# Sets

$\mathbb{N} = \{0, 1, 2, \dots\}$ natural numbers

$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ integers

$\mathbb{Q} = \{ \frac{k}{n} \mid k, n \in \mathbb{N},\ n \neq 0 \}$

rationals

$\mathbb{R}$ = real numbers

$\mathbb{C} = \{ a + ib \mid a, b \in \mathbb{R} \}$ = complex

$i$ = "imaginary unit"
characterized by $i^2 = -1$.

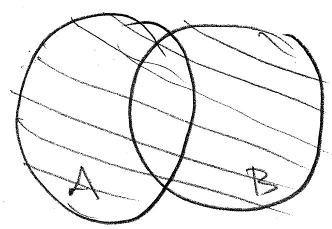
$\mathbb{R}^d = \{ (a_1, \dots, a_d) \mid a_i \in \mathbb{R} \}$ d-dimensional space

$\mathbb{R}^\infty = \{ (a_0, a_1, a_2, \dots) \mid a_i \in \mathbb{R} \}$
space of <u>infinite sequences</u> of reals

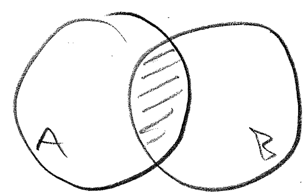
$A, B$ sets. <u>Operations</u>:

$A \cup B = \{ c \mid c \in A \text{ or } c \in B \}$ <u>union</u>

$A \cap B = \{ c \mid c \in A \text{ and } c \in B \}$ <u>intersection</u>



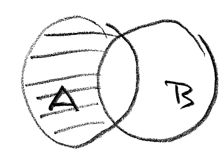

$A \cup B$




$A \cap B$

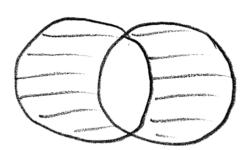
$A \setminus B = \{ c \in A \mid c \notin B \}$ :

"<u>difference</u>"




$A \setminus B$

$A \Delta B = A \setminus B \cup B \setminus A$ :



$= (A \cup B) \setminus A \cap B$ etc.

"<u>symmetric difference</u>"

Let $(A_\alpha)_{\alpha \in I}$ be an indexed family of sets (index set $I$).

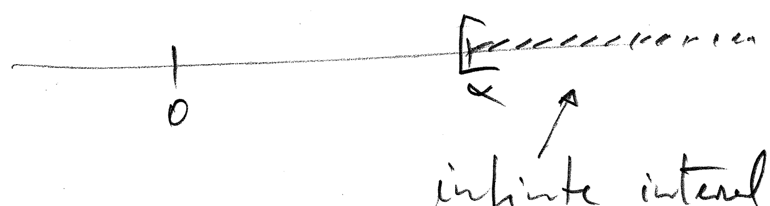
EX ① $I = \mathbb{N}$,

$$A_\alpha := \{\alpha, \alpha+1\}$$

definition

② $I = \mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$

$$A_\alpha := [\alpha, \infty)$$




union: $\bigcup_{\alpha \in I} A_\alpha := \{a \mid a \in A_\alpha \text{ for some } \alpha \in I\}$

$= \{a \mid \exists\, \alpha \in I \text{ such that } a \in A_\alpha\}$

intersectn $\bigcap_{\alpha \in I} A_\alpha = \{a \mid a \in A_\alpha \underbrace{\text{ for every } \alpha \in I}_{\forall \alpha \in I}\}$.

## Maps in general.

Map = function = mapping (= transformation)

Let $A$, $B$ be sets ($\neq \emptyset$). A map $f$

$$\begin{aligned} f : A &\longrightarrow B \\ a &\longmapsto f(a) \end{aligned}$$

is an assignment : to $\forall a \in A$ $f$ assign a unique $b \in B$ which is called the value of $f$ at $a$ and is denoted by $f(a)$.

Note: $f$ assigns to $\forall a \in A$ a value, but there may exists $b \in B$ st $f(a) \neq b$ $\forall a \in A$.

$A$ is called the domain (of definition) of $f$

$B$ is — " — the target space of $f$.

If $A' \subseteq A$, then the map $f'$

$$f' : A' \to B$$
$$a \mapsto f'(a) := f(a)$$

is called the restriction of $f$ to $A'$

( denoted by $f' = f\restriction_{A'}$ )

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EX: ① $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$ → $f = \sin$ fct.

