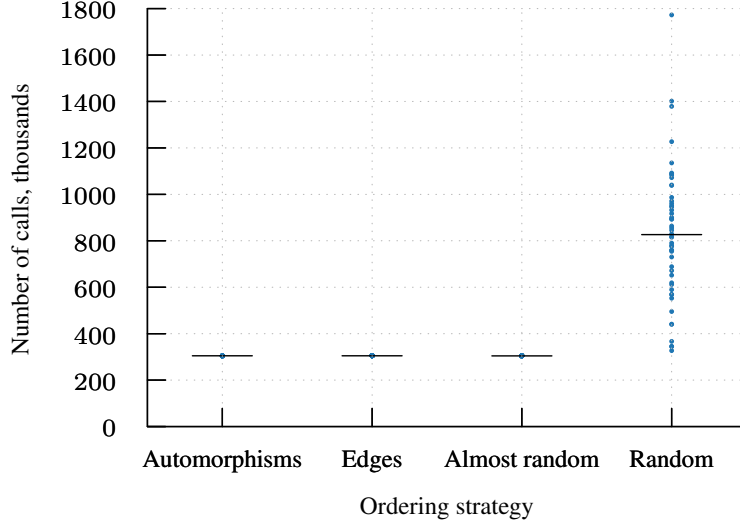


Figure 3: Using each of our four order strategies, we ran `AllInducedUniversalGraphs($\mathcal{F}(5), 9$)` 50 times. The plot shows the number calls to the subgraph isomorphism algorithm, in thousands. Horizontal lines show average (mean) number of calls.

Note that the two graphs in the second row are similar to the graphs in the first row; the first graphs in each of the first two rows, for example, have a maximum common induced subgraph of size 4. It seems likely that if a graph contains both of the graphs in the first row of the table as induced subgraphs, then it may also contain the graphs in the second row. The almost-random strategy is likely to choose graphs that are quite different from a clique and an independent set after the first two graphs, perhaps increasing the probability that one of these graphs will not be contained in the larger graph.

3.2 Testing the strategies: $\mathcal{F} \subset \mathcal{F}(5)$

While the almost-random strategy was very effective when searching for induced universal graphs of *all* graphs of a given order, it is plausible that it will be less effective if \mathcal{F} is a strict subset of $\mathcal{F}(k)$, and in particular if it includes neither I_k nor K_k . Figure 4 examines this claim. We generated 50 subsets \mathcal{F} of $\mathcal{F}(5)$ by random selection, with each subset containing exactly half of the elements of $\mathcal{F}(5)$. For each of these, we ran `AllInducedUniversalGraphs($\mathcal{F}, 9$)` using each of the four strategies. The ‘automorphisms’ strategy was the best overall in this case, requiring on average 8%, 27%, and 52% fewer calls to the subgraph isomorphism function than the edges, almost-random, and random strategies respectively.


































Automorphisms	Graphs
120	 
24	 
12	     
10	
8	   
6	 
4	     
2	         

Table 2: The 34 graphs of order 5, classified by automorphism group size

To help us understand why the almost-random strategy performs poorly for these instances, we connect the four solver runs on each instance together on the plot with blue lines. The almost-random strategy, on the instances where it performs poorly, requires exactly the same number of subgraph isomorphism calls as the random strategy. These are the instances in which \mathcal{F} contains neither K_5 nor I_5 ; thus, the almost-random and random strategies are equivalent. Notably, for these instances it is also the case that the automorphisms strategy tends to outperform the edges strategy.

3.3 Testing the strategies: a family of trees

As a final test case, we let \mathcal{F} be the family of all trees on 6 vertices, and searched for all graphs of order 8 that are induced universal for \mathcal{F} . Figure 5 shows the results. The edges, almost-random, and random strategies had identical results, since all members of \mathcal{F} have five edges and the family contains neither K_6 nor I_6 . The automorphisms strategy requires fewer subgraph isomorphism calls on average than the other three strategies, although on seven of the 50 instances it does require marginally more calls than the other strategies.

The ‘automorphisms’ strategy performs consistently well across our three test cases; therefore, we use this strategy for the remainder of the paper.

