# CS2040S
# Data Structures and Algorithms
(e-learning edition)

## Hashing!
## (Part 1)

# Semester Roadmap

Where are we?

Part 1

- Searching
- Sorting
- Trees
- **Hash Tables**

You are here

Part 2

- Graphs
  - Shortest paths
  - Spanning trees
  - Directed acyclic graphs
- Advanced material

# Plan: this week and next

Three Days of Hashing

- Applications
- Basic theory
- Handling collisions
- (Hashing in Java)
- Amortized analysis (doubling/shrinking)
- Sets and Bloom filters

# Topic of the Week: Hash Tables

# Abstract Data Types

## Symbol Table

```
public interface  SymbolTable

         void   insert(Key k, Value v)   insert (k,v) into table
        Value   search(Key k)            get value paired with k
         void   delete(Key k)            remove key k (and value)
      boolean   contains(Key k)          is there a value for k?
          int   size()                   number of (k,v) pairs
```

Note: no successor / predecessor queries.

# Symbol Table

## Examples:

| | |
|---|---|
| Dictionary: | key = word |
| | value = definition |
| Phone Book | key = name |
| | value = phone number |
| Internet DNS | key = website URL |
| | value = IP address |
| Java compiler | key = variable name |
| | value = type and value |

Implement a symbol table with a Linked List:
($C_I$= cost insert, $C_S$=cost search)

1. $C_I$ =O(1), $C_S$=O(1)
2. $C_I$ =O(1), $C_S$=O(log *n*)
3. $C_I$ =O(1), $C_S$=O(*n*)
4. $C_I$ =O(log *n*), $C_S$=O(log *n*)
5. $C_I$ =O(*n*), $C_S$=O(log *n*)
6. $C_I$ =O(*n*), $C_S$=O(*n*)

Implement symbol table with an AVL tree:
($C_I$= cost insert, $C_S$=cost search)

1. $C_I = O(1)$, $C_S = O(1)$
2. $C_I = O(1)$, $C_S = O(\log n)$
3. $C_I = O(1)$, $C_S = O(n)$
4. ✓ $C_I = O(\log n)$, $C_S = O(\log n)$
5. $C_I = O(n)$, $C_S = O(\log n)$
6. $C_I = O(n)$, $C_S = O(n)$

# Symbol Table

Implement a symbol table with:

- $C_I = O(1)$
- $C_S = O(1)$

Fast, fast, fast….

# Dictionaries vs. Symbol Tables

What can you do with a dictionary but not a symbol table?

# Dictionaries vs. Symbol Tables

Sorting with a dictionary:

1) Insert every item into the dictionary.

2) Search for the minimum item.

3) Repeat: find successor

Running time to implement sorting:

With an AVL tree/dictionary?

With a symbol table?

# Dictionaries vs. Symbol Tables

Sorting with a dictionary:

1) Insert every item into the dictionary.
2) Search for the minimum item.
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Running time to implement sorting:

With an AVL tree/dictionary? O(n log n)

With a symbol table?

# Dictionaries vs. Symbol Tables

Sorting with a dictionary:

1) Insert every item into the dictionary.

2) Search for the minimum item.

3) Repeat: find successor

Running time to implement sorting:

With an AVL tree/dictionary? $O(n \log n)$

With a symbol table? $O(n^2)$

# Sorting (aside)

Isn't O(1) impossible?

Sorting takes $\Omega(n \log n)$ comparisons.

- How do you sort with a symbol table?
- Only search/insert/delete.

# Sorting (aside)

## Isn't O(1) impossible?

Sorting takes $\Omega(n \log n)$ comparisons.

- How do you sort with a symbol table?
- Only search/insert/delete.

(Binary) search takes $\Omega(\log n)$ comparisons.

- Impossible to search in fewer than $\log(n)$ comparisons.
- But a symbol table finds an item in O(1) steps!!
- Conclusion: symbol table is not ***comparison-based***.

# Symbol Tables in Java

# Symbol Tables in Java

## java.util.Map

| `public interface` | `java.util.Map<Key, Value>` | |
|---|---|---|
| `void` | `clear()` | *removes all entries* |
| `boolean` | `containsKey(Object k)` | *is k in the map?* |
| `boolean` | `containsValue(Object v)` | *is v in the map?* |
| `Value` | `get(Object k)` | *get value for k* |
| `Value` | `put(Key k, Value v)` | *adds (k,v) to table* |
| `Value` | `remove(Object k)` | *remove mapping for k* |
| `int` | `size()` | *number of entries* |

Note: no successor / predecessor queries.

# Symbol Tables in Java

- Parameterized by key and value.
- Does not extend Comparable

## java.util.Map

```
public interface  java.util.Map<Key, Value>

         void  clear()                      removes all entries
      boolean  containsKey(Object k)        is k in the map?
      boolean  containsValue(Object v)      is v in the map?
        Value  get(Object k)                get value for k
        Value  put(Key k, Value v)          adds (k,v) to table
        Value  remove(Object k)             remove mapping for k
          int  size()                       number of entries
```

Note: no successor / predecessor queries.

# Symbol Tables in Java

## java.util.Map

- Search by key.

```
public interface  java.util.Map<Key, Value>

     void   clear()                    removes all entries
  boolean   containsKey(Object k)      is k in the map?
  boolean   containsValue(Object v)    is v in the map?
    Value   get(Object k)              get value for k
    Value   put(Key k, Value v)        adds (k,v) to table
    Value   remove(Object k)           remove mapping for k
      int   size()                     number of entries
```

Note: no successor / predecessor queries.

# Symbol Tables in Java

## java.util.Map

- Search by key.
- Search by value.

(May not be efficient.)

```
public interface  java.util.Map<Key, Value>

   void   clear()                       removes all entries
boolean   containsKey(Object k)         is k in the map?
boolean   containsValue(Object v)       is v in the map?
  Value   get(Object k)                 get value for k
  Value   put(Key k, Value v)           adds (k,v) to table
  Value   remove(Object k)              remove mapping for k
    int   size()                        number of entries
```

Note: no successor / predecessor queries.

# Symbol Tables in Java

## java.util.Map

• Can use any Object as key?

**`public interface java.util.Map<Key, Value>`**

| | | |
|---|---|---|
| `void` | `clear()` | *removes all entries* |
| `boolean` | `containsKey(Object k)` | *is k in the map?* |
| `boolean` | `containsValue(Object v)` | *is v in the map?* |
| `Value` | `get(Object k)` | *get value for k* |
| `Value` | `put(Key k, Value v)` | *adds (k,v) to table* |
| `Value` | `remove(Object k)` | *remove mapping for k* |
| `int` | `size()` | *number of entries* |

Note: no successor / predecessor queries.

# Symbol Tables in Java

## java.util.Map

• Put new (key, value) in table.

```
public interface  java.util.Map<Key, Value>

           void   clear()                     removes all entries
        boolean   containsKey(Object k)       is k in the map?
        boolean   containsValue(Object v)     is v in the map?
          Value   get(Object k)               get value for k
          Value   put(Key k, Value v)         adds (k,v) to table
          Value   remove(Object k)            remove mapping for k
            int   size()                      number of entries
```

Note:  no successor / predecessor queries.

# Map Interface in Java

java.util.Map<Key, Value>

- No duplicate keys allowed.
- No *mutable* keys
  - If you use an *object* as a key, then you can't modify that object later.

# Symbol Table

What time does this plane depart it?

## Key Mutability

```
SymbolTable<Time, Plane> t =
           new SymbolTable<Time, Plane>();

Time  t1 = new Time(9:00);
Time  t2 = new Time(9:15);

t.insert(t1, "SQ0001");
t.insert(t2, "SQ0002");

t1.setTime(10:00);

x = new Time(9:00);
t.search(x);
```

# Symbol Table

## Key Mutability

Moral: Keys should be immutable.

Examples: Integer, String

```
SymbolTable<Time, Plane> t =
           new SymbolTable<Time, Plane>();

Time  t1 = new Time(9:00);
Time  t2 = new Time(9:15);

t.insert(t1, "SQ0001");
t.insert(t2, "SQ0002");

t1.setTime(10:00);

x = new Time(9:00);
t.search(x);
```

# Symbol Tables in Java

## java.util.Map

| | | |
|---|---|---|
| | **public interface** | **java.util.Map<Key, Value>** |
| `Set<Map.Entry<Key, Value>` | `entrySet()` | *set of all mappings* |
| `Set<Key>` | `keySet()` | *set of all keys* |
| `Collection<Value>` | `values()` | *collection of all values* |

Note: not sorted

not necessarily efficient to work with these sets/collections.

# What is wrong here?

Example:

There is a bug here!

```
Map<String, Integer> ageMap = new Map<String, Integer>();

ageMap.put(“Alice”, 32);
ageMap.put(“Bernice”, 84);
ageMap.put(“Charlie”, 7);

Integer age = ageMap.get(“Alice”)
```

- Key-type: String
- Value-type: Integer

# What is wrong here?

Example:

Map is an interface!
Cannot instantiate an interface.

```
Map<String, Integer> ageMap = new Map<String, Integer>();

ageMap.put(“Alice”, 32);
ageMap.put(“Bernice”, 84);
ageMap.put(“Charlie”, 7);

Integer age = ageMap.get(“Alice”)
```

- Key-type: String
- Value-type: Integer

# Map Class in Java

## Example: HashMap

```java
Map<String, Integer> ageMap = new HashMap<String, Integer>();

ageMap.put("Alice", 32);
ageMap.put("Bernice", 84);
ageMap.put("Charlie", 7);

Integer age = ageMap.get("Alice");
System.out.println("Alice's age is: " + age + ".");
```

- Key-type: String
- Value-type: Integer

# Map Class in Java

## Example: HashMap

```
Map<String, Integer> ageMap = new HashMap<String, Integer>();

ageMap.put("Alice", 32);
ageMap.put("Bernice", null);
ageMap.put("Charlie", 7);

Integer age = ageMap.get("Bob");
if (age==null){
    System.out.println("Bob's age is unknown.");
}
```

- Returns “null” when key is not in map.
- Returns “null” when value is null.

# Map Classes in Java

## HashMap

Symbol Table

- containsKey
- containsValue
- entrySet
- get
- isEmpty
- keySet
- put
- putAll
- remove
- values

## TreeMap

Dictionary

- containsKey
- containsValue
- entrySet
- get
- isEmpty
- keySet
- put
- putAll
- remove
- values

# Map Classes in Java

HashMap

Symbol Table

TreeMap

Dictionary

- ceilingEntry
- ceilingKey
- descendingKeySet
- firstEntry
- firstKey
- floorEntry
- floorKey
- headMap
- higherEntry
- higherKey
- … (and more)

Lots of functionality

Wide Interfaces

vs.

Narrow Interfaces

Limited functionality

Lots of functionality

- Java
- No guarantee of efficiency.
- Easy to use (badly).