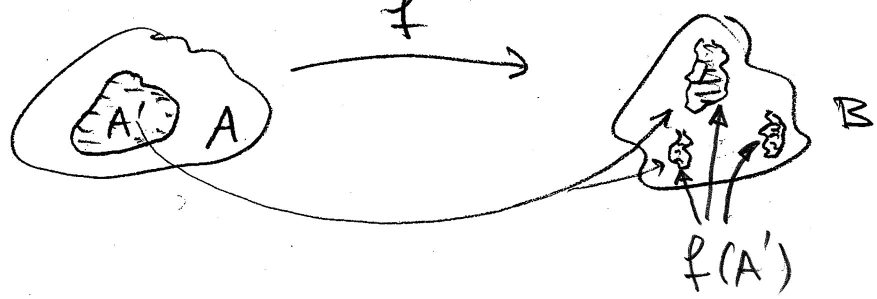
② $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $x \mapsto 1/x$

③ $g : \mathbb{R} \to \mathbb{R}$
$$x \mapsto \begin{cases} 1/x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

④ $f : V \to$ set of subspaces of $V = \mathcal{S}$
$$\vec{v} \mapsto \mathrm{span}(\vec{v})$$

⑤ $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ derivative
$$p(x) \mapsto \frac{d}{dx} p(x) = p'$$

Let




$$f(A') := \{ b \in B \mid \exists a \in A : f(a) = b \} =$$

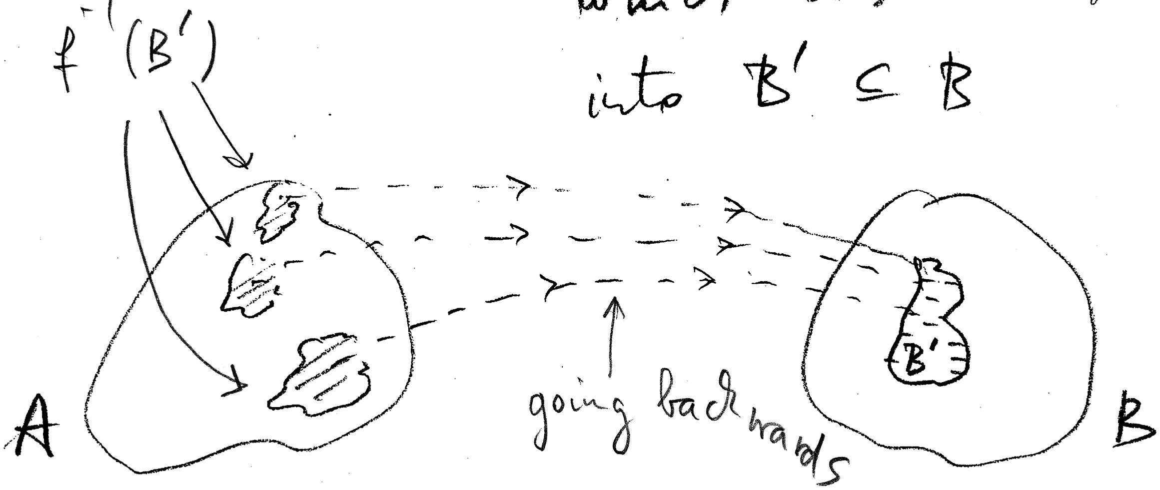
[the image of the subset $A' \subseteq A$] $= \{ f(a) \mid a \in A' \}$

image = "forward image"

Similarly we define the inverse image (backwards image) of $B' \subseteq B$

$$f^{-1}(B') := \{ a \in A \mid f(a) \in B' \}$$

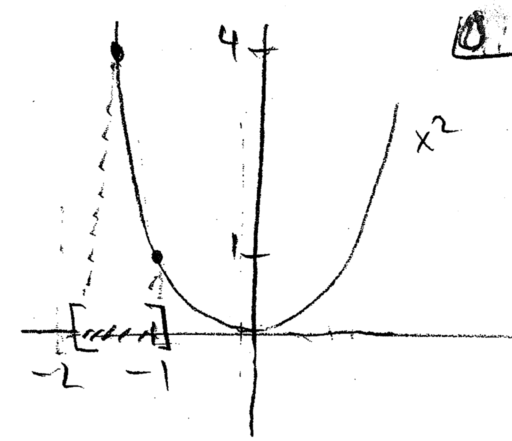
all the elements $a$ in $A$ which has been mapped into $B' \subseteq B$




⚠ $f^{-1}$ here is NOT the inverse map!