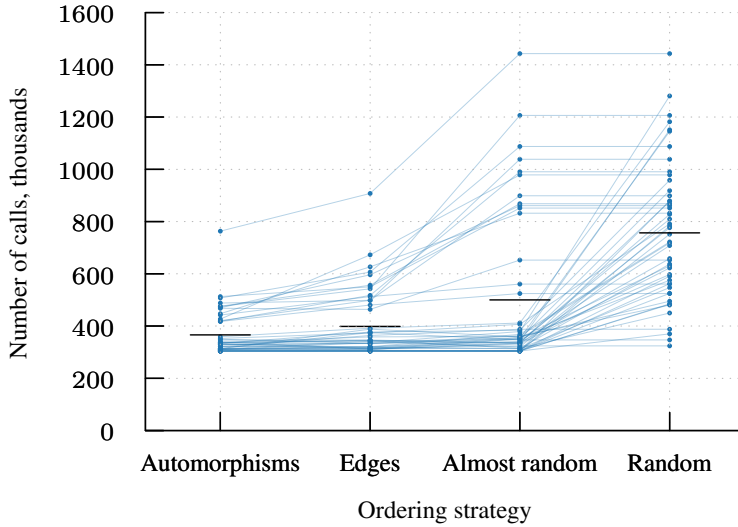


Figure 4: Using each of our four order strategies, we ran $\text{AllInducedUniversalGraphs}(\mathcal{F}, 9)$ 50 times, where \mathcal{F} is a random subset of $\mathcal{F}(5)$ containing 17 graphs. This parallel coordinates plot shows the number calls to the subgraph isomorphism algorithm, in thousands. Horizontal black lines show average (mean) number of calls. The four runs for each instance are connected by blue lines.

4 Results for $k \leq 5$

Recall that $\mathcal{F}(k)$ is the family of all graphs of order k , $f(k)$ is the order of a minimal induced universal graph for $\mathcal{F}(k)$, and $F(k)$ is the number of non-isomorphic graphs of that order that are induced universal for $\mathcal{F}(k)$. This section presents values of $f(k)$ and $F(k)$ for $0 \leq k \leq 5$, computed using Algorithm 1. The results are shown in Table 3. To my knowledge, the value of $f(5)$ and the values of $F(k)$ have not been published previously.

The values $f(1)$, $f(2)$ and $f(3)$ are equal to the simple lower bound $2k - 1$ given in a question by “Chain Markov” on Mathematics Stack Exchange [3]. The values $f(4)$ and $f(5)$ are equal to a lower bound given in a comment by “bof” on the same question: $f(k) \geq 2k$ if $k \geq 4$. (For $k < 10$, this bound improves upon Moon’s lower bound $f(k) \leq 2^{(k-1)/2}$.) To briefly summarise the proof, if $f(k) \leq 2k$ then G must be a split graph (that is, a graph whose vertices can be partitioned into a clique and an independent set); therefore, G cannot contain the cycle C_4 as an induced subgraph. An example of an 8-vertex induced universal graph for the family of 4-vertex graphs was given by “Chain Markov” as a comment on the same question.

Figure 6 shows examples of minimal induced universal graphs for the families of

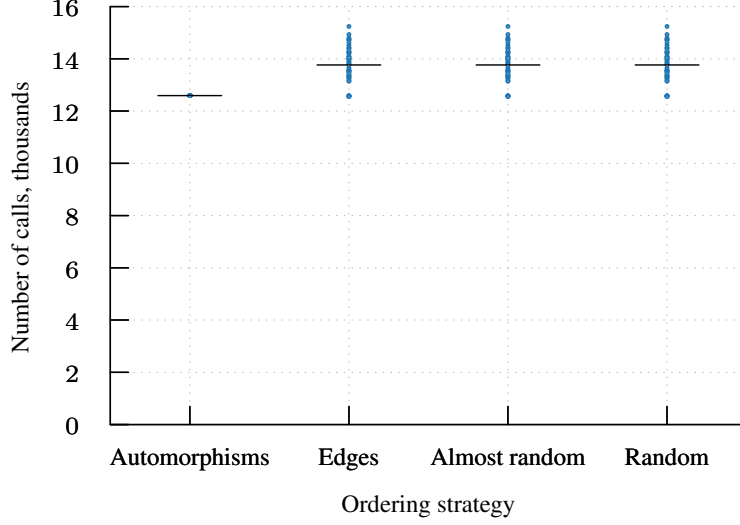


Figure 5: Using each of our four order strategies, we ran `AllInducedUniversalGraphs($\mathcal{F}, 8$)` 50 times, where \mathcal{F} is the set of all trees of order 6. The plot shows the number calls to the subgraph isomorphism algorithm, in thousands. Horizontal black lines show average (mean) number of calls.

