CS2040S

Data Structures and Algorithms

(e-learning edition)

Welcome!

Announcements

Lectures are now available via video (only).

Tutorials and Recitations continue in person.

Watch the lecture before recitation!

Plan of the Day

Trees

- Terminology
- Traversals
- Operations

Balanced Trees

- Height-balanced binary search trees
- AVL trees
- Rotations

Dictionary Interface

A collection of (key, value) pairs:

interface	**IDictionar**	
`void`	`insert(Key k, Value v)`	*insert (k,v) into table*
`Value`	`search(Key k)`	*get value paired with k*
`Key`	`successor(Key k)`	*find next key > k*
`Key`	`predecessor(Key k)`	*find next key < k*
`void`	`delete(Key k)`	*remove key k (and value)*
`boolean`	`contains(Key k)`	*is there a value for k?*
`int`	`size()`	*number of (k,v) pairs*

Dictionary

Implementation

Option 1: Sorted array

- insert: add to middle of array --- O(n)
- search : binary search through array --- O(log n)

Option 2: Linked list

- insert: add to middle of array --- O(n)
- search : no binary search in array --- O(n)

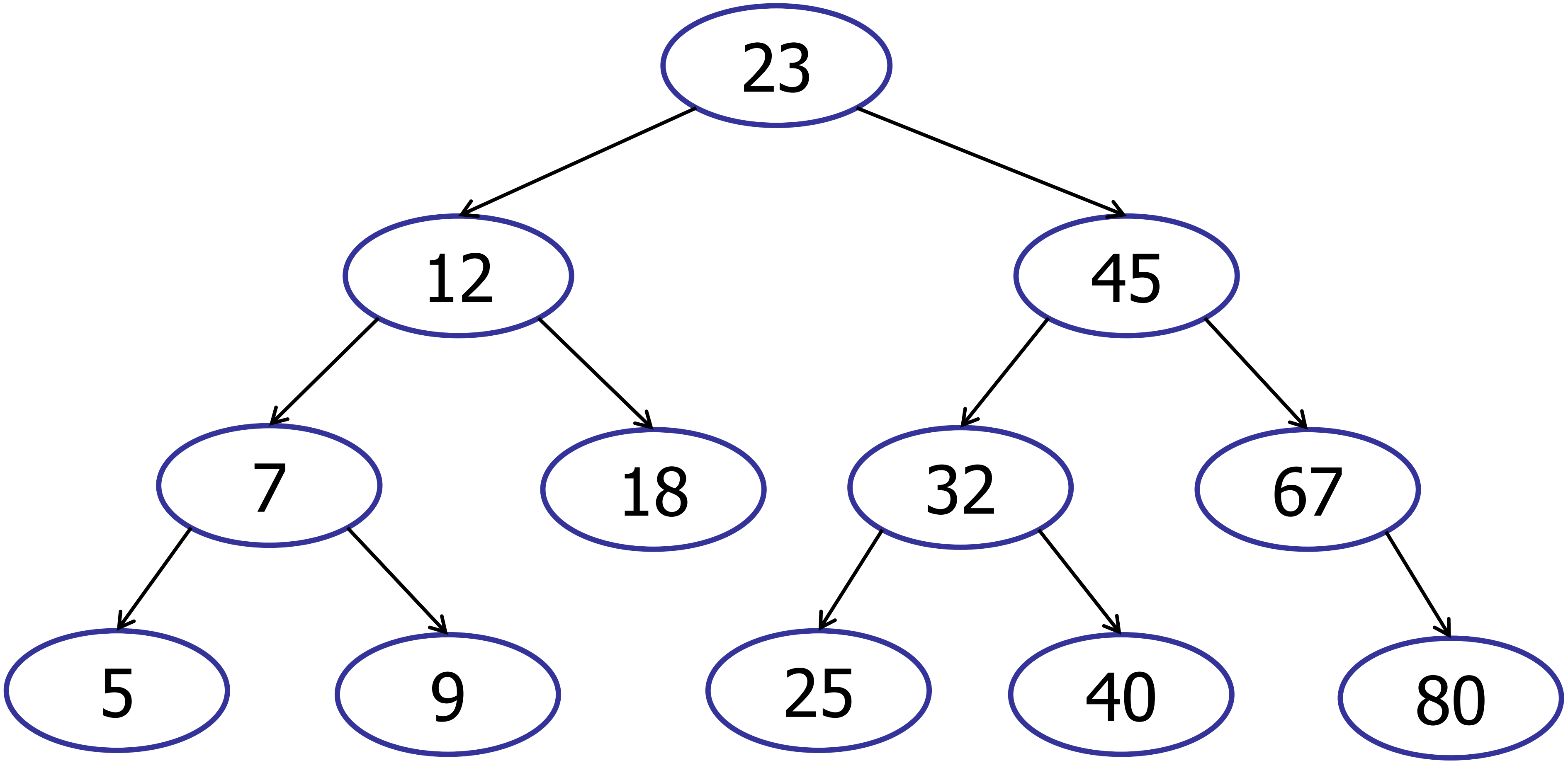
Dictionary Implementation

Possible Choices:

- Implement using an array (see: java.util.ArrayList).
- Implement using an array (see: java.util.Vector).
- Implement using a queue.
- Implement using a LinkedList
- …
- Implement using a tree.

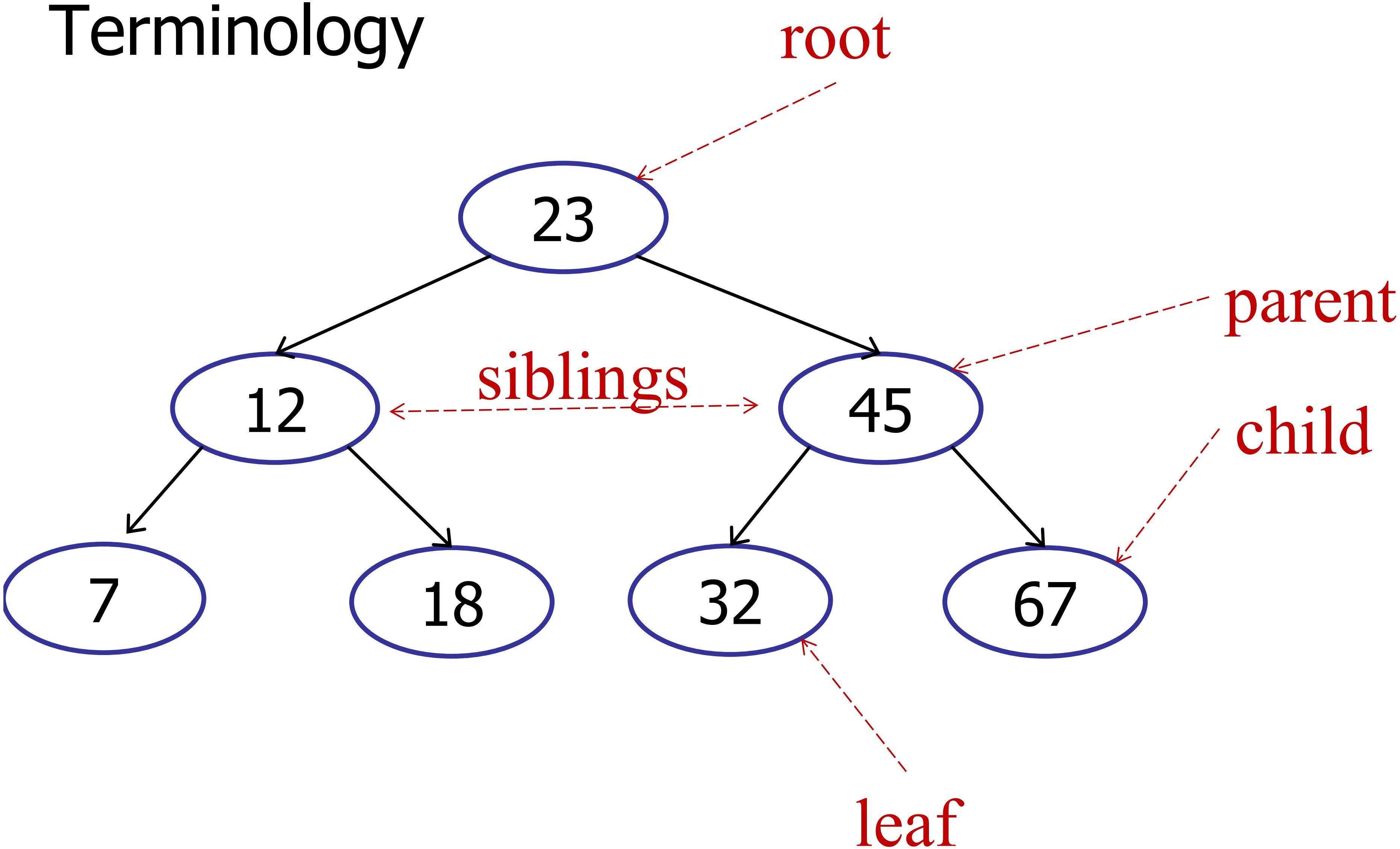
Dictionary

Implementation idea: Tree

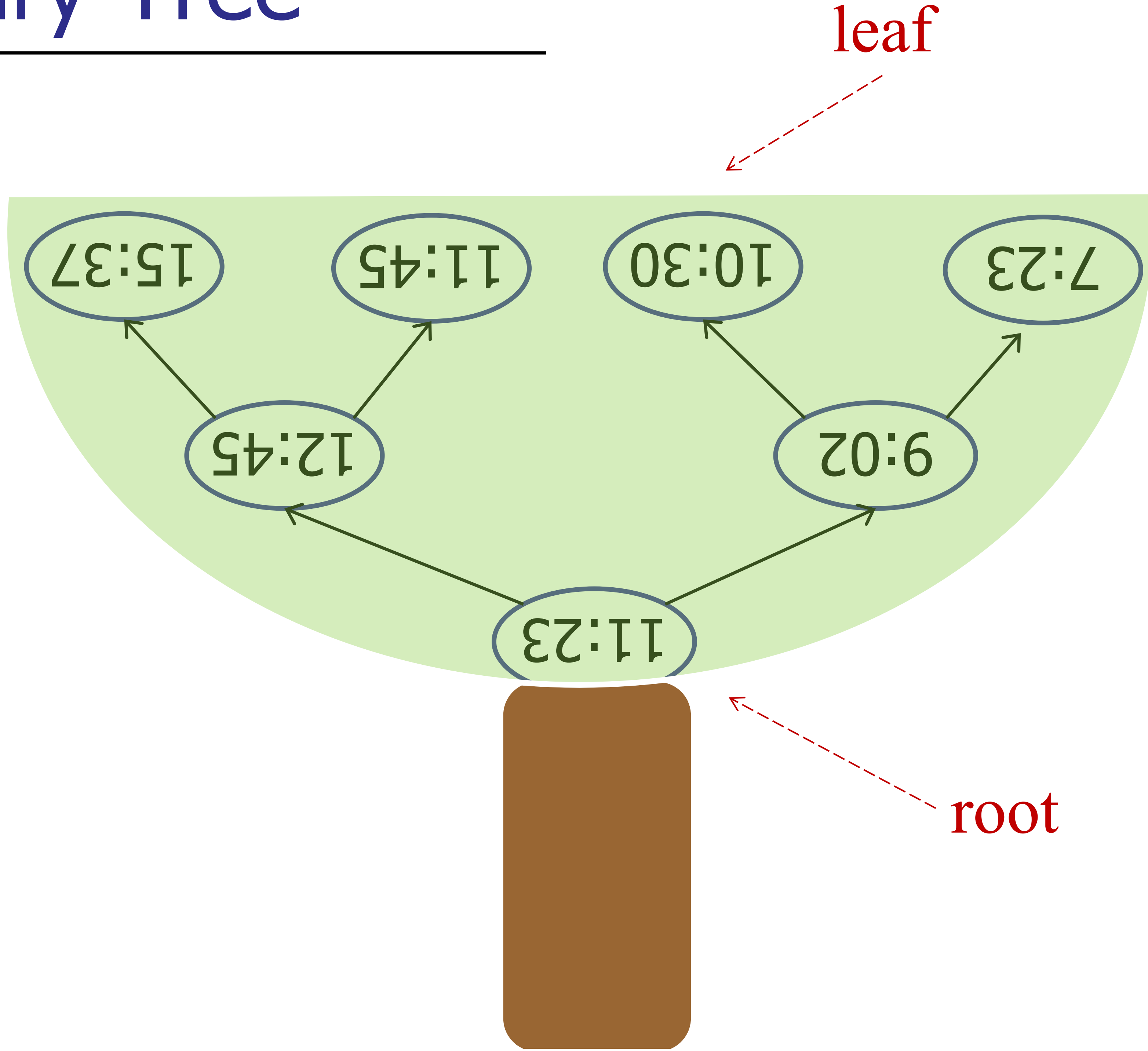


Binary Tree

Terminology

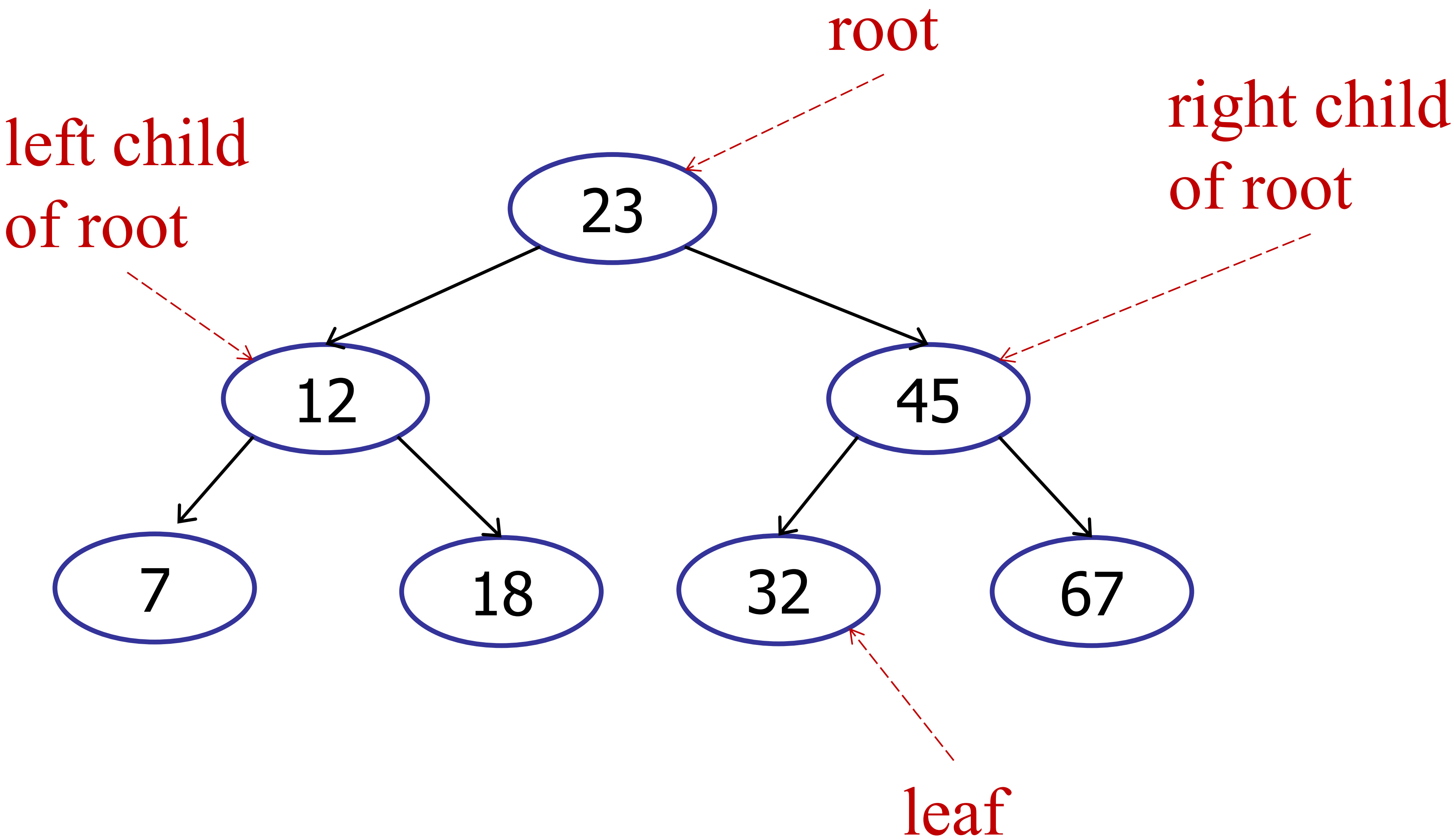


Binary Tree



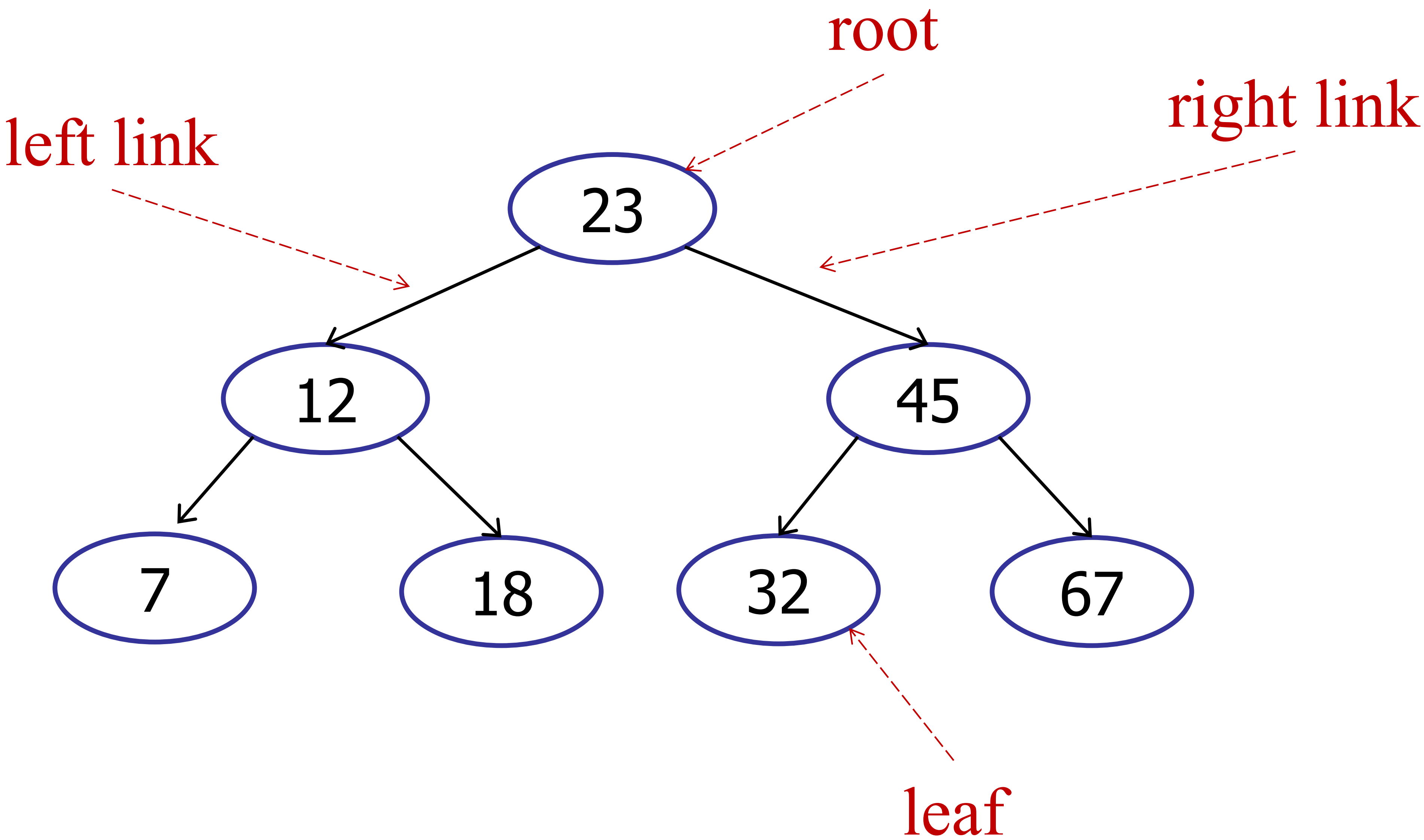
Binary Tree

Terminology



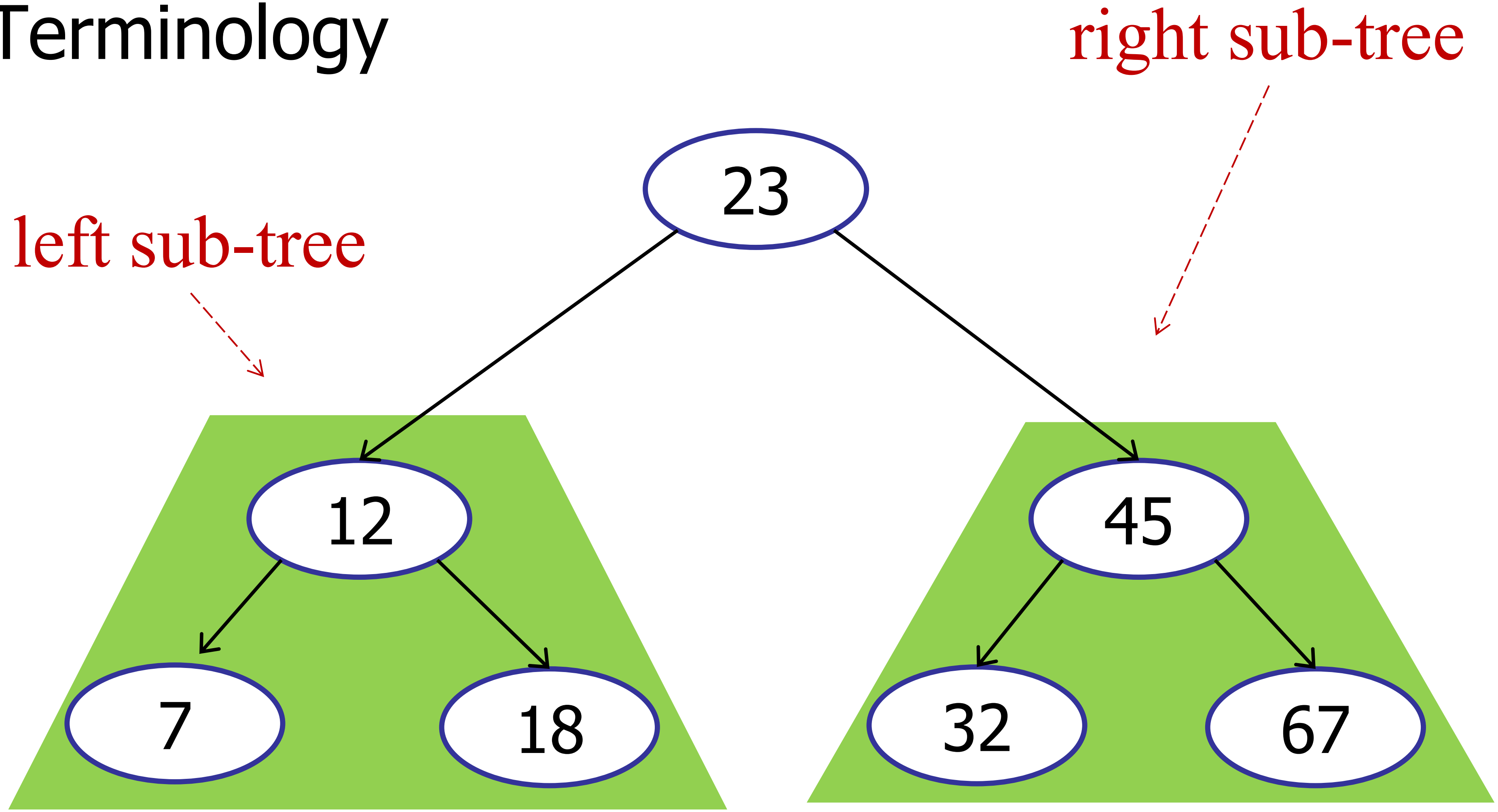
Binary Tree

Terminology



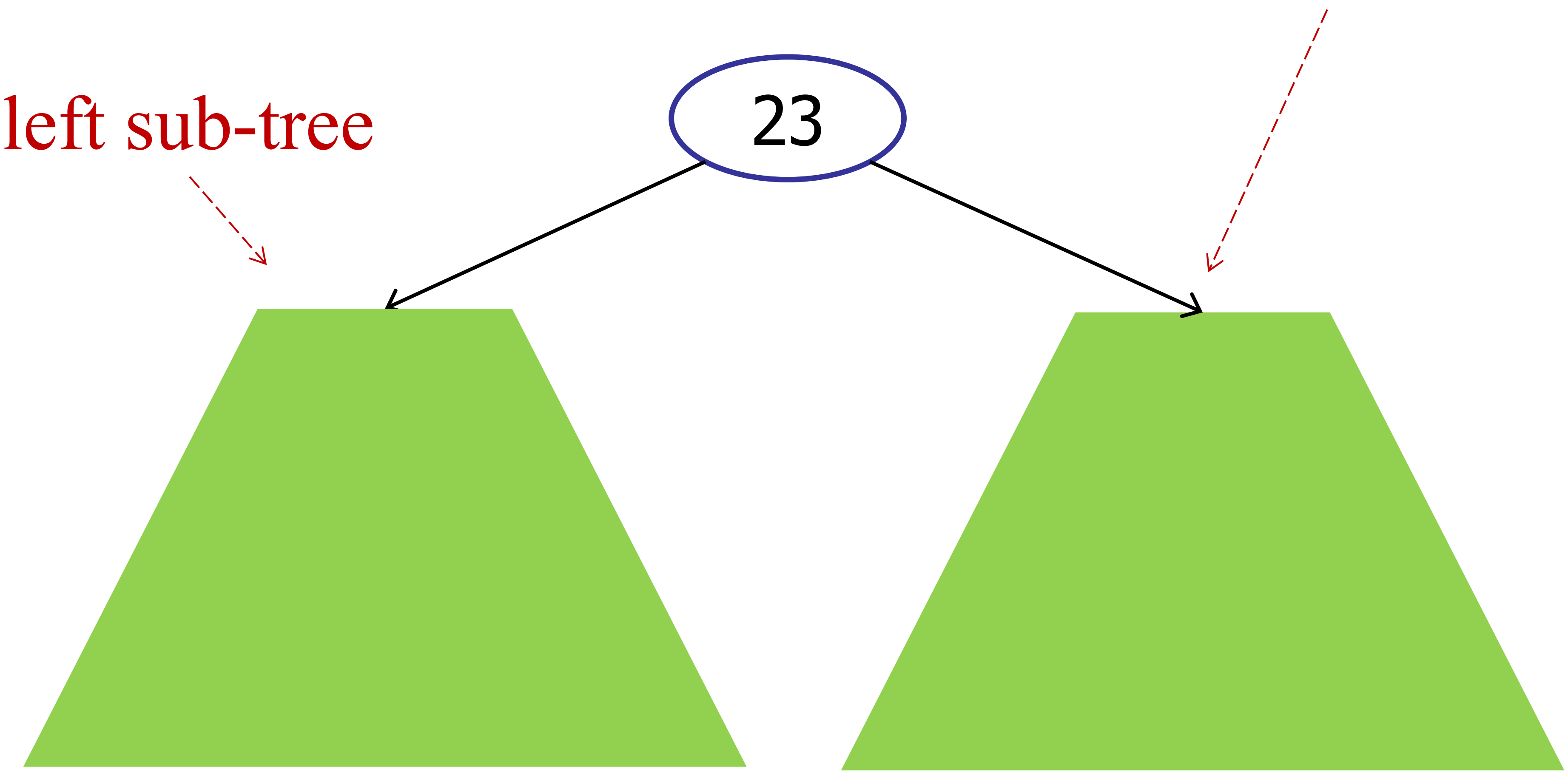
Binary Tree

Terminology



Binary Tree

Recursive Definition



A binary tree is either: **(a) empty**

(b) a node pointing to two binary trees

Binary Tree

Java??

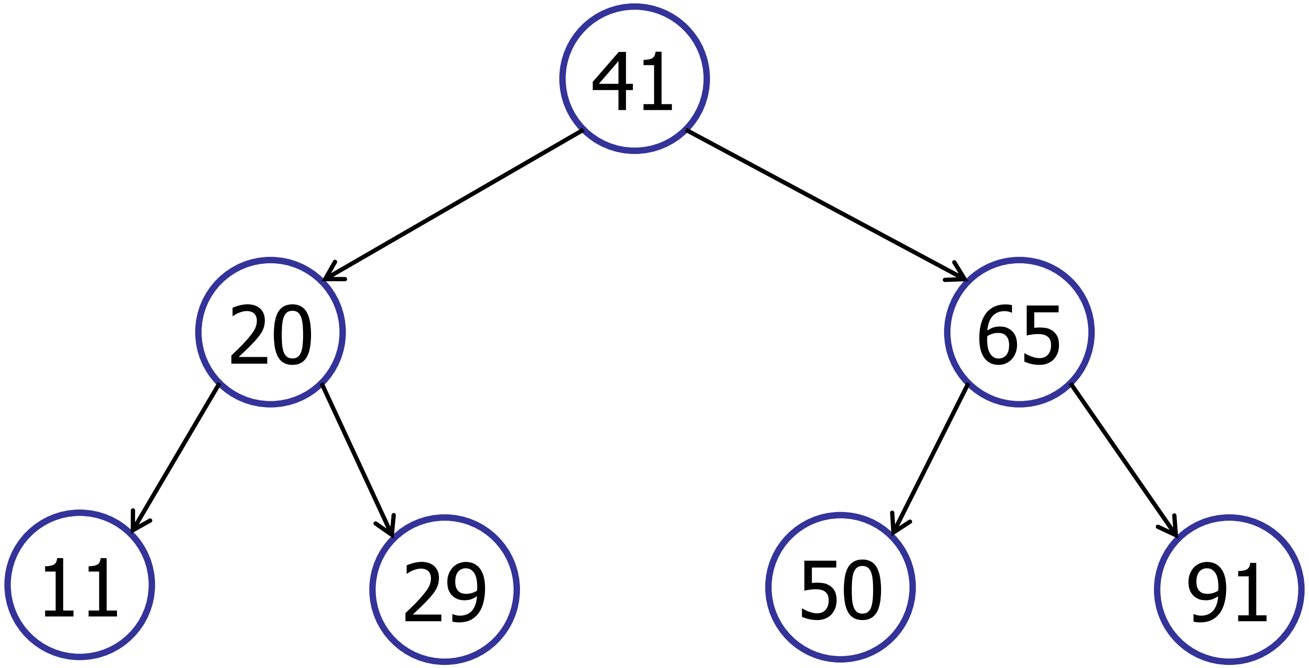
```java
public class BinaryTree {

        private BinaryTree leftTree;
        private BinaryTree rightTree;

        private KeyType key;
        private ValueType value;

        // Remainder of binary tree implementation
}
```

Binary **Search** Trees (BST)

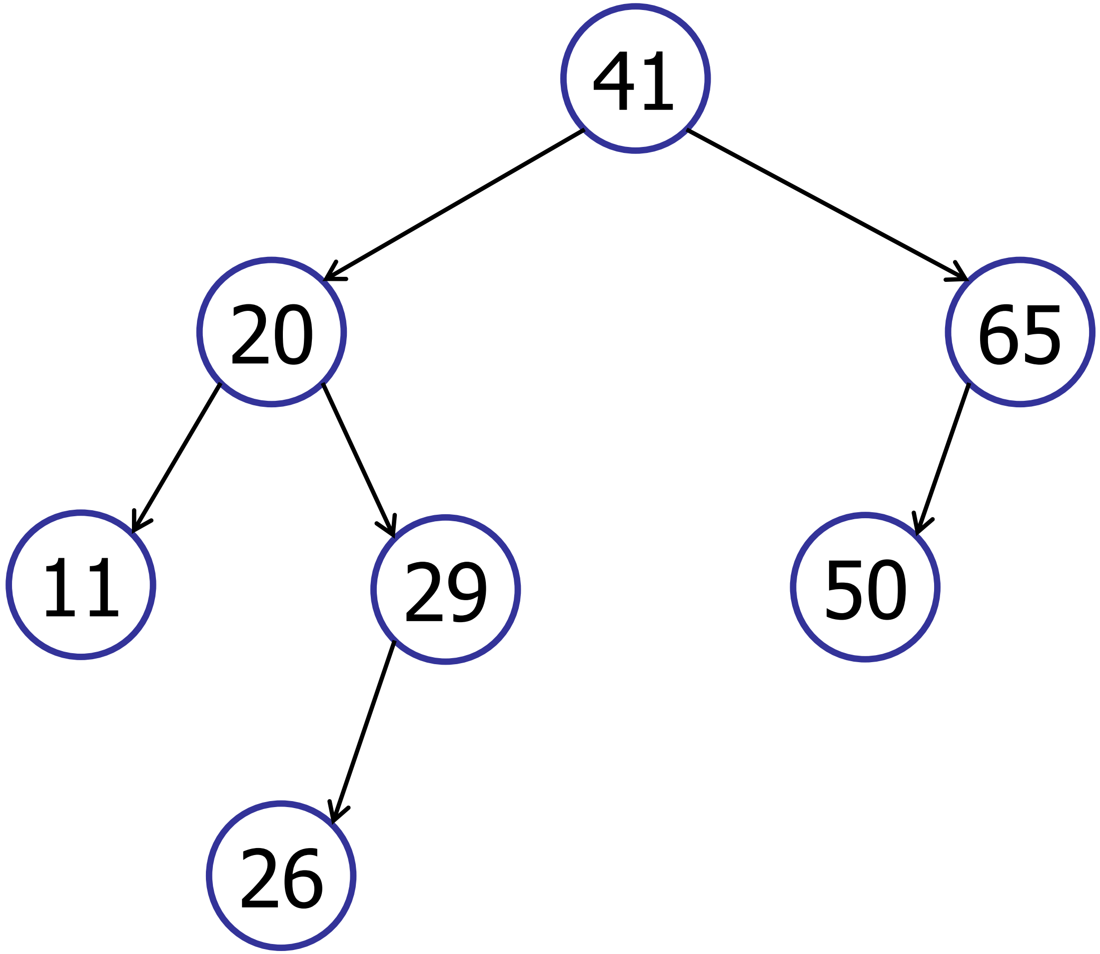


BST Property:

all in left sub-tree < key < all in right sub-right

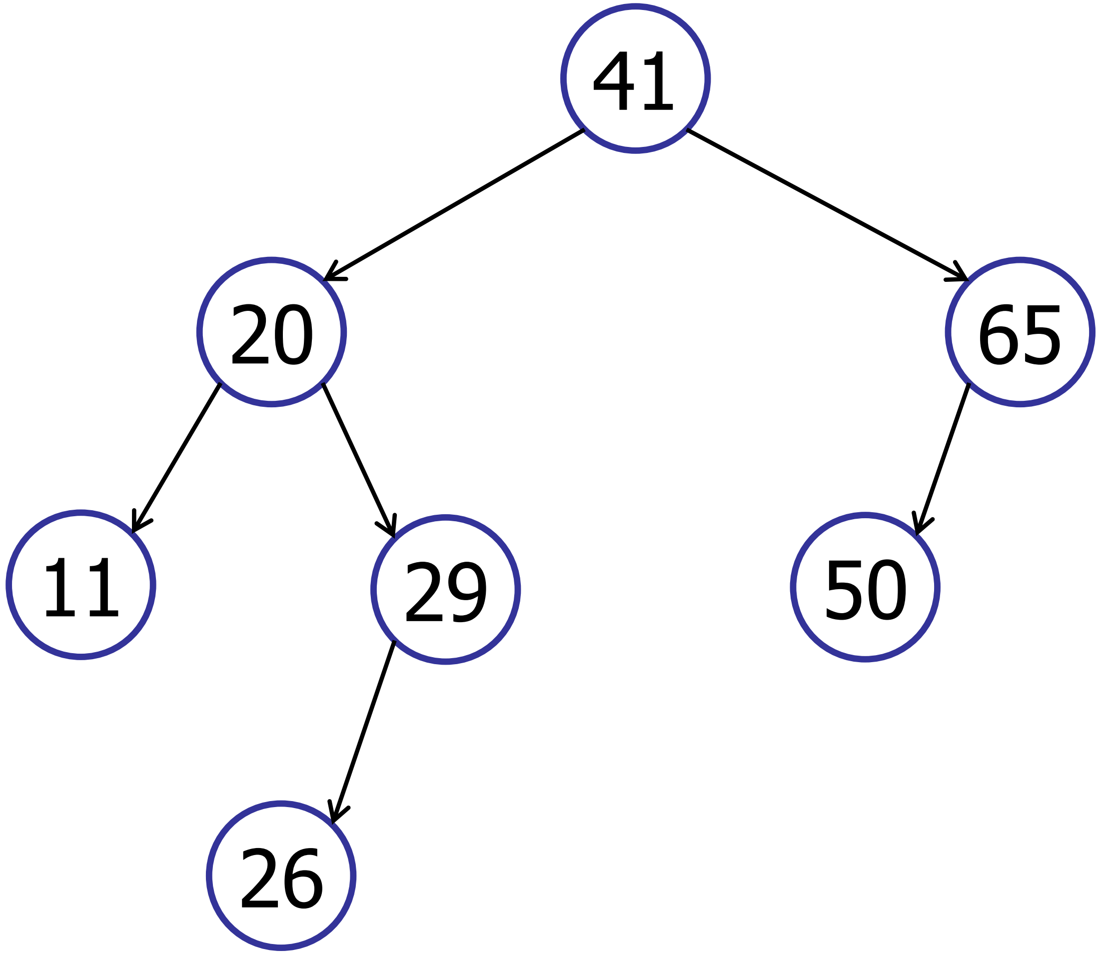
Is this a binary search tree?