... $f^{-1}$ does not exist!

<u>Ex</u> $f : \mathbb{R} \to \mathbb{R}$
$x \longmapsto x^2$




$A' = [-2, -1]$

$f(A') = [1, 4]$

let $B' = [1, 4]$

$f^{-1}(B') = \{ x \in \mathbb{R} \mid x^2 \in [1, 4] \}$

$= [-2, -1] \cup [1, 2]$

( $\to$ in general $f^{-1}(f(A')) \supseteq A'$ )

etc.

Lemma: $(B_\alpha)_{\alpha \in I}$, $B_\alpha \subseteq B$.

then

1) $f^{-1}\left(\bigcup_\alpha B_\alpha\right) = \bigcup_\alpha f^{-1} B_\alpha$

2) $f^{-1}\left(\bigcap_\alpha B_\alpha\right) = \bigcap_\alpha f^{-1} B_\alpha$

3) $f^{-1}(B^c) = (f^{-1}B)^c$

⌜ HW. ⌟

Special case: $I = \{1,2\}$.

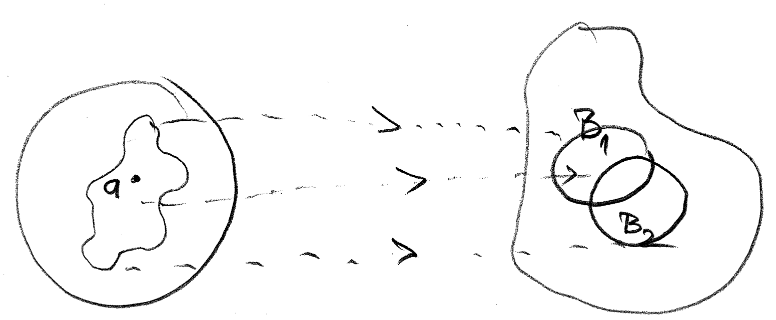
1) $f^{-1}(B_1 \cup B_2) = f^{-1}B_1 \cup f^{-1}B_2$

⌜ Let $a \in f^{-1}(B_1 \cup B_2) \iff$

$f(a) \in B_1 \cup B_2 \iff \underbrace{f(a) \in B_1}_{\iff a \in f^{-1}B_1}$ OR $\underbrace{f(a) \in B_2}_{a \in f^{-1}B_2}$

$\iff a \in f^{-1}B_1 \cup f^{-1}B_2$ ⌟

etc.



**Def** $f : A \to B$ is called

(1) <u>injective</u> (one to one) if $f(a) = f(a') \Rightarrow a = a'$
(distinct elements have distinct values)

(2) <u>surjective</u> (onto) : if $f(A) = B$, $\Leftrightarrow$
$\forall b \in B \ \exists a$ s.t. $f(a) = b$.

(3) <u>bijective</u> if <u>injective</u> + <u>surjective</u>.

In this case each $a \in A$ corresponds to exactly one $b \in B$ (and vice versa) and we can define the <u>inverse</u> $f^{-1}$

$$f^{-1} : B \to A$$
$$b \mapsto \text{the unique } a \in A \text{ with } f(a) = b$$

Then
$$f^{-1} f(a) = a \qquad \forall a \in A.$$
$$f f^{-1}(b) = b \qquad \forall b \in B$$

EX.

① $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ is <u>not injective</u>

(since $f(-1) = f(1) = 1$ and $-1 \neq 1$)

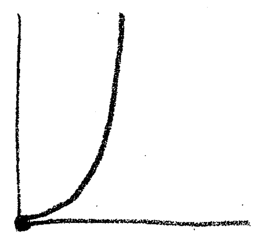
is <u>not surjective</u> since

$\nexists\, x \in \mathbb{R}$ st $f(x) = x^2 = -1$,

(and $-1 \in \mathbb{R}$).

② $f : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto x^2$ is <u>bijective</u>

and the inverse map is given



$f^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$, $x \mapsto \sqrt{x}$.