k	$f(k)$	$F(k)$
0	0	1
1	1	1
2	3	2
3	5	5
4	8	438
5	10	22

Table 3: For each k , $f(k)$ is the minimum order of a graph containing all k -vertex graphs as induced subgraphs, and $F(k)$ is the number of distinct $f(k)$ -vertex graphs that contain all k -vertex graphs as induced subgraphs.

all graphs with four and five vertices respectively.

5 $f(6) = 14$

This section shows that $f(6) = 14$. We begin with the lower bound. For $k \geq 6$, we can increase by 2 the lower bound — $f(k) \geq 2k$ for $k \geq 4$ — cited in the previous

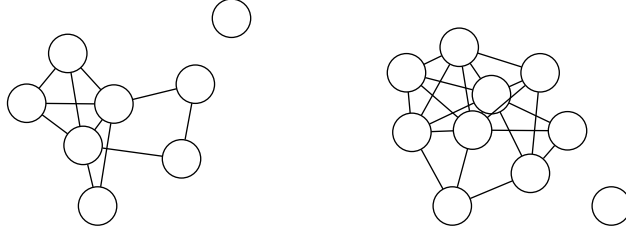


Figure 6: Examples of minimal induced universal graphs for $\mathcal{F}(4)$ and $\mathcal{F}(5)$

section.

Proposition 1. $f(k) \geq 2k + 2$ for all $k \geq 6$.

Proof. Suppose that G is an induced universal graph for the family of all graphs on k vertices, and that G has no more than $2k + 1$ vertices. Graphs K_k and I_k are both members of $\mathcal{F}(k)$, and therefore must be induced subgraphs of G . Clearly, this clique and independent set may overlap by no more than one vertex, so their union must contain at least $2k - 1$ vertices. Therefore it is possible to partition the vertices of G into three sets: a clique S_1 , an independent set S_2 , and a third set S_3 containing at most 2 vertices.

We will show that G cannot contain as induced subgraphs both of the graphs in Figure 7. These graphs are complements of each other, and we refer to them as H and H' respectively.

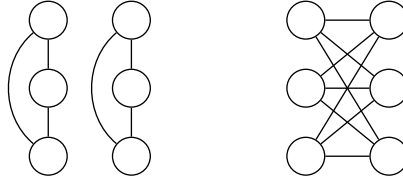


Figure 7: If $k > 5$ and G is a graph on $2k + 1$ or fewer vertices, then G cannot have as induced subgraphs all four of the following graphs: the clique K_k , the independent set I_k , and the two graphs shown. See the the proof of Proposition 1.

Let G_1 be an induced subgraph of G that is isomorphic to H . Since there are no edges between the two three-vertex cliques in G_1 , it must be the case that the vertex set of at least one of these cliques does not intersect S_1 . Since S_2 is an independent set in G , this clique must have exactly one vertex in S_2 and two vertices in S_3 . We can deduce, then, that S_3 contains exactly two vertices, and that these vertices are adjacent in G .

Now consider graph H' . Since H' is an induced subgraph of G , it follows by taking complements of H' and G that H is an induced subgraph of G' (the complement of

G). We can repeat the argument of the previous paragraph with the roles of S_1 and S_2 reversed to show that the two vertices in S_3 must be adjacent in the complement of G , and therefore must not be adjacent in G . Since we previously showed that these vertices are adjacent in G , we have a contradiction. \square

Proposition 1 implies that $f(6) \geq 14$.

To find an induced universal graph for $\mathcal{F}(6)$ in order to give an upper bound on $f(6)$, we use the following local search algorithm, which is shown in Algorithm 2. The entry point to the algorithm is function `FindInducedUniversalGraph()`, which takes as its parameter the maximum number of attempted improvement steps to take before restarting from scratch. The main search function, which is called until timeout, is `FindInducedUniversalGraph'()`.

