Small induced universal graphs

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Abstract

Using the results of brute-force computational experiments, we give for $0 \leq k \leq 5$ the minimum number of vertices $f(k)$ in a graph containing all k-vertex graphs as induced subgraphs. We give the corresponding results for $0 \leq k \leq 6$ for graphs containing all trees on k vertices. We also show that $13 \leq f(6) \leq 15$.

1 Introduction

Given a collection $\mathcal{F}$ of graphs, graph G is *induced universal* for $\mathcal{F}$ if and only if each element of $\mathcal{F}$ is an induced subgraph of G. Graph G is a *minimal* induced universal graph for $\mathcal{F}$ if it has as few vertices as possible. We write $f(k)$ to denote the order (number of vertices) of a minimal induced universal graph for the family of all graphs on k vertices. Moon showed that $f(k) \leq 2^{(k-1)/2}$ [4], and Alon showed that $f(k) = (1 + o(1))2^{(k-1)/2}$ [1]. There is an extensive literature on bounds on the order of minimal induced universal subgraphs for many families of graphs; see the references in [1]. However, I am not aware of any existing systematic attempts to find exact values for families of small graphs. This paper presents exact values of $f(k)$ for $k \leq 5$, obtained by brute-force computational search. The corresponding counts of minimal induced universal subgraphs, and results for families of trees, are also presented.

2 Method

Given values of k and n, a simple brute-force approach was used to find the set of n-vertex graphs that are induced universal for the family $\mathcal{F}$ of all k-vertex graphs.

Each possible n-vertex graph G from Brendan McKay's lists of graphs[1] was tested in turn. We used a Python implementation of the McSplit algorithm [3] to test whether each k-vertex graph is an induced subgraph of G. Although the McSplit algorithm was designed for the more general maximum common induced subgraph problem, it performs well on the subgraph isomorphism problem if both input graphs are small.

For those graphs G that do not contain a copy of each element of $\mathcal{F}$, it is desirable to iterate over $\mathcal{F}$ in an order that allows us to quickly reject G by finding a graph that is not an induced subgraph of G. The approach we used was to sort $\mathcal{F}$ in descending order of number of isomorphisms. In particular, this means that the clique K_k and the independent set I_k are the first two graphs tested.

The full set of experiments described in this paper, run sequentially, required around 14 million calls to the subgraph isomorphism algorithm and took around 12 minutes to complete on a laptop. [2].

3 Results

Table 1 shows, for $0 \leq k \leq 5$, the value of $f(k)$. The table also shows $F(k)$, the number of minimal induced universal graphs for the family of all k-vertex graphs. To my knowledge, the value of $f(10)$ and the values of $F(k)$ have not been published previously.

The values $f(1)$, $f(2)$ and $f(3)$ are equal to to the simple lower bound $2k-1$ given in a question by "Chain Markov" on Mathematics Stack Exchange [2]. The values $f(4)$ and $f(5)$ are equal to a lower bound given in a comment by "bof" on that question: $f(k) \geq 2k$ if $k \geq 4$. (For $k < 10$, this bound improves upon Moon's lower bound.) To briefly summarise the proof, if $k \leq 2k$ then G must be a split graph (that is, a graph whose vertices can be partitioned into a clique and an independent set); therefore, G cannot contain the cycle C_4 as an induced subgraph. An example of an 8-vertex induced universal graph for the family of 4-vertex graphs was given by "Chain Markov" as a comment on the same question.

Figure 3 shows three examples of minimal induced universal graphs. These are for the families of all graphs with three, four and five vertices respectively.

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[1] `https://users.cecs.anu.edu.au/~bdm/data/graphs.html`

[2] The code and results from this paper, including lists of minimal induced universal graphs graph6 format, are available from `https://github.com/jamestrimble/small-universal-graphs`

k	$f(k)$	$F(k)$
0	0	1
1	1	1
2	3	2
3	5	5
4	8	438
5	10	22

Table 1: For each k, $f(k)$ is the minimum order of a graph containing all k-vertex graphs as induced subgraphs, and $F(k)$ is the number of distinct $f(k)$-vertex graphs that contain all k-vertex graphs as induced subgraphs.