③ $D : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ ← polynomials

$p(x) \mapsto p'(x)$ is <u>surjective</u>

Let $p \in \mathcal{P}(\mathbb{R})$. Must find $q \in \mathcal{P}(\mathbb{R})$ with $Dq = p$

if $p = \sum_{k=0..n} a_k x^k$ and we set $q = \sum_{k=1}^{n+1} \frac{a_{k-1}}{k} x^k$

$\Rightarrow Dq = \sum_{k=1..n+1} a_{k-1} x^{k-1} = \sum_{k=0..n} a_k x^k = p$ qed $\lrcorner$

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$D$ is <u>not injective</u> since

$$D(p_1) = 0 = D(p_2)$$

where $p_1(x) \equiv 1$, $p_2(x) \equiv 2$ (constant polynomials)

but $p_1 \neq p_2$.

## <u>Composition of maps</u>

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, dashed, "h = g\circ f"'] & B \arrow[r, "g"] & C \end{tikzcd}$$

$A, B, C$ sets, $f, g$ maps. Then

$$h : A \to C$$
$$a \mapsto g(f(a)) =: h(a)$$

is called the <u>composition</u> of $f, g$ and is denoted by $\boxed{g \circ f.}$

## Size (cardinality) of sets.

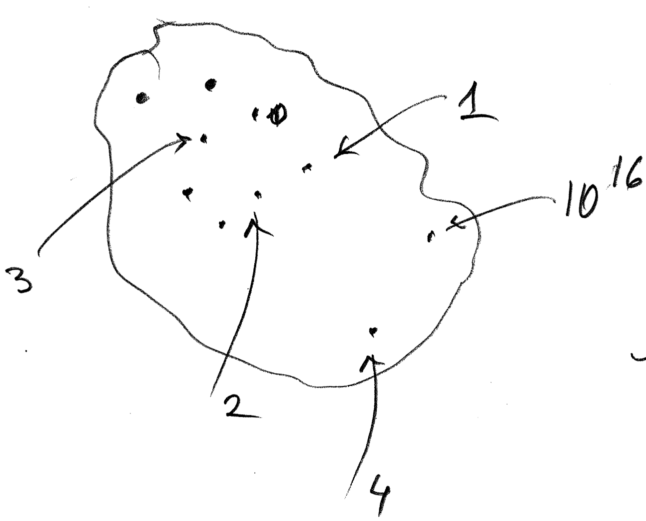
$|A| :=$ "number of elements of A"

- well-defined for finite sets

if $A$ infinite, $|A| = \infty$.

Def $A$ is called countably infinite if we can enumerate all its elements

i.e. if $\exists\ \varphi : \mathbb{N} \to A$ bijective.

$A =$ 


etc.

<u>Def</u> $|A| = |B|$ ($A, B$ have the same size)