1. Yes
2. No
3. I don't know.



Is this a binary search tree?

✓ 1. Yes
2. No
3. I don't know.



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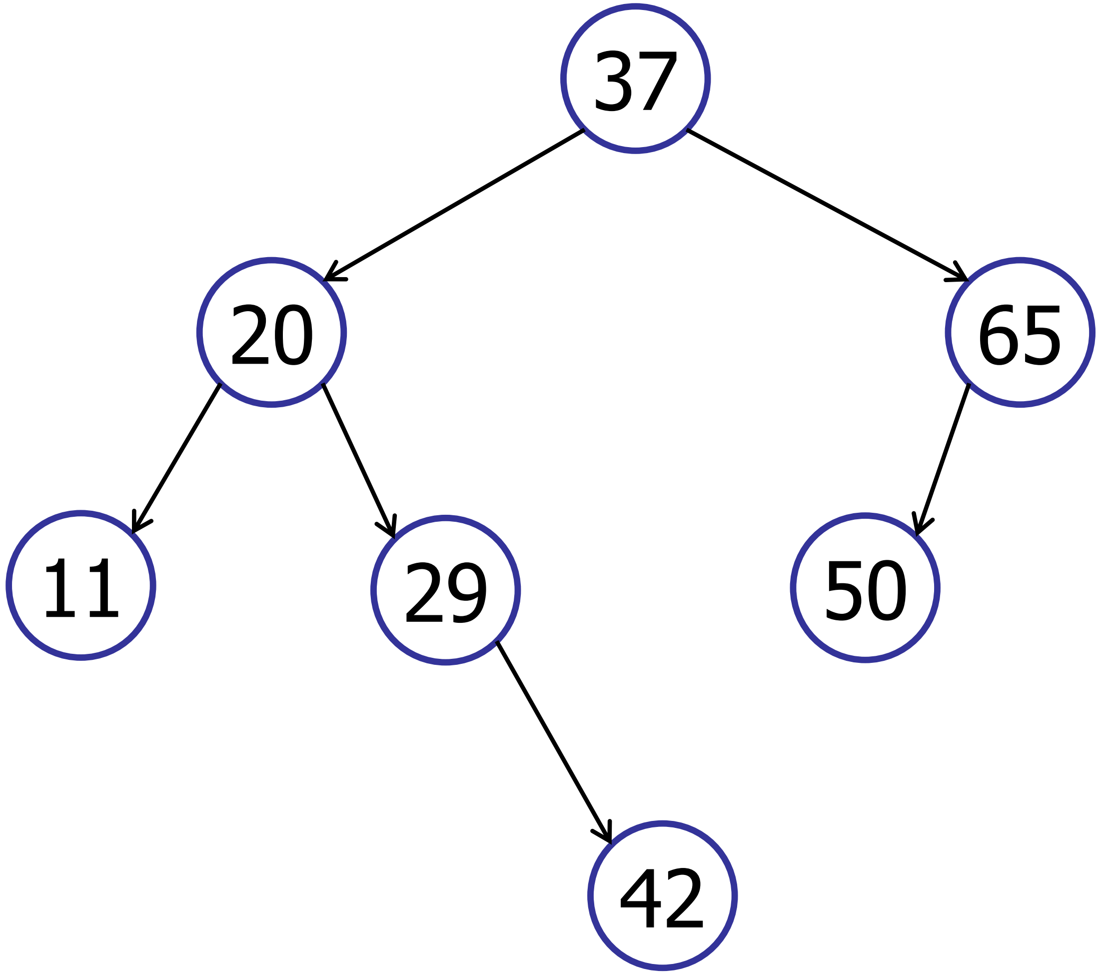

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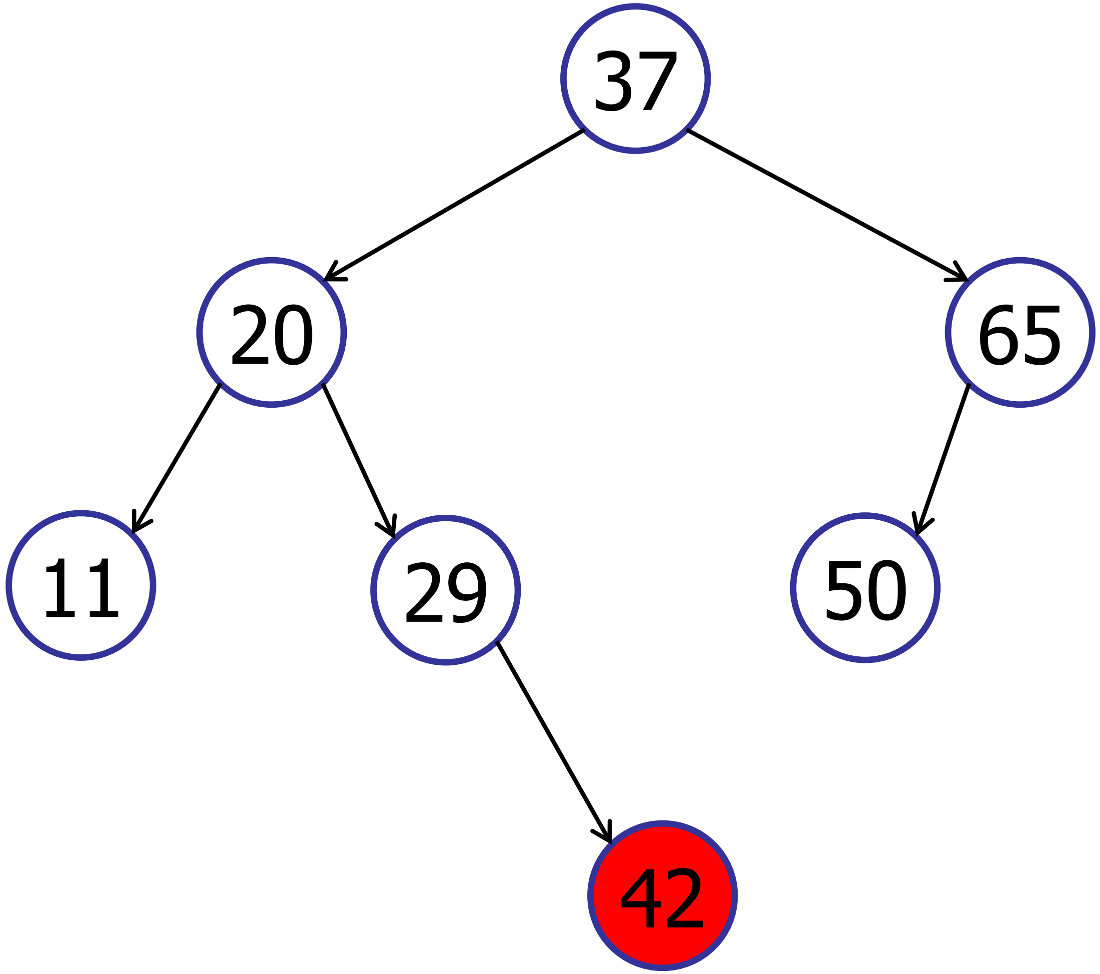


Binary Search Trees

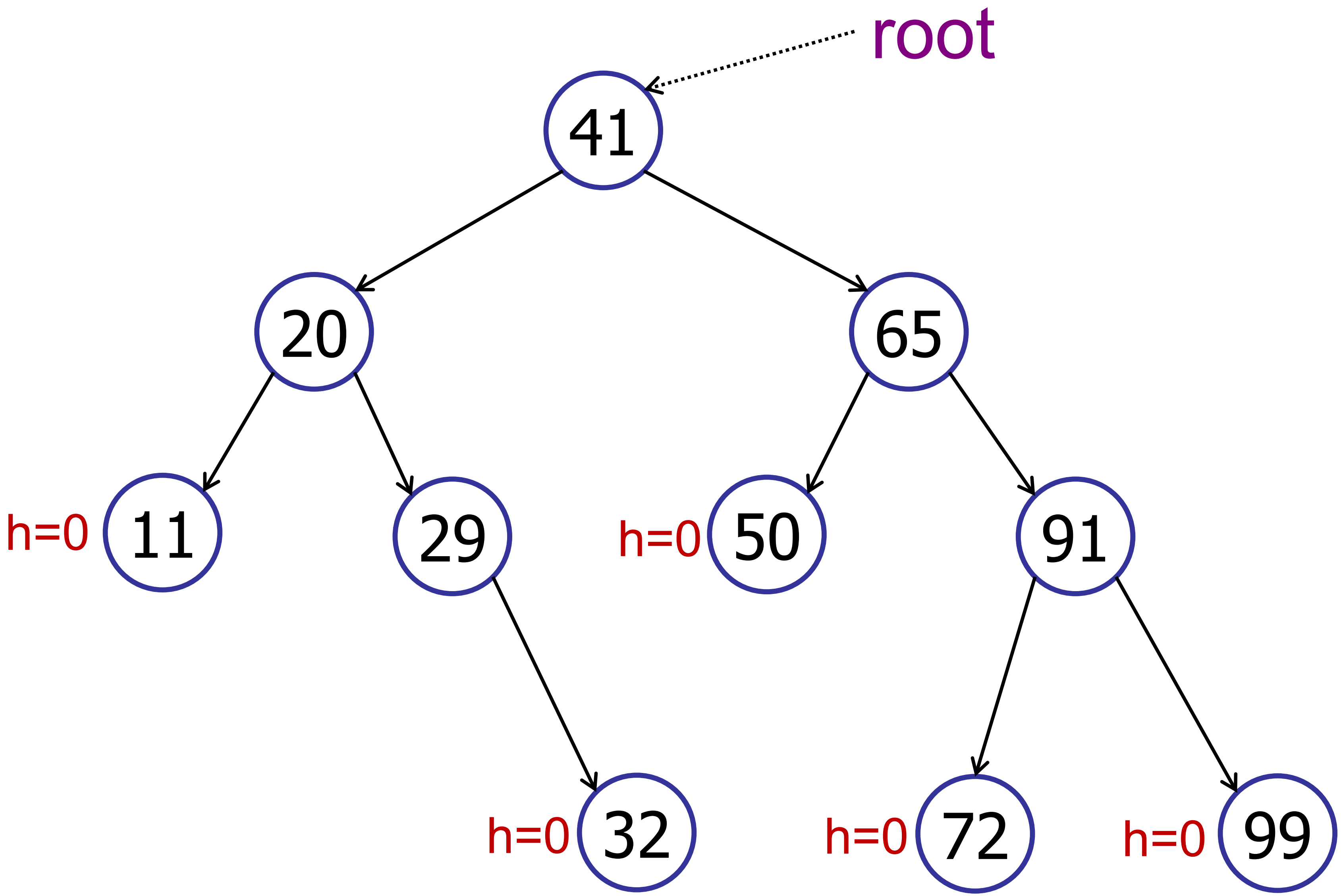
1. Terminology and Definitions

2. Basic operations:
   - height
   - search, insert
   - searchMin, searchMax

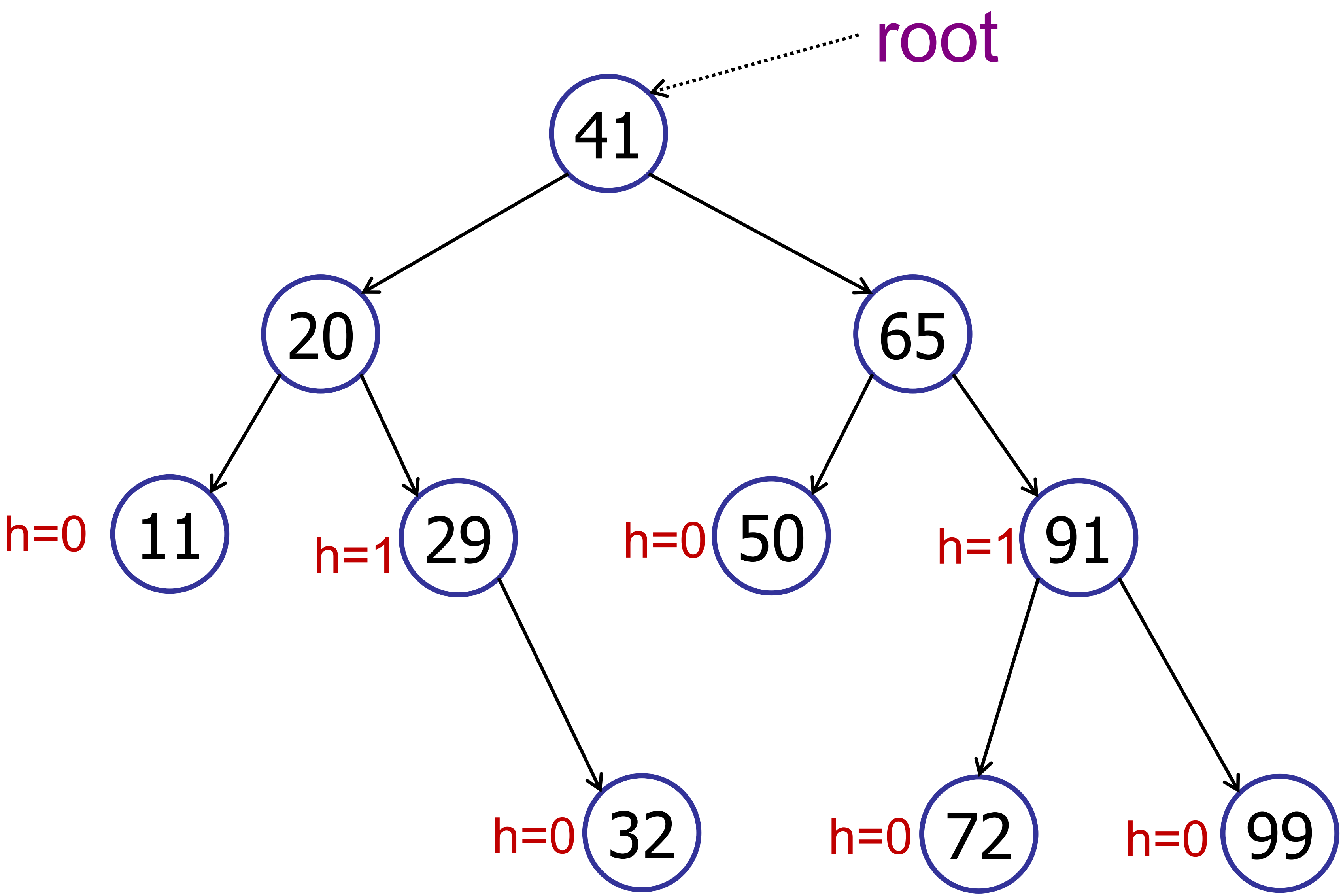
3. Traversals
   - in-order, pre-order, post-order

4. Other operations

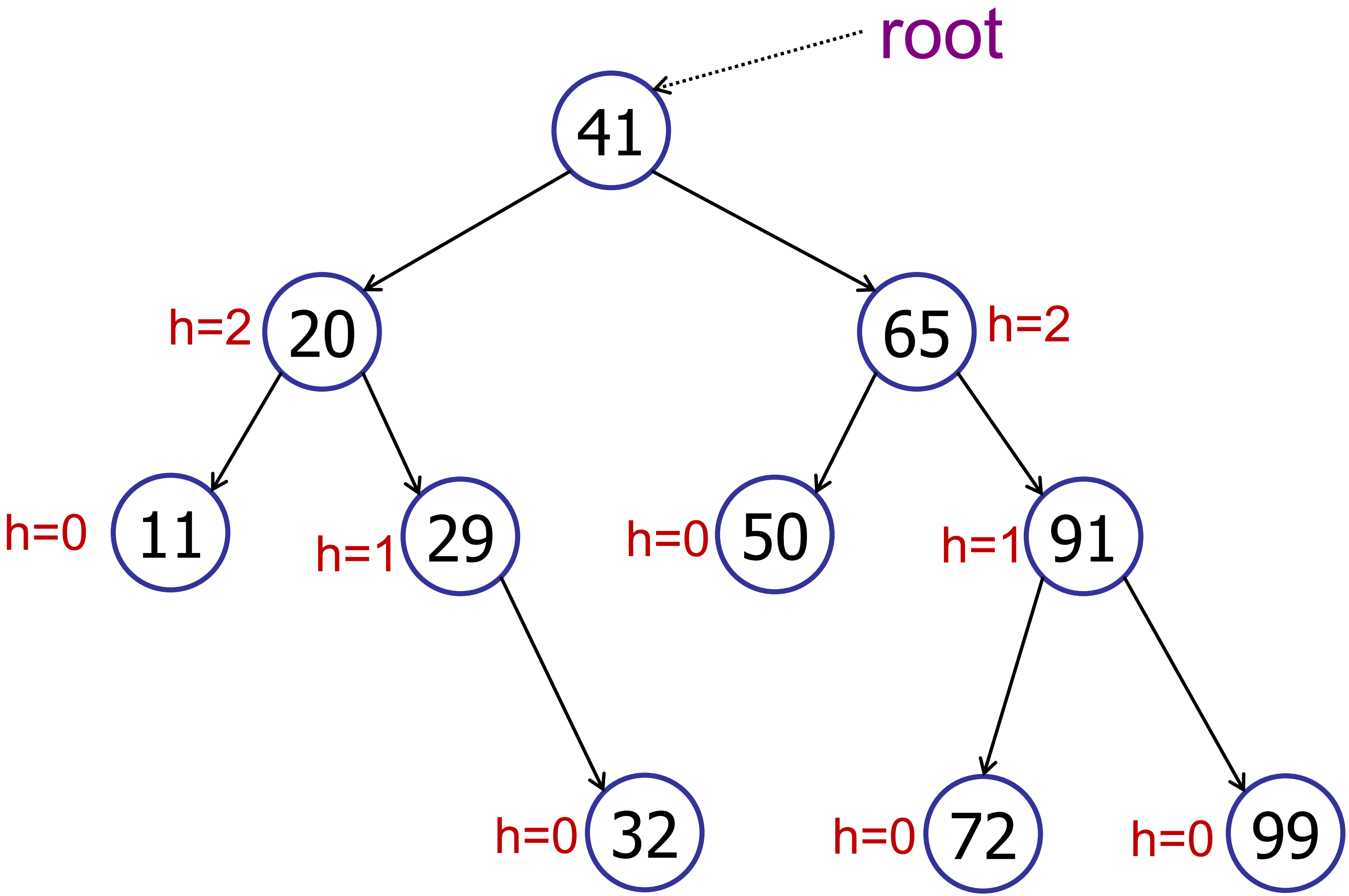
Height of a Binary Tree



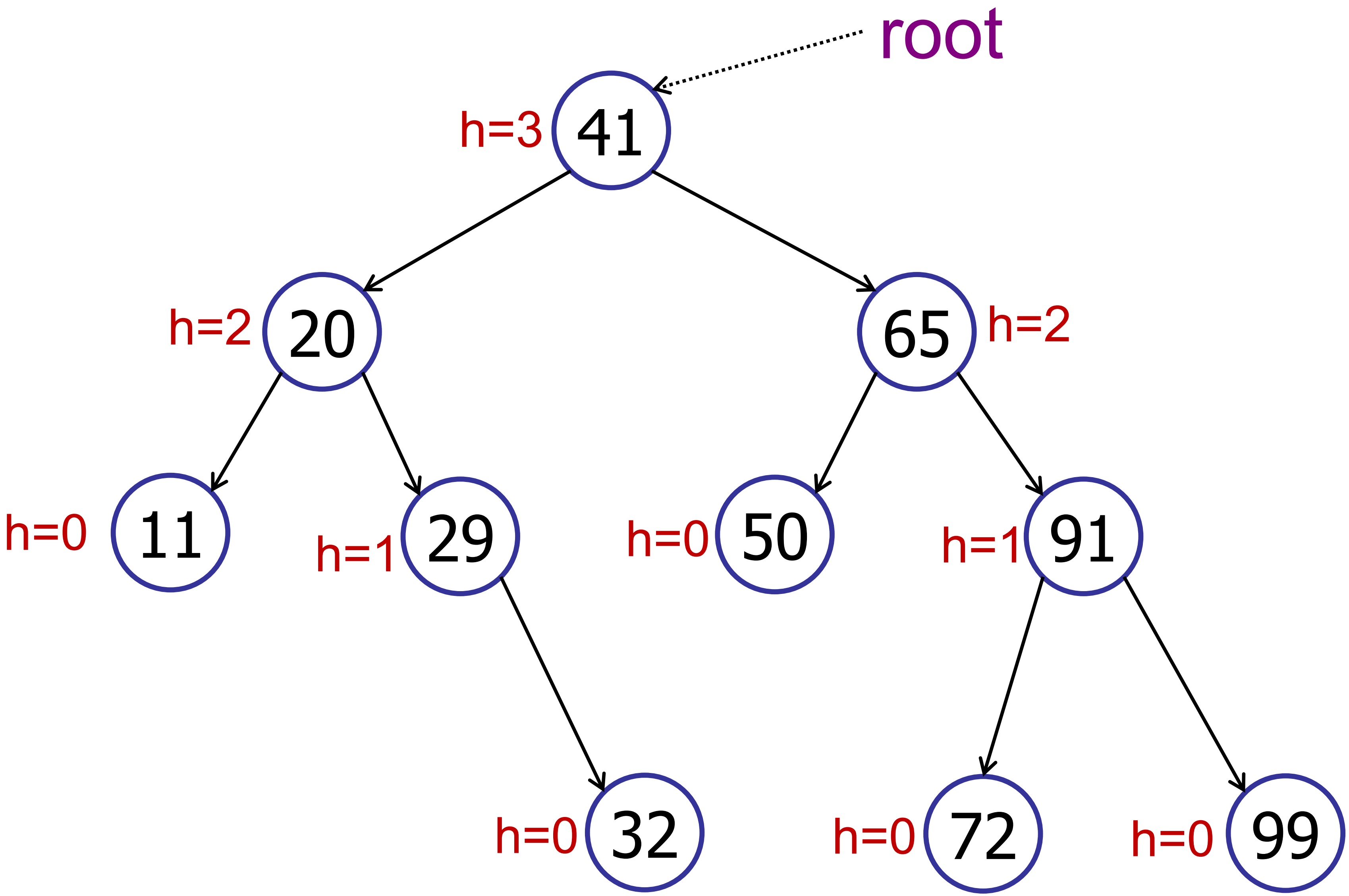
Height of a Binary Tree



Height of a Binary Tree



Height of a Binary Tree



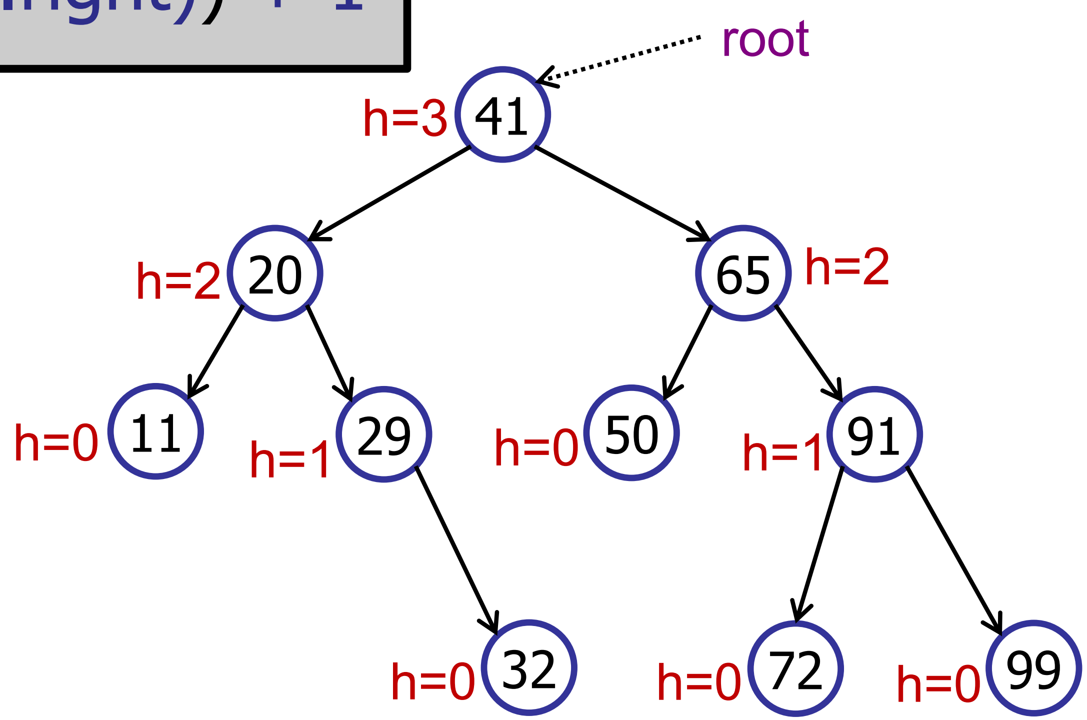
Height of a Binary Tree

Height:

Number of edges on longest path from root to leaf.

$h(v) = 0$ (if v is a leaf)

$h(v) = \max(h(v.left), h(v.right)) + 1$



(For simplicity: $h(null) = -1$)

Binary Tree

Calculating the heights

check for null

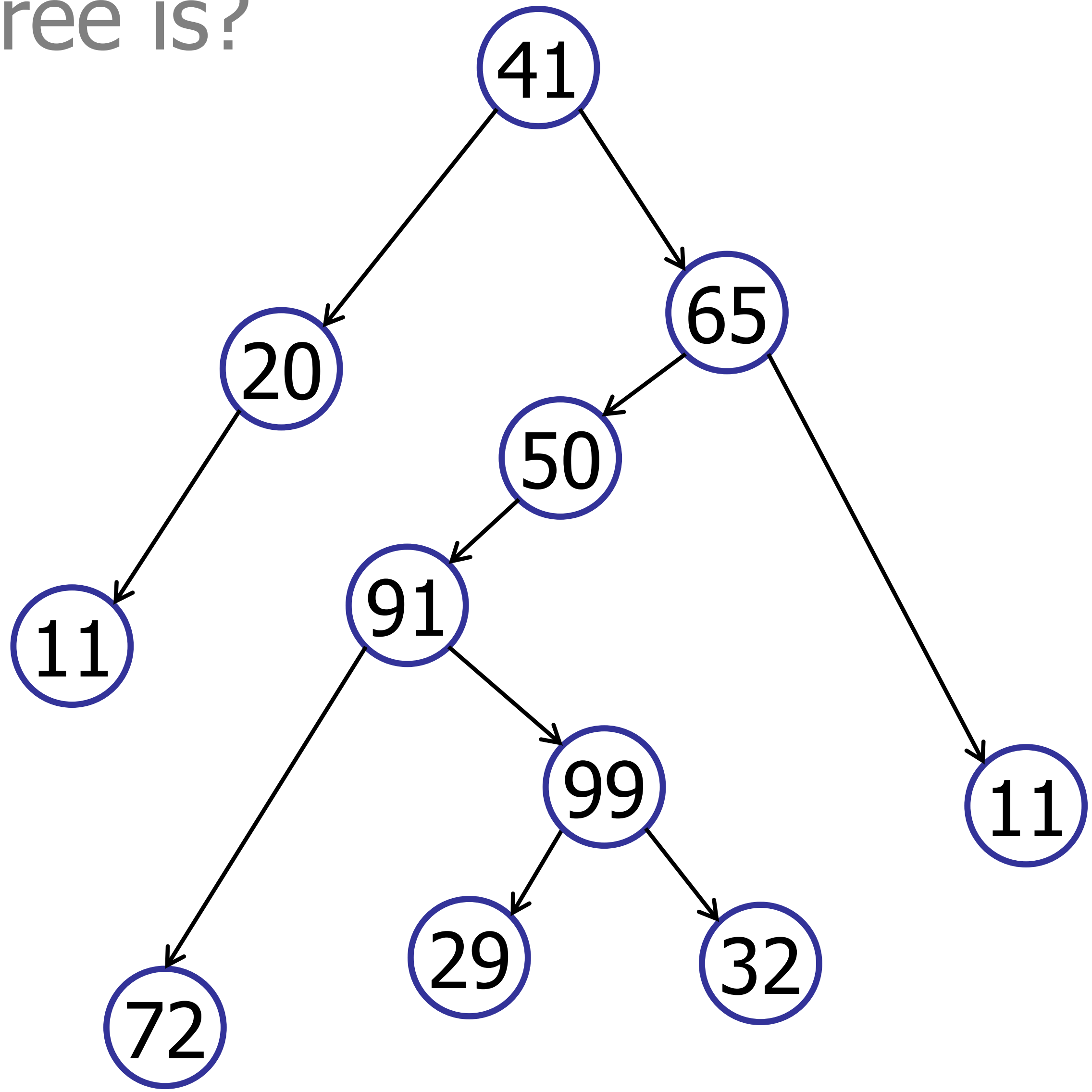
```
public int height(){
    int leftHeight = -1;
    int rightHeight = -1;
    if (m_leftTree != null)
        leftHeight = m_leftTree.height();
    if (m_rightTree != null)
        rightHeight = m_rightTree.height();
    return max(leftHeight, rightHeight) +  1;
}
```

max of subtrees

add 1

The height of this tree is?