# Wide Interfaces

# vs.

# Narrow Interfaces

Limited functionality

- Enforces proper use.
- Restricts usage.

# Symbol Tables are Useful

Examples:

1. Spelling correction (key=misspelled word, data=word)
2. Scheme interpreter (key=variable, data=value)
3. Web server
   - Lots of simultaneous network connections.
   - When a packet arrives, give it to the right process to handle the connection.
   - key=ip address, data = connection handler

In this cases, O(log n) often isn't fast enough!

# Symbol Tables are Useful

## Example 1: Pilot Scheduling

1. Check to see if feasible to schedule at time *t*.

   No two airplanes can land with 3 minutes of each other.

2. Find schedule of pilot *p*.

   Get a list of all the planes that are being flown by a specified pilot.

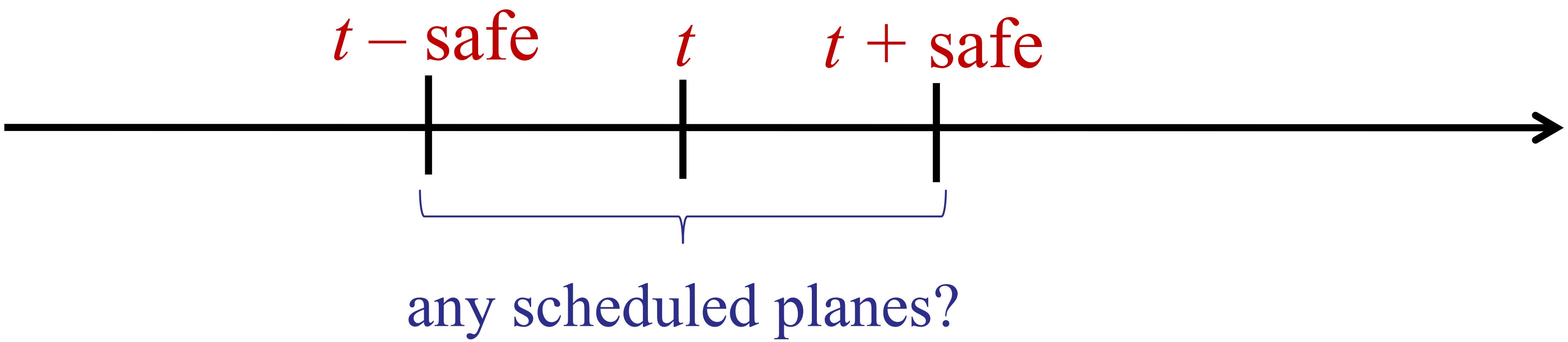
## Which can be efficiently solved with a symbol table?

1. Both: scheduling and pilots info.
2. Only scheduling.
3. Only pilot info.
4. Neither.

# Symbol Tables are Useful

## Example 1: Pilot Scheduling

1. Check to see if feasible to schedule at time *t*.
    - Hard with a symbol table!
    - Need to find out if there are any planes scheduled in the interval [*t*, *t* ± safe_distance]



# Symbol Tables are Useful

## Example 1: Pilot Scheduling

1. Check to see if feasible to schedule at time *t*.
2. Find schedule of pilot *p*.
   - Perfect for a symbol table!
   - Can insert new pilots.
   - Can search for (and update) existing pilots.
   - Listing all pilots?

# Symbol Tables are Useful

Example 2: Document Distance

- Given two documents A and B, how similar are they?
  - Two documents are *similar* if they have similar words in similar frequencies.
  - Formally, define each text as a vector with one entry per word.
  - The distance between the two texts is the angle between the two vectors.

# Symbol Tables are Useful

Example 2: Document Distance

- Step 1: Read in each document
    - Read the file as a string.
    - Parse the file into words.
    - Sort the list of words.
    - Count the frequency of each word.
- Step 2: Compare the two documents
    - Calculate the norm $|A|$ and $|B|$ of each vector
    - Calculate the dot product $AB$.
    - Calculate the angle between $A$ and $B$.

# Symbol Tables are Useful

Example 2: Document Distance

- Step 1: Read in each document

$O(n)$ • Read the file as a string.

$O(n)$ • Parse the file into words.

$O(n \log n)$ • Sort the list of words.

$O(n)$ • Count the frequency of each word.

- Step 2: Compare the two documents

$O(n)$ • Calculate the norm $|A|$ and $|B|$ of each vector

$O(n)$ • Calculate the dot product $AB$.

$O(n)$ • Calculate the angle between $A$ and $B$.

# Performance Profiling (Sorting)

*(Dracula* vs. *Lewis & Clark)*

| Step | Function | Running Time |
|---|---|---|
| Create vectors: | Read each file | 1.03s |
| | Parse each file | 1.23s |
| | Sort words in each file | 2.04s |
| | Count word frequencies | 0.41s |
| Dot product: | | 6.10s |
| Norm: | | 0.01s |
| Angle: | | 0.02s |
| **Total:** | | **12.75s** |

# Performance Profiling (Symbol Table)

(*Dracula* vs. *Lewis & Clark*)

| Step | Function | Running Time |
|---|---|---|
| Create vectors: | Read each file | 1.19s |
| | Parse each file | 1.37s |
| | Sort words in each file | 0 |
| | Count word frequencies | 0 |
| Dot product: | | 0.03s |
| Norm: | | 0.01s |
| Angle: | | 0.02s |
| **Total:** | | **2.43s** |

# Symbol Tables are Useful

Example 2: Document Distance

– Step 1: Read in each document
  - Read and parse the file.
  - Put each (word, count) in a HashMap.

Symbol Table:

- key (String) = word
- value (Integer) = count (# times in doc)

# Symbol Tables are Useful

Example 2: Document Distance

- Step 1: Read in each document
  - Read and parse the file.
  - Put each (word, count) in a map.

```
if (word != "")
{
    if (m_WordList.containsKey(word))
    {
        int count = m_WordList.get(word)+1;
        m_WordList.put(word, count);
    }
    else
    {
        m_WordList.put(word, 1);
    }
    word = "";
}
```

# Symbol Tables are Useful

– Step 2: Compare documents (dot-product)

```
// Get an iterator for all the keys stored in A
Set<String> ASet = A.m_WordList.keySet();
Iterator<String> AIterator = ASet.iterator();

// Iterate through all the keys in A
while (AIterator.hasNext())
{
    String Key = AIterator.next();

    // If the key from A is also in B
    if (B.m_WordList.containsKey(Key))
    {
        // Add the product of the counts to the sum.
        int AValue = A.m_WordList.get(Key);
        int BValue = B.m_WordList.get(Key);
        sum += AValue*BValue;
    }
}
```

# Document Distance

*(Dracula* vs. *Lewis & Clark)*

| Version | Change | Running Time |
|---|---|---|
| Version 1 | | 4,311.00s |
| Version 2 | Better file handling | 676.50s |
| Version 3 | Faster sorting | 6.59s |
| Version 4 | Symbol Table | 2.35s |

# Building a Symbol Table

# Direct Access Tables

Attempt #1: Use a table, indexed by keys.

| | |
|---|---|
| 0 | null |
| 1 | null |
| 2 | item1 |
| 3 | null |
| 4 | null |
| 5 | item3 |
| 6 | null |
| 7 | null |
| 8 | item2 |
| 9 | null |

Universe U={0..9} of size $m = 10$.

(key, value)

(2, item1)
(8, item2)
(5, item3)

Assume keys are distinct.

# Direct Access Tables

Attempt #1: Use a table, indexed by keys.

| | |
|---|---|
| 0 | null |
| 1 | null |
| 2 | item1 |
| 3 | null |
| 4 | null |
| 5 | item3 |
| 6 | null |
| 7 | null |
| 8 | item2 |
| 9 | null |

Example: insert(4, Seth)

# Direct Access Tables

Attempt #1: Use a table, indexed by keys.

| | |
|---|---|
| 0 | null |
| 1 | null |
| 2 | item1 |
| 3 | null |
| 4 | Seth |
| 5 | item3 |
| 6 | null |
| 7 | null |
| 8 | item2 |
| 9 | null |

Example: insert(4, Seth)

Time: O(1) / insert, O(1) / search

# Direct Access Tables

Problems:

- Too much space
  - If keys are integers, then table-size > 4 billion

- What if keys are not integers?
  - Where do you put the key/value “**(hippopotamus, bob)**”?
  - Where do you put 3.14159…?

# Direct Access Tables

Pythagoras said, “Everything is a number.”

“The School of Athens” by Raphael

# Direct Access Tables

Pythagoras said, “Everything is a number.”

- Everything is just a sequence of bits.
- Treat those bits as a number.

- English:
  - 26 letters => 5 bits/letter
  - Longest word = 28 letters (antidisestablishmentarianism?)
  - 28 letters * 5 bits = 140 bits
  - So we can store any English word in a direct-access array of size $2^{140}$.

# Direct Access Tables

Pythagoras said, "Everything is a number."

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- English:
  - 26 letters => 5 bits/letter
  - Longest word = 28 letters (antidisestablishmentarianism?)
  - 28 letters * 5 bits = 140 bits
  - So we can store any English word in a direct-access array of size $2^{140}$. ≈ number of atoms in observable universe

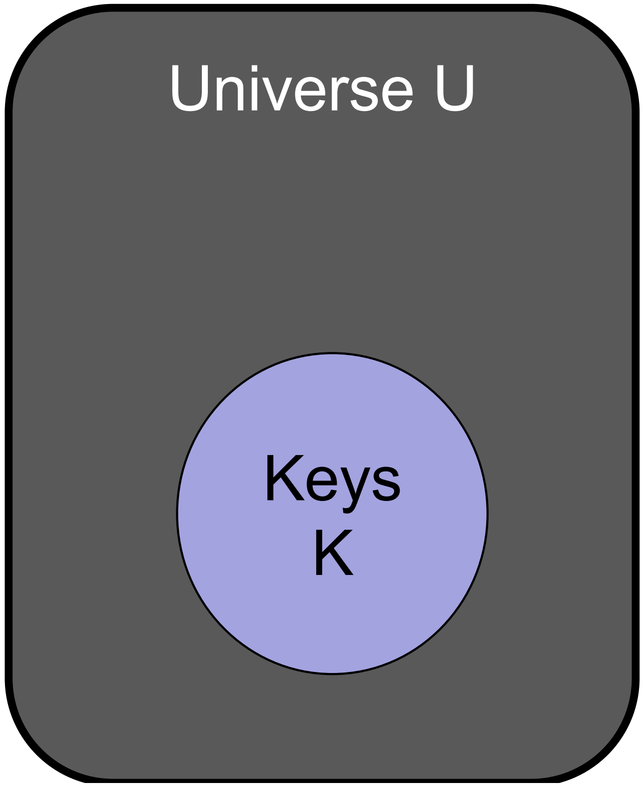
# Hash Functions

Problem:

– Huge universe $U$ of possible keys.