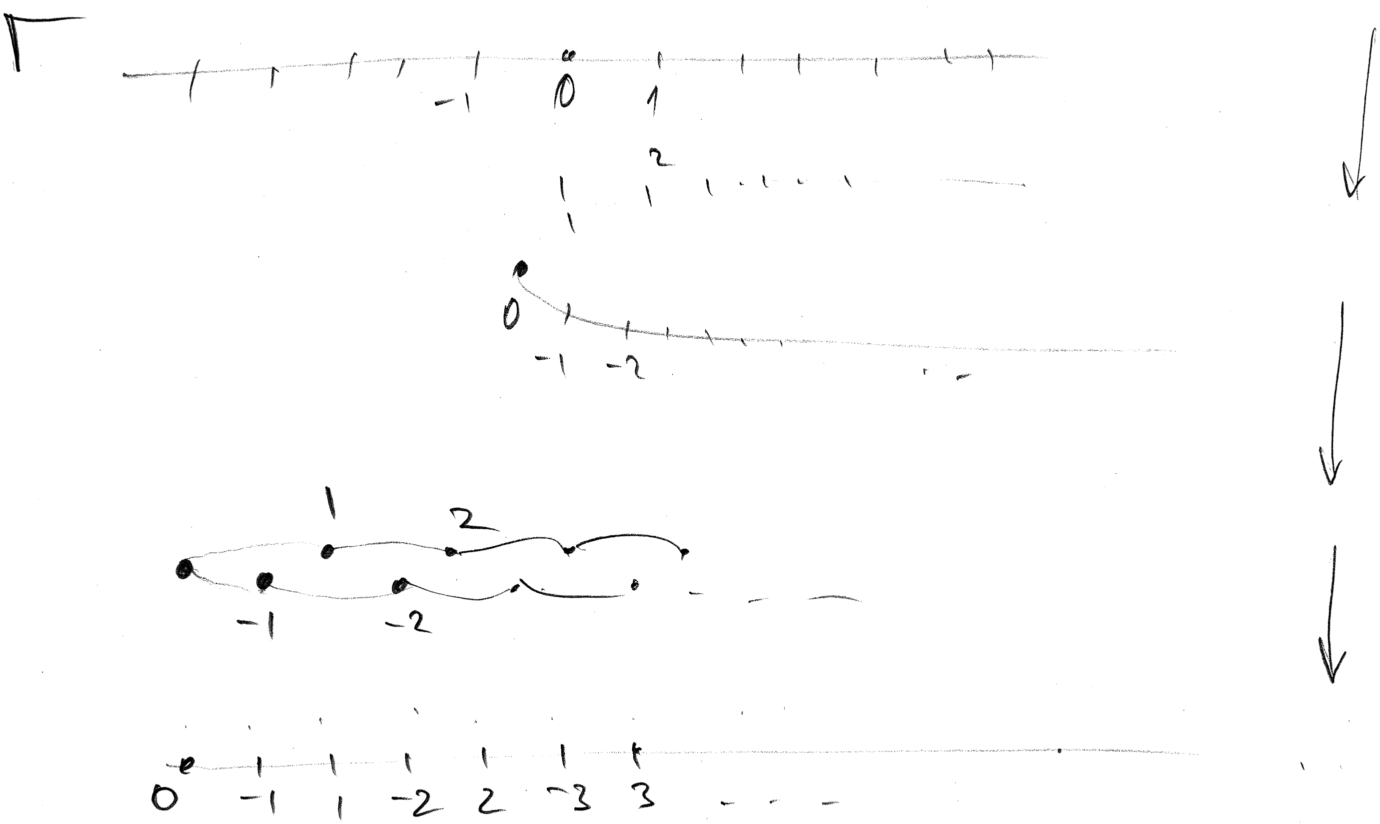
$\iff \exists\ \varphi : A \to B$ bijective.

So $A$ is countably infinite

$\iff$ has the cardinality of $\mathbb{N}$.

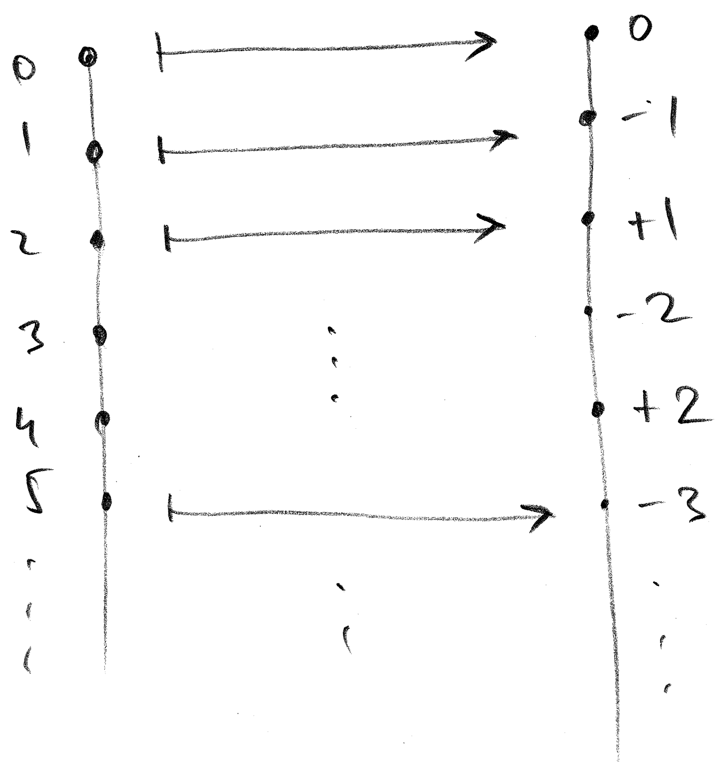
<u>EX</u> is $|\mathbb{N}| \stackrel{?}{=} |\mathbb{Z}|$ ?