1. 2
2. 4
3. 5
4. 6
5. 7
6. 42



The height of this tree is?

1. 2
2. 4
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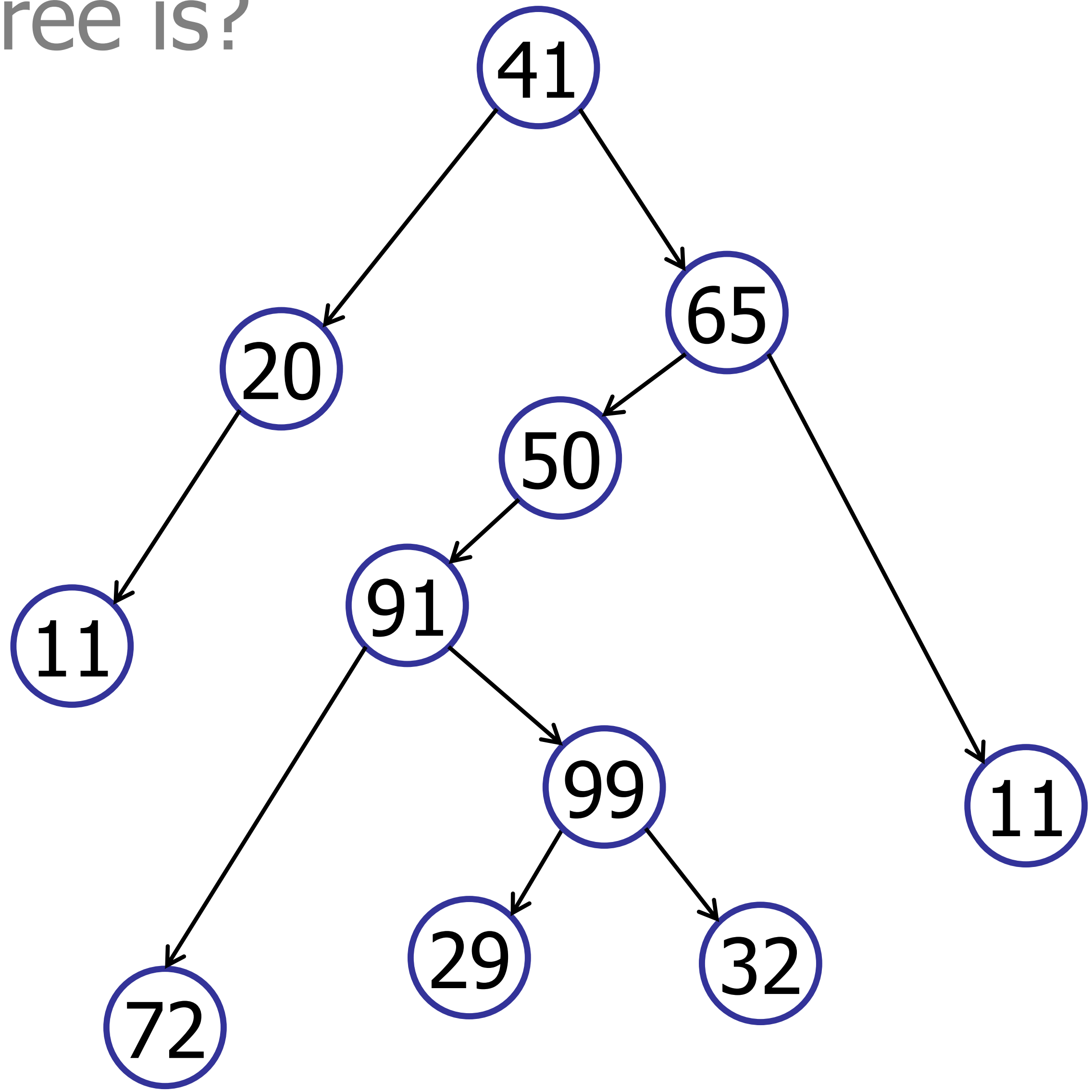
6. 42



Binary Search Trees

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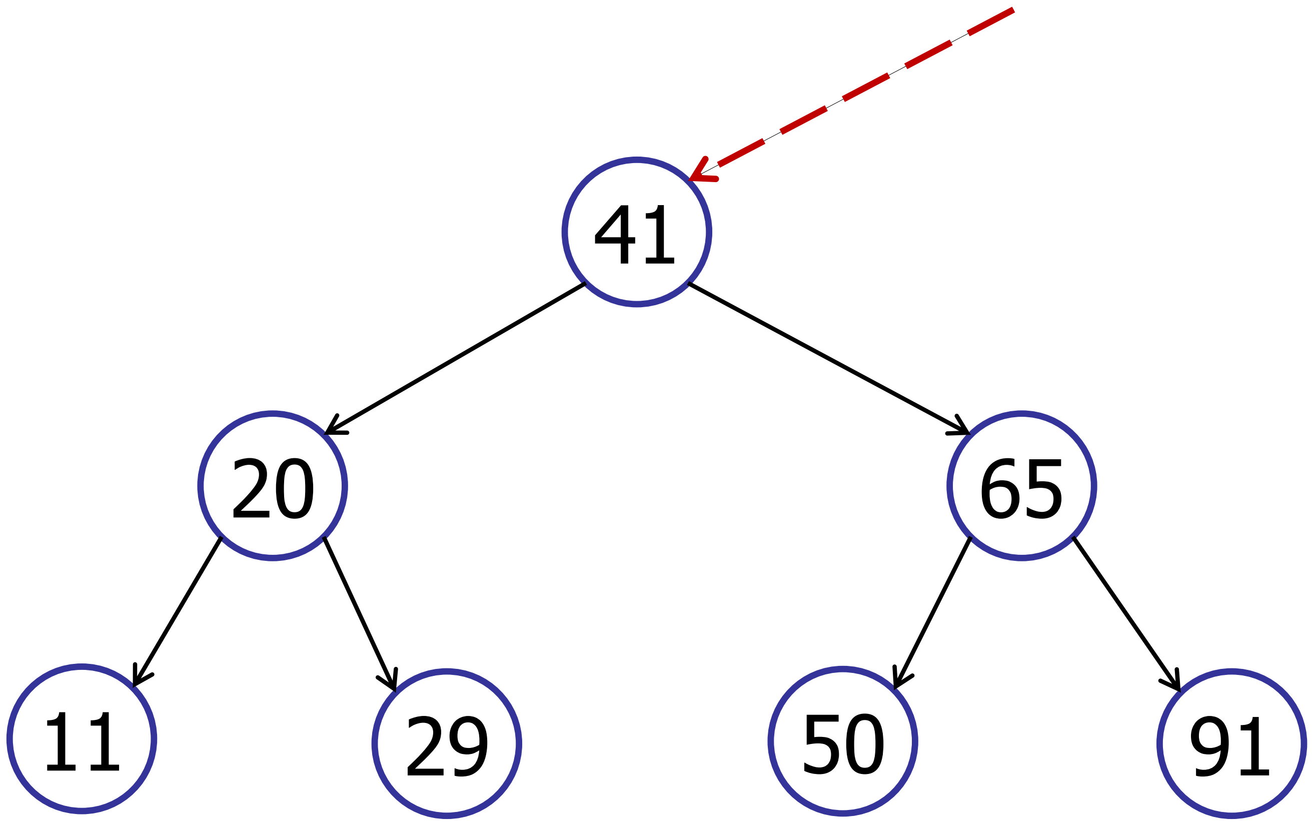
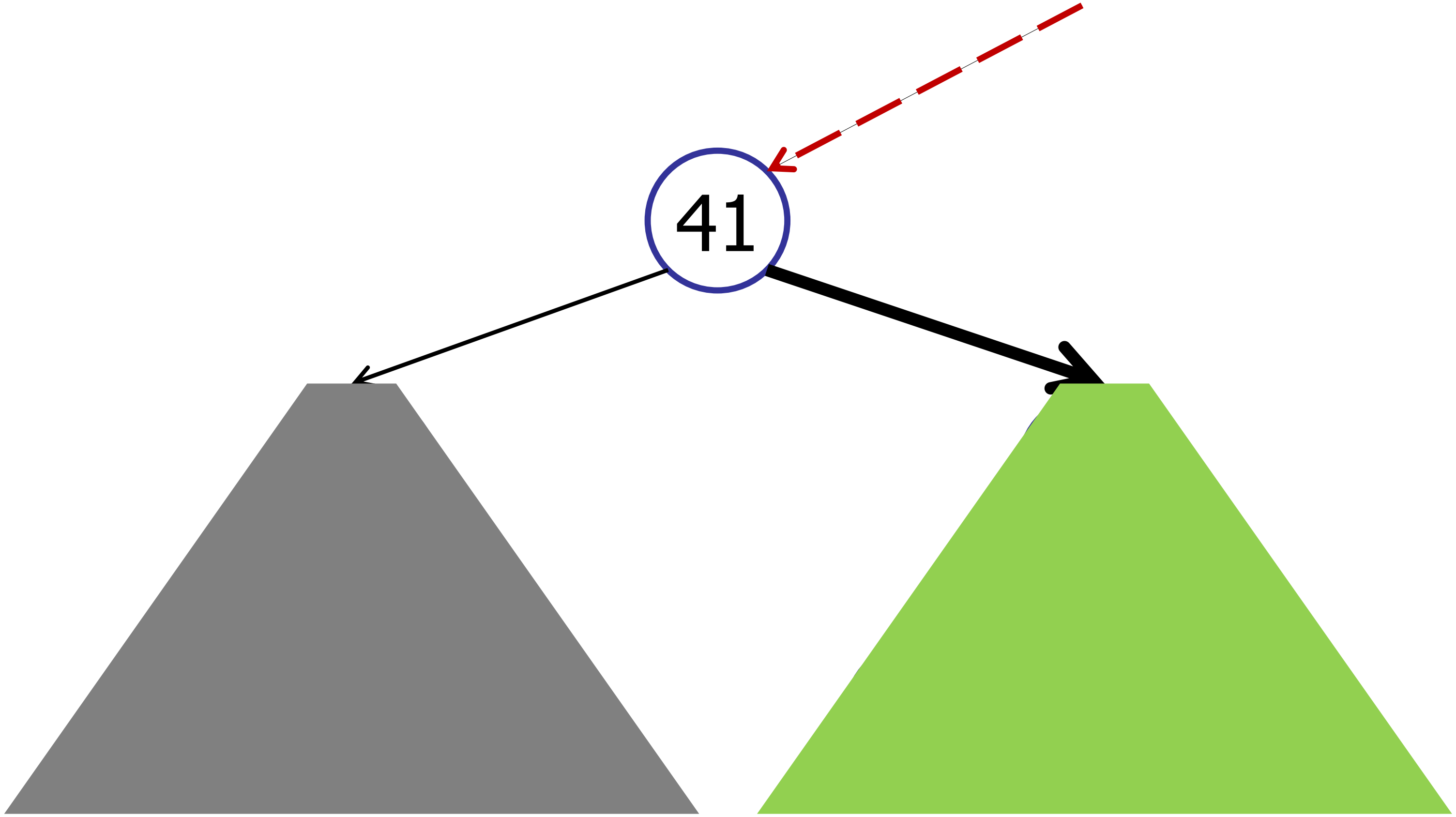
Binary Search Trees

Search for the maximum key:



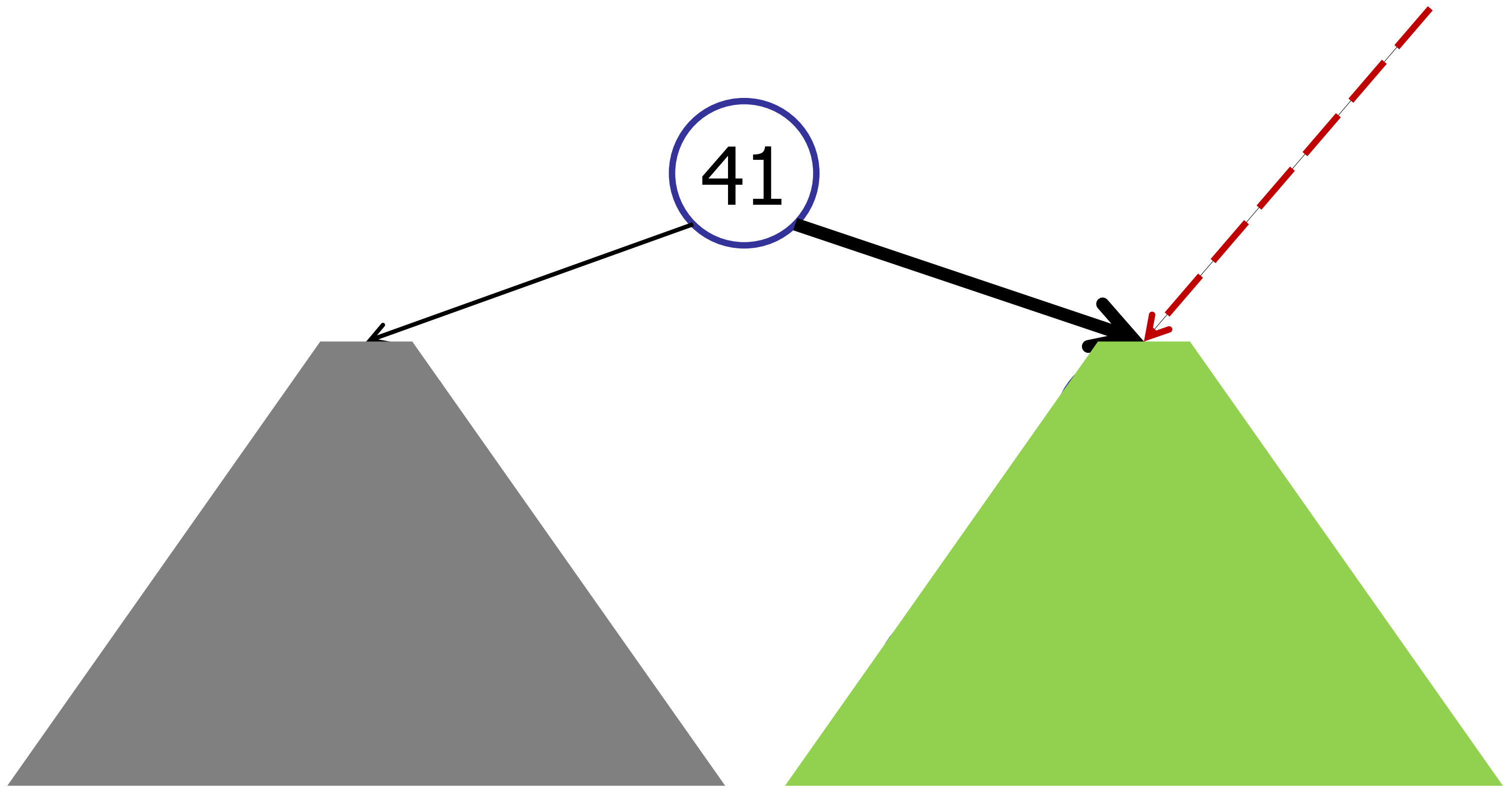
Binary Search Trees

Search for the maximum key:



Binary Search Trees

Search for maximum key



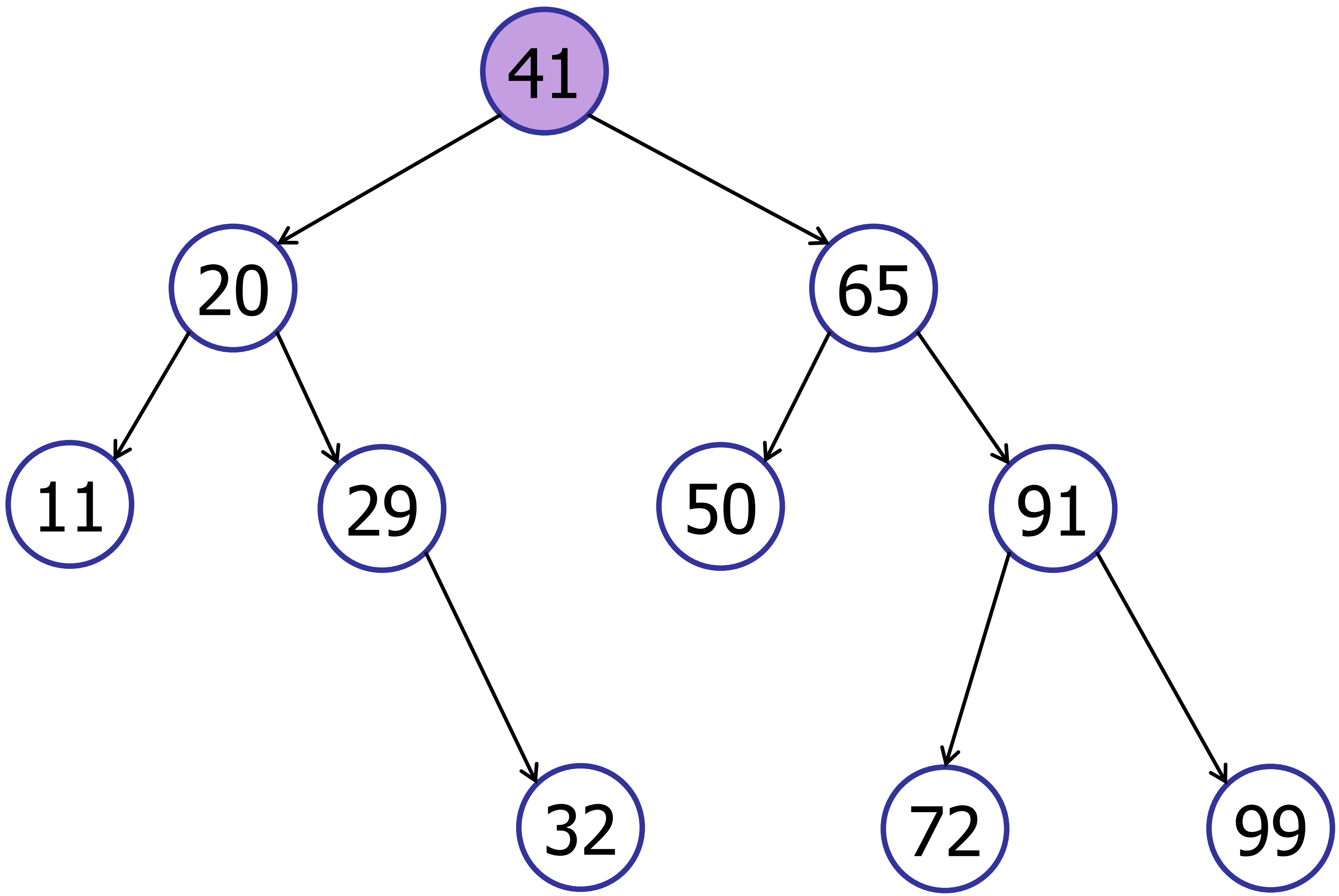
Binary Tree

Searching for the maximum key

```
public BinaryTree searchMax(){
        if (rightTree != null) {
                return rightTree.searchMax();
        }
        else return this; // Key is here!
}
```

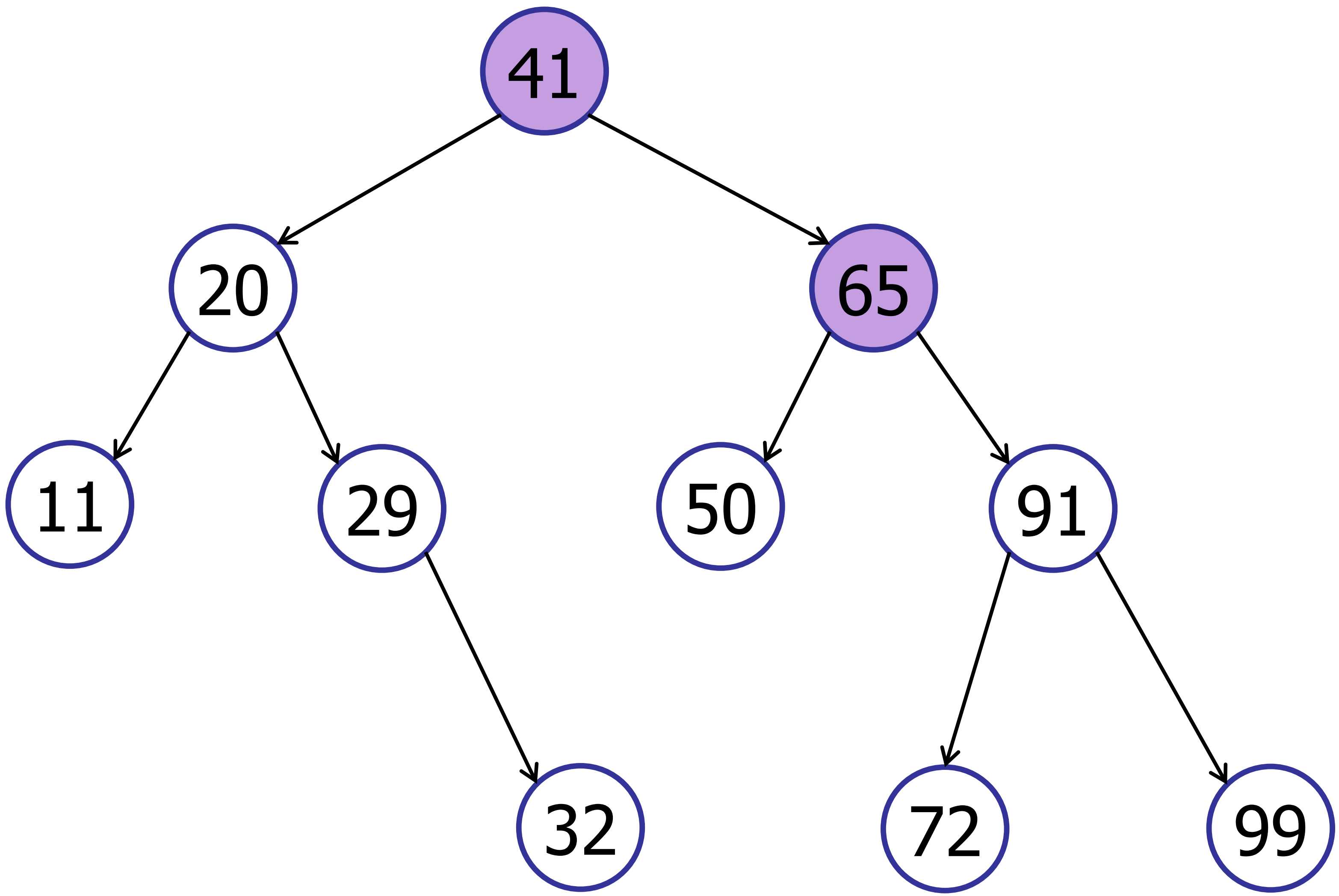
Binary Search Trees

searchMax()



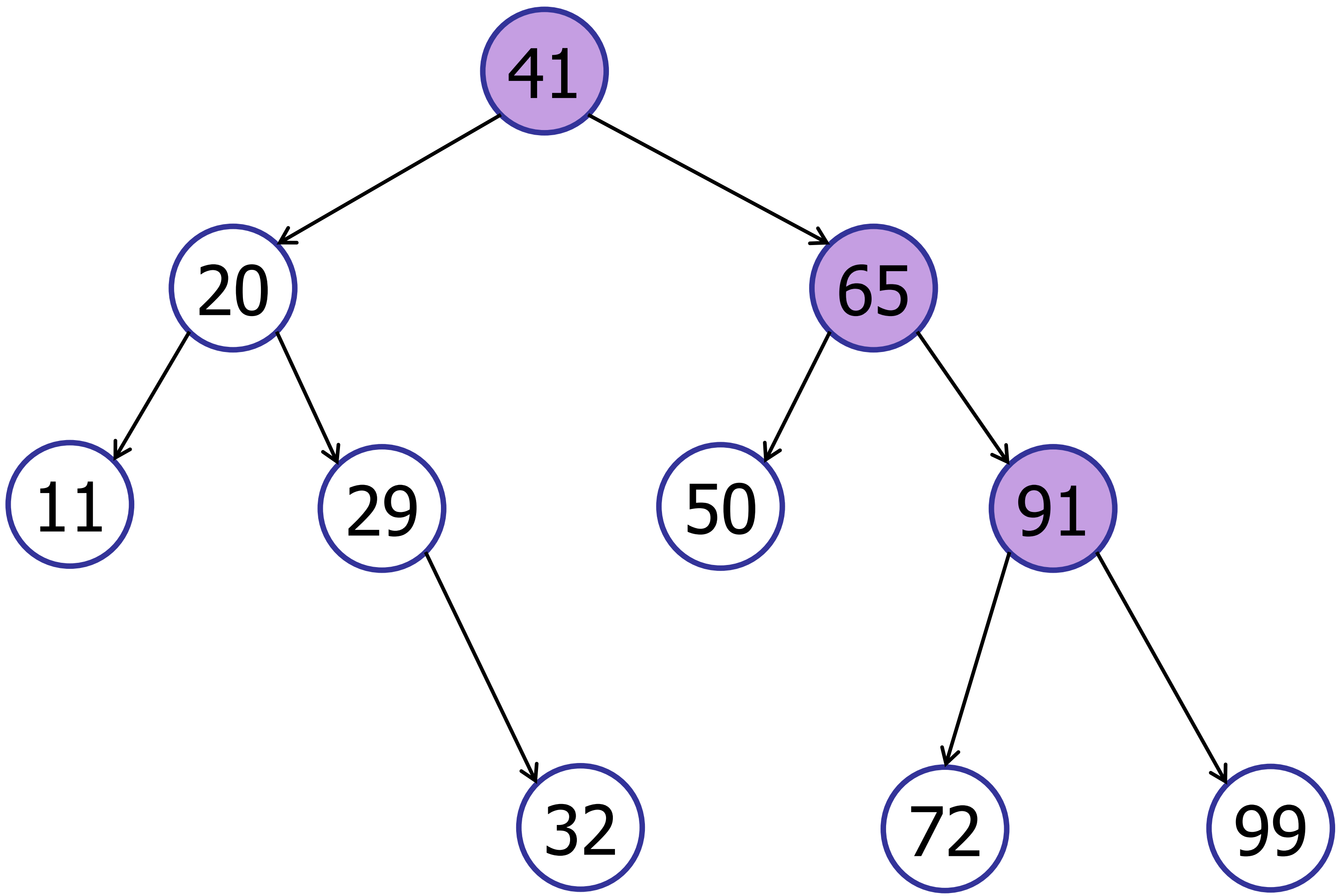
Binary Search Trees

searchMax()



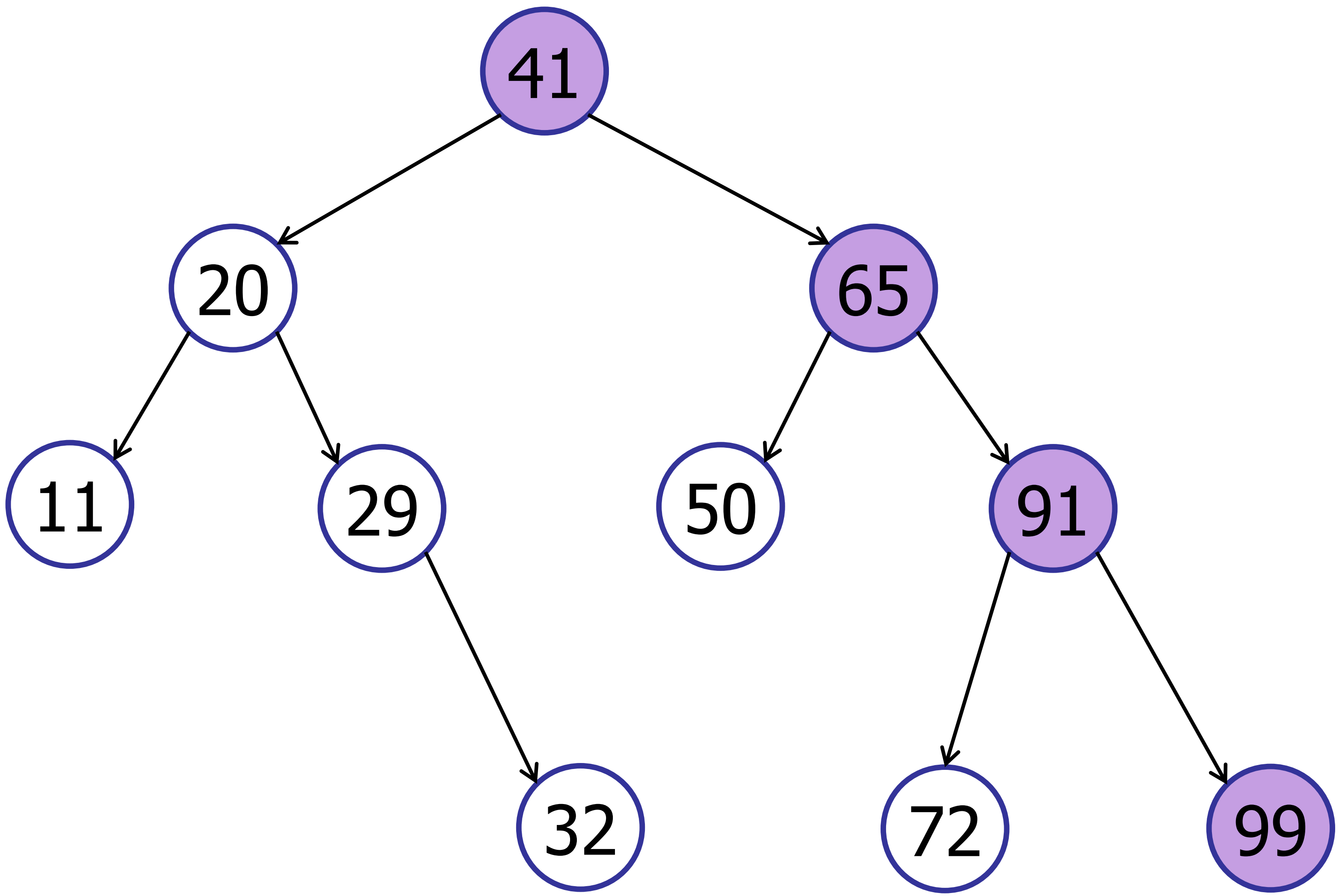
Binary Search Trees

searchMax()



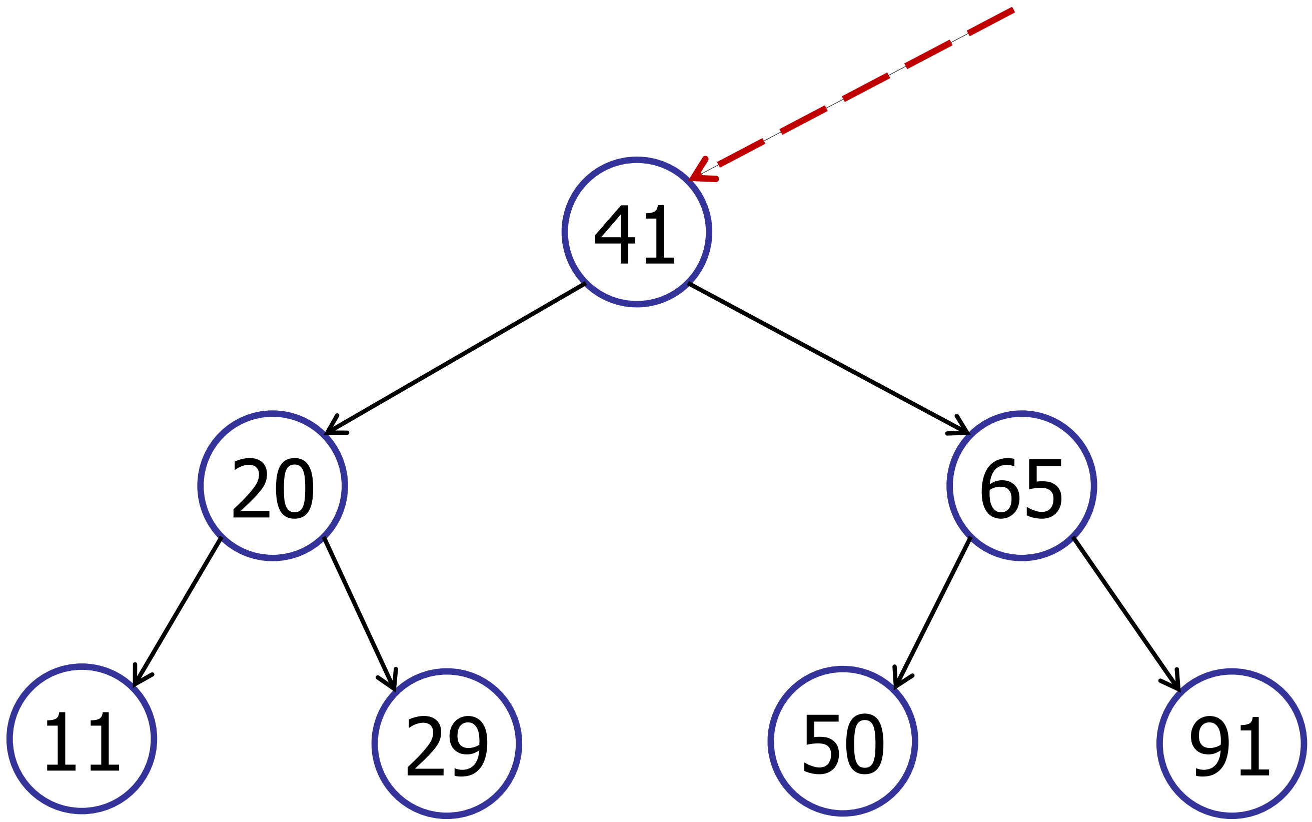
Binary Search Trees

searchMax()