e.g., $2^{140}$

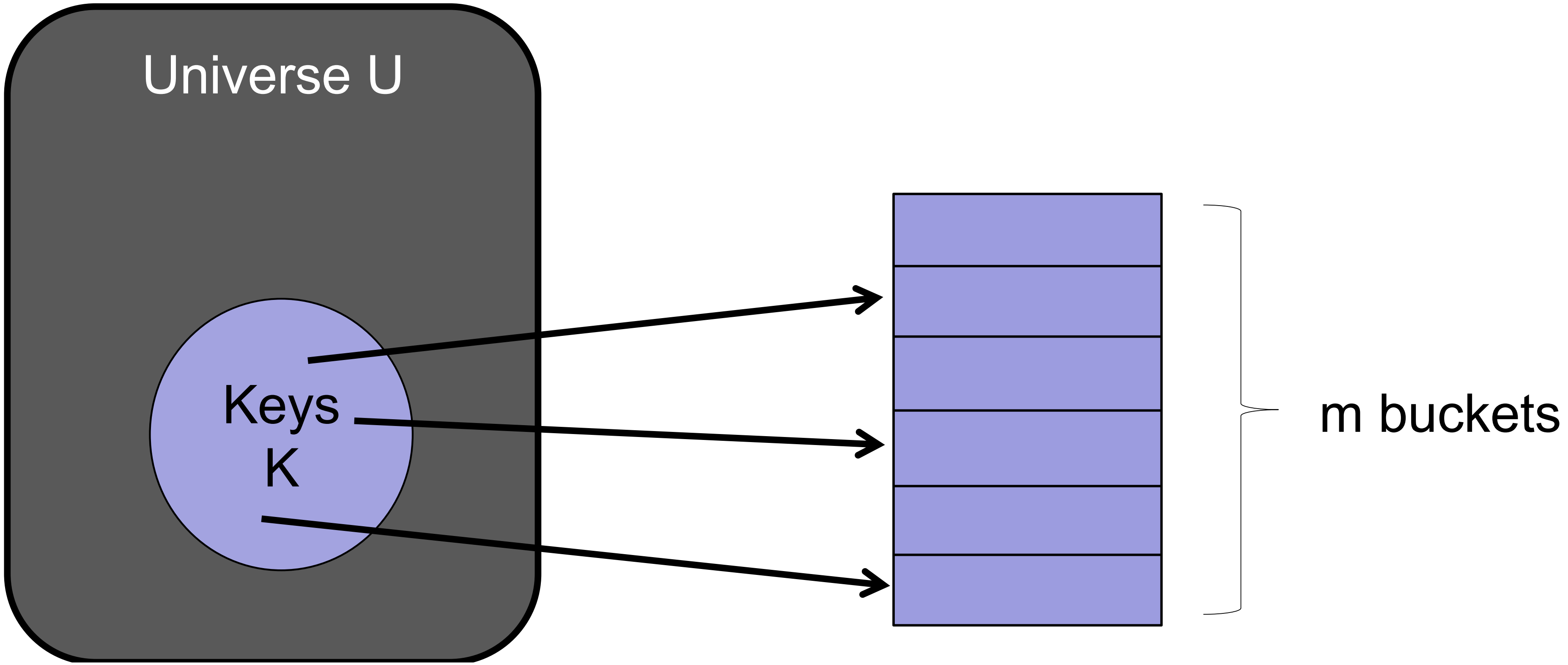
– Smaller number $n$ of actual keys.



# Hash Functions

Problem:

- Huge universe $U$ of possible keys.
- Smaller number $n$ of actual keys.
- How to map $n$ keys to $m \approx n$ buckets?



# Hash Functions

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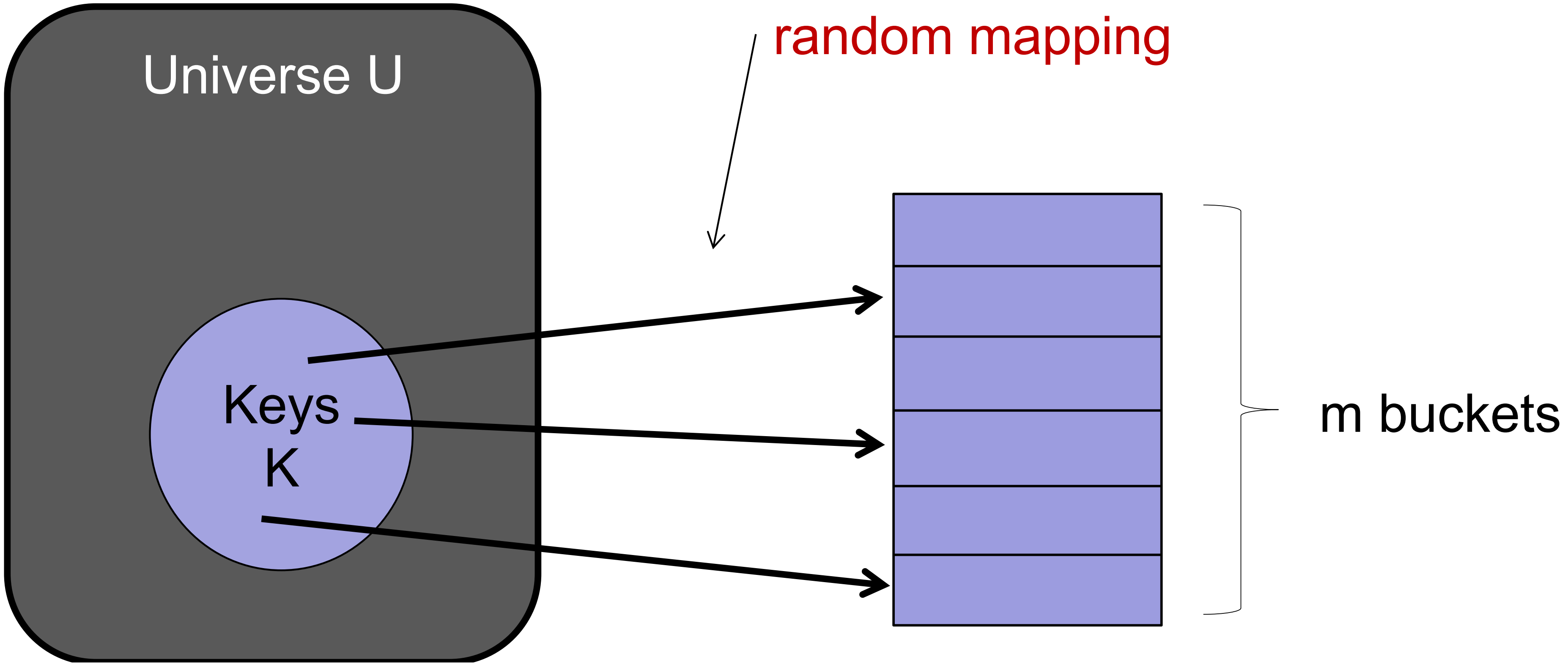
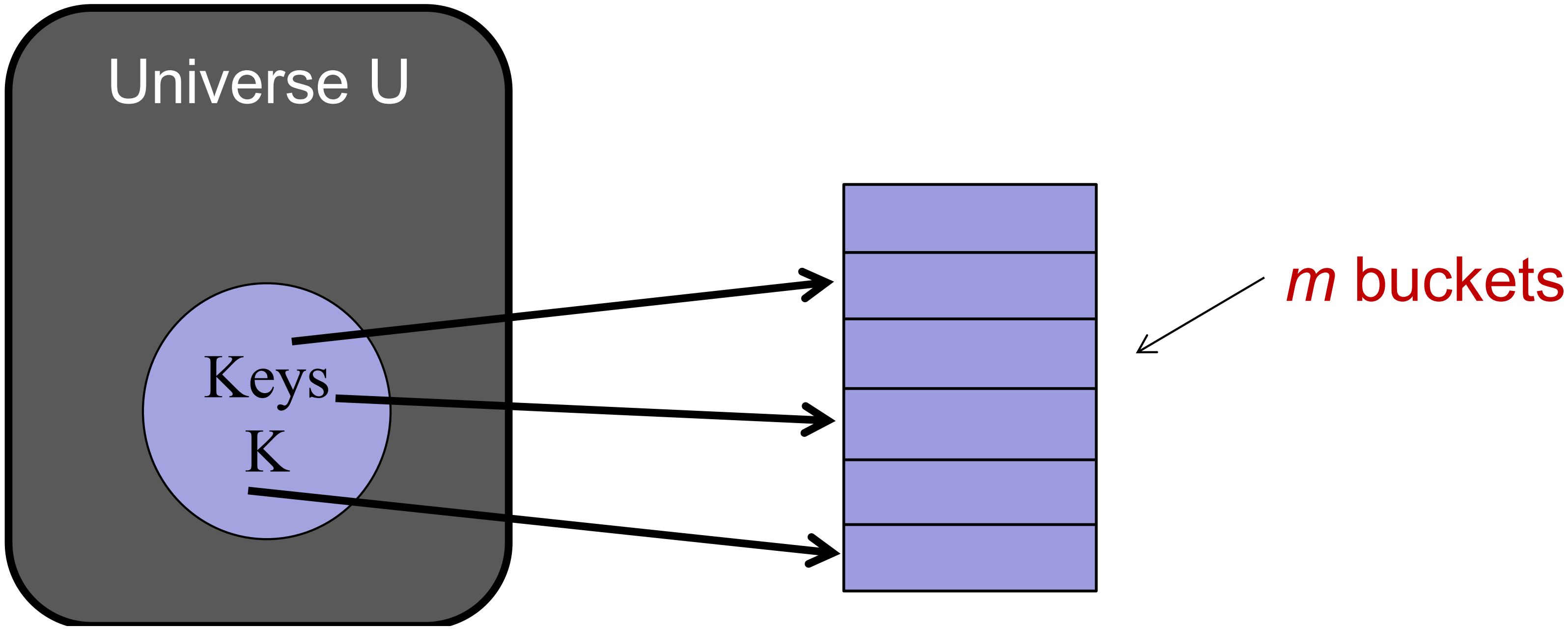