The function `FindInducedUniversalGraph'()` begins by generating a random graph G on 14 vertices as follows (see Figure 8). Numbering the vertices from zero, we make the k vertices numbered 0 to $k - 1$ a clique, and the k vertices numbered $k - 1$ to $2k - 2$ an independent set.⁴

0	1	1	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	0	1
1	1	1	1	1	1	0	0	0	0	0	0	.	.
.	0	0	0	0	0	0	.	.
.	0	0	0	0	0	0	.	.
.	0	0	0	0	0	0	.	.
.	0	0	0	0	0	0	.	.
.	0	0	0	0	0	0	.	.
.	0	.
.	0	.
.	0	0

Figure 8: The adjacency matrix used in our heuristic algorithm when searching for an induced universal graph of order 14 for $\mathcal{F}(6)$. The blue region (a 6-vertex clique) and the yellow region (a 6-vertex independent set) remain unchanged as the algorithm runs. The remaining possible edges, marked \cdot , are initially set to 0 or 1 at random, and are flipped between 0 and 1 as the algorithm progresses.

Each possible edge that is not involved in either the clique or the independent set — shown in the figure with a dot — is added with probability $1/2$. The loop

⁴The clique and the independent set thus have one vertex in common. The proof of Proposition 1 can be modified straightforwardly to show that there is no 14-vertex induced universal graph for this family of graphs that contains a 6-vertex clique and a 6-vertex independent set as induced subgraphs with disjoint vertex sets.

Algorithm 2: A hill-climbing algorithm to find an induced universal graph for family $\mathcal{F}(6)$

```

1 FlipEdge( $G, v, w$ )
2 begin
3   if  $G$  has edge  $\{v, w\}$  then
4     return  $(V(G), E(G) \setminus \{v, w\})$ 
5   else
6     return  $(V(G), E(G) \cup \{v, w\})$ 
7 Score( $G$ )
8 begin
9   return  $|\{H \in \mathcal{F}(6) \mid H \text{ is an induced subgraph of } G\}|$ 
10 FindInducedUniversalGraph'( $maxIter$ )
11 begin
12    $G \leftarrow (\{0, \dots, 13\}, \emptyset)$   $\triangleright$  A graph on 14 vertices with no edges
13   Add edges to make vertices  $\{0, \dots, 5\}$  of  $G$  a clique
14   Add to  $G$  each possible edge marked  $\bullet$  in Figure 8 with probability  $1/2$ 
15   for  $i \in \{1, \dots, maxIter\}$  do
16      $v, w \leftarrow$  vertices corresponding to a randomly-selected position marked
17        $\bullet$  in Figure 8
18      $G' \leftarrow \text{FlipEdge}(G, v, w)$ 
19     if Score( $G'$ ) =  $|\mathcal{F}(6)|$  then return  $G' \triangleright$  Success!
20     if Score( $G'$ )  $\geq$  Score( $G$ ) then  $G \leftarrow G'$ 
21   return null
22 FindInducedUniversalGraph( $maxIter$ )
23 begin
24   while time limit is not exceeded do
25      $G \leftarrow \text{FindInducedUniversalGraph}'(maxIter)$ 
26     if  $G \neq null$  then return  $G$ 
27   return null

```

in lines 15 to 19 then carries out up to $maxIter$ iterations of a step that randomly modifies graph G , and accepts the modification if it causes the number of graphs in $\mathcal{F}(6)$ that are induced subgraphs of G to increase. The attempted-improvement step chooses a random pair of vertices $\{v, w\}$, both of which are not in the same coloured square in Figure 8, and “flips” the status of the edge between these two vertices: the

edge is added if it is not present, and removed otherwise. After each flip, we count the number of graphs on 6 vertices (from a total of 156 graphs) that are isomorphic to an induced subgraph of our 14-vertex graph. If the most recent flip increased this number or left it unchanged, we accept the modification to the graph. After *maxIter* flips, we restart the algorithm with a new randomly-generated graph.

Our implementation has two optimisations that are not shown in Algorithm 2. First, scores are cached in a dictionary data structure, so that we can quickly look up the score of a graph that has already been visited. This data structure is cleared on return from `FindInducedUniversalGraph'`, in order to bound the program's memory use. (An alternative approach, which we have not implemented but which may be useful in order to explore more of the space of possible graphs, would be to use a tabu list [6] to avoid revisiting graphs that have been visited recently.)