A: yes



and the corresponding bijection $\varphi$ (called the "enumeration") is

$$\varphi : \mathbb{N} \longrightarrow \mathbb{Z}$$




etc.

an explicite formula:

$$\varphi(k) = (-1)^k \left[ \frac{k+1}{2} \right]$$

integer part of $\frac{k+1}{2}$

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**EX.** $|\mathbb{Q}| = ?$

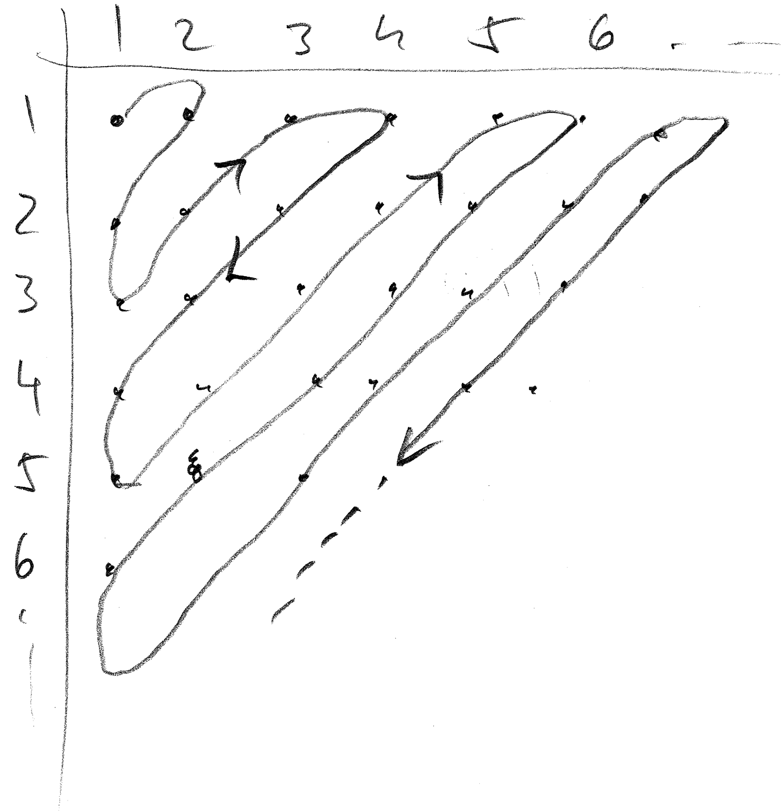
Claim $|\mathbb{Q}| = |\mathbb{N}|$.

$q \in \mathbb{Q} \Rightarrow q = \frac{k}{n}$ ← relative prime (fraction is simplified

| $n \downarrow$ \ $k \rightarrow$ | 1 | 2 | 3 | 4 | 5 | |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | $\frac{1}{2}$ | $\cancel{\frac{2}{2}}$ | $\frac{3}{2}$ | $\cancel{4/2}$ | $\frac{5}{2}$ | |
| 3 | $\frac{1}{3}$ | $\frac{2}{3}$ | $\cancel{3/3}$ | $4/3$ | | |
| 4 | $\frac{1}{4}$ | $\cancel{\frac{2}{4}}$ | $\frac{3}{4}$ | | | |
| 5 | $\frac{1}{5}$ | $\frac{2}{5}$ | | | | |

$\otimes$ means not rel. prime.

How to enumerate them?




!

so the enumeration is:

$$1,\ 2,\ \frac{1}{2},\ \frac{1}{3},\ 3,\ 4,\ 3/2,\ \frac{2}{3},\ \frac{1}{4},\ \frac{1}{5},\ 5,\ 6 \ldots$$

first second third ... etc.

**Q**: Is every infinite set countable?

**A**: No.

**Thm** (Cantor) Let $A$ be a set and define the **power set** $\mathcal{P}(A)$ as

$$\mathcal{P}(A) := \{B \mid B \subseteq A\}$$

the set of all subsets of $A$.

Then $|\mathcal{P}(A)| \underset{\neq}{>} |A|$.

⌜no proof⌟
(easy...).

In particular $|\mathcal{P}(\mathbb{N})| > |\mathbb{N}|$

"**uncountable**"

**Fact**: $\mathbb{R}$ is uncountable.

if $|A| = n \in \mathbb{N}$ i.e. $A$ is finite

then $|\mathcal{P}(A)| = 2^n$.

| $a_1$ | $a_2$ | $a_3$ | $\dots$ | $a_n$ |
|---|---|---|---|---|
| 0 | 1 | 1 | 0 1 $\dots$ | 1 |

this sequence corresponds to the subset $\{a_2, a_3, a_5, \dots, a_n\}$

Indeed there are as many subsets as 0-1 sequences of length $n$

$\Rightarrow 2^n$