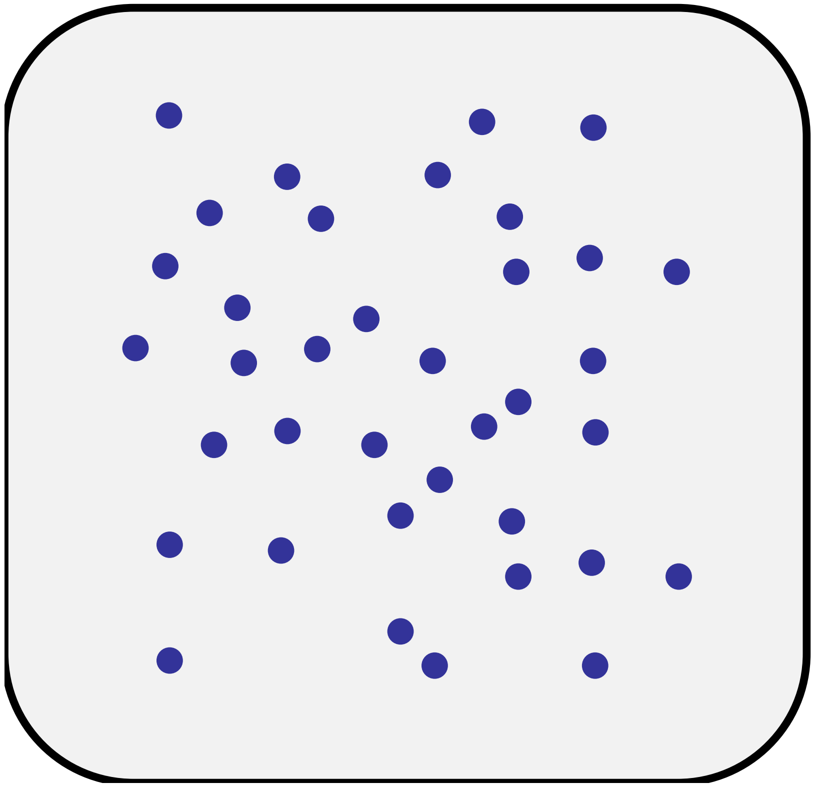
# Hash Functions

Define hash function $h : U \rightarrow \{1..m\}$

- Store key $k$ in bucket $h(k)$.
- Time complexity:
  - Time to compute h + Time to access bucket
- For now: assume hash function has cost 1 to compute.



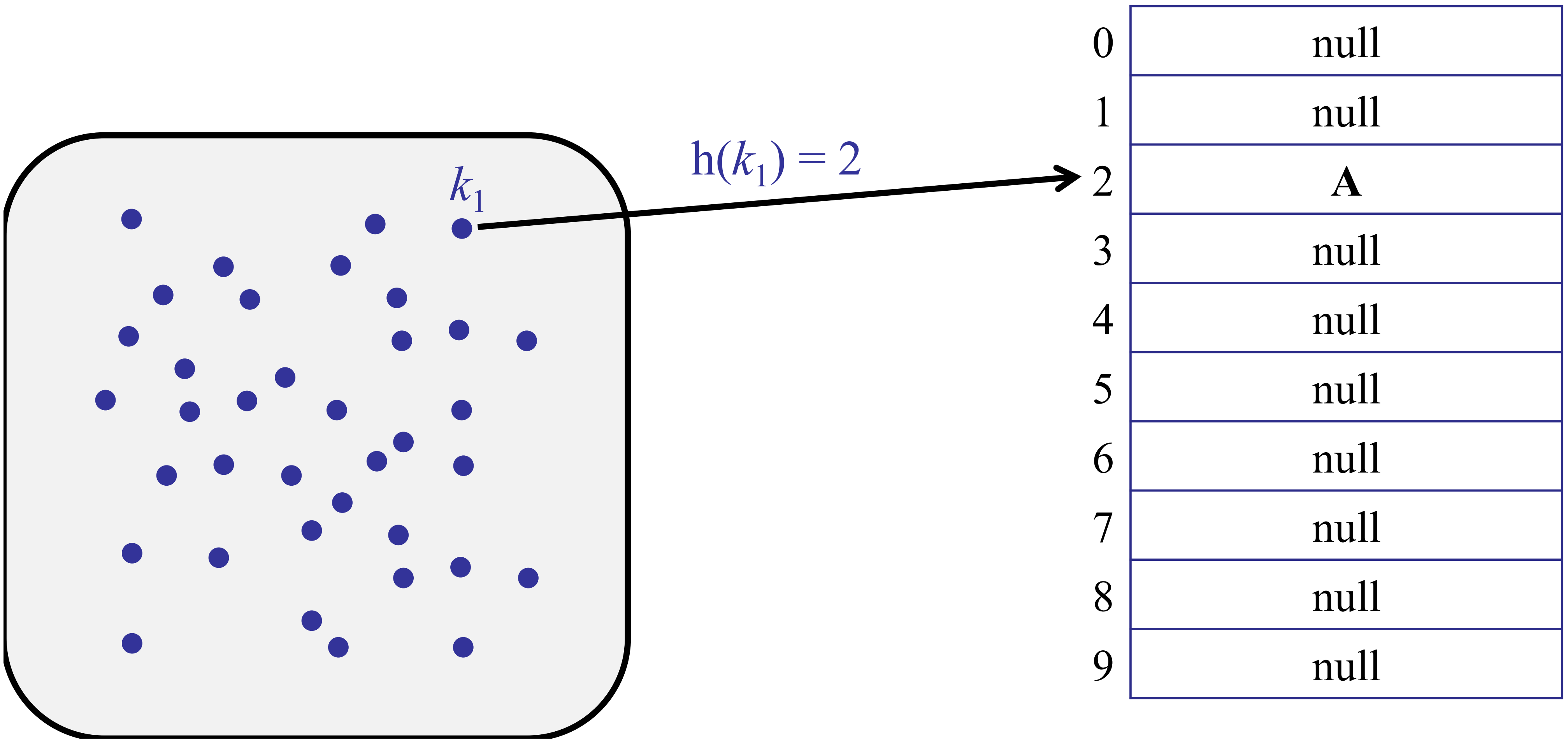
# Hash Functions



| | |
|---|---|
| 0 | null |
| 1 | null |
| 2 | null |
| 3 | null |
| 4 | null |
| 5 | null |
| 6 | null |
| 7 | null |
| 8 | null |
| 9 | null |

# Hash Functions

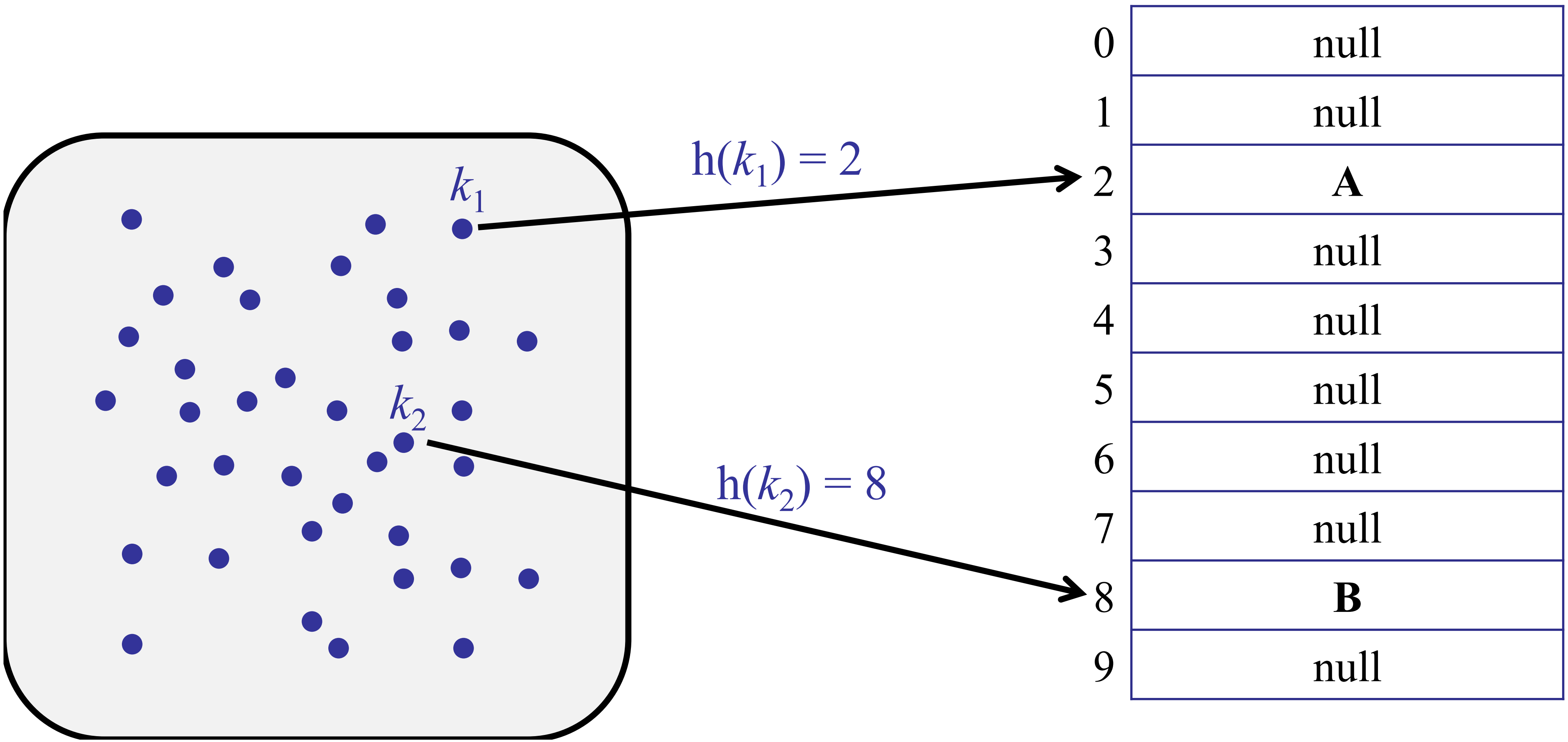
insert($k_1$, A)



# Hash Functions

insert($k_1$, A)

insert($k_2$, B)