The second optimisation often avoids the need to visit every graph in $\mathcal{F}(6)$ during calls to `Score()`. Rather than beginning our count at 0 and incrementing it for every graph in $\mathcal{F}(6)$ that is an induced subgraph of G , we begin our count at $|\mathcal{F}(6)|$ and decrement it for every graph that is *not* an induced subgraph of G . We can then return early from the function with a score of -1 as soon as the counter variable falls below the score obtained by the graph prior to the most recent edge-flip. This does not affect the behaviour of the hill-climbing search, since any score below the previous graph's score results in the most recent change being rejected.

By running this search algorithm, we quickly found the graph of order 14 whose adjacency matrix is shown in Figure 9. This is induced universal for $\mathcal{F}(6)$; therefore, $f(6) \leq 14$. Combining our two bounds, we have $f(6) = 14$.

```

0 1 1 1 1 1 0 1 1 0 0 0 1 0
1 0 1 1 1 1 1 1 1 1 1 0 0 1
1 1 0 1 1 1 0 1 1 0 1 0 1 0
1 1 1 0 1 1 1 0 0 0 1 1 0 0
1 1 1 1 0 1 0 1 0 0 0 0 0 0
1 1 1 1 1 0 0 0 0 0 0 1 0 1
0 1 0 1 0 0 0 0 0 0 0 1 1 1
1 1 1 0 1 0 0 0 0 0 0 1 1 0
1 1 1 0 0 0 0 0 0 0 0 0 1 0
0 1 0 0 0 0 0 0 0 0 0 1 0 1
0 1 1 1 0 0 0 0 0 0 0 1 1 0
0 0 0 1 0 1 1 1 0 1 1 0 0 1
1 0 1 0 0 0 1 1 1 0 1 0 0 0
0 1 0 0 0 1 1 0 0 1 0 1 0 0

```

Figure 9: The adjacency matrix of a 14-vertex induced universal graph for the family of all six-vertex graphs

6 Bounds on $f(7)$

By Proposition 1, we have $f(7) \geq 16$. Figure 10 shows the adjacency matrix of an 18-vertex induced universal graph for the family of all seven-vertex graphs. This was generated with the heuristic described in Section 5, with two modifications. First, 10,000 rather than 1000 edge-flips were permitted before each restart, as this was found to be more effective in a preliminary run of the experiment. Second, the overlapping six-vertex clique and independent set were replaced with a clique on seven vertices and an independent set on seven vertices. Again, these had one vertex in common. (We also tried making the clique and independent set vertex-disjoint, but did not find an 18-vertex solution in four hours with this approach.)

Thus we have $16 \leq f(7) \leq 18$.

0	1	1	1	1	1	1	1	1	0	1	1	1	1	0	0
1	0	1	1	1	1	0	1	1	1	0	1	1	0	1	0
1	1	0	1	1	1	1	0	1	0	1	0	0	0	0	1
1	1	1	0	1	1	1	0	1	0	0	1	0	0	0	0
1	1	1	1	0	1	1	1	1	0	0	0	0	0	1	0
1	1	1	1	1	0	1	0	1	0	0	0	0	1	1	1
1	1	1	1	1	1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	1	0	0	0	0	0	0	0	0	1	1	0
1	1	0	1	1	1	0	0	0	0	0	0	0	1	1	0
1	1	1	0	0	0	0	0	0	0	0	0	0	1	1	1
0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
1	0	1	1	0	0	0	0	0	0	0	0	0	1	1	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1
1	1	0	0	0	1	1	0	1	0	1	0	0	0	0	0
1	0	0	0	1	1	0	1	1	1	0	1	0	0	0	0
1	1	0	0	0	1	0	1	1	1	0	1	1	0	0	1
0	0	0	0	0	1	0	1	0	1	1	1	1	0	0	1
0	0	1	0	1	1	0	1	1	0	0	1	1	1	1	0

Figure 10: The adjacency matrix of an 18-vertex induced universal graph for the family of all seven-vertex graphs

7 Trees

To generate induced universal graphs for families of all trees on k vertices, we tried two approaches. The first approach was to run Algorithm 1, with the family \mathcal{F} of order- k trees loaded from a list published by Brendan McKay.⁵ The second approach, described in the following subsection, generates only candidate graphs that contain a star — one of the elements of \mathcal{F} — as an induced subgraph. The final part of this section gives results generated using these two methods, along with additional upper bounds generated using a specialised version of our hill-climbing algorithm.

