Spatial Ecological Models

Agenda I

A sampling of Ecological questions

Modelling Animal Movement

Some classic results

Modelling Animal Movement

Brownian motion and random walks

model individual movement as a random process

Brownian motion and random walks

model individual movement as a random process

position jump process
: individuals take steps of fixed or random length in random directions

velocity jump process
: individuals randomly change speed and/or direction

Brownian motion and random walks

model individual movement as a random process

position jump process individuals take steps of fixed or random length in random directions

velocity jump process individuals randomly change speed and/or direction

kinesis change in speed or direction of movement in response to concentration

taxis change in direction in response to gradient

Response to Habitat Boundaries

Figure 7: change direction in response to habitat boundary

A single random walker on a homogeneous landscape

Let $X(t)$ be the position of the walker at time t.

Suppose

$$X(t+\tau) = \begin{cases} X(t)+\delta & \text{with probabilityp/2} \\ X(t)-\delta & \text{with probabilityp/2} \\ X(t) & \text{with probability1-p} \end{cases}$$

A single random walker on a homogeneous landscape

Let $X(t)$ be the position of the walker at time t.

Suppose

$$X(t+\tau) = \begin{cases} X(t)+\delta & \text{with probabilityp/2} \\ X(t)-\delta & \text{with probabilityp/2} \\ X(t) & \text{with probability1-p} \end{cases}$$

Let $u(t,x)$ be the probability the walker is at position x at time t

$$u(t+\tau,x) = (1-p)u(t,x) + \frac{p}{2}(u(t,x+\delta) + u(t,x-\delta)$$

A single random walker on a homogeneous landscape

Let $X(t)$ be the position of the walker at time t.

Suppose

$$X(t+\tau) = \begin{cases} X(t)+\delta & \text{with probabilityp/2} \\ X(t)-\delta & \text{with probabilityp/2} \\ X(t) & \text{with probability1-p} \end{cases}$$

Let $u(t,x)$ be the probability the walker is at position x at time t

$$u(t+\tau,x) = (1-p)u(t,x) + \frac{p}{2}(u(t,x+\delta)+u(t,x-\delta)$$

Take a particular limit as $(\tau,\delta)\to(0,0)$ et voila:

$$u_t = \mu u_{xx}$$

Many random walkers - Individual based models

- probabilistic model of position (and other aspects) of each individual
$$\mathsf{Pr}\left(X(t+\tau)=x \mid |X(t)=y\right)$$
- good way to describe individual behaviours, heterogeneity between individuals
- continuous time/space, or discrete time/space
- easy to add (too much) detail to state of each individual
- also referred to as the Lagrangian viewpoint (particle tracking)

Many random walkers - Population based models

- Compartmental/Density based models
- also referred to as the Eulerian viewpoint (flow past a point)
- track density of individuals as function of position

cellular automata (CA) discrete time and space

integro-difference equation (IDE) discrete time, continuous space, dispersal kernel

$$N_{t+1}(x) = \int_{\Omega} k(x,y) F(y, N_t(y))\, dy$$

integro-differential equation (IDE) continuous time, continuous space, dispersal kernel

$$\frac{\partial u}{\partial t}(t,x) = \int_{\Omega} k(x,y) F(y, u(t,y))\, dy$$

partial differential equation PDE continuous time and space, diffusion limit

$$u_t = \mu u_{xx} + f(u)$$

Some classic results

Travelling waves and Fisher's equation

Add diffusion to the logistic equation

$$N_t = \mu N_{xx} + rN\left(1 - \frac{N}{K}\right)$$

Travelling waves and Fisher's equation

Add diffusion to the logistic equation

$$N_t = \mu N_{xx} + rN\left(1 - \frac{N}{K}\right)$$

Look for a **travelling wave** solution

$$N(t,x) = Ku(x - ct)$$

Travelling waves and Fisher's equation

Add diffusion to the logistic equation

$$N_t = \mu N_{xx} + rN\left(1 - \frac{N}{K}\right)$$

Look for a **travelling wave** solution

$$N(t,x) = Ku(x-ct)$$

$$-cu' = \mu u'' + ru(1-u), \qquad u(-\infty) = 1, \qquad u(\infty) = 0$$

Travelling waves and Fisher's equation

Add diffusion to the logistic equation

$$N_t = \mu N_{xx} + rN\left(1 - \frac{N}{K}\right)$$

Look for a **travelling wave** solution

$$N(t,x) = Ku(x - ct)$$

$$-cu' = \mu u'' + ru(1-u), \qquad u(-\infty) = 1, \qquad u(\infty) = 0$$

$$\begin{aligned} u' &= v \\ \mu v' &= -ru(1-u) - cv \end{aligned}$$

$$\begin{aligned} u' &= v \\ \mu v' &= -ru(1-u) - cv \end{aligned}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ -r/\mu & -c/\mu \end{pmatrix}$$

$$J(1,0) = \begin{pmatrix} 0 & 1 \\ r/\mu & -c/\mu \end{pmatrix}$$

$$
\begin{aligned}
u' &= v \\
\mu v' &= -ru(1-u) - cv
\end{aligned}
$$

$$
J(0,0) = \begin{pmatrix} 0 & 1 \\ -r/\mu & -c/\mu \end{pmatrix}
$$

$$
J(1,0) = \begin{pmatrix} 0 & 1 \\ r/\mu & -c/\mu \end{pmatrix}
$$

Travelling wave solutions if $c > \sqrt{2r\mu}$.

Critical Patch Size

Again consider a PDE model with diffusion and growth,

$$N_t = \mu N_{xx} + f(x, N)$$

Determine the stability of the trivial equilibrium, $N = 0$, as a function of a patch size, L.

Linearize:

$$N_t = \mu N_{xx} + r(x)N$$

Remember, r is the net growth rate, $r = b - d$.

Consider $r(x) < 0$ for $|x| > \frac{L}{2}$

See, for example, Cantrell and Cosner [https://www.math.miami.edu/~rsc/docs/2001spatial.pdf]

Chemotaxis - Keller-Segel model

Our random walk is biased in the direction of an increasing chemical concentration

$$
\begin{aligned}
\text{critter:} \qquad & \frac{\partial u}{\partial t} = f(u,v) + \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x} - \chi u \frac{\partial v}{\partial x}\right) \\
\text{chemical:} \qquad & \frac{\partial v}{\partial t} = g(u,v) + \frac{\partial^2 v}{\partial^2 x}
\end{aligned}
$$

See recent overview of models by Kevin J. Painter [https://doi.org/10.1016/j.jtbi.2018.06.019]

Chemokinesis

Jump distance or speed in random walk is influenced by a chemical concentration

$$\text{critter:} \qquad \frac{\partial u}{\partial t} = f(u,v) + \frac{\partial^2}{\partial x^2}\mu(v)u$$

$$\text{chemical:} \qquad \frac{\partial v}{\partial t} = g(u,v) + \frac{\partial^2 v}{\partial^2 x}$$

Stage-structured Integro-difference Equations

Cellular Automata

Discrete-time, discrete-space, discrete-state

Example: Fitzhugh-Nagumo moel

Approximate by discrete-state

Resting – Excited – Recovering – Resting