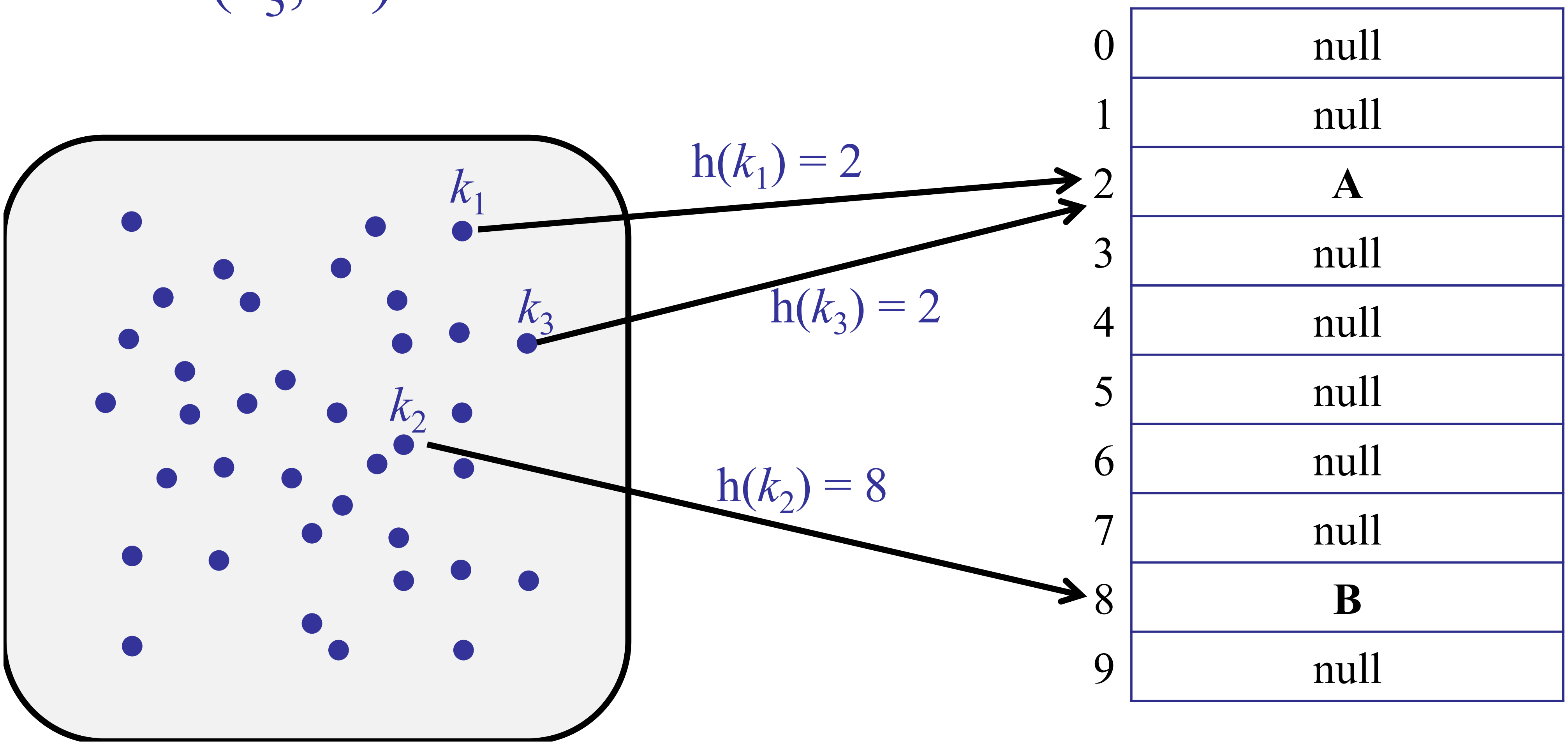
# Hash Functions

insert($k_1$, A)

insert($k_2$, B)

insert($k_3$, C)

Collision!



# Hash Functions

Collisions:

– We say that two distinct keys $k_1$ and $k_2$ collide if:

$$h(k_1) = h(k_2)$$

## Can we choose a hash function with no collisions?

1. Yes
2. Sometimes, if we choose carefully
3. No, impossible

# Hash Functions

Collisions:

- We say that two distinct keys $k_1$ and $k_2$ collide if:

$$h(k_1) = h(k_2)$$

- Unavoidable!
  - The table size is smaller than the universe size.
  - The pigeonhole principle says:
    - There must exist two keys that map to the same bucket.
    - Some keys must collide!

# Coping with Collision

- Idea: choose a new, better hash functions

# Coping with Collision

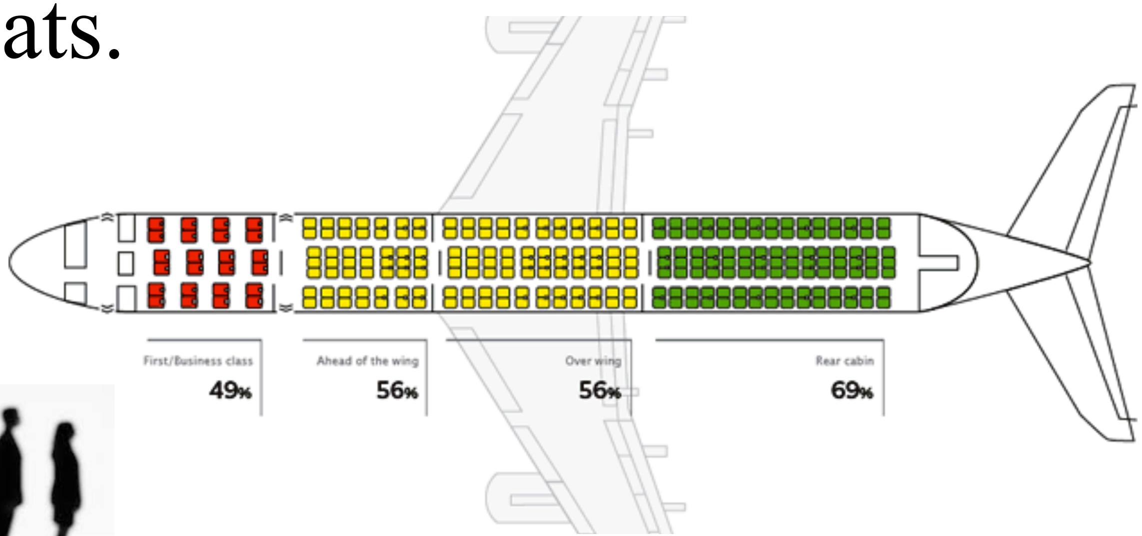
- Idea: choose a new, better hash functions
  - Hard to find.
  - Requires re-copying the table.
  - Eventually, there will be another collision.

# Coping with Collision

- Idea: choose a new, better hash functions
  - Hard to find.
  - Requires re-copying the table.
  - Eventually, there will be another collision.
- Idea: chaining (today)
  - Put both items in the same bucket!
- Idea: open addressing (next week)
  - Find another bucket for the new item.

# Puzzle Break

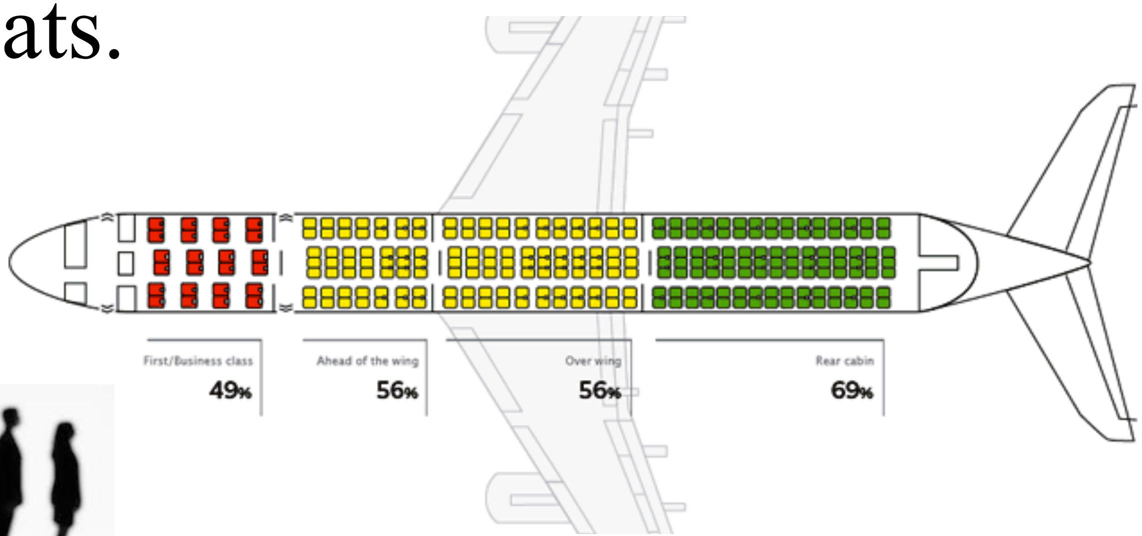
An airplane has 100 seats.




100 passengers board the airplane in a random order.

# Puzzle Break

An airplane has 100 seats.




100 passengers board the airplane in a random order.

Passenger 1 is Mr. Burns.

Mr. Burns sits in a random seat.

# Puzzle Break

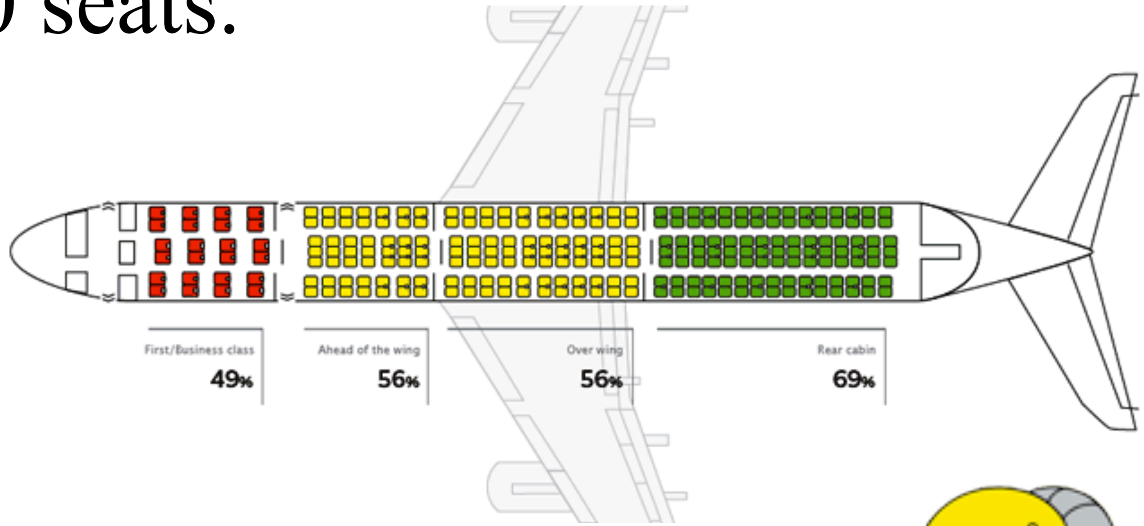
An airplane has 100 seats.


Every other passenger:

- Sits in their assigned seat, if it is free.
- Otherwise, sits in a random seat.

# Puzzle Break

An airplane has 100 seats.