⁵<https://users.cecs.anu.edu.au/~bdm/data/trees.html>

7.1 The completion method

In our second approach to finding induced universal graphs for the family of all trees on k vertices, we embed the star S_{k-1} — which must be present since it is a tree on k vertices — in the top left position of the adjacency matrix, then systematically try all possible ways to complete the adjacency matrix. For each completed adjacency matrix, we test whether the graph contains as an induced subgraph each tree in our family. If it does, and if the graph is not isomorphic to any induced universal graph that has already been found, we add it to our collection of graphs. We will refer to this as the “naive completion” method.

Because the naive completion method has to test a huge number of graphs, many pairs of which are isomorphic, we created an improved version of the algorithm, which we will refer to as the “symmetry breaking completion” method. Our description refers to the adjacency matrix in Figure 11. This adds two symmetry-breaking constraints on the adjacency matrix. The first of these constraints concerns the grey shaded region at the bottom-right of the adjacency matrix, which has $n - k$ rows and columns. Rather than trying all $2^{\binom{n-k}{2}}$ possible entries for this region of the adjacency matrix, we use a list containing a canonical adjacency matrix for each possible graph of order $n - k$, and we require that the shaded region be equal to one of these adjacency matrices.

0	1	1	1	1	1	1	x_{11}	x_{12}	x_{13}	x_{14}
1	0	0	0	0	0	0	x_{21}	x_{22}	x_{23}	x_{24}
1	0	0	0	0	0	0	x_{31}	x_{32}	x_{33}	x_{34}
1	0	0	0	0	0	0	x_{41}	x_{42}	x_{43}	x_{44}
1	0	0	0	0	0	0	x_{51}	x_{52}	x_{53}	x_{54}
1	0	0	0	0	0	0	x_{61}	x_{62}	x_{63}	x_{64}
1	0	0	0	0	0	0	x_{71}	x_{72}	x_{73}	x_{74}
x_{11}	x_{21}	x_{31}	x_{41}	x_{51}	x_{61}	x_{71}	0	.	.	.
x_{12}	x_{22}	x_{32}	x_{42}	x_{52}	x_{62}	x_{72}	.	0	.	.
x_{13}	x_{23}	x_{33}	x_{43}	x_{53}	x_{63}	x_{73}	.	.	0	.
x_{14}	x_{24}	x_{34}	x_{44}	x_{54}	x_{64}	x_{74}	.	.	.	0

Figure 11: The adjacency-matrix regions used in the “symmetry breaking completion” method. For this example, $n = 11$ and $k = 7$.

The second symmetry-breaking constraint concerns the yellow shaded region in Figure 11. Each row of this region has $n - k$ entries, named $x_{i1}, \dots, x_{i,n-k}$. For our symmetry-breaking constraint, we view each row as the binary number $x_{i1}, \dots, x_{i,n-k}$

(with $x_{i,n-k}$ as the least significant digit), and we insist that these numbers are in nondecreasing order; thus $x_{21}, \dots, x_{2,n-k} \leq \dots \leq x_{k1}, \dots, x_{k,n-k}$. In other words, we impose a lexicographic ordering constraint on the rows of this region.

The algorithm with these symmetry-breaking constraints is shown in Algorithm 3. The call to `MakeGraph()` on line 14 constructs a graph by building its adjacency matrix as shown in Figure 11. The first k vertices are connected in the form of a star; for each i such that $1 \leq i \leq j$, the digits of the binary representation of x_i are assigned to $x_{i1}, \dots, x_{i,n-k}$, and these values are placed in the adjacency matrix as shown in the figure. Finally, the small adjacency matrix M is copied to the bottom-right of the adjacency matrix of the constructed graph.