Binary Search Trees

Search for the minimum key:



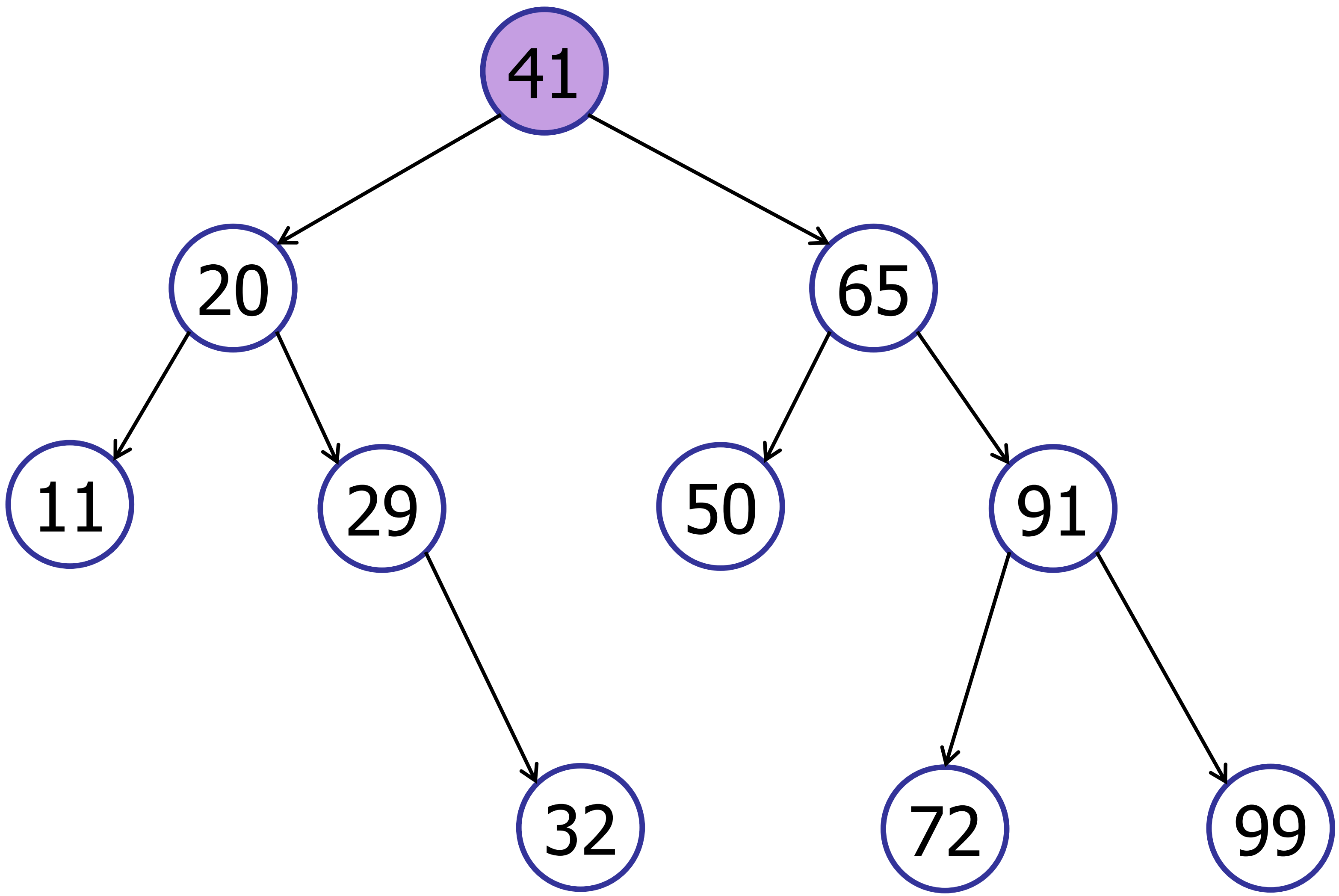
Binary Tree

Searching for the minimum key

```
public BinaryTree searchMin(){
        if (m_leftTree != null) {
                return leftTree.searchMin();
        }
        else return this; // Key is here!
}
```

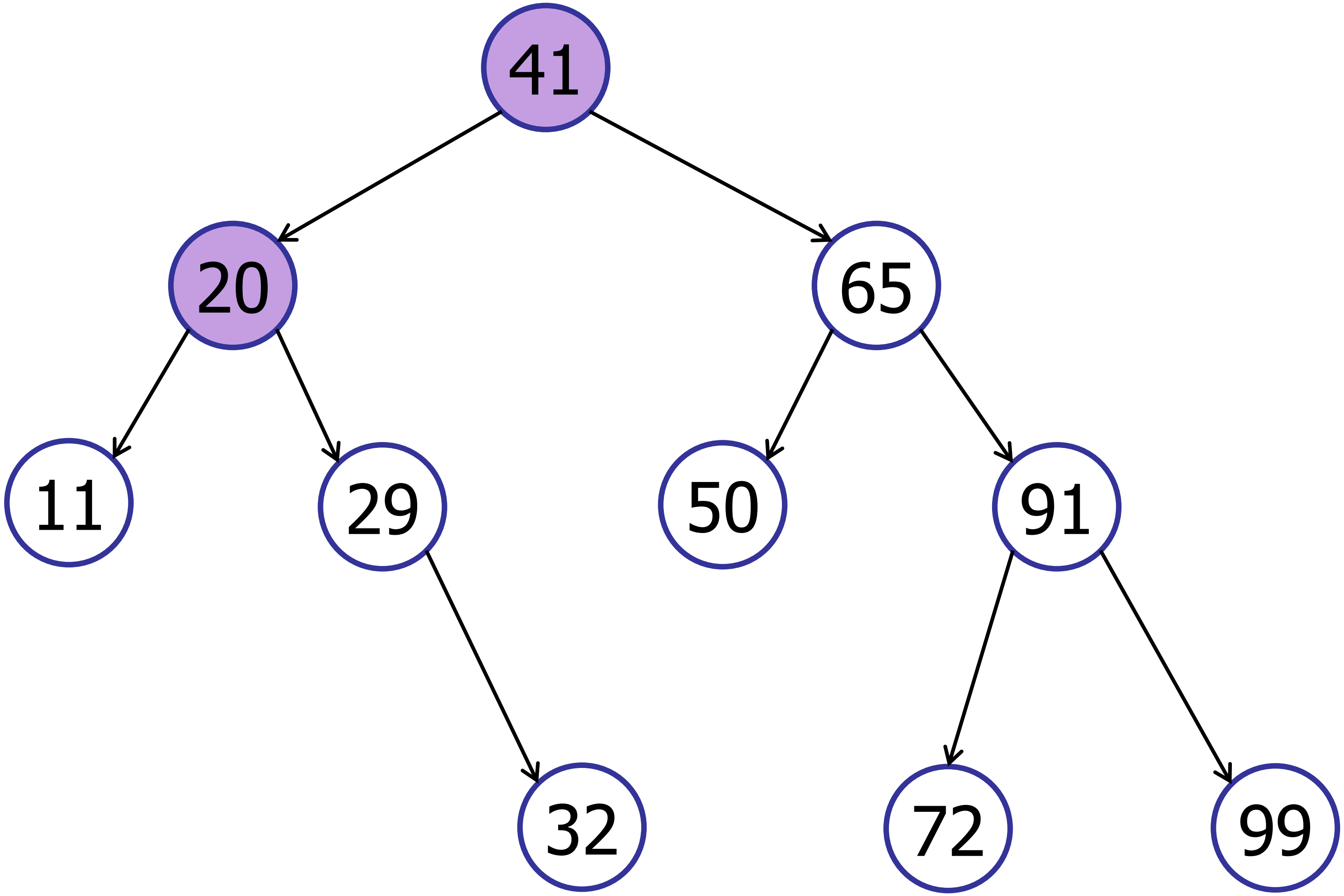
Binary Search Trees

searchMin()



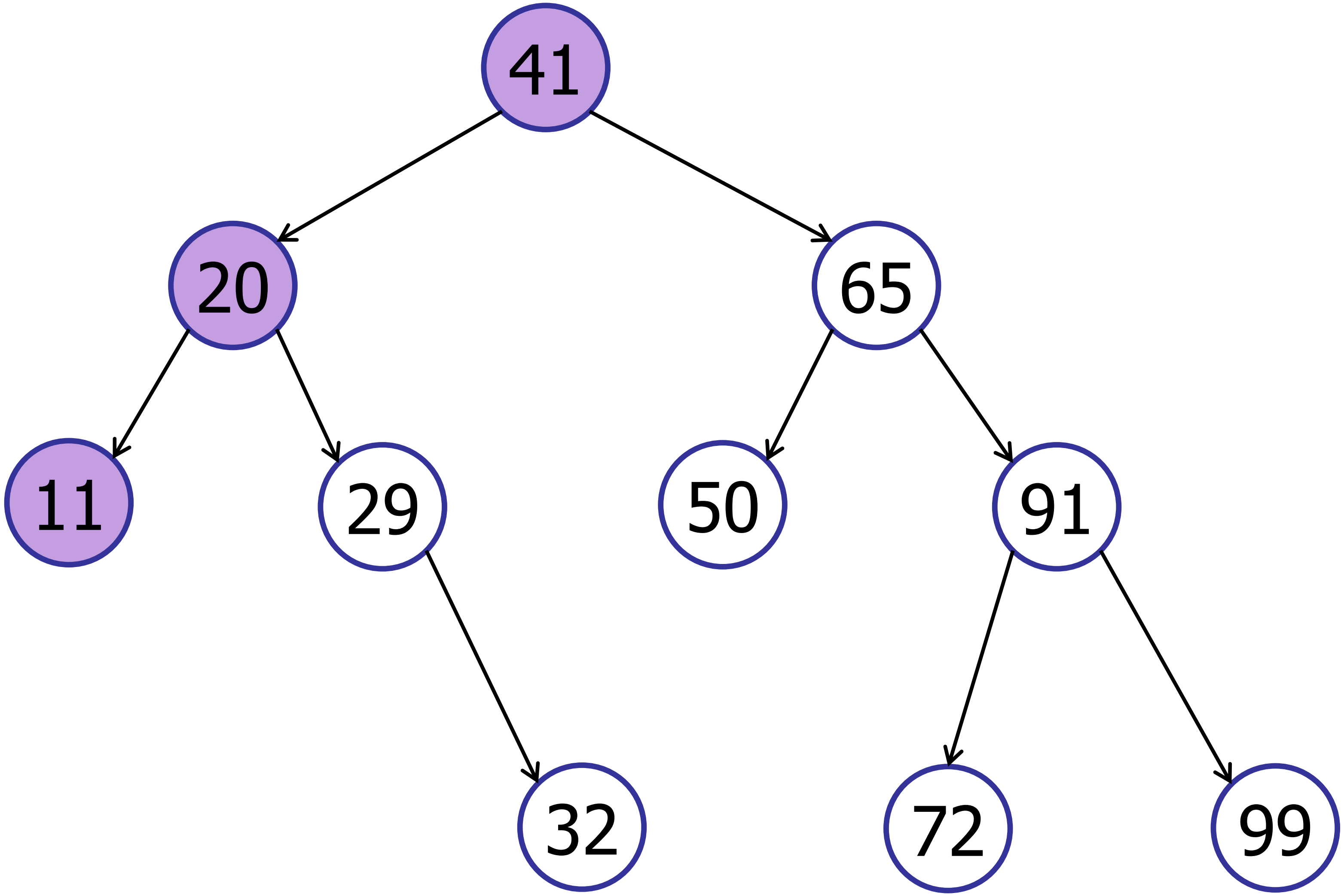
Binary Search Trees

searchMin()



Binary Search Trees

searchMin()



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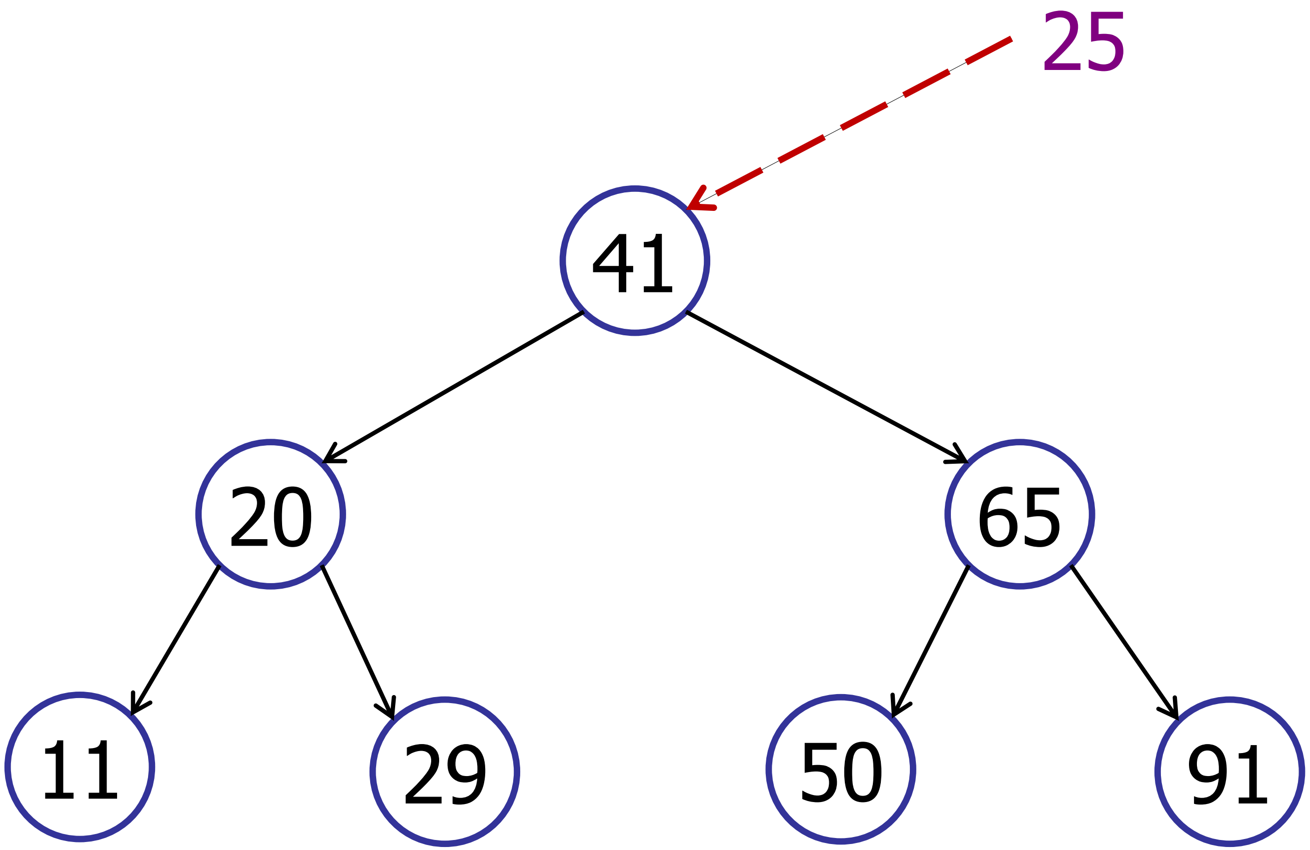
Binary Search Trees

Search for a key:



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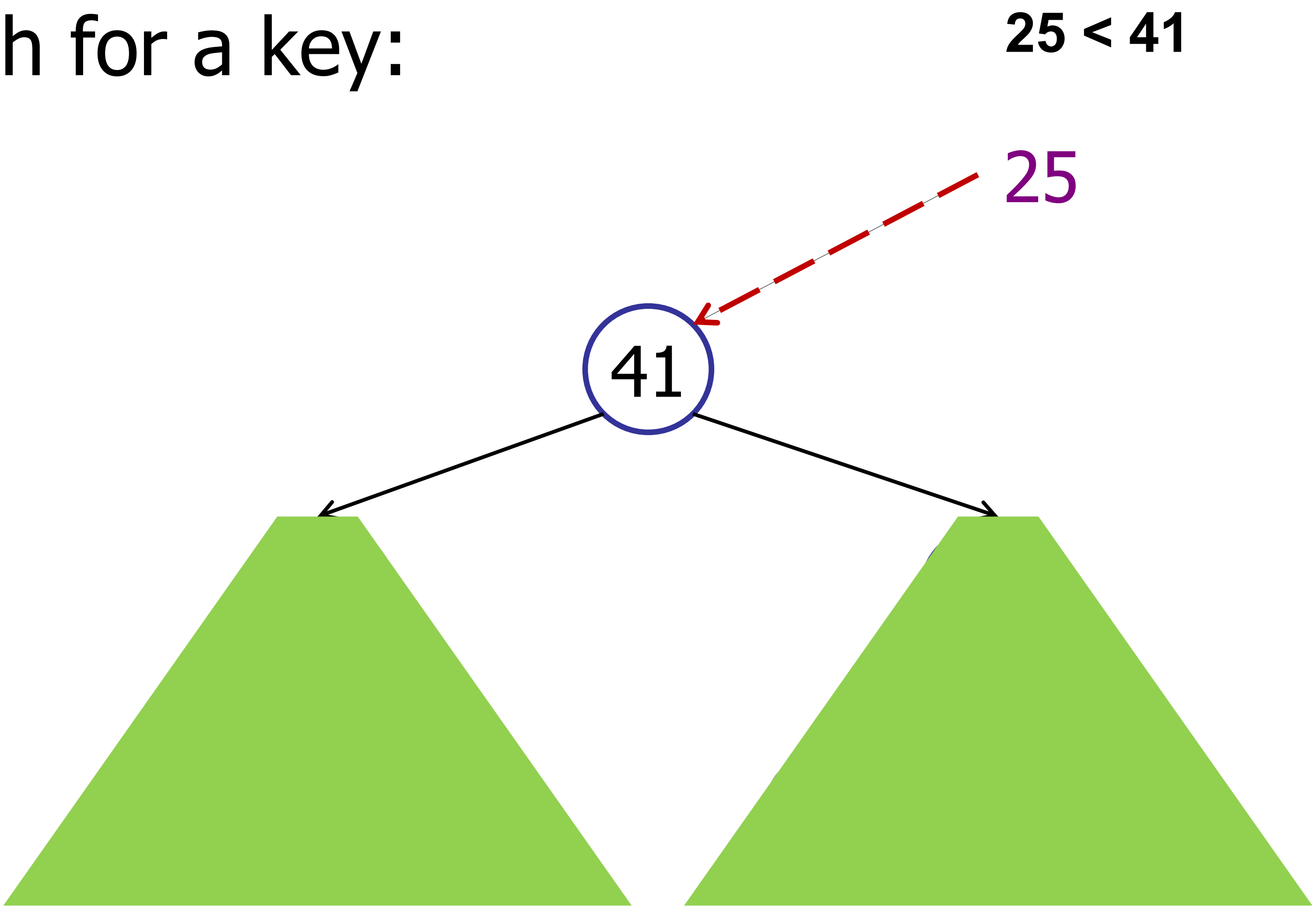
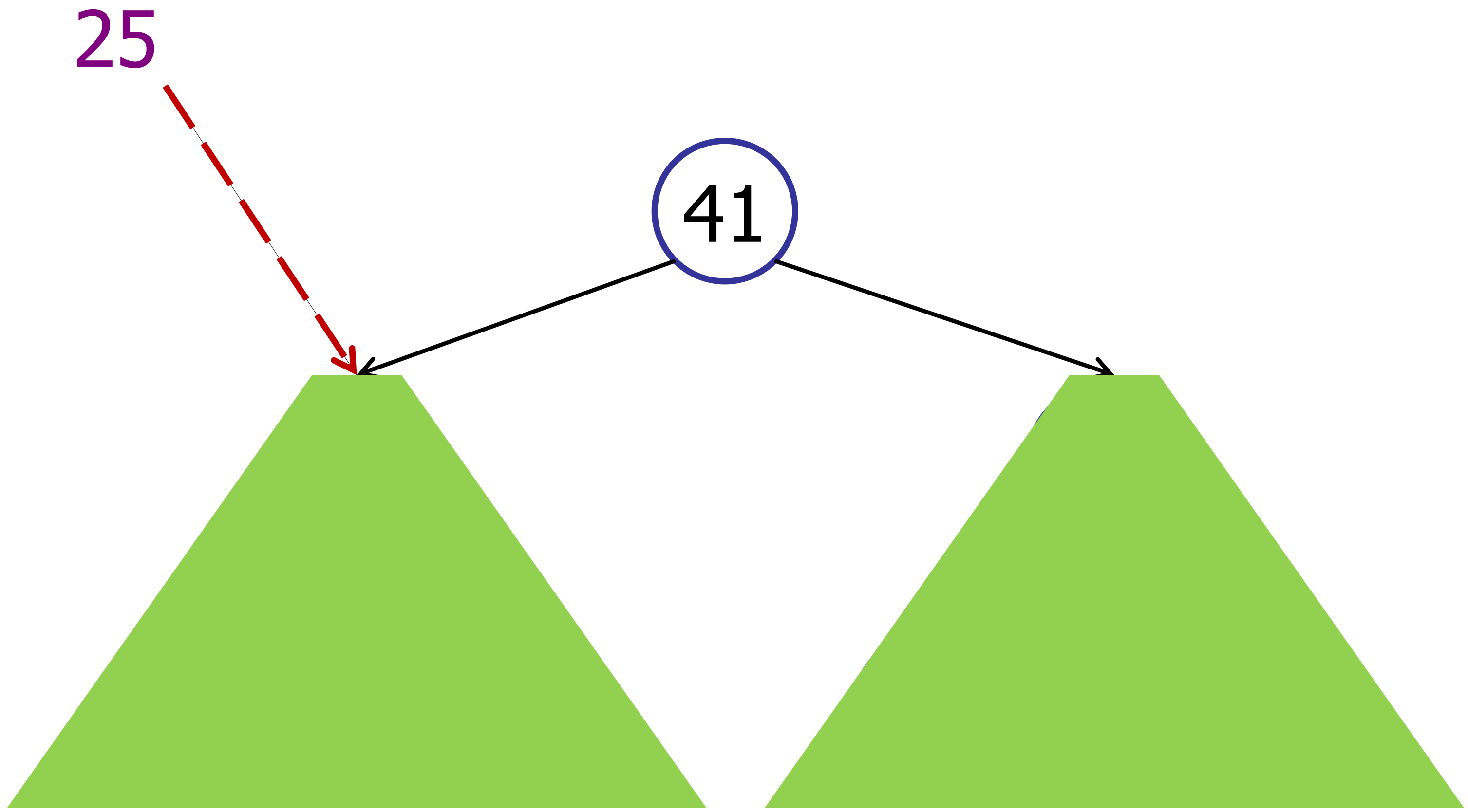


Binary Search Trees

Search for a key:



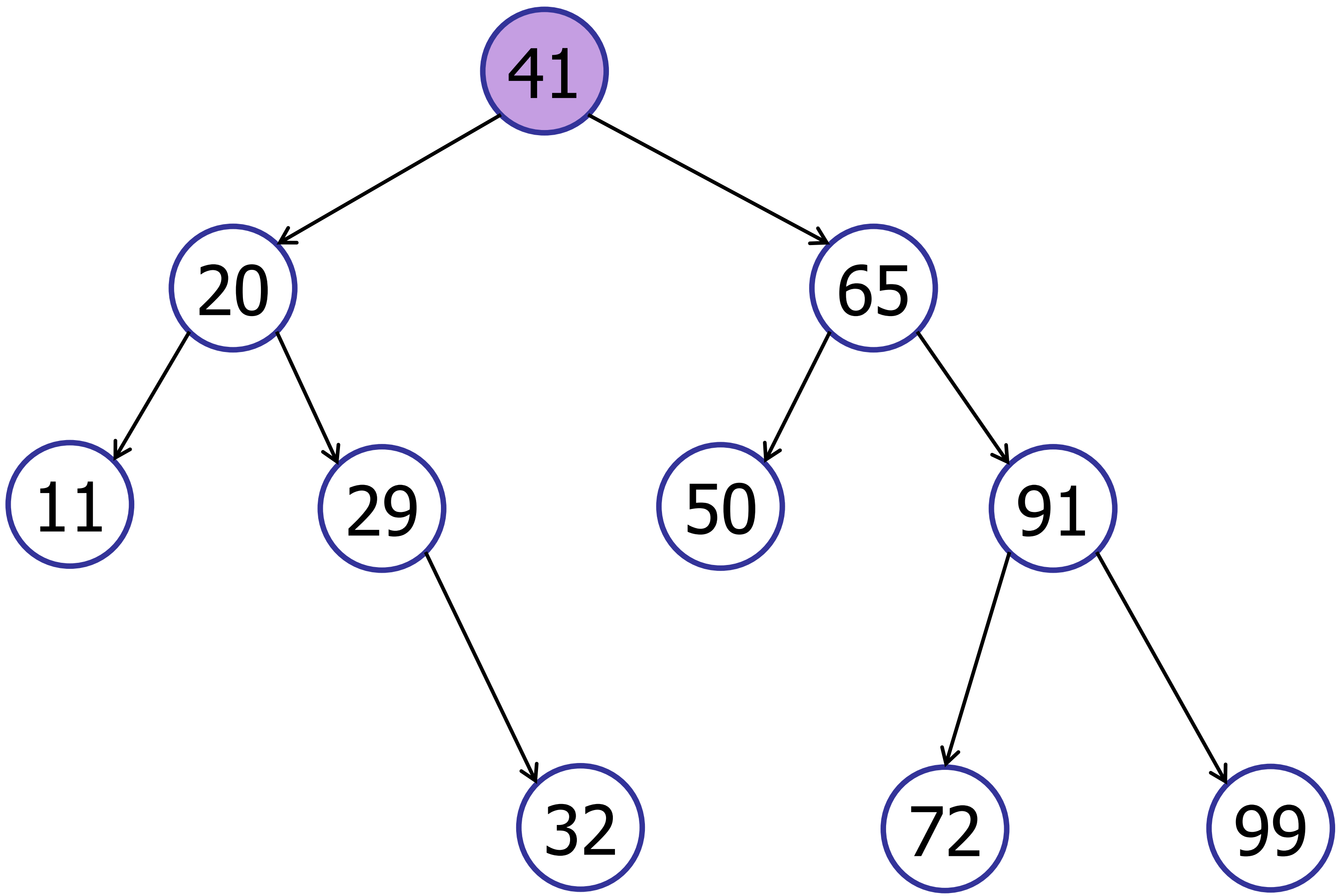
Binary Tree

Inserting a new key

```
public BinaryTree search(KeyType queryKey){
        if (queryKey.compareTo(key) < 0) {
                if (leftTree != null)
                        return leftTree.search(key);
                else return null;
        }
        else if (queryKey.compareTo(key) > 0) {
                if (rightTree != null)
                        return rightTree.search(key);
                else return null;
        }
        else return this; // Key is here!
}
```

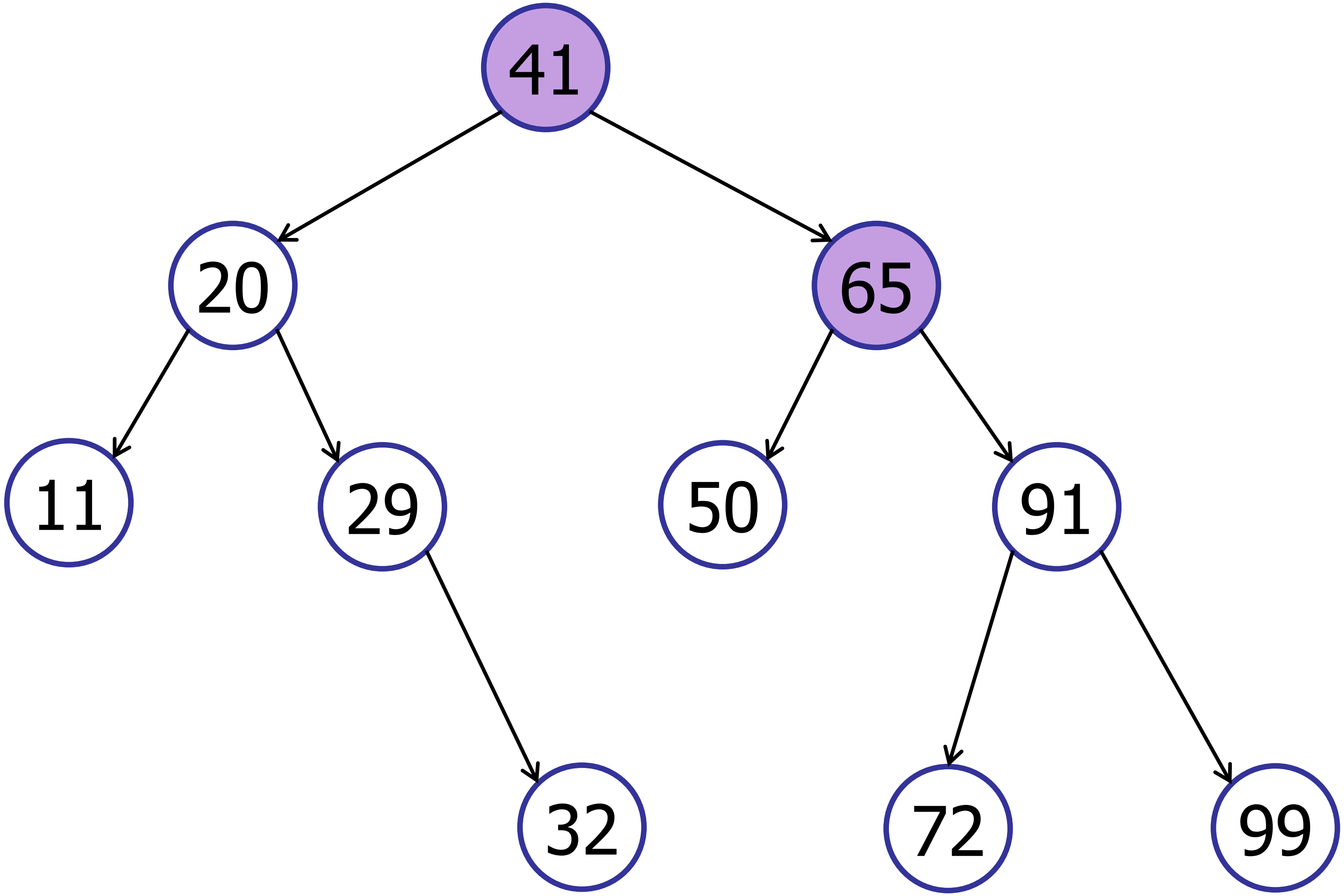
Binary Search Trees

search(72)



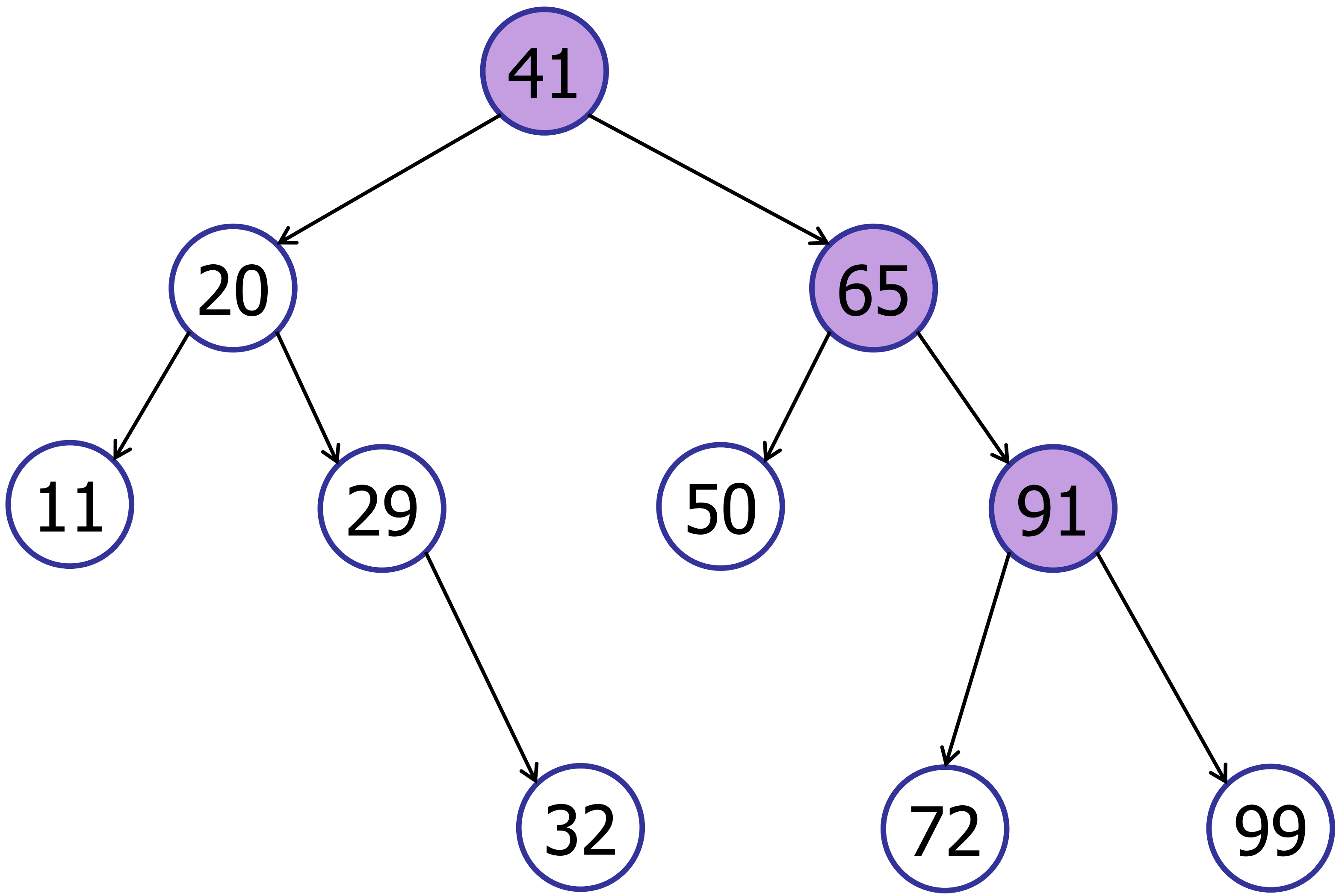
Binary Search Trees

search(72)