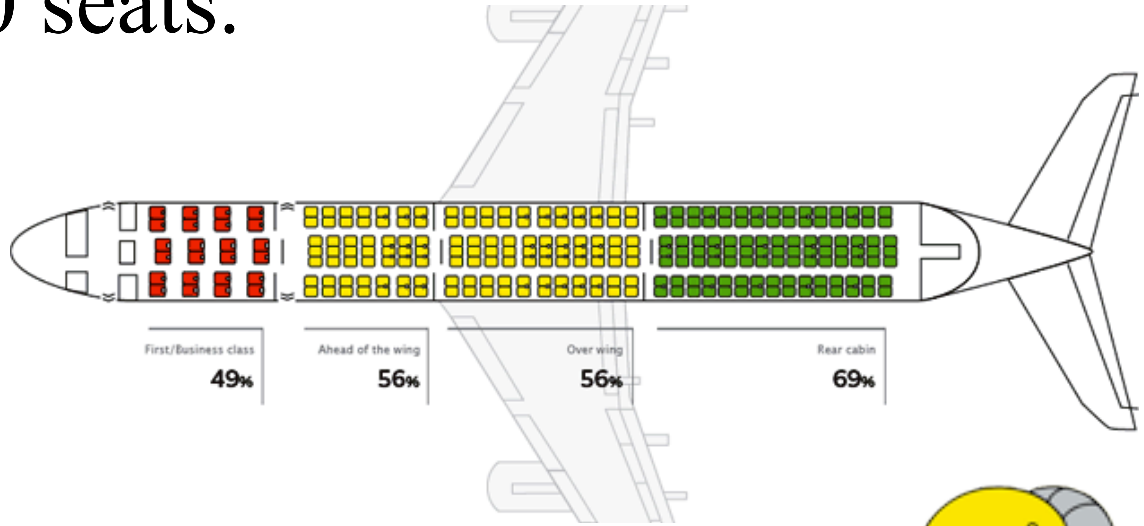



You are passenger #100.

What is the probability your seat is free when you board?

# Puzzle Break

An airplane has 100 seats.




What is the probability your seat is free when you board?

*Problem Solving techniques*:

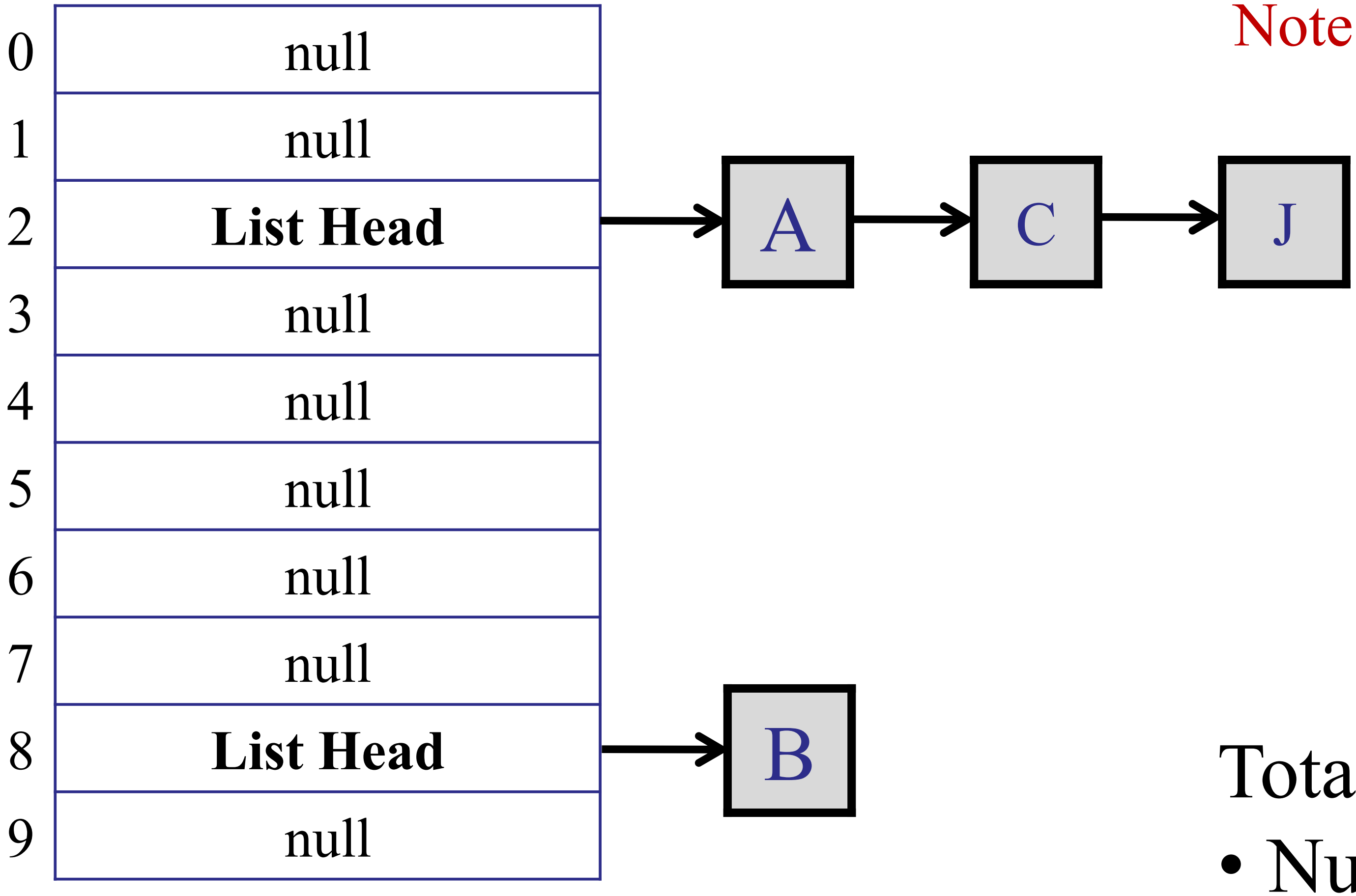
Try a plane with 2 seats.  Try a plane with 3 seats.

# Coping with Collision

- Idea: choose a new, better hash functions
  - Hard to find.
  - Requires re-copying the table.
  - Eventually, there will be another collision.
- Idea: chaining (today)
  - Put both items in the same bucket!
- Idea: open addressing (next week)
  - Find another bucket for the new item.

# Chaining

Each bucket contains a linked list of items.



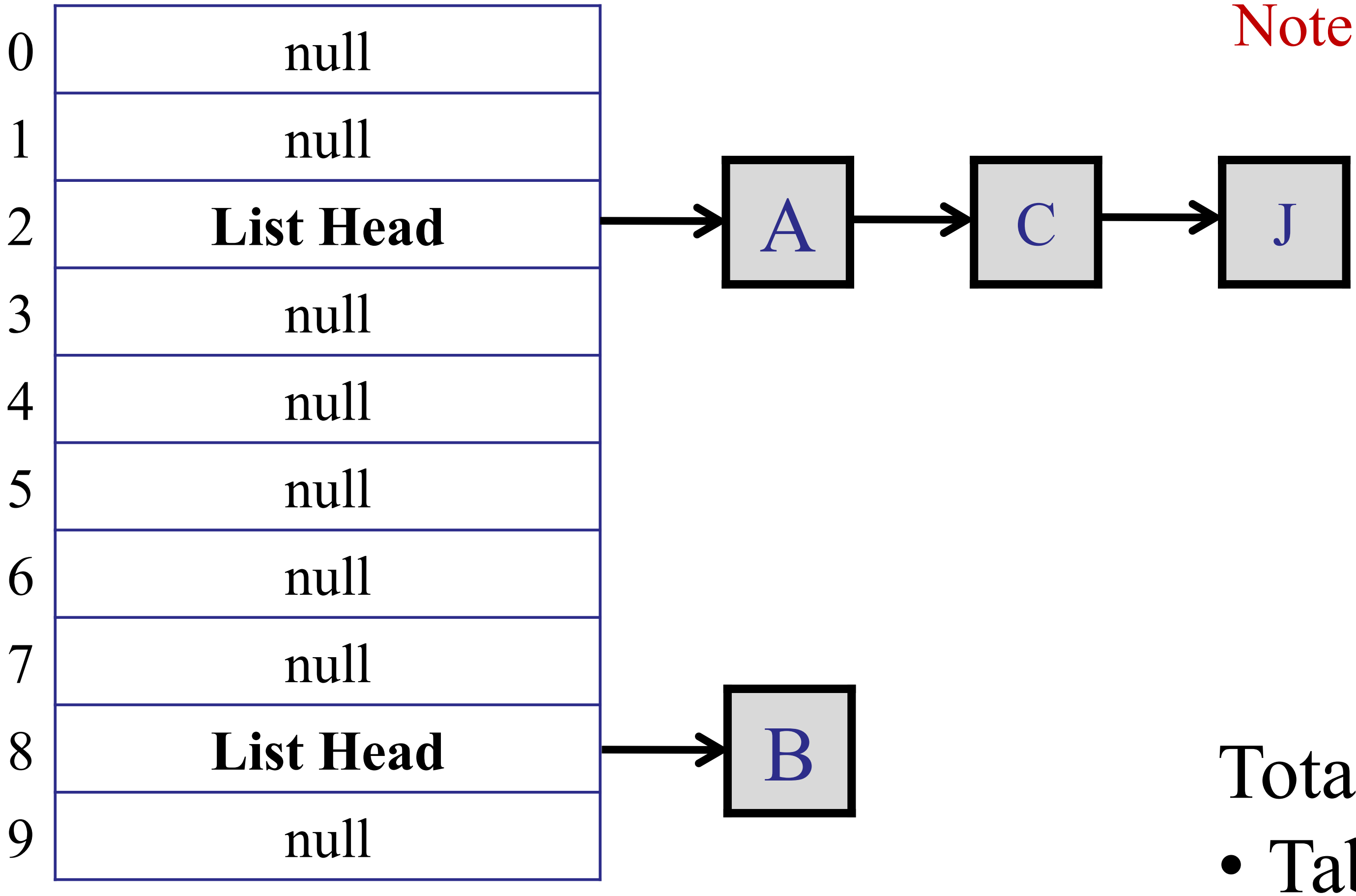

Note: h(A) == h(C) == h(J)

Total space:

- Number buckets: $m$
- Number entries: $n$

# Chaining

Each bucket contains a linked list of items.




Note: h(A) == h(C) == h(J)

Total space: $O(m + n)$
- Table size: $m$
- Linked list size: $n$

# Hashing with Chaining

Operations:

- insert(key, value)
  - Calculate h(key)
  - Lookup h(key) and add (key,value) to the linked list.

- search(key)
  - Calculate h(key)
  - Search for (key,value) in the linked list.

## What is the worst-case cost of inserting a (key, value)?

✓ 1. $O(1 + \text{cost}(h))$
2. $O(\log n + \text{cost}(h))$
3. $O(n + \text{cost}(h))$
4. $O(n\ \text{cost}(h))$
5. We cannot determine it without knowing h.

# Hashing with Chaining

Operations:

- insert(key, value)
  - Calculate h(key)
  - Lookup h(key) and add (key,value) to the linked list.

Always O(1).

- search(key)
  - Calculate h(key)
  - Search for (key,value) in the linked list.

What is the worst-case cost of searching a (key, value)?

1. O(1 + cost(h))
2. O(log $n$ + cost(h))
3. O($n$ + cost(h))
4. O($n$*cost(h))
5. We cannot determine it without knowing h.