For the collection of graphs \mathcal{G} defined on line 5, isomorphic graphs are considered identical. Thus, we only add a graph to \mathcal{G} if it is not isomorphic to any existing member of the collection.⁶

We claim that although the symmetry breaking completion method does not visit every adjacency matrix visited by the naive completion method, it finds, up to isomorphism, all induced universal graphs for the family of trees on k vertices. This is shown by the following proposition.

Proposition 2. *Let n, k be given, and let G be a graph on n vertices that is induced universal for the family of trees of order k . Let \mathcal{M} be a set of canonical adjacency matrices for $\mathcal{F}(n - k)$. There is a graph on n vertices numbered $1, \dots, n$ that is isomorphic to G and satisfies the two symmetry-breaking constraints described above.*

Proof. Since G is induced universal for the family of trees of order k , it contains the star S_{k-1} as an induced subgraph. Arbitrarily choose one such star in G and give the label 1 to its vertex of degree $k - 1$. Its remaining vertices will be given the labels $\{2, \dots, k\}$, in an order that will be specified in the next paragraph. The remaining $n - k$ vertices of G induce a subgraph isomorphic to one of the adjacency matrices in \mathcal{M} ; call this matrix M . By choosing an appropriate relabelling of the $n - k$ vertices of G , the region of the adjacency matrix corresponding to these vertices can be made equal to M . Thus, the first symmetry-breaking constraint is satisfied.

The second symmetry-breaking constraint can now be satisfied by assigning labels $\{2, \dots, k\}$ to the degree-1 vertices of the star S_{k-1} in such an order that their adjacencies to the vertices outside the star, viewed as binary numbers, are in lexicographic order. \square

⁶An efficient method to ensure that isomorphic graphs are not stored in \mathcal{G} would be label the vertices of a graph G canonically before adding it to the collection [11], and to store the collection as a hash set. Because this part of the program is not a performance bottleneck, and to minimise software dependencies, our implementation uses the simpler approach of testing whether G is isomorphic to each existing member of \mathcal{G} , in turn.

Algorithm 3: The symmetry-breaking completion algorithm for finding all graphs of order n that are induced universal for the family of all order- k trees

```

1 Search( $n, k, (x_1, \dots, x_j)$ )
2 Data: Natural numbers  $n$  and  $k$ , and a sequence  $(x_1, \dots, x_j)$  of nonnegative
   integers such that the binary representation of each  $x_i$  represents
   entries  $x_{i1}, \dots, x_{i,n-k}$  of an adjacency matrix, as shown in Figure 11
3 Result: The set of all order- $n$  graphs that are induced universal for the
   family of trees of order  $k$ , and have an adjacency matrix given by
   MakeGraph( $n, k, X, M$ ) for some completion  $X$  of  $(x_1, \dots, x_j)$  and
   some  $M \in \mathcal{M}(n - k)$ 
4 begin
5    $\mathcal{G} \leftarrow \emptyset$ 
6   if  $j \leq 1$  then
7     for  $i \in \{0, \dots, 2^{n-k} - 1\}$  do
8        $\mathcal{G} \leftarrow \mathcal{G} \cup \text{Search}(n, p, (x_1, \dots, x_j, i))$ 
9   else if  $j \leq k$  then
10    for  $i \in \{0, \dots, x_j\}$  do
11       $\mathcal{G} \leftarrow \mathcal{G} \cup \text{Search}(n, p, (x_1, \dots, x_j, i))$ 
12  else
13    for  $M \in \mathcal{M}(n - k)$  do
14       $G \leftarrow \text{MakeGraph}(n, k, (x_1, \dots, x_j), M)$ 
15      if  $G$  contains every tree of order  $k$  as an induced subgraph then
16         $\mathcal{G} \leftarrow \mathcal{G} \cup \{G\}$ 
17  return  $\mathcal{G}$ 

18 FindAllInducedUniversalGraphsForTrees( $n, k$ )
19 Data: Natural numbers  $n$  and  $k$  such that  $k < n$ 
20 Result: The set of all graphs of order  $n$  that are induced universal for the
   family of trees of order  $k$ .
21 begin
22   return Search( $n, k, ()$ )

```

7.2 Results for families of trees

The symmetry-breaking completion method is much faster than Algorithm 1. For $n = 9$ and $k = 6$, for example, the completion method takes 9 seconds whereas the brute-force algorithm takes 46 seconds. The difference in run-times becomes even

greater for larger values of n and k . Using the symmetry-breaking completion method, we were able to find the values of $t(k)$ and $T(k)$ for $k \leq 7$. We were also able to show that $t(8) > 12$. We confirmed the results for $k \leq 6$ using Algorithm 1.