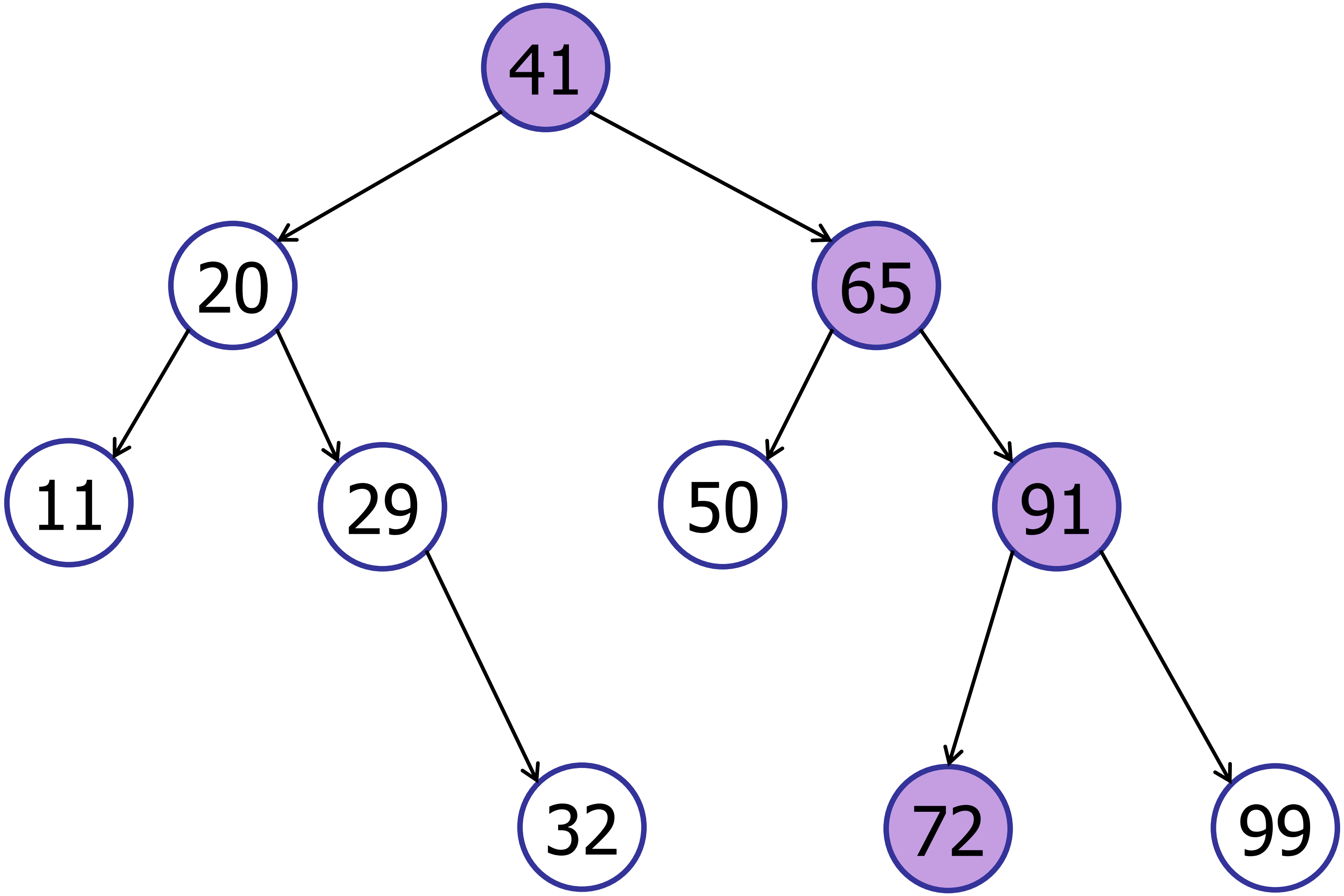
Binary Search Trees

search(72)



Binary Search Trees

search(72)



Binary Search Trees

1. Terminology and Definitions

2. Basic operations:
   - height
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   - search, insert

3. Traversals
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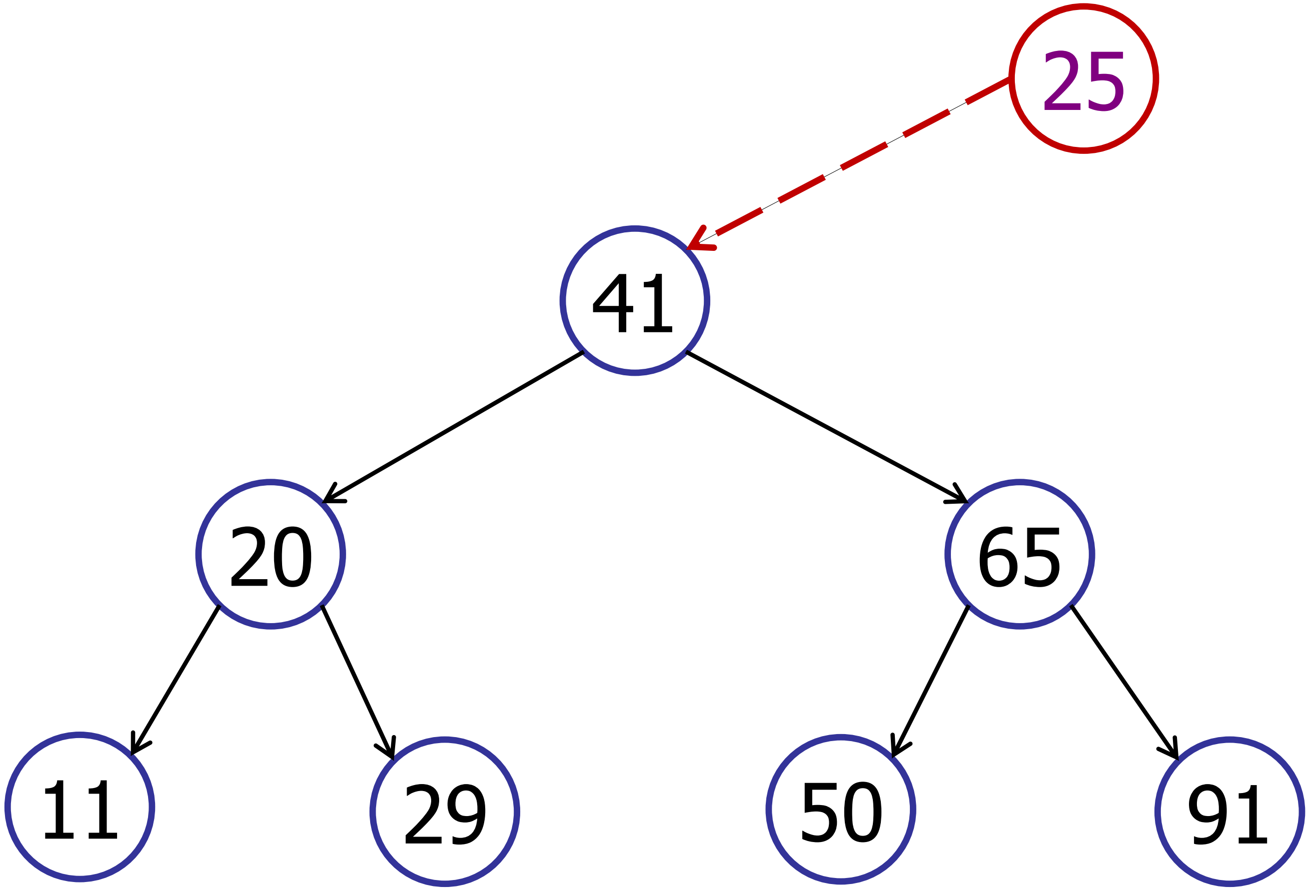
4. Other operations

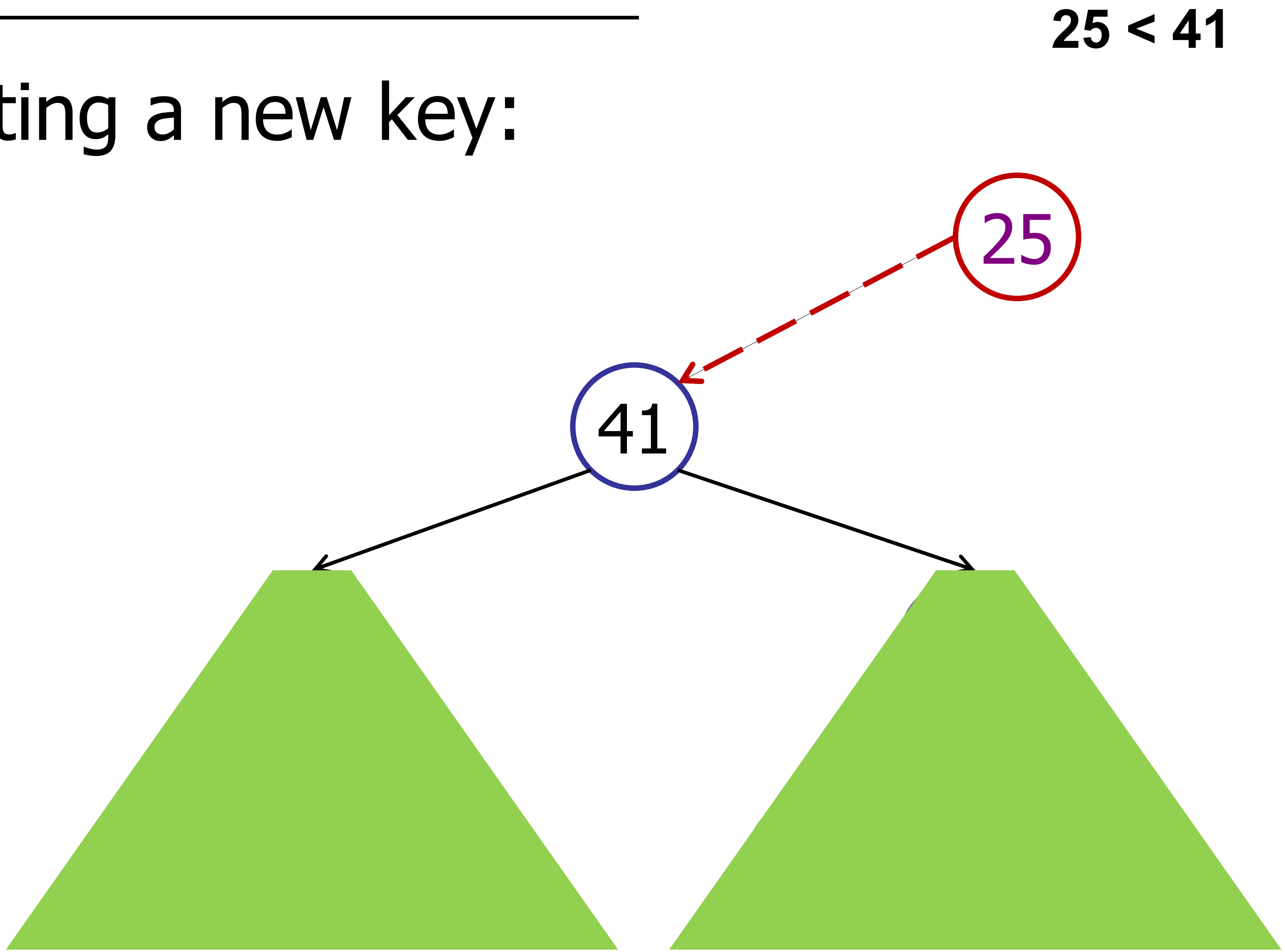
Binary Search Trees

Inserting a new key:



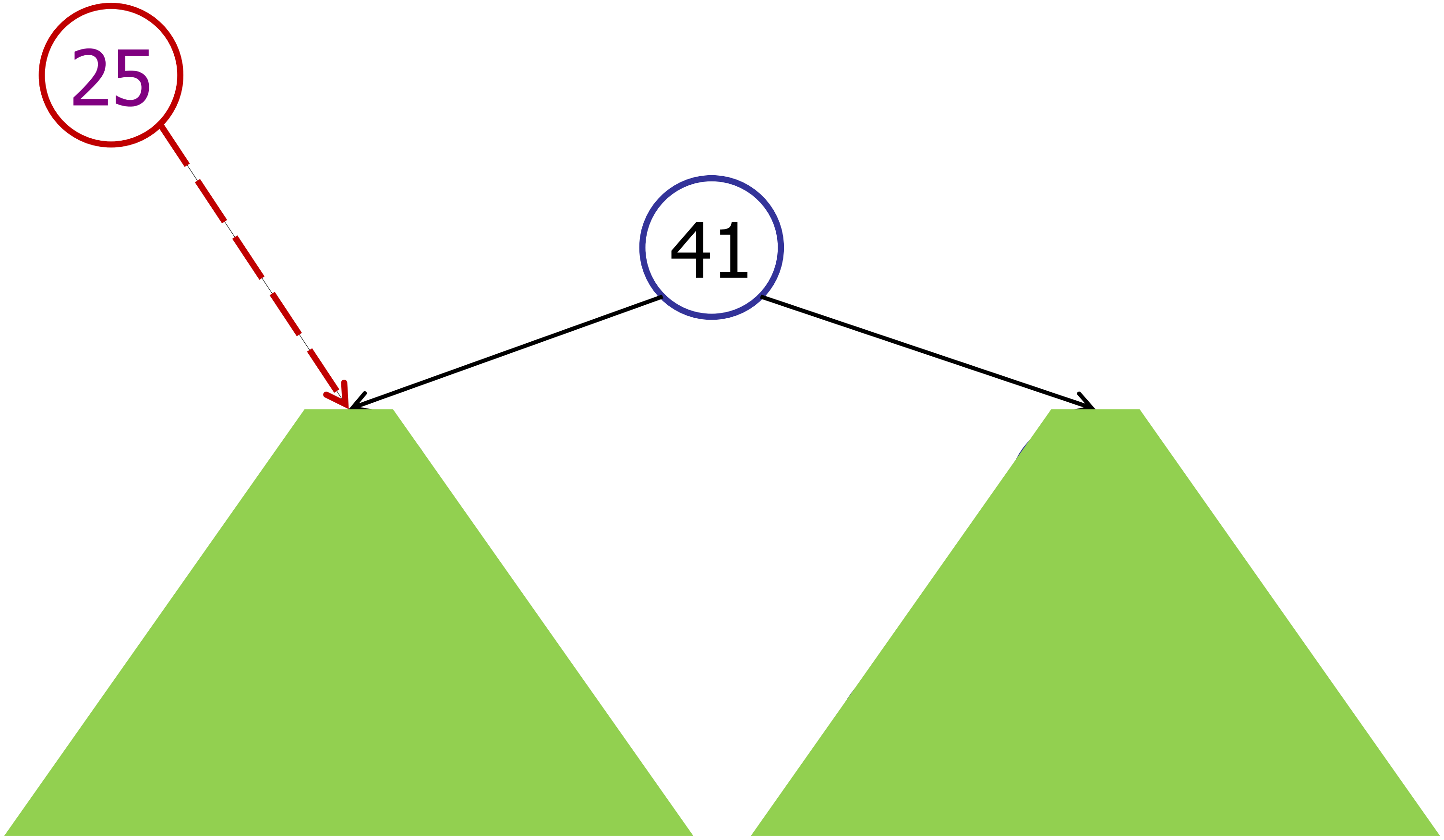
Binary Search Trees

Inserting a new key:



Binary Search Trees

Inserting a new key:



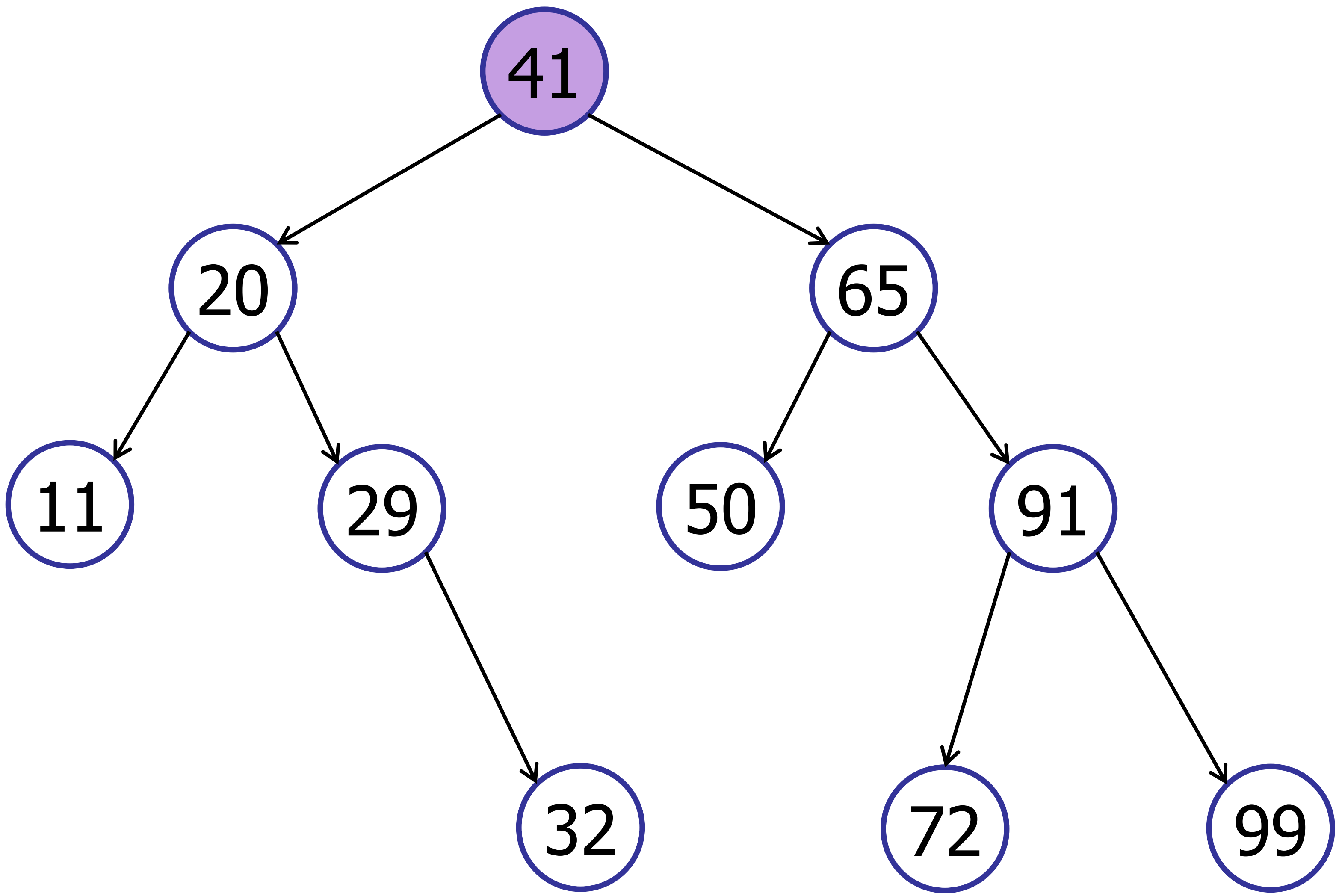
Binary Tree

Inserting a new key

```
public void insert(Key insKey){
        if (insKey.compareTo(key) < 0) {
                if (leftTree != null)
                        leftTree.insert(insKey);
                else leftTree = new BinaryTree(insKey);
        }
        else if (insKey.compareTo(key) > 0) {
                if (rightTree != null)
                        rightTree.insert(insKey);
                else rightTree = new BinaryTree(insKey);
        }
        else return; // Key is already in the tree!
}
```

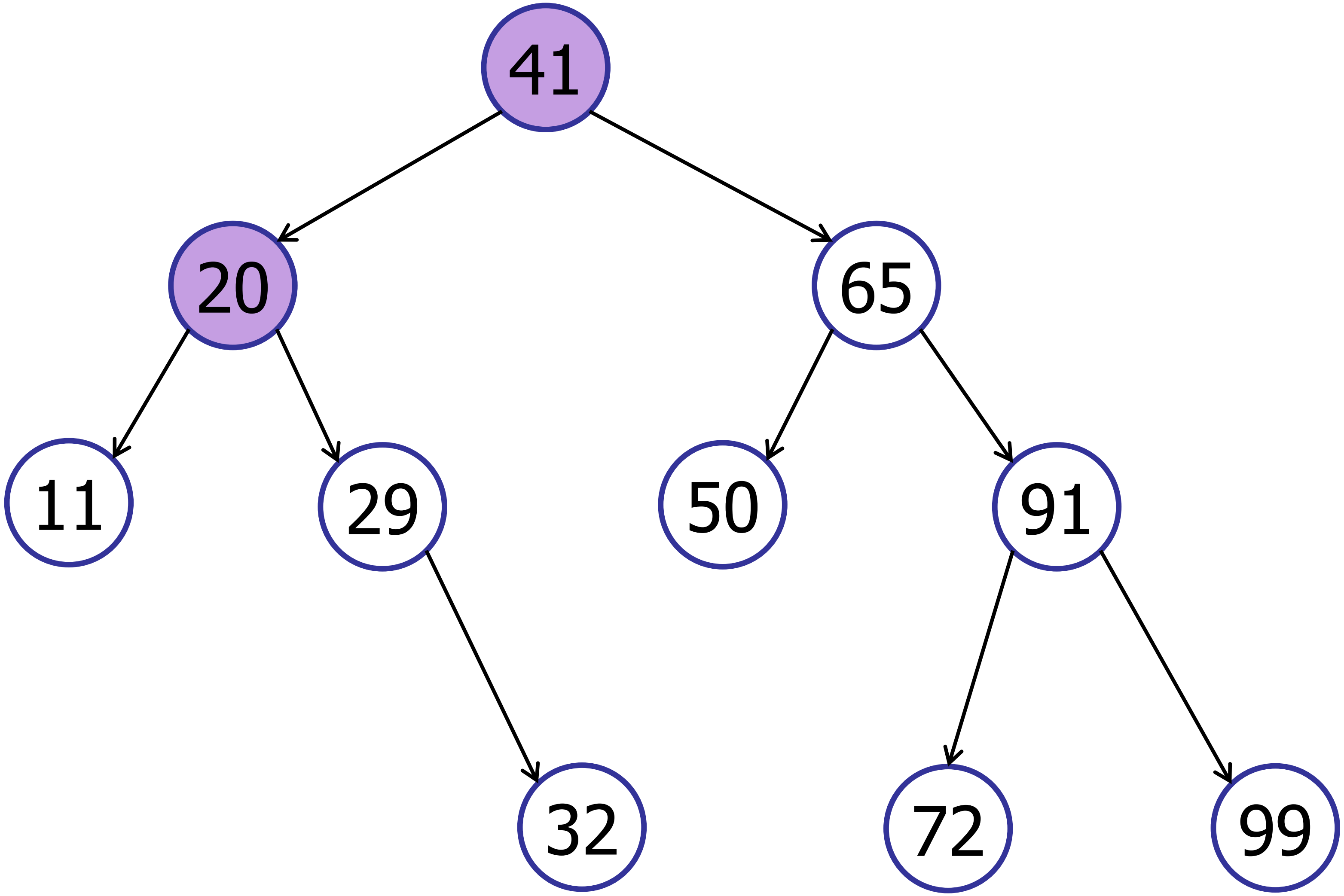
Binary Search Trees

insert(27)



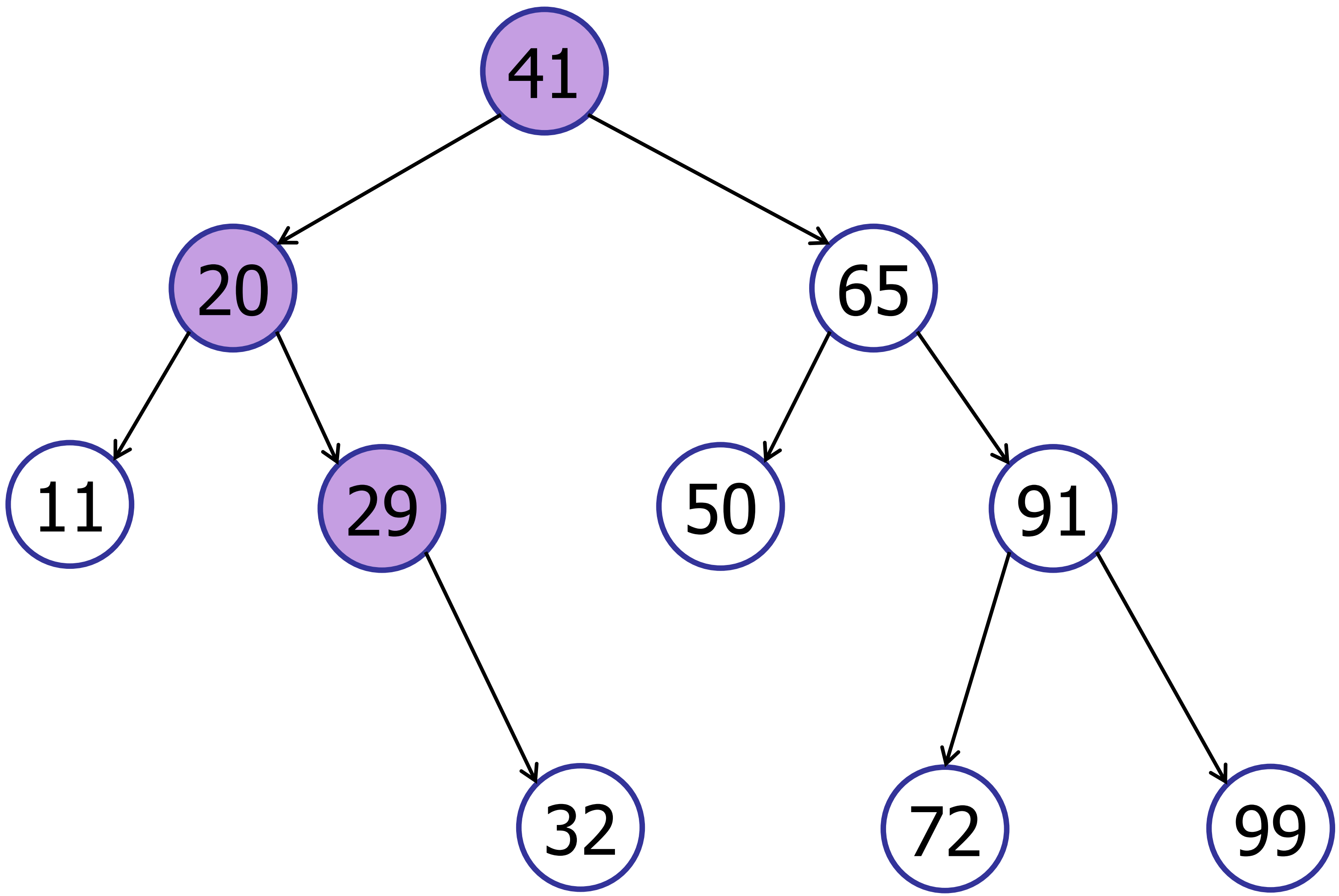
Binary Search Trees

insert(27)



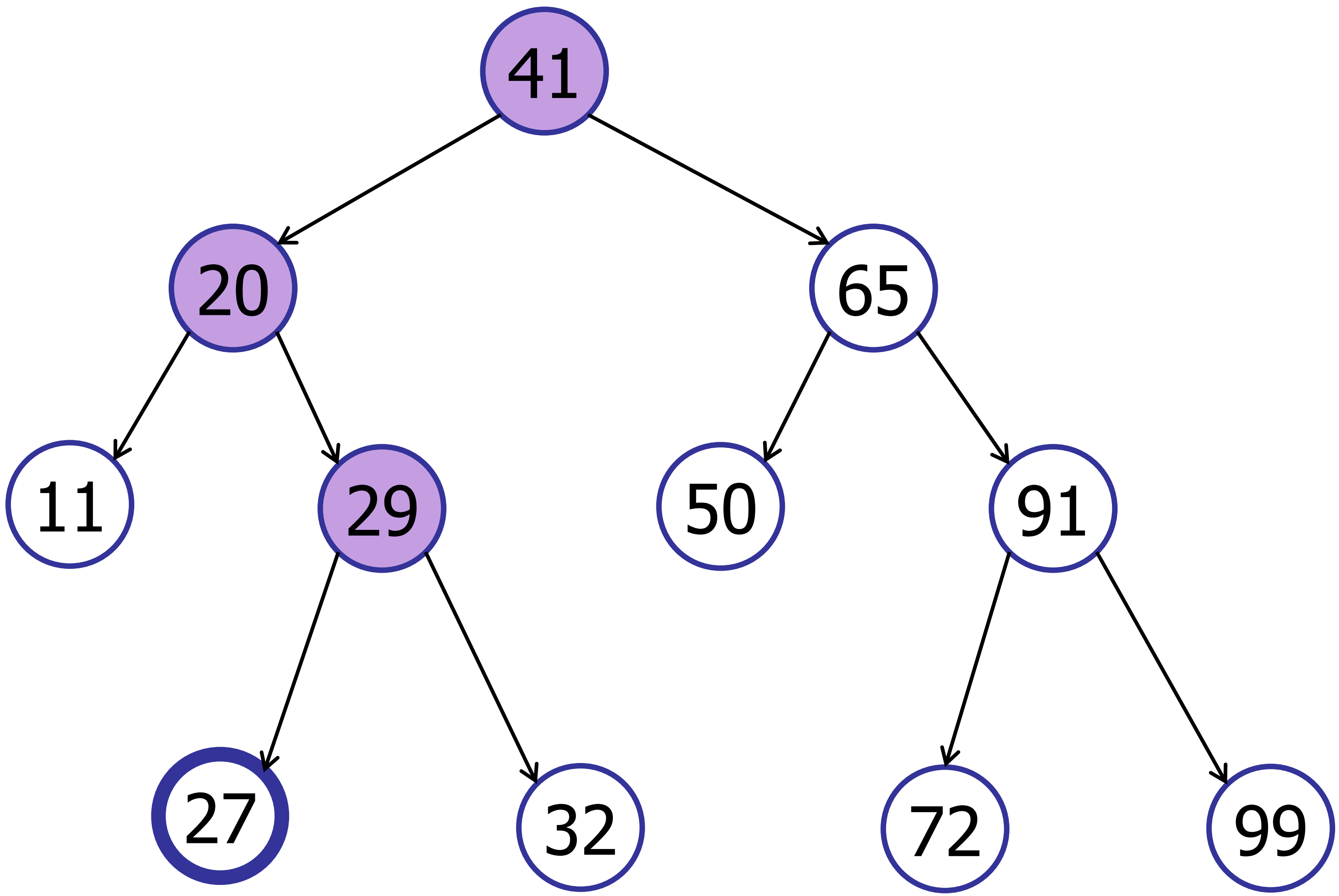
Binary Search Trees

insert(27)



Binary Search Trees

insert(27)



Binary Search Tree

What is the worst-case running time of search in a BST?