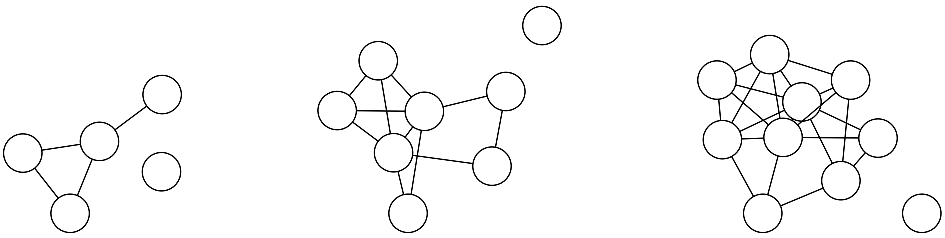

Figure 1: Minimal induced universal graphs for the families of all graphs with 3, 4, and 5 vertices

4 Bounds on $f(6)$

To give an upper bound of 15 on $f(6)$, Figure 2 shows the adjacency matrix of a 15-vertex graph that is induced universal for the family of all graphs on six vertices. This was generated using a specialised random graph generator. We number the vertices from zero; the generator makes the k vertices numbered 0 to $k-1$ a clique, and the k vertices numbered $k-1$ to $2k-2$ an independent set. Each possible edge that is not involved in either the clique or the independent set is added with probability 1/2. It was only necessary to generate 53 graphs with this generator to find a solution.

For $k > 5$, it is straightforward to increase by 1 the lower bound by "bof" from the previous section.

Proposition 1. *$f(k) > 2k+1$ for all $k > 5$.*

Proof. Suppose, for contradition, that G is an induced universal graph on $2k$ vertices for the family of all graphs on k vertices. Graph G must have K_k and I_k as induced sugraphs. This clique and independent set must overlap by exactly one vertex (otherwise, as in the proof by "bof" from the previous section, G would be a split graph

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0 1 1 1 1 1 0 0 0 1 1 0 0 0 1
1 0 1 1 1 1 0 0 0 0 1 1 1 1 1
1 1 0 1 1 1 1 0 0 1 0 0 1 0 0
1 1 1 0 1 1 1 1 1 1 0 0 0 0 0
1 1 1 1 0 1 1 1 0 1 1 1 1 0 0
1 1 1 1 1 0 0 0 0 0 0 0 0 0 1
0 0 1 1 1 0 0 0 0 0 0 0 0 1 0
0 0 0 1 1 0 0 0 0 0 0 1 1 1 1
0 0 0 1 0 0 0 0 0 0 0 1 1 1 0
1 0 1 1 1 0 0 0 0 0 0 1 0 0 0
1 1 0 0 1 0 0 0 0 0 0 0 1 0 1
0 1 0 0 1 0 0 1 1 1 0 0 0 1 0
0 1 1 0 1 0 0 1 1 0 1 0 0 1 1
0 1 0 0 0 0 1 1 1 0 0 1 1 0 0
1 1 0 0 0 1 0 1 0 0 1 0 1 0 0
```

Figure 2: The adjacency matrix of a 15-vertex induced universal graph for the family of all six-vertex graphs

and would not have C_4 as an induced subgraph). Thus we can partition the vertices of G into three sets: a clique S_1, an independent set S_2, and a set S_3 containing the single remaining vertex.

Let S be a set of vertices such that the subgraph of G induced by S is isomorphic to the cycle graph C_k. S may contain at most two vertices from S_1, and these must be adjacent in the induced cycle. Trivially, S can contain at most one vertex from S_3. Hence, S must contain at least $k-3$ vertices from S_2. At least two of these vertices must be adjacent in the induced C_k. Since S_2 is an independent set in G, we have a contradiction. $\square$

Thus, we have that $13 \leq f(6) \leq 15$.

5 Trees

Table 2 gives the order $t(k)$ of a minimal induced universal graph for the family of k-vertex trees, and the number $T(k)$ of such graphs. Figure 5 shows one of the 66 minimal induced universal graphs for the family of 6-vertex trees.

k	$t(k)$	$T(k)$
0	0	1
1	1	1
2	2	1
3	3	1
4	5	2
5	7	18
6	9	66

Table 2: For each k, $t(k)$ is the minimum order of a graph containing all k-vertex trees as induced subgraphs, and $T(k)$ is the number of distinct $t(k)$-vertex graphs that contain all k-vertex trees as induced subgraphs.

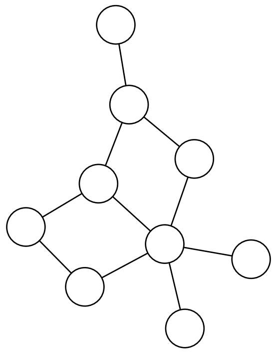

Figure 3: A universal graph for the family of all trees with 6 vertices

References

[1] N. Alon. Asymptotically optimal induced universal graphs. *Geometric and Functional Analysis*, 27(1):1–32, 2017.

[2] Chain Markov. What is the minimal possible size of an n-universal graph? Mathematics Stack Exchange. URL:https://math.stackexchange.com/q/3246874 (version: 2019-06-01).

[3] C. McCreesh, P. Prosser, and J. Trimble. A partitioning algorithm for maximum common subgraph problems. In C. Sierra, editor, *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI 2017, Melbourne, Australia, August 19-25, 2017*, pages 712–719. ijcai.org, 2017.

[4] J. W. Moon. On minimal n-universal graphs. *Proceedings of the Glasgow Mathematical Association*, 7(1):32–33, 1965.