# Hashing with Chaining

Operations:

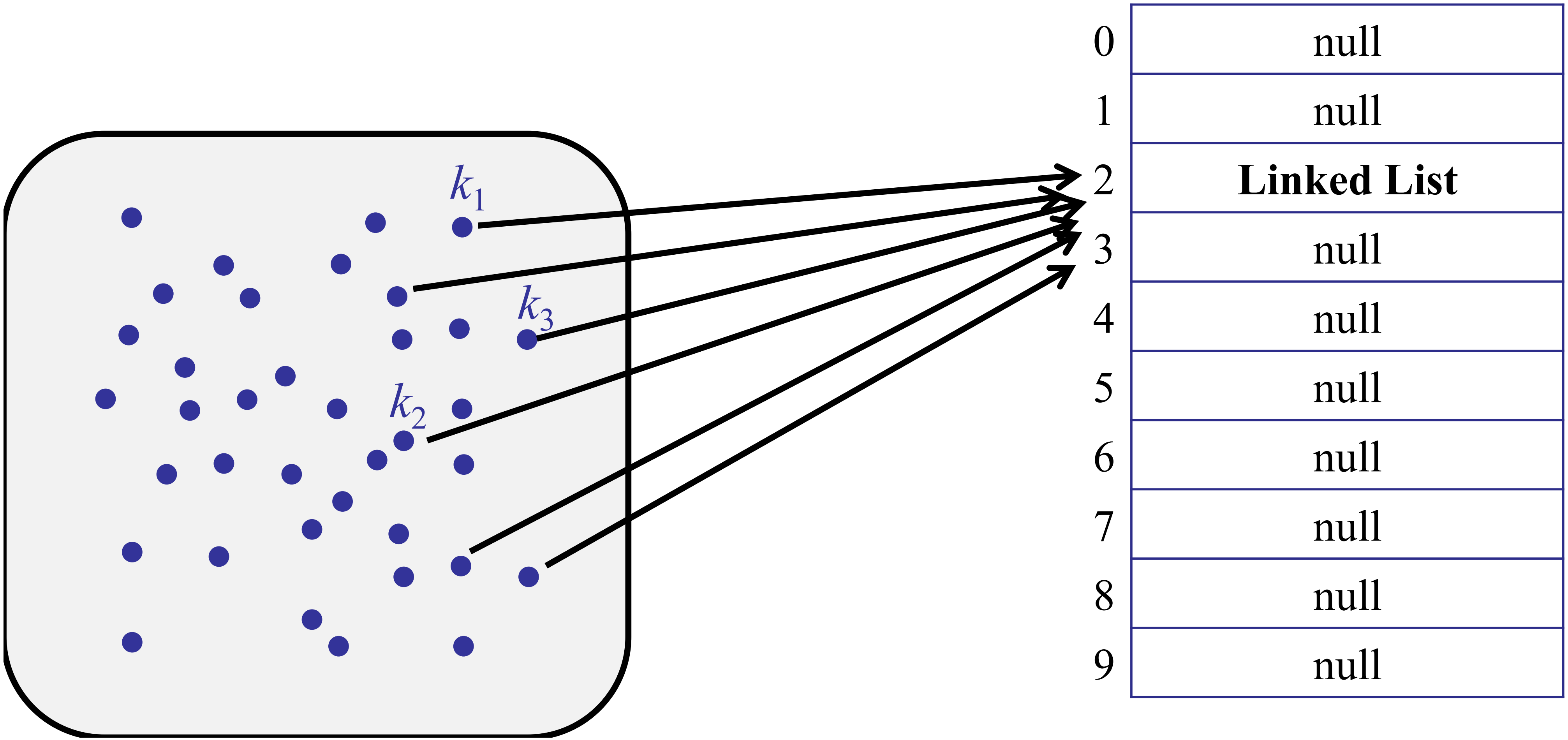
- insert(key, value)
  - Calculate h(key)
  - Lookup h(key) and add (key,value) to the linked list.

- search(key) → time depends on length of linked list
  - Calculate h(key)
  - Search for (key,value) in the linked list.

# Hashing with Chaining

Assume all keys hash to the same bucket!

- Search costs O($n$)
- Oh no!



# Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Every key is equally likely to map to every bucket.
- Keys are mapped independently.

Intuition:

- Each key is put in a random bucket.
- Then, as long as there are enough buckets, we won't get too many keys in any one bucket.

# Why don't we just insert each key into a random bucket (instead of using h)?

1. It would be slow to insert.
2. Computers don't have a real source of randomness.
3. By choosing the keys carefully, a user could force the random choices to create many collisions.
4. Searching would be very slow.
5. None of the above.

# Let's be optimistic today.

The Simple Uniform Hashing Assumption

– Assume:
  - $n$ items
  - $m$ buckets
– Define: load(hash table) = $n/m$

  = average # items / bucket.

– Expected search time = 1 + *expected # items per bucket*

hash function + array access

linked list traversal

# Probability Theory

Set of outcomes for $X = (e_1, e_2, e_3, \ldots, e_k)$:

- $Pr(e_1) = p_1$
- $Pr(e_2) = p_2$
- …
- $Pr(e_k) = p_k$

Expected outcome:

$$E[X] = e_1p_1 + e_2p_2 + \ldots + e_kp_k$$

# Probability Theory

Linearity of Expectation:

- $E[A + B] = E[A] + E[B]$

Example:

- A = # heads in 2 coin flips
- B = # heads in 2 coin flips
- A + B = # heads in 4 coin flips

$E[A+B] = E[A] + E[B] = 1 + 1 = 2$

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# A little more probability

Indicator random variables

$$
\begin{aligned}
X(i, j) &= 1 \quad \text{if item } i \text{ is put in bucket } j \\
&= 0 \quad \text{otherwise}
\end{aligned}
$$

$\Pr( X(i, j) == 1) = ?$

1. $1/m$
2. $1/n$
3. $1/(m+n)$
4. $m/n$
5. $n/m$
6. $\log(n)$

# Let’s be optimistic today.

The Simple Uniform Hashing Assumption

- Every key is equally likely to map to every bucket.
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Intuition:

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# A little probability

Indicator random variables

$$X(i, j) = 1 \quad \text{if item } i \text{ is put in bucket } j$$
$$= 0 \quad \text{otherwise}$$

$$\mathbf{Pr}(X(i, j) == 1) = 1/m$$

# A little probability

Indicator random variables

$X(i, j) = 1$ if item $i$ is put in bucket $j$

$= 0$ otherwise

$\mathbf{Pr}(X(i, j) == 1) = 1/m$

$\mathbf{E}(X(i, j)) = \mathbf{??}$

# A little probability

Indicator random variables

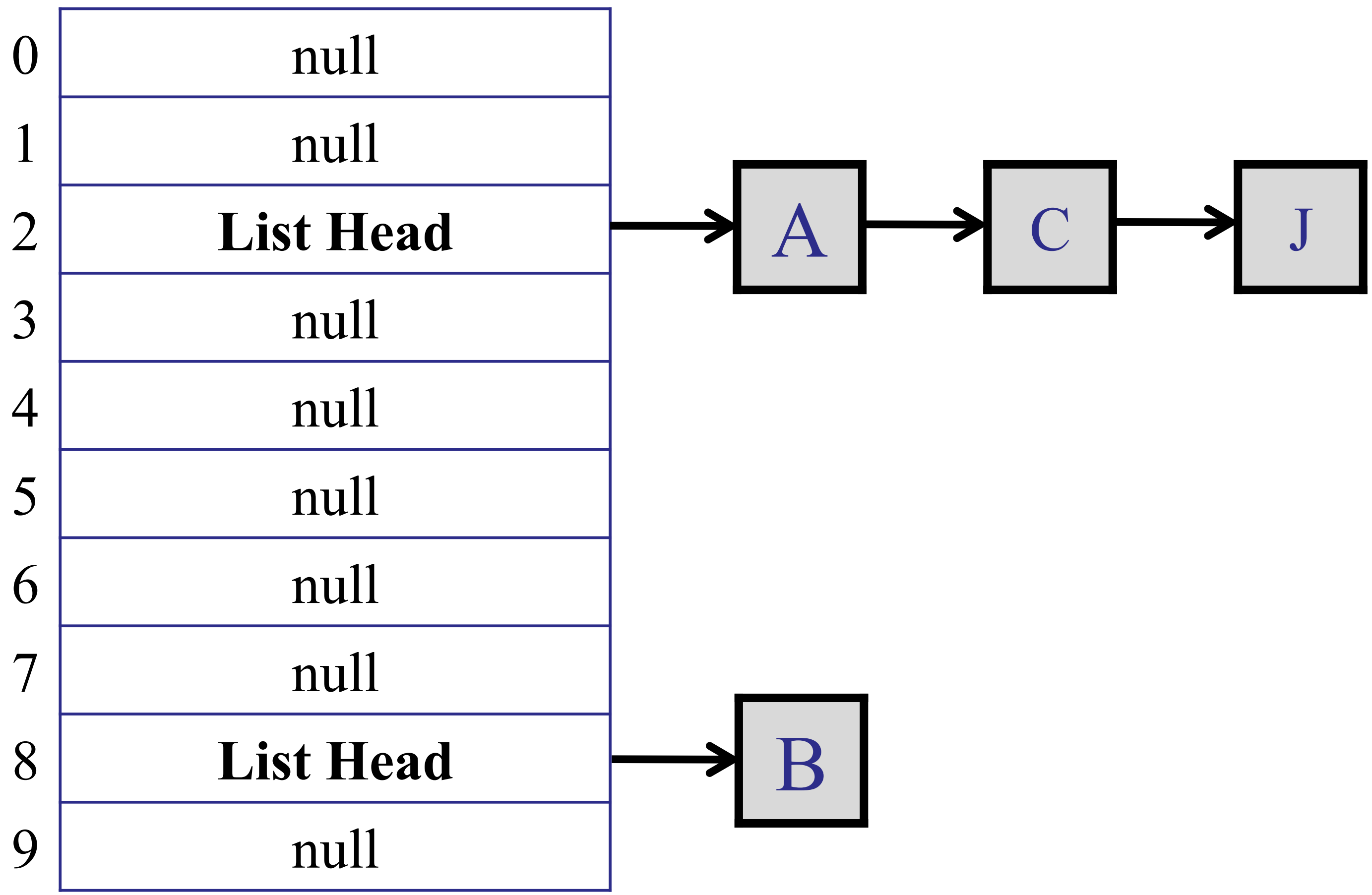
$$X(i, j) = 1 \quad \text{if item } i \text{ is put in bucket } j$$
$$= 0 \quad \text{otherwise}$$

$$\mathbf{Pr}(X(i, j)==1) = 1/m$$

$$\mathbf{E}(X(i, j)) = \mathbf{Pr}(X(i, j)==1)*1 + \mathbf{Pr}(X(i, j)==0)*0$$
$$= \mathbf{Pr}(X(i, j)==1)$$
$$= 1/m$$

# A little probability

What is the expected number of items in a bucket?