1. $O(1)$
2. $O(\log n)$
3. $O(n)$
4. $O(n^2)$
5. $O(n^3)$
6. $O(2^n)$

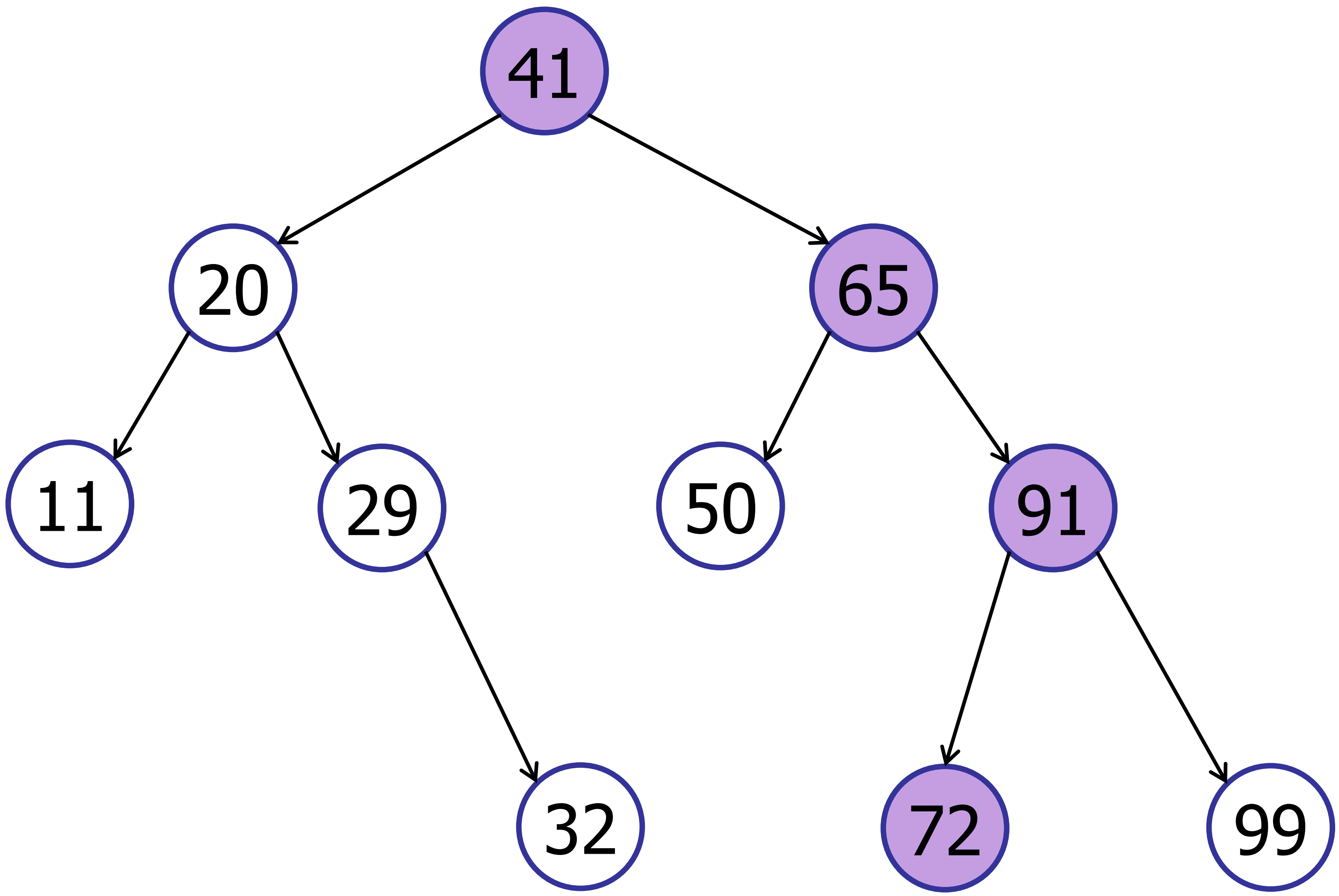
Binary Search Tree

What is the worst-case running time of **search** in a BST?

1. $O(1)$
2. $O(\log n)$
3. $O(n)$ ✓
4. $O(n^2)$
5. $O(n^3)$
6. $O(2^n)$

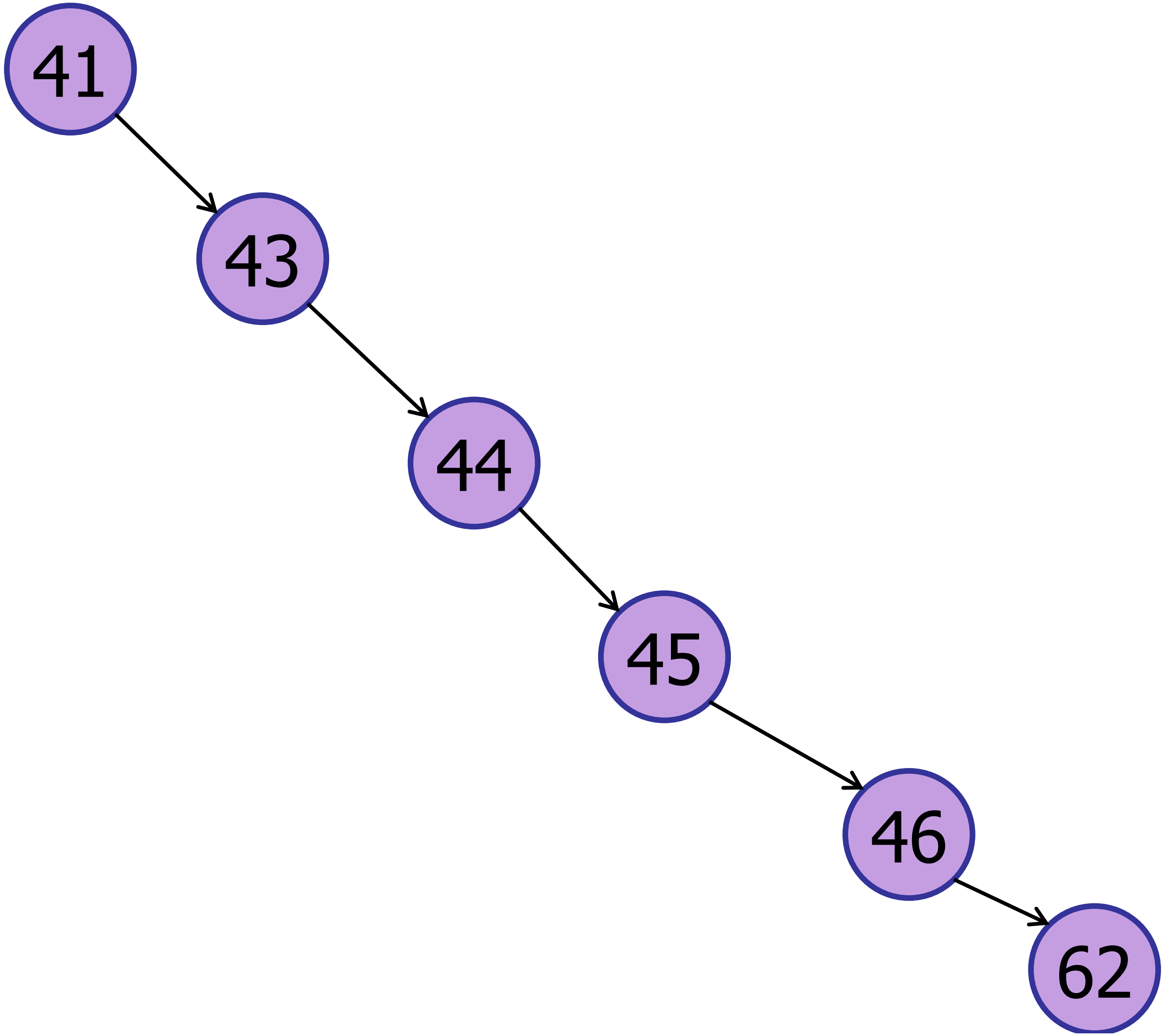
Binary Search Trees

search(72) : O(h)



Binary Search Trees

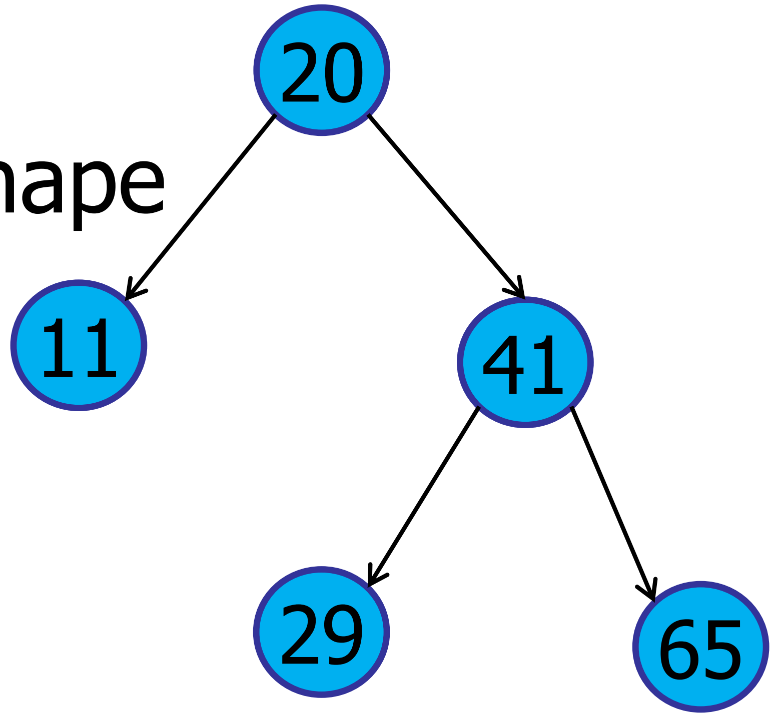
search(72) : O(h)

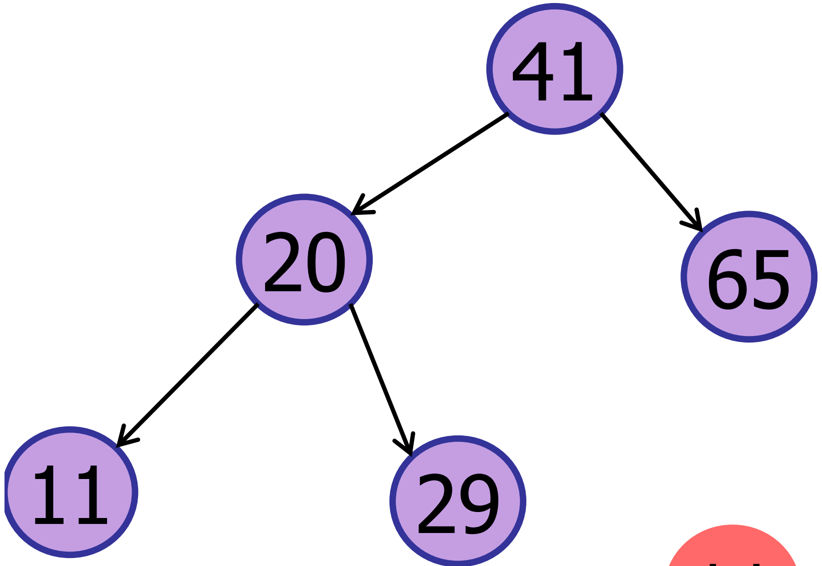


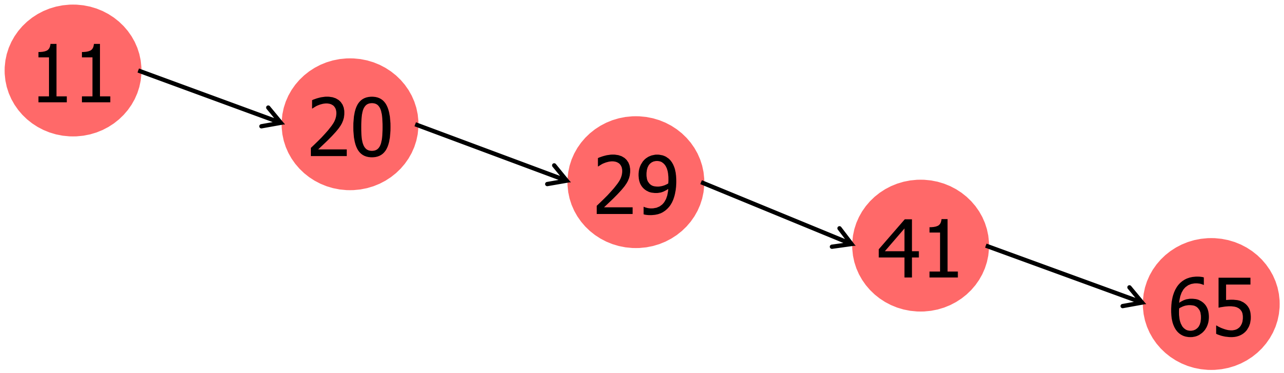
Tree Shape

Trees come in many shapes

- same keys ≠ same shape
- performance depends on shape



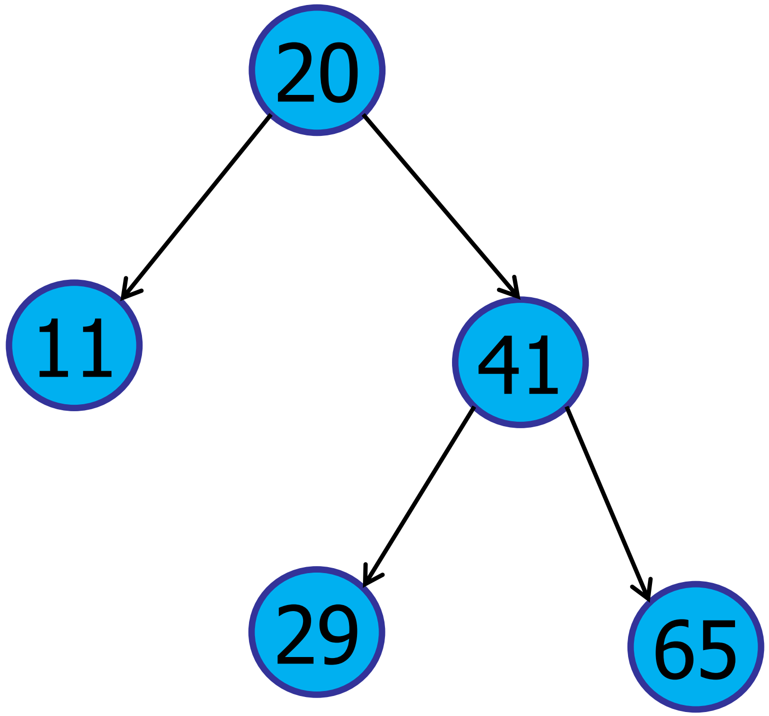


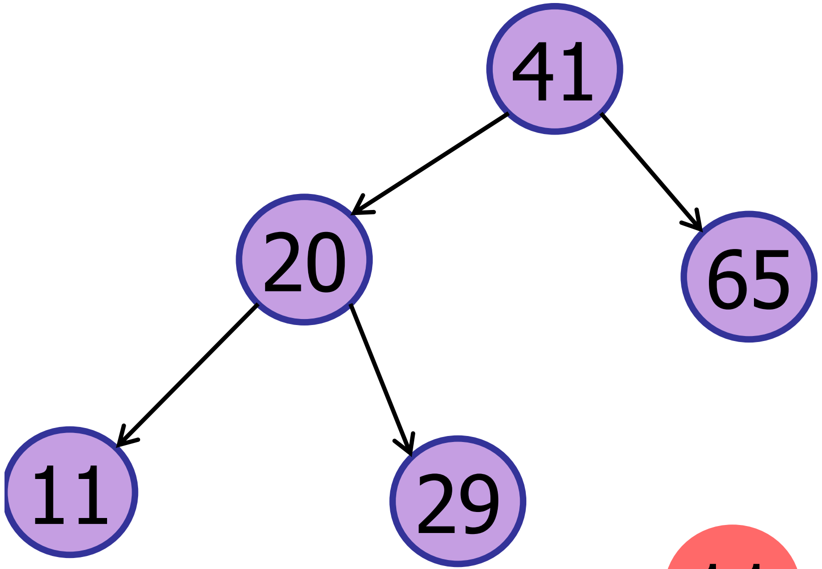


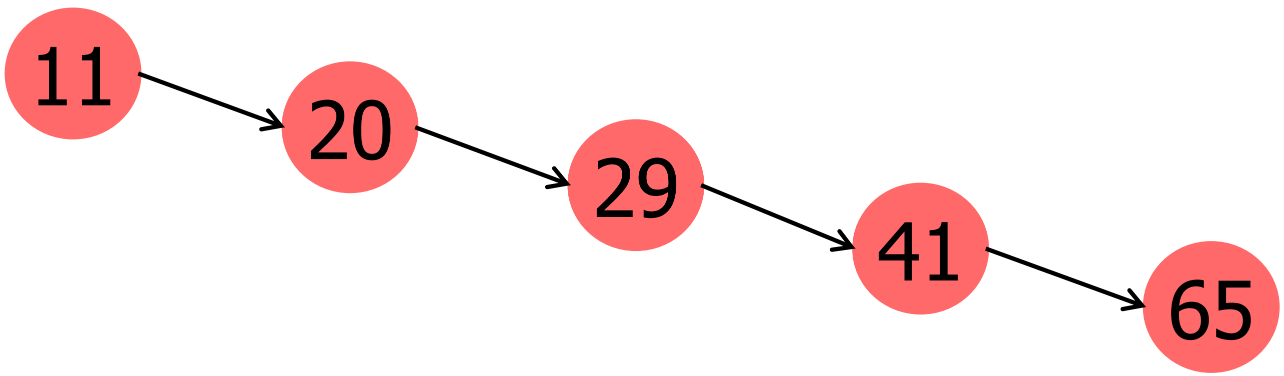
Tree Shape

What determines shape?

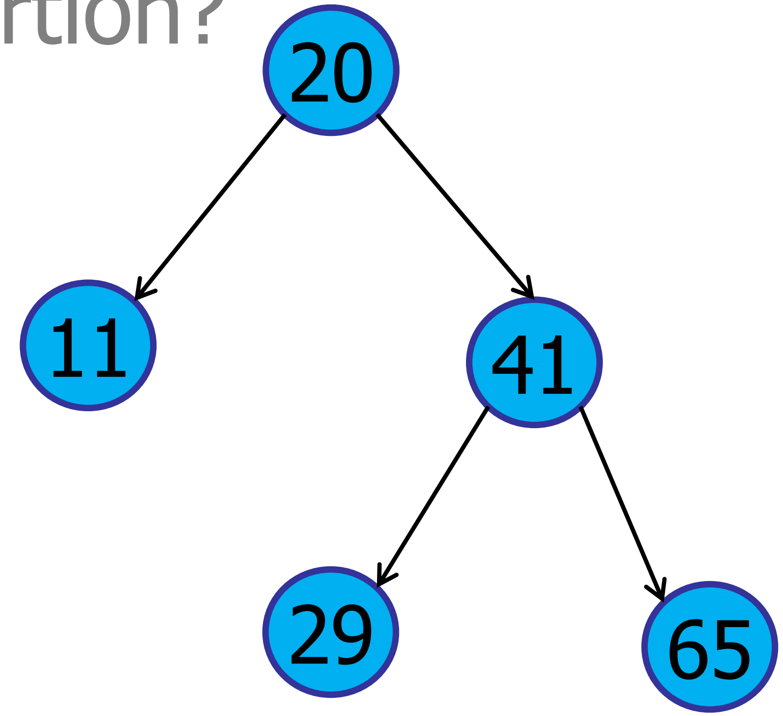
- Order of insertion





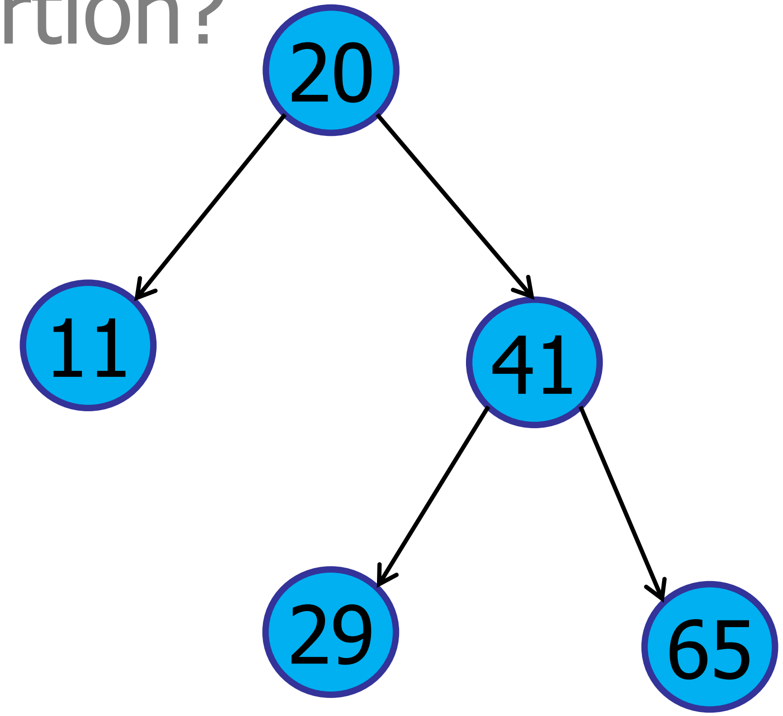


What was the order of insertion?



1. 11, 20, 29, 41, 65
2. 20, 11, 41, 29, 65
3. 11, 20, 41, 29, 65
4. 65, 41, 29, 20, 11
5. Impossible to tell.

What was the order of insertion?

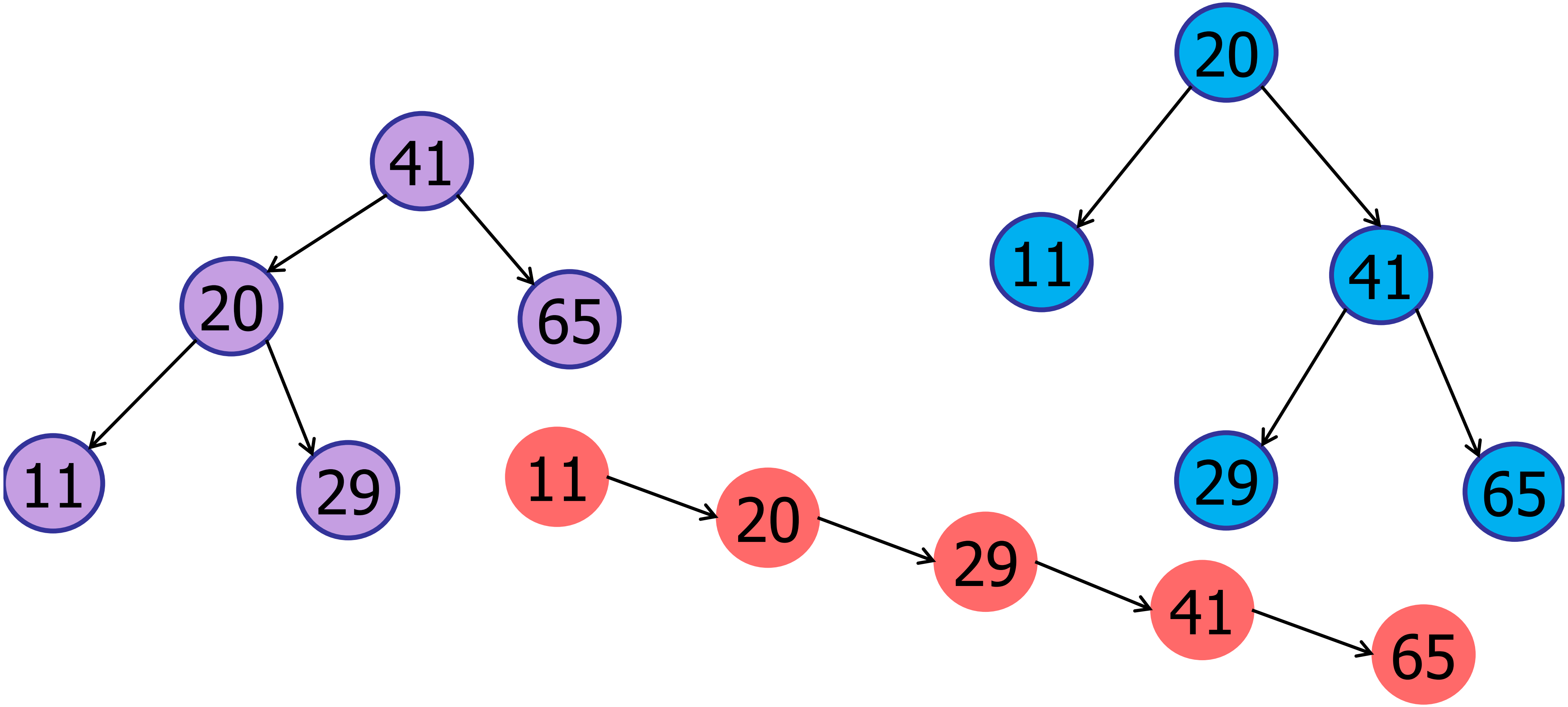


1. 11, 20, 29, 41, 65
2. 20, 11, 41, 29, 65
3. 11, 20, 41, 29, 65
4. 65, 41, 29, 20, 11
5. Impossible to tell.

Tree Shape

What determines shape?

- Order of insertion
- Does each order yield a unique shape?



Tree Shape

What determines shape?

- Order of insertion
- Does each order yield a unique shape? NO
  - # ways to order insertions: n!
  - # shapes of a binary tree? $\sim 4^n$

Catalan Numbers

Tree Shape

Catalan Numbers

C_n = # of trees with (n+1) leaves

C_n = # expressions with n pairs of matched parentheses

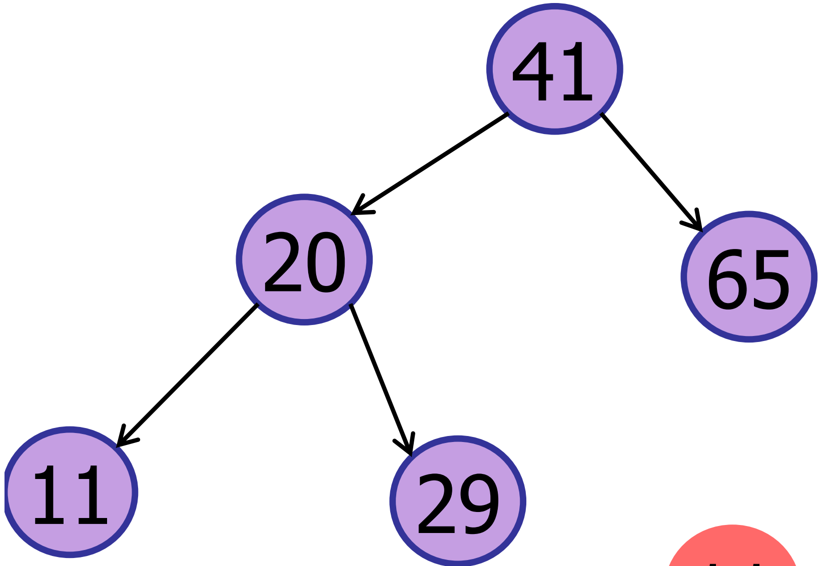
((()))   ()(())   (()())   (())()   ()()()

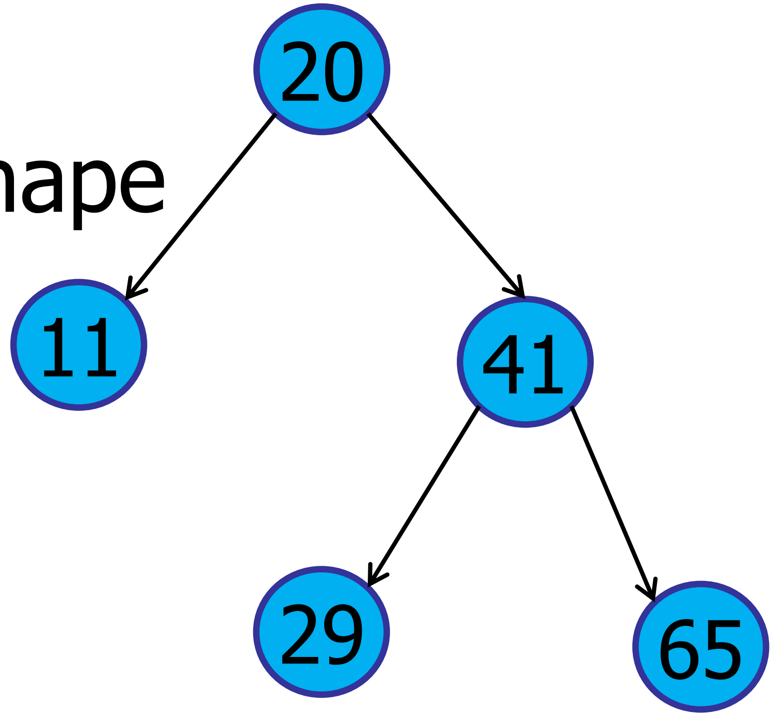
Why are these the same?

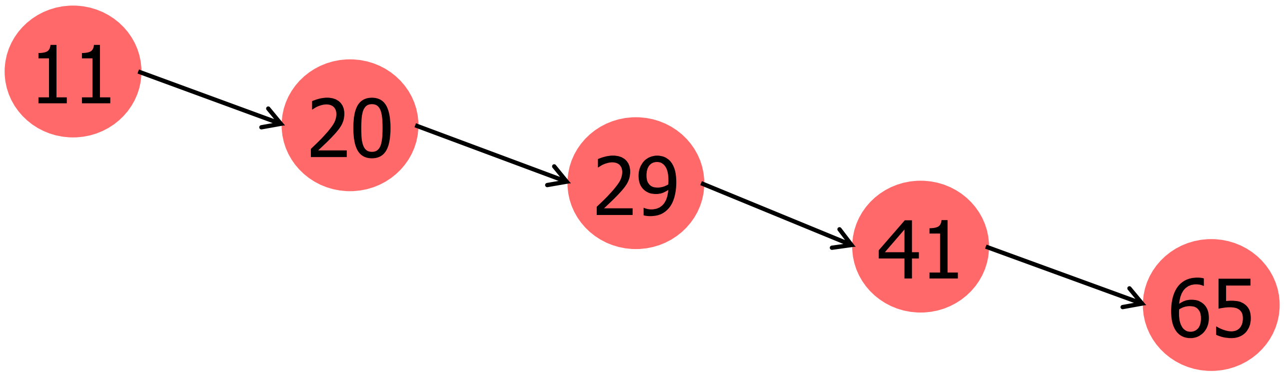
Tree Shape

Trees come in many shapes

- same keys ≠ same shape
- performance depends on shape



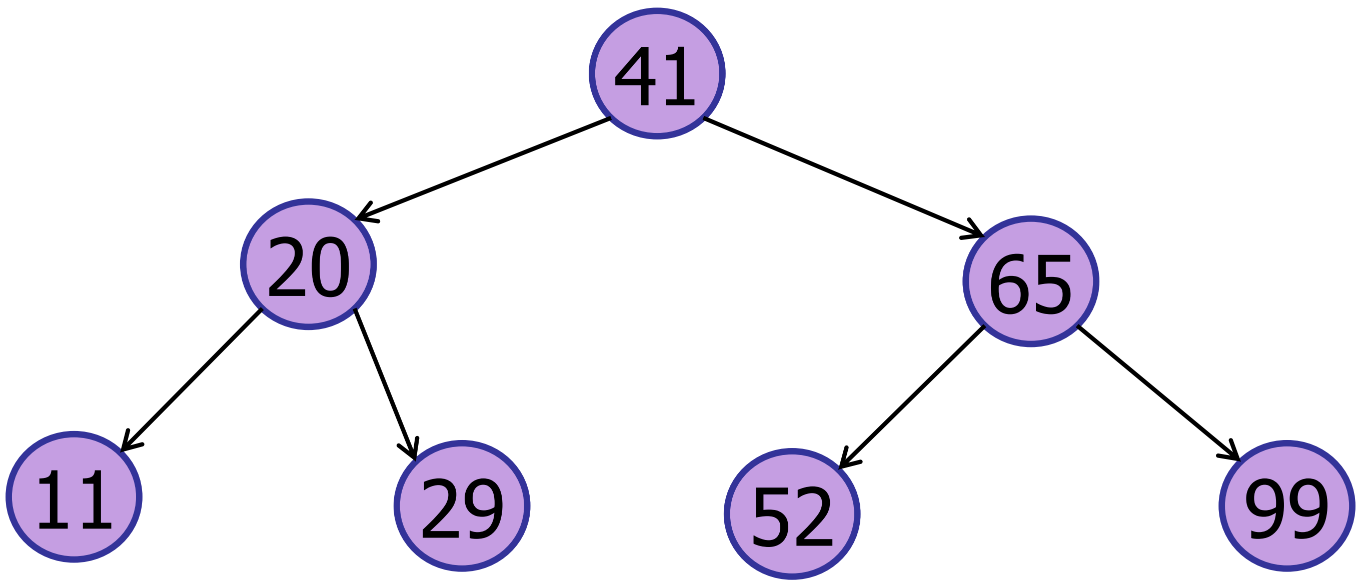




Tree Shape

Trees come in many shapes

- same keys $\neq$ same shape
- performance depends on shape
- insert keys in a *random* order $\Rightarrow$ balanced