Table 4 gives the order $t(k)$ of a minimal induced universal graph for the family of k -vertex trees, and the number $T(k)$ of such graphs. Figure 12 shows one of the 687 minimal induced universal graphs for the family of 7-vertex trees.

k	$t(k)$	$T(k)$
1	1	1
2	2	1
3	3	1
4	5	2
5	7	18
6	9	66
7	11	687

Table 4: For each k , $t(k)$ is the minimum order of a graph containing all k -vertex trees as induced subgraphs, and $T(k)$ is the number of distinct $t(k)$ -vertex graphs that contain all k -vertex trees as induced subgraphs.

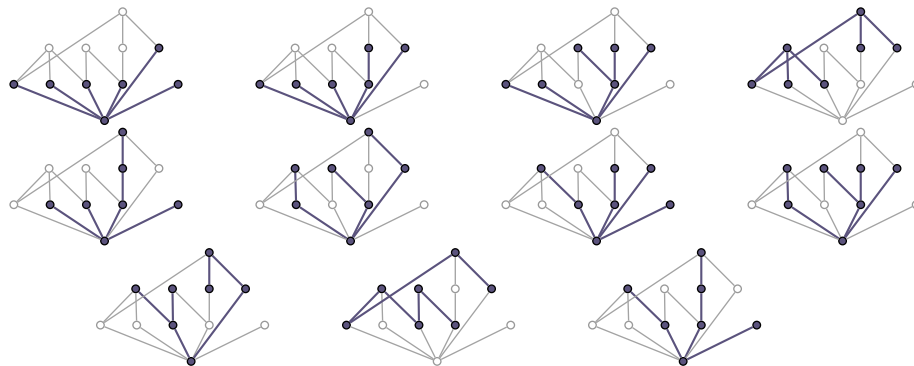


Figure 12: An induced universal graph for the family of all trees on 7 vertices. In each copy of the graph, one of the 11 trees of order 7 is highlighted.

To obtain upper bounds on $t(k)$ for larger values of k , we use a version of Algorithm 2 with minor modifications. Instead of initially embedding a clique and an independent set as in Figure 8, we embed a single tree — the k -vertex star S_{k-1} — as shown in Figure 13. Each possible edge marked with a dot in the figure is added with probability 0.1.

0	1	1	1	1	1	·	·	·
1	0	0	0	0	0	·	·	·
1	0	0	0	0	0	·	·	·
1	0	0	0	0	0	·	·	·
1	0	0	0	0	0	·	·	·
1	0	0	0	0	0	·	·	·
·	·	·	·	·	·	0	·	·
·	·	·	·	·	·	·	0	·
·	·	·	·	·	·	·	·	0

Figure 13: The adjacency matrix used in our heuristic algorithm when searching for an induced universal graph of order 9 for the set of all trees of order 6. The blue region, a star with 6 vertices, remains unchanged as the algorithm runs. The remaining possible edges, marked \cdot , are flipped between 0 and 1 as the algorithm progresses.

Using this heuristic algorithm, we were able to find induced universal graphs to demonstrate that $t(8) \leq 13$, $t(9) \leq 15$ and $t(10) \leq 18$. Since we have matching lower and upper bounds for $t(8)$, we can state that $t(8) = 13$.

8 Verification

To verify correctness of the results in Table 3 and Table 4, we repeated every call to the McSplit subgraph isomorphism solver using the LAD algorithm [14] as implemented in the `igraph` library [5].

We verified that the graphs shown as adjacency matrices in Figure 9 and Figure 10 are induced universal graphs for their corresponding families using a shell script that calls the Glasgow Subgraph Solver [7, 9] to check for the inclusion of every graph of k vertices. This script shares no code with the Python program that generated the graphs.

Finally, we verified all but the final row in each of Table 3 and Table 4 using a second shell script that calls the Glasgow Subgraph Solver. Again, this shares no code with the program used to generate the results in the tables.

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