# A little probability

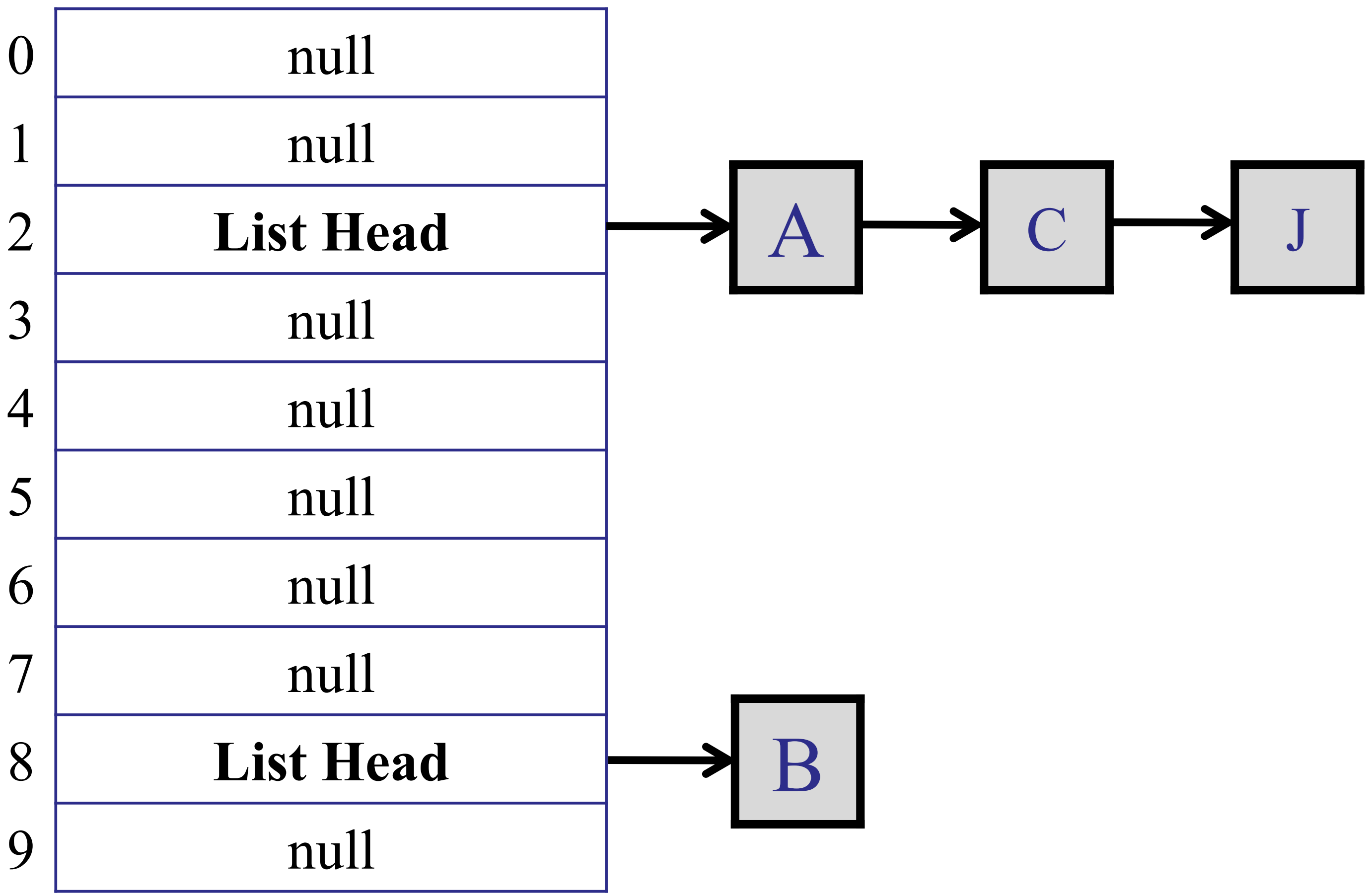
Indicator random variables

$$X(i, j) = 1 \quad \text{if item } i \text{ is put in bucket } j$$
$$= 0 \quad \text{otherwise}$$

$$\Sigma_i\, X(i, b) = \text{number of items in bucket } b$$

# A little probability

Each item contributes `1' to the bucket it is in..



# A little probability

Indicator random variables

$X(i, j) = 1$ if item $i$ is put in bucket $j$

$\quad\quad\quad = 0$ otherwise

$\Sigma_i X(i, b) =$ number of items in bucket $b$

# A little probability

Calculate expected number of items per bucket:

Expected ($\Sigma_i$ X(i, b)) =

# A little probability

Calculate expected number of items per bucket:

$$\mathbf{E}(\Sigma_i X(i, b)) = \Sigma_i \mathbf{E}(X(i, b))$$

Linearity of expectation: $E(A + B) = E(A) + E(B)$

# A little probability

Calculate expected number of items per bucket:

$$\mathbf{E}(\Sigma_i X(i, b)) = \Sigma_i \mathbf{E}(X(i, b))$$

$$= \Sigma_i 1/m$$

$$= n/m$$

# Let's be optimistic today.

The Simple Uniform Hashing Assumption

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- Define: load(hash table) = $n/m$

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- Expected search time = 1 + $n/m$

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# Let's be optimistic today.

The Simple Uniform Hashing Assumption

- Assume:
  - $n$ items
  - $m = \Omega(n)$ buckets, e.g., $m = 2n$

- Expected search time $= 1 + n/m$

  $$= O(1)$$

# Hashing with Chaining

Searching:

- Expected search time = 1 + $n/m$ = O(1)
- Worst-case search time = O(n)

Inserting:

- Worst-case insertion time = O(1)

# Hashing with Chaining

What if you insert n elements in your hash table?

What is the expected *maximum* cost?

# Hashing with Chaining

What if you insert $n$ elements in your hash table?

What is the expected *maximum* cost?

- Analogy:
  - Throw $n$ balls in $m = n$ bins.
  - What is the maximum number of balls in a bin?

Cost: $O(\log n)$

# Hashing with Chaining

What if you insert n elements in your hash table?

What is the expected *maximum* cost?

- Analogy:
  - Throw n balls in $m = n$ bins.
  - What is the maximum number of balls in a bin?

Cost: $\Theta(\log n / \log\log n)$

# Hashing: Recap

Problem: coping with large universe of keys

- Number of possible keys is very, very large.
- Direct Access Table takes too much space

Hash functions

- Use hash function to map keys to buckets.
- Sometimes, keys collide (inevitably!)
- Use linked list to store multiple keys in one bucket.

Analyze performance with simple uniform hashing.

- Expected number of keys / bucket is $O(n/m) = O(1)$.

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An airplane has 100 seats.


100 passengers board the airplane in a random order.

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An airplane has 100 seats.


100 passengers board the airplane in a random order.

Passenger 1 is Mr. Burns.

Mr. Burns sits in a random seat.

# Puzzle Break

An airplane has 100 seats.


Every other passenger:

- Sits in their assigned seat, if it is free.
- Otherwise, sits in a random seat.

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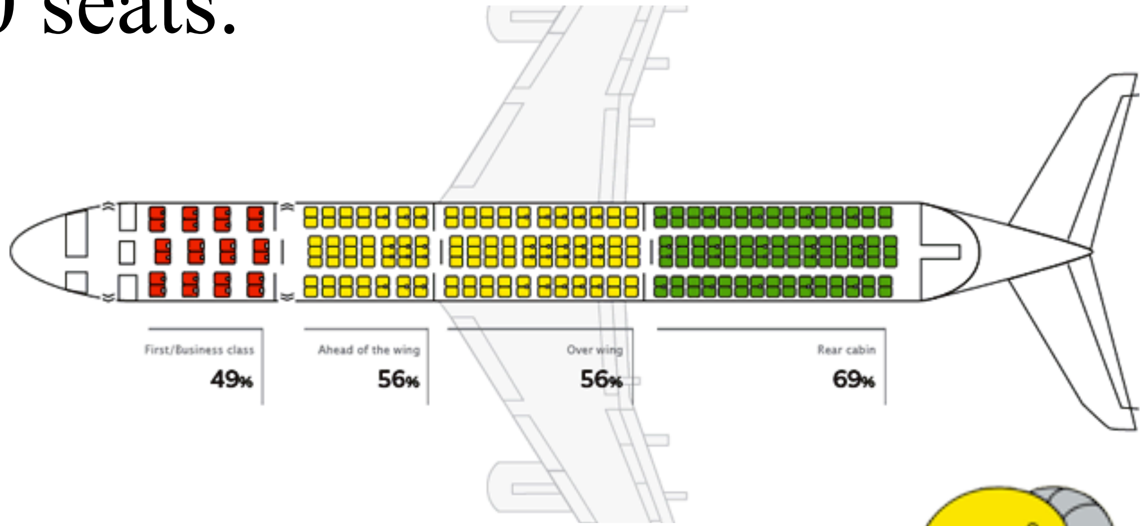


You are passenger #100.

What is the probability your seat is free when you board?

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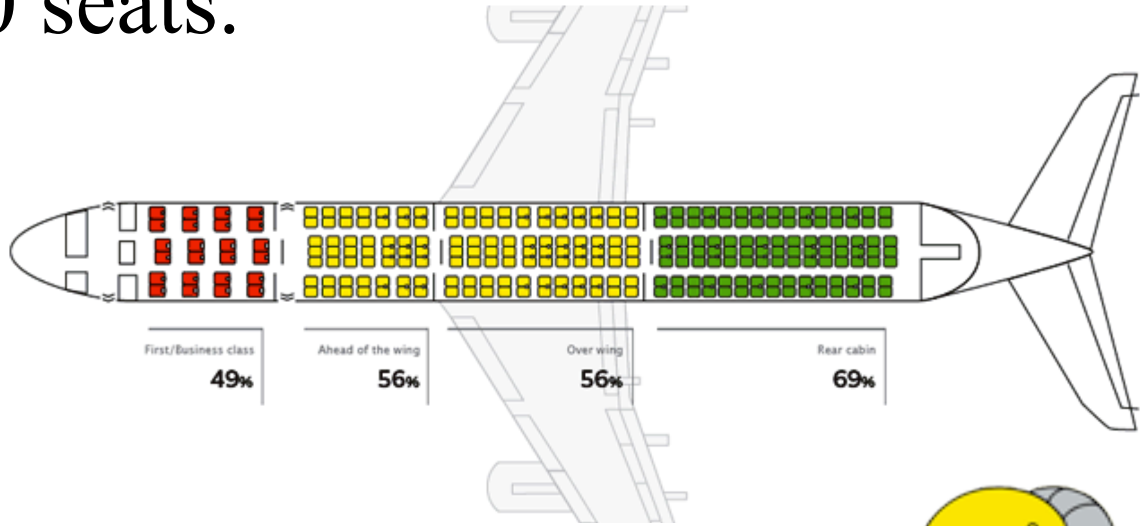


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