Binary Search Trees

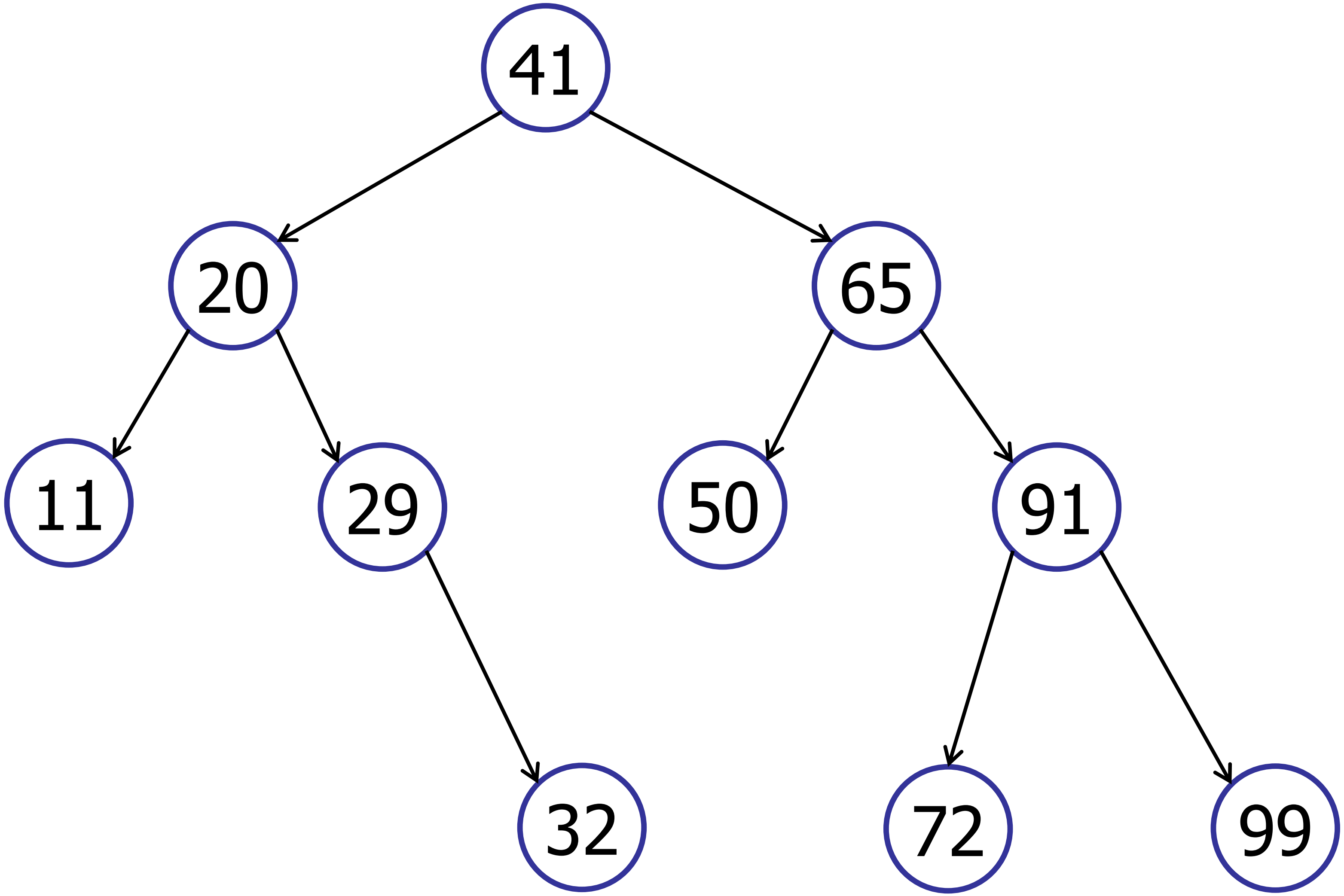
1. Terminology and Definitions

2. Basic operations:
   - height
   - searchMin, searchMax
   - search, insert

3. Traversals
   - in-order, pre-order, post-order

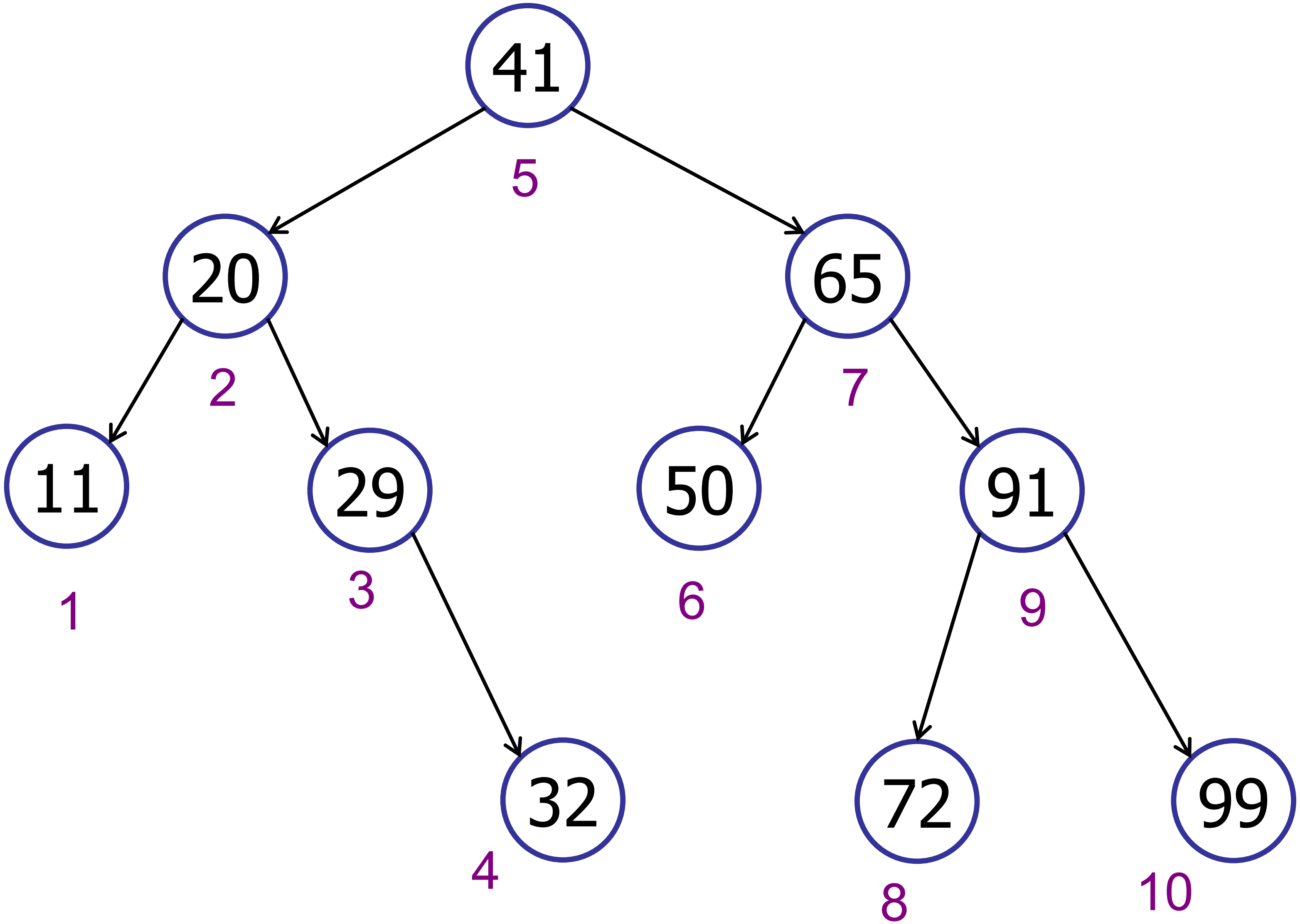
4. Other operations

Tree Traversal



11 20 29 32 41 50 65 72 91 99

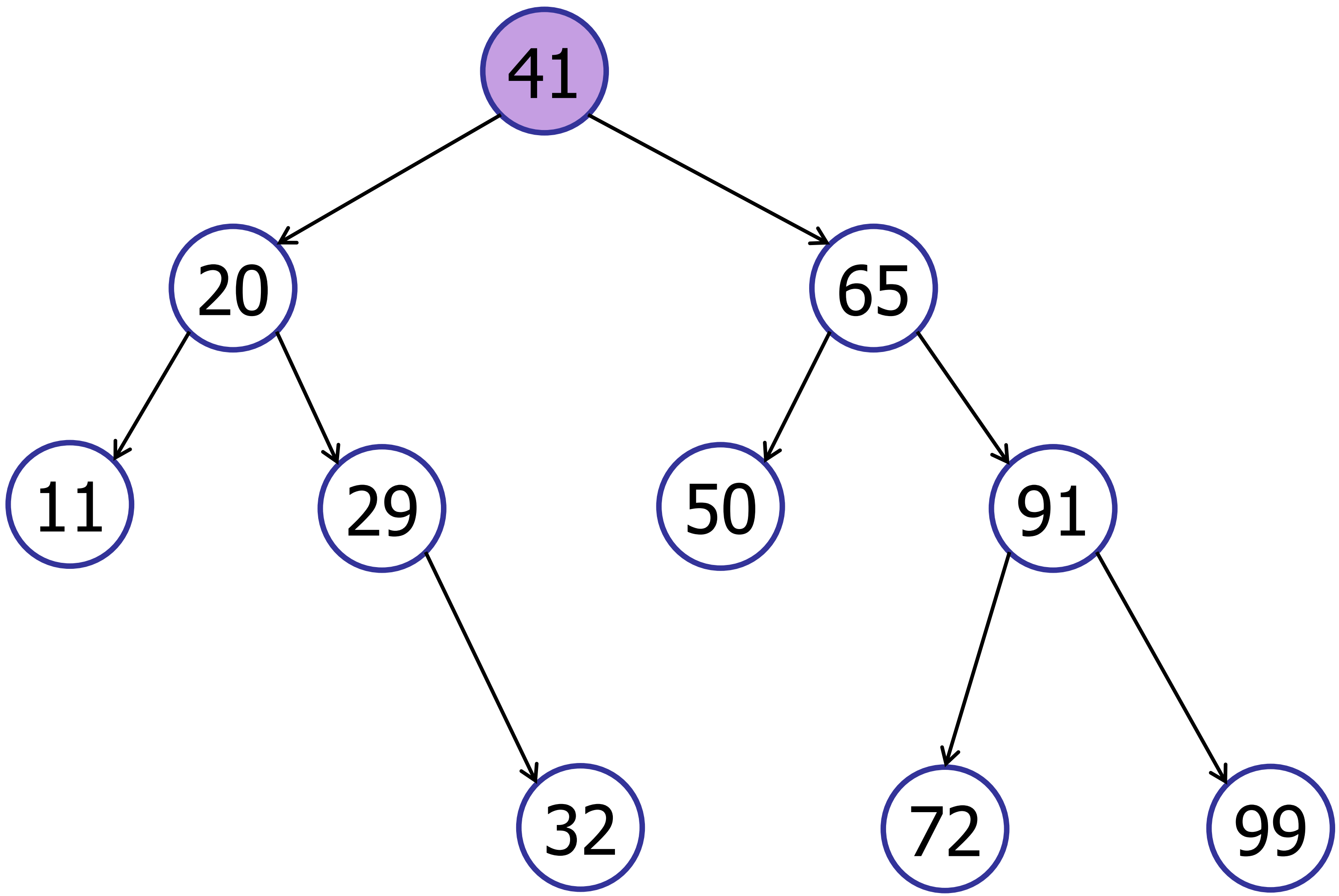
Tree Traversal



11 20 29 32 41 50 65 72 91 99

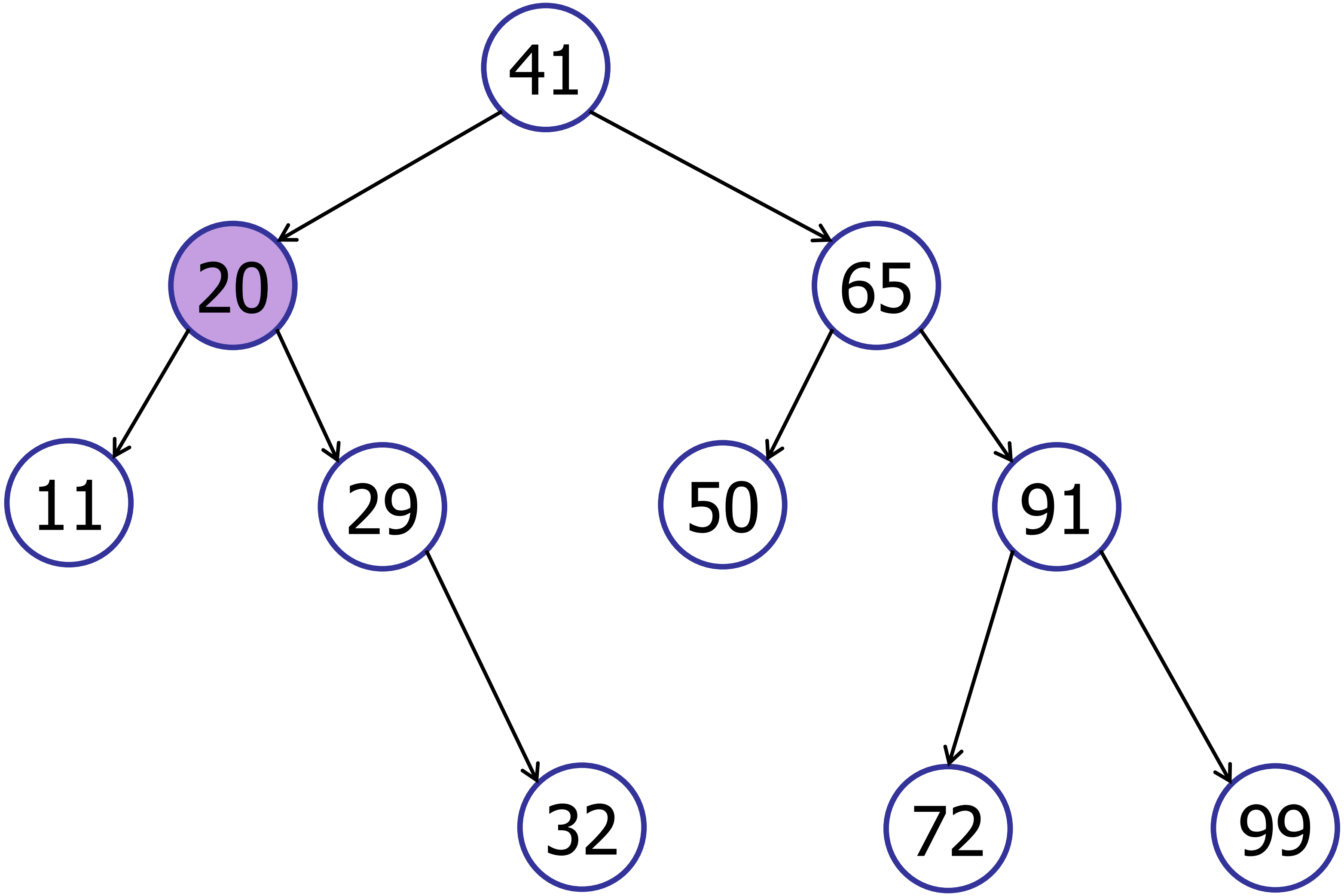
Tree Traversal

in-order-traversal



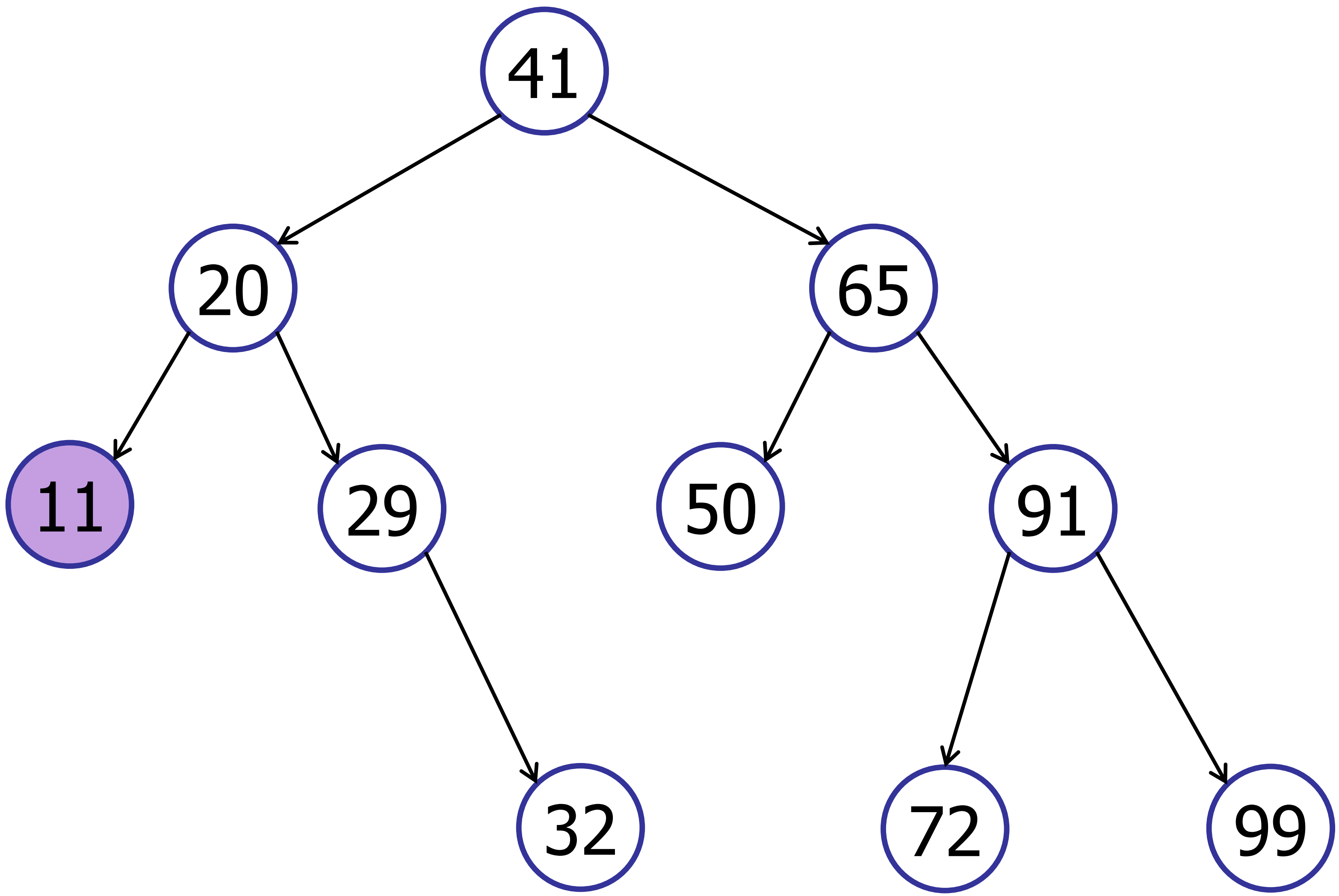
Tree Traversal

in-order-traversal



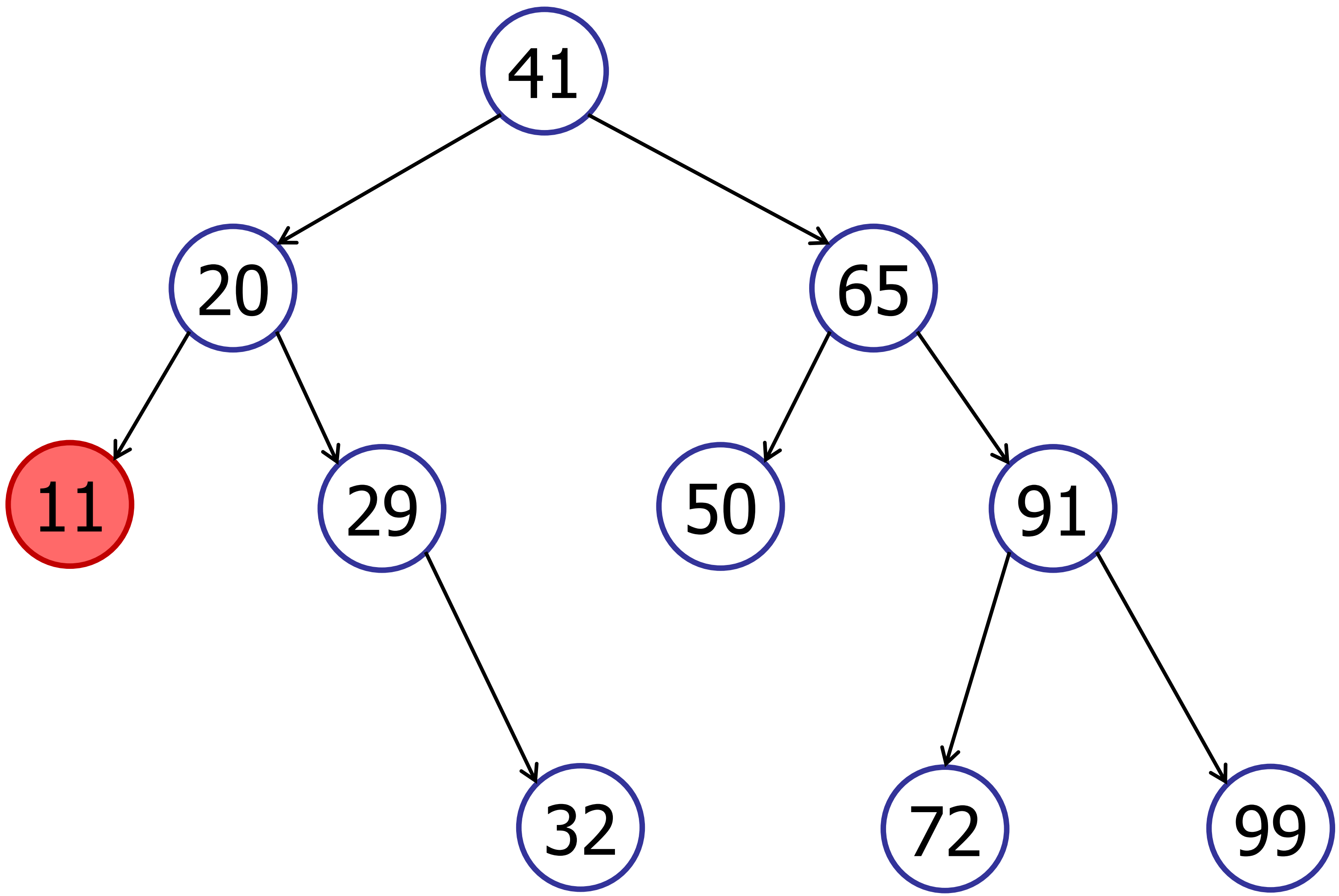
Tree Traversal

in-order-traversal



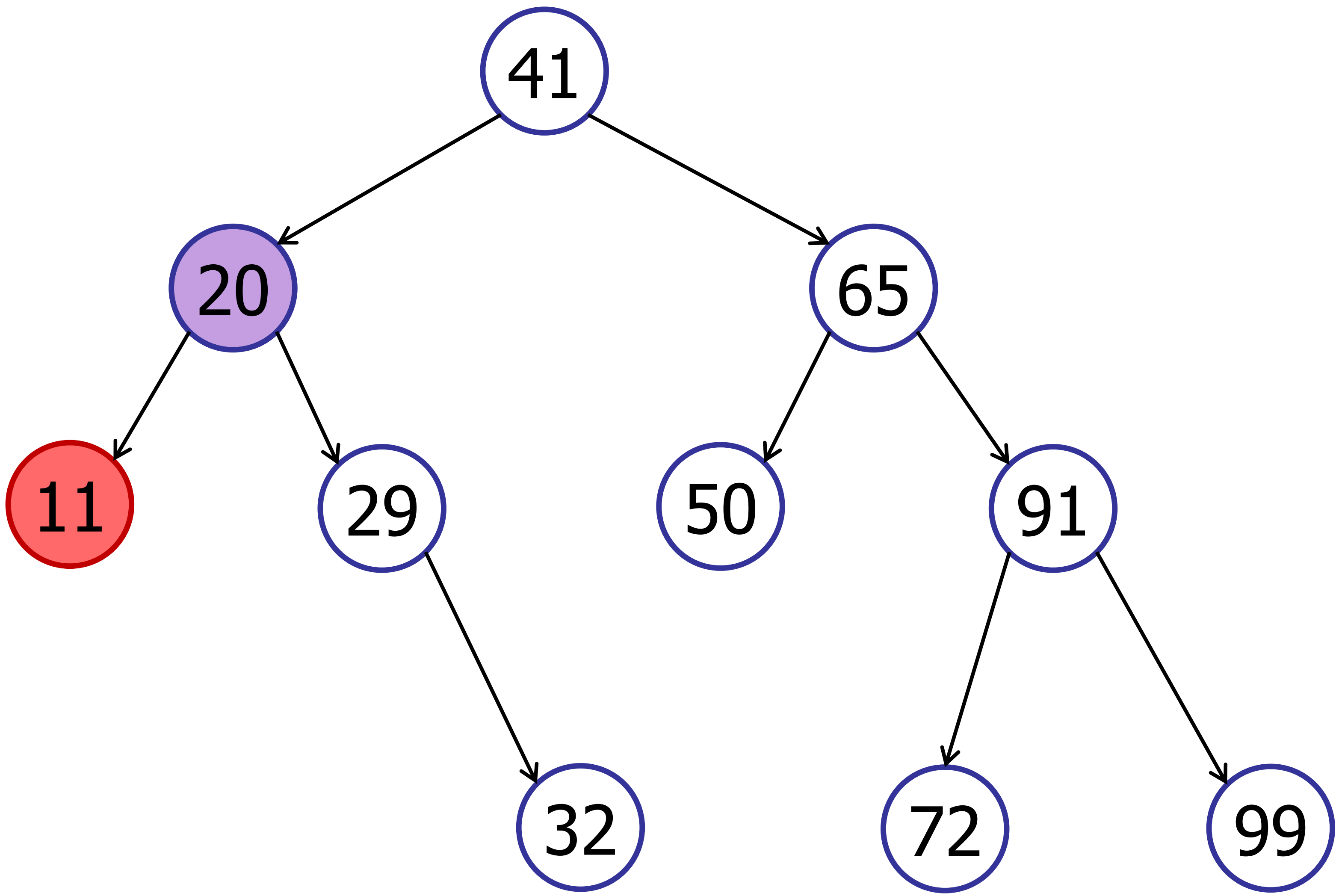
Tree Traversal

in-order-traversal



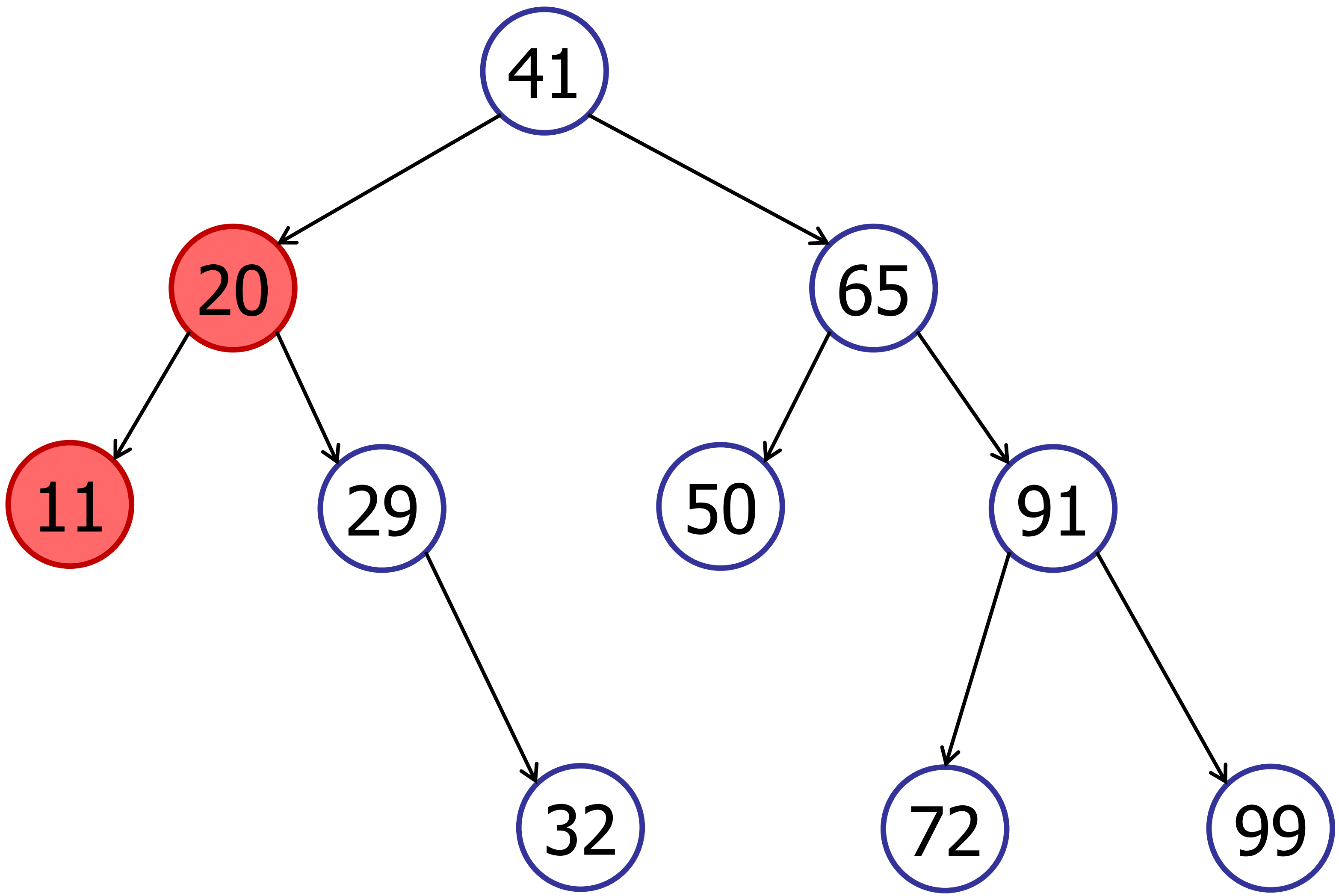
Tree Traversal

in-order-traversal



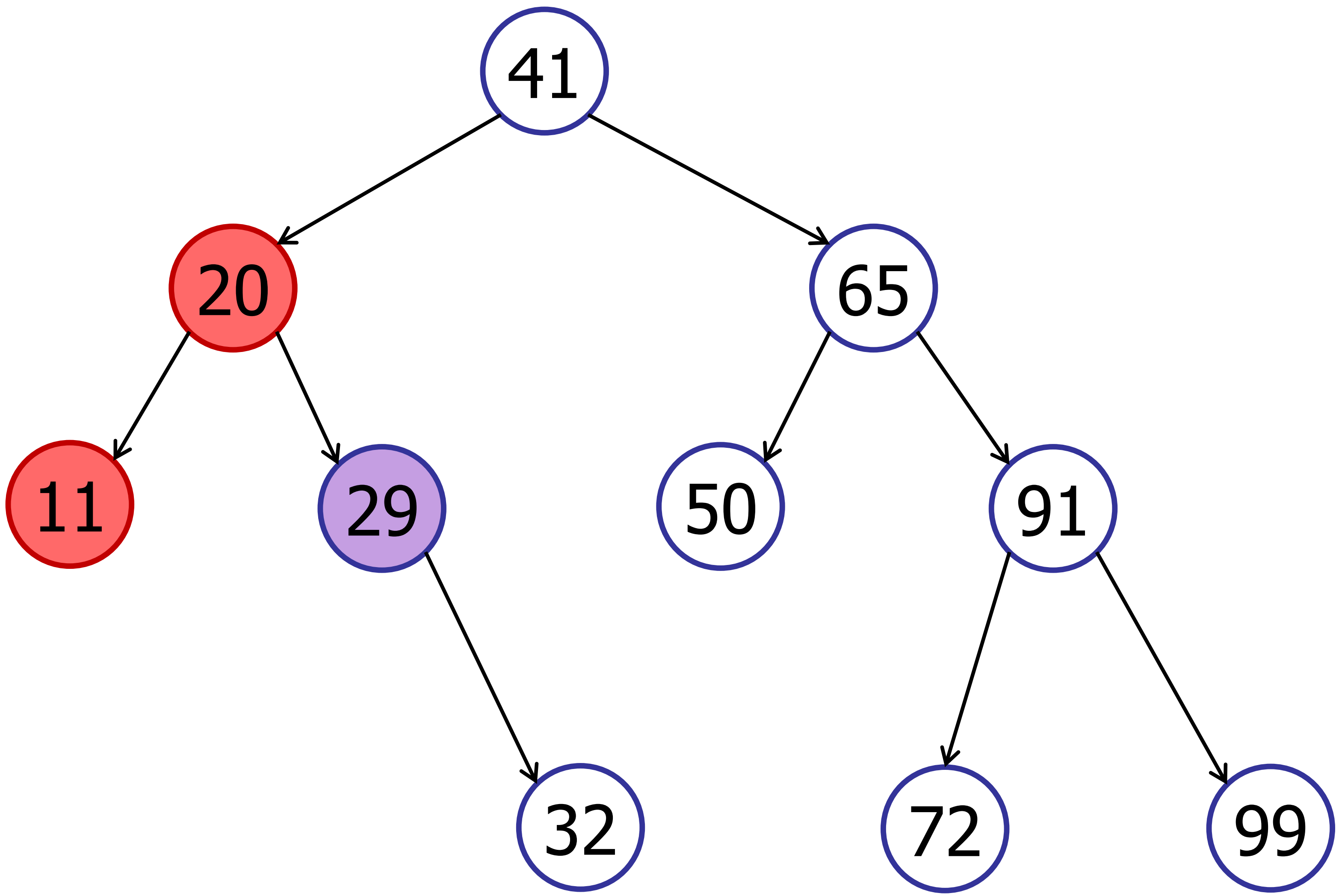
Tree Traversal

in-order-traversal



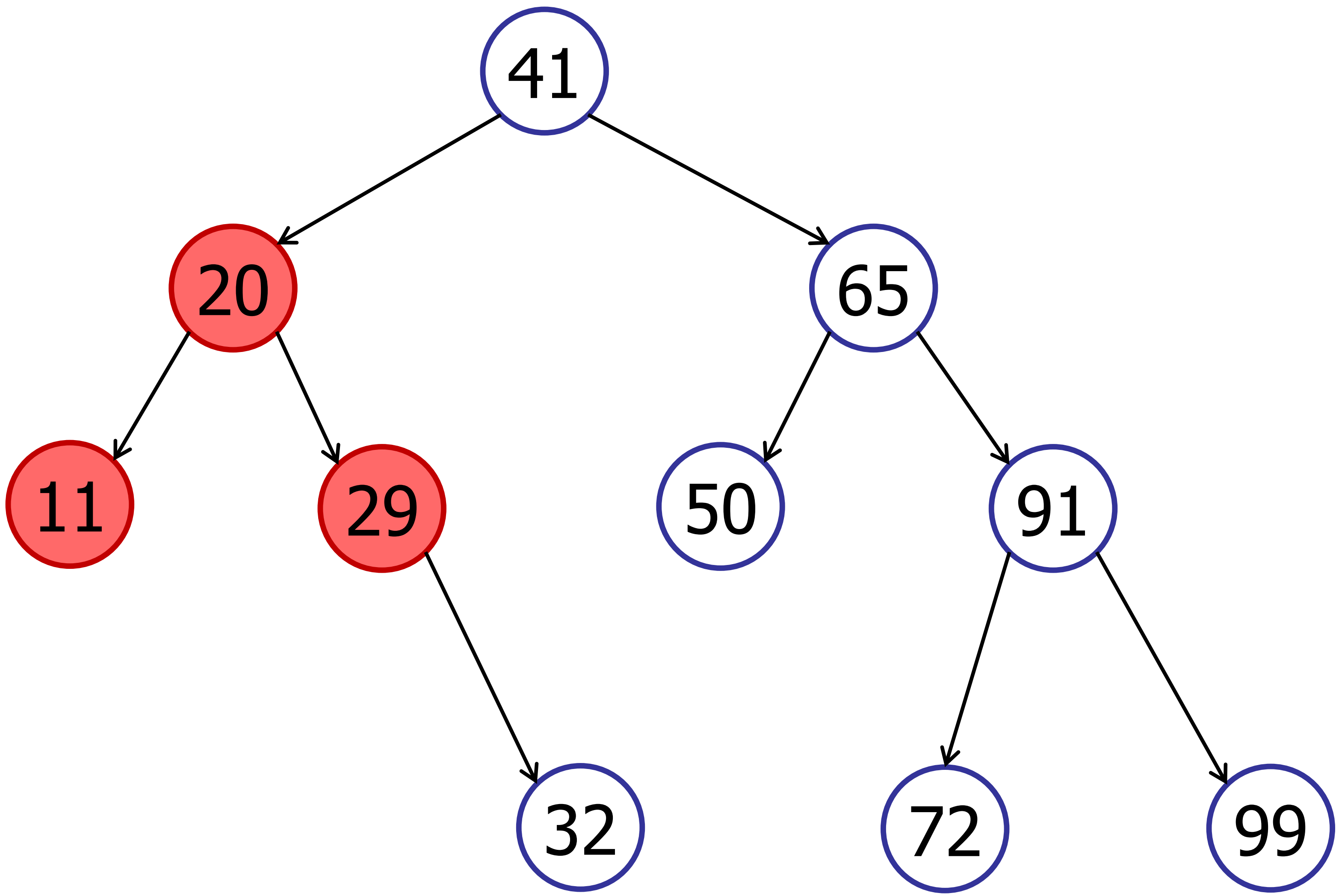
Tree Traversal

in-order-traversal



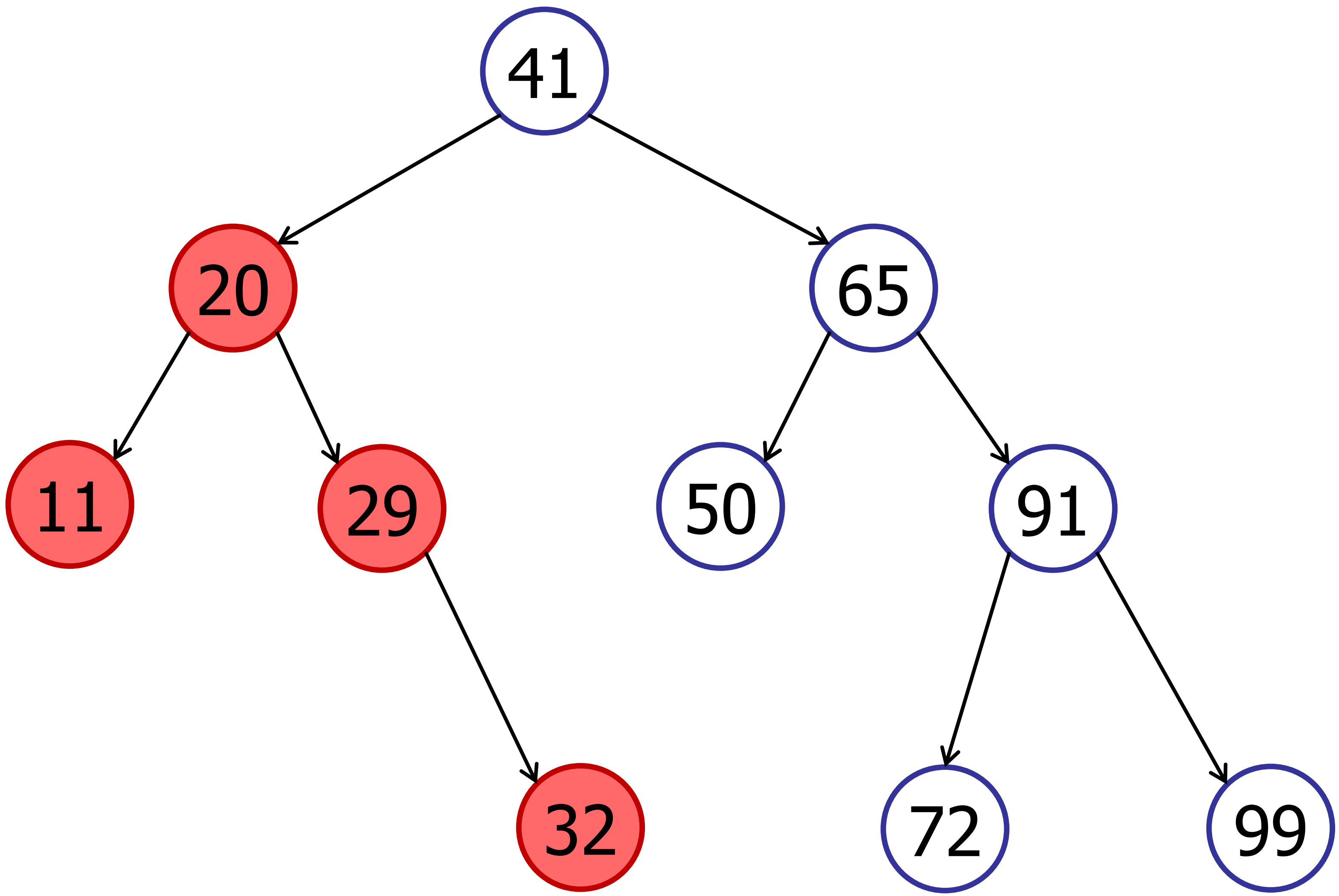
Tree Traversal

in-order-traversal



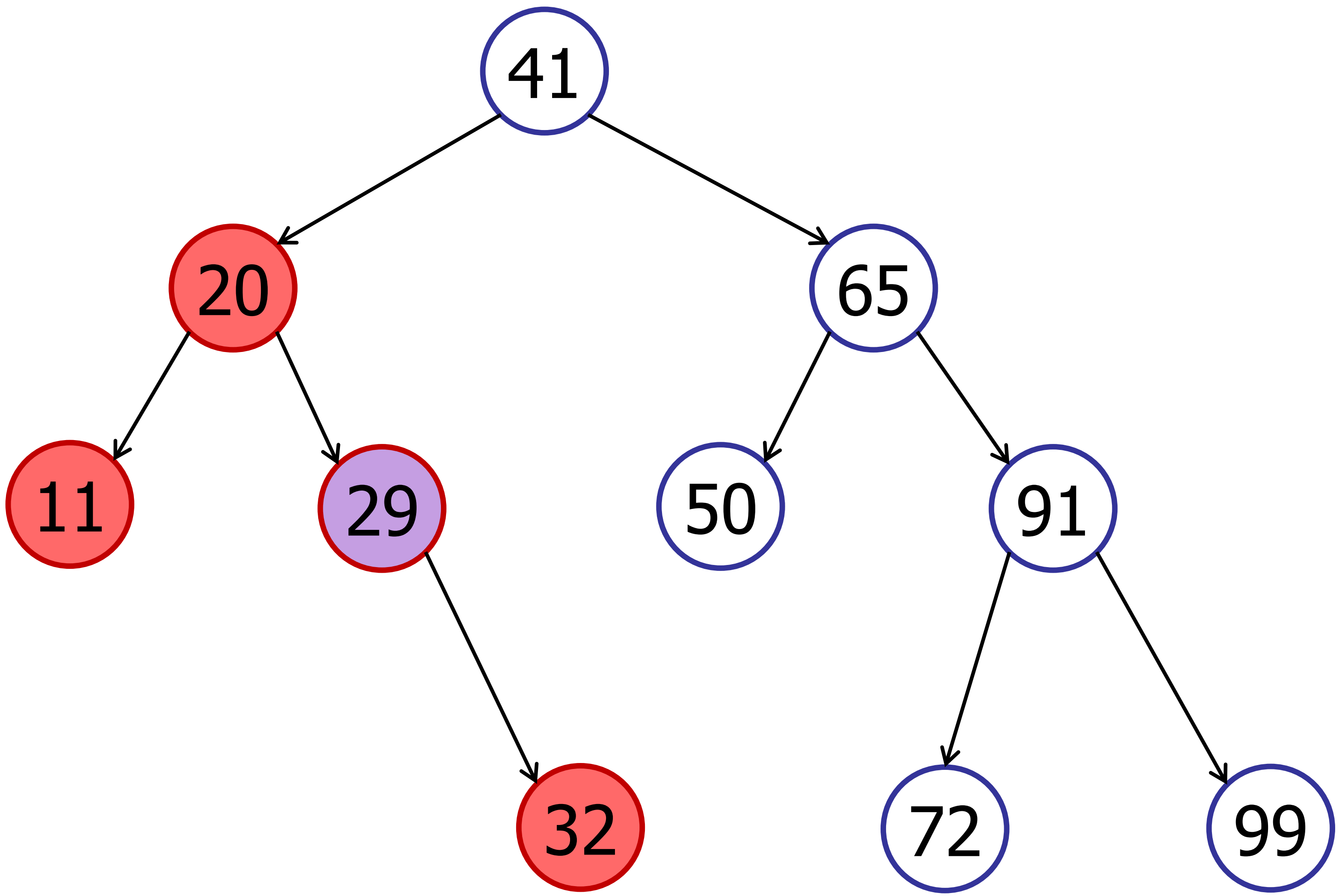
Tree Traversal

in-order-traversal



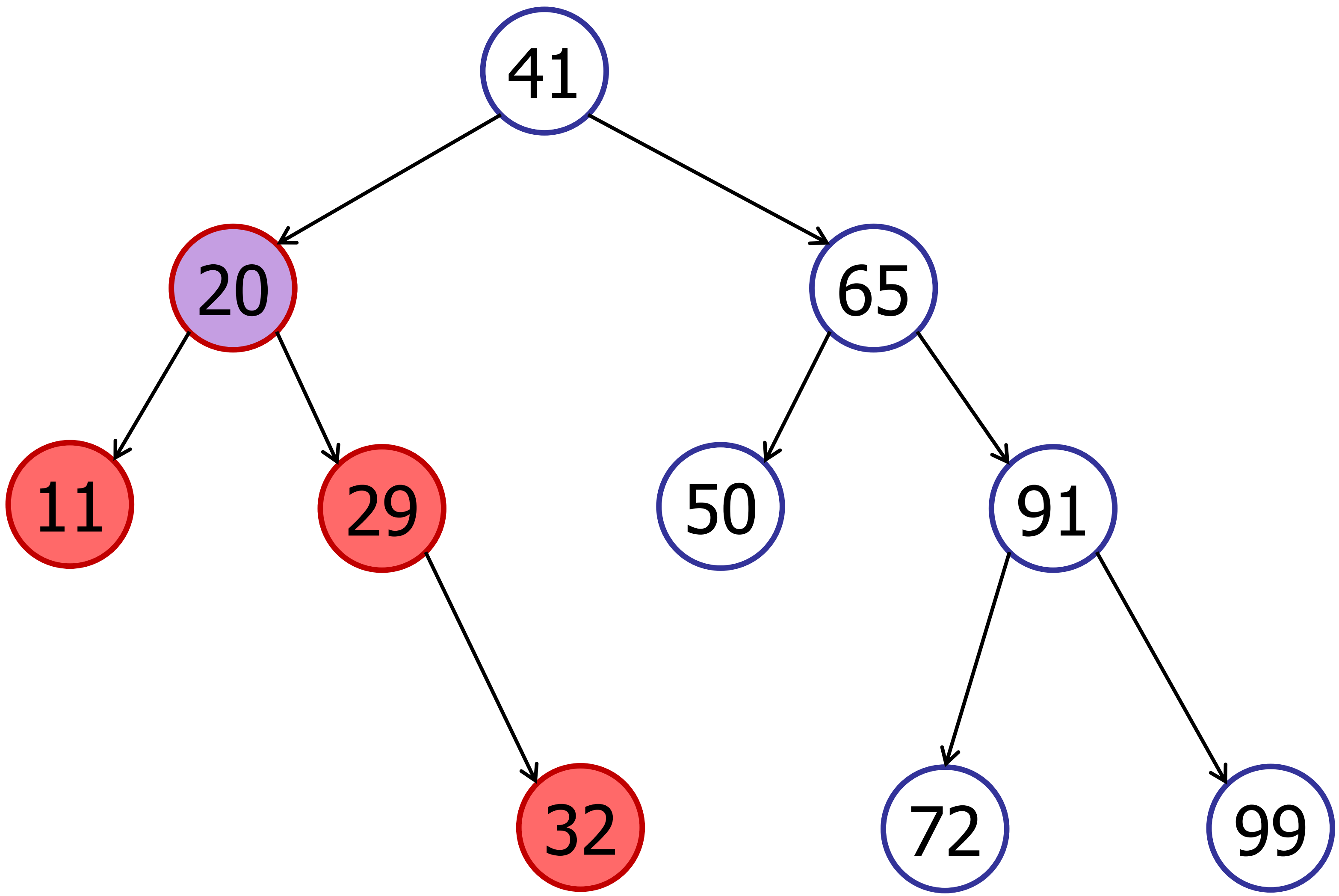
Tree Traversal

in-order-traversal



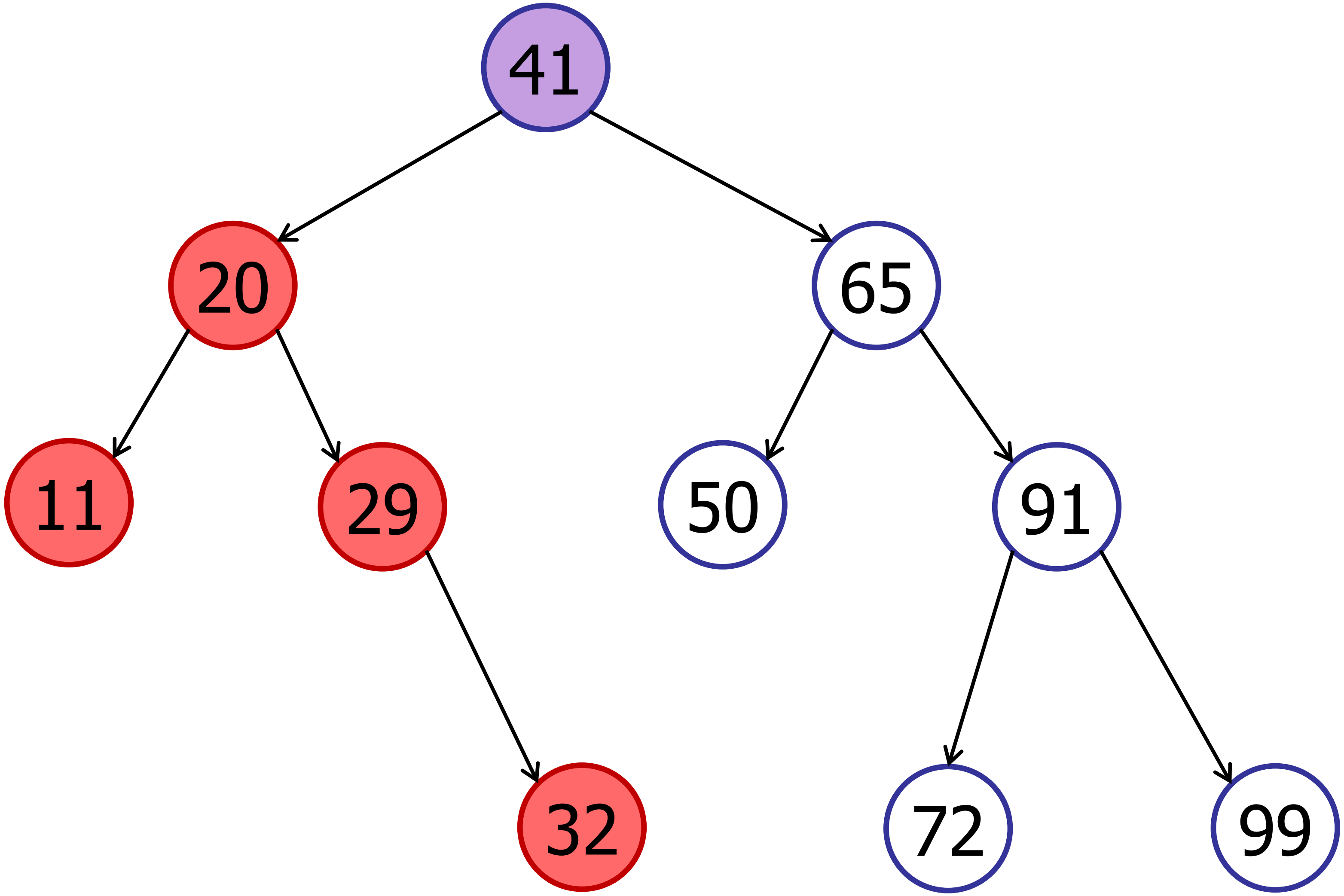
Tree Traversal

in-order-traversal



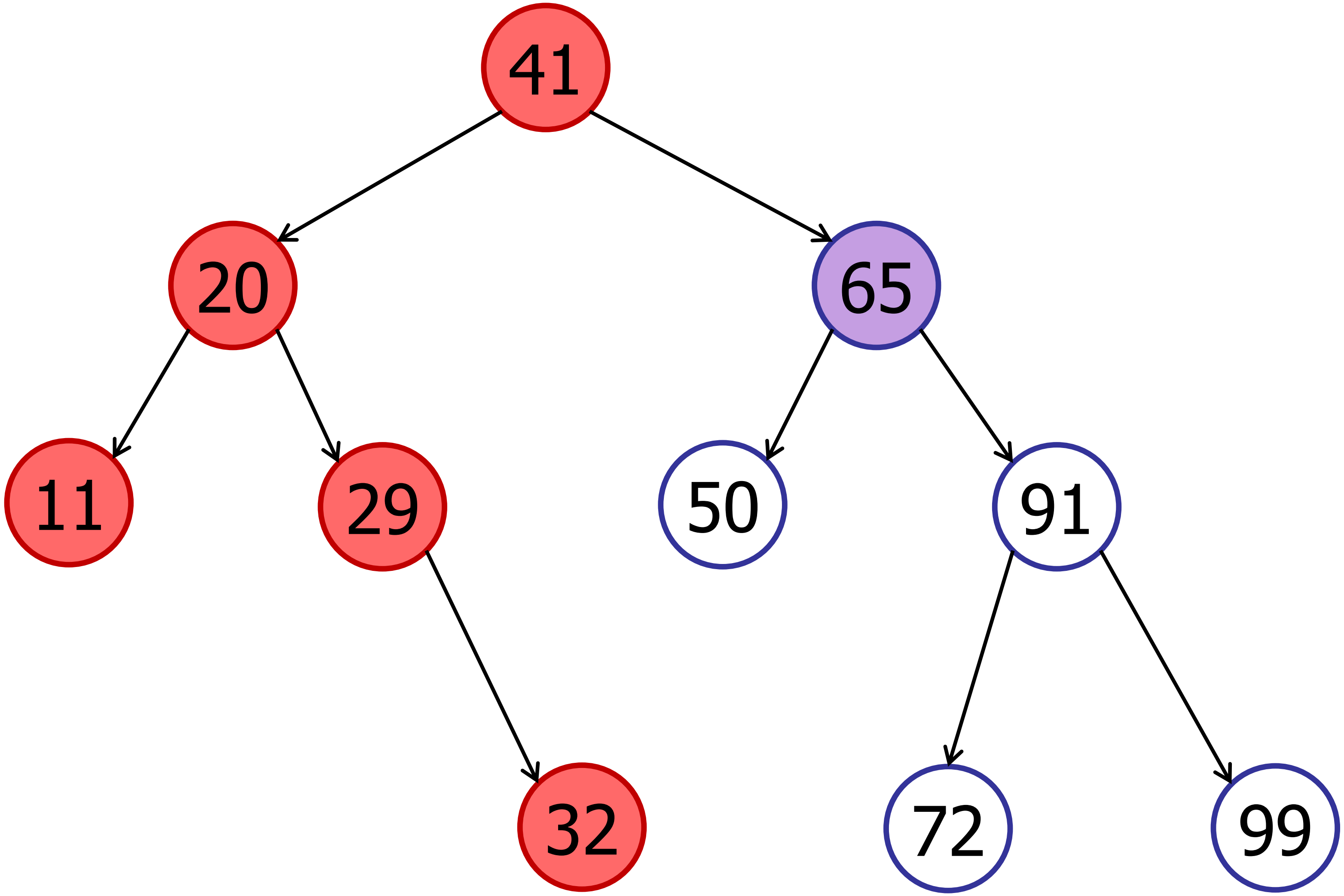
Tree Traversal

in-order-traversal



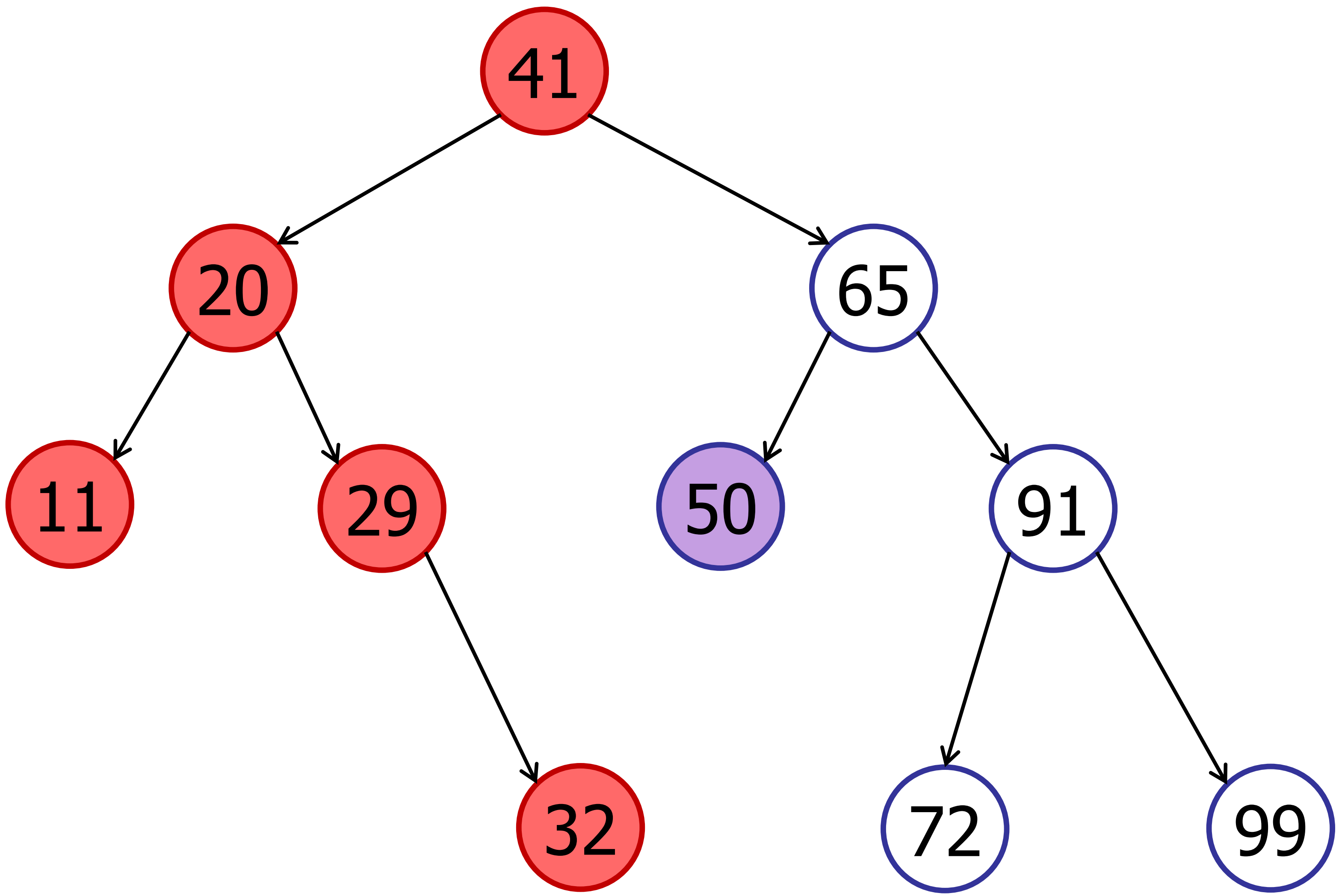
Tree Traversal

in-order-traversal



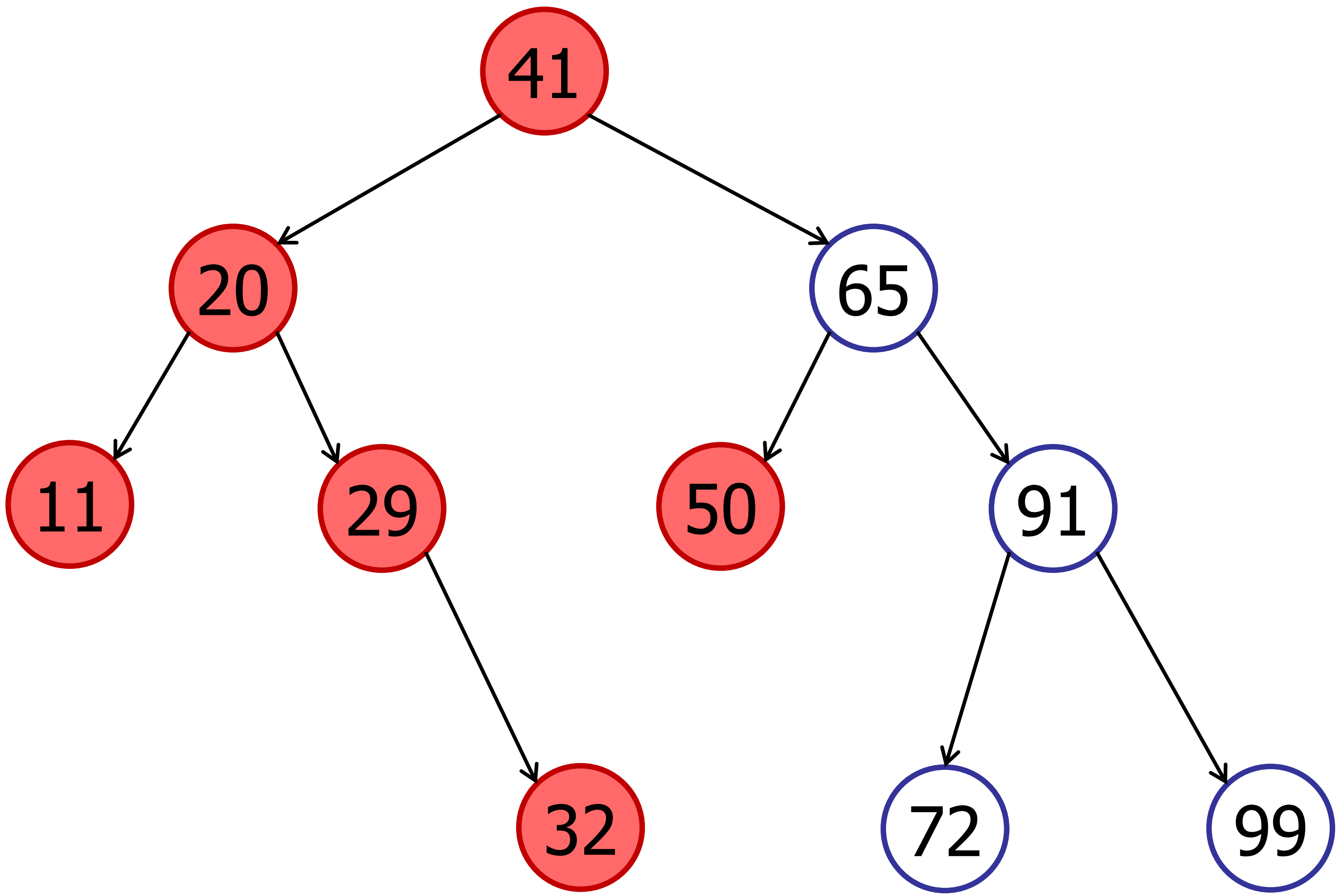
Tree Traversal

in-order-traversal



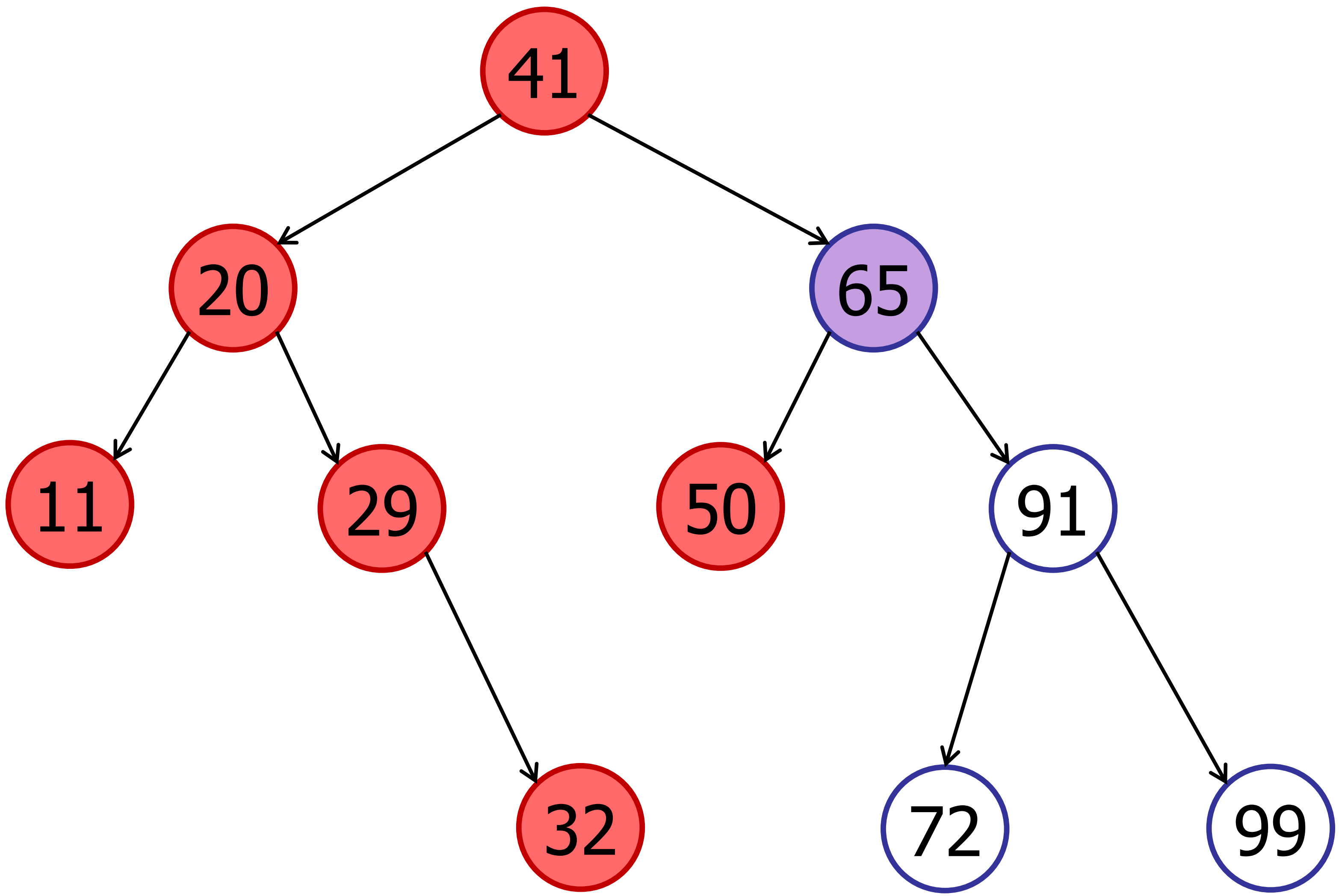
Tree Traversal

in-order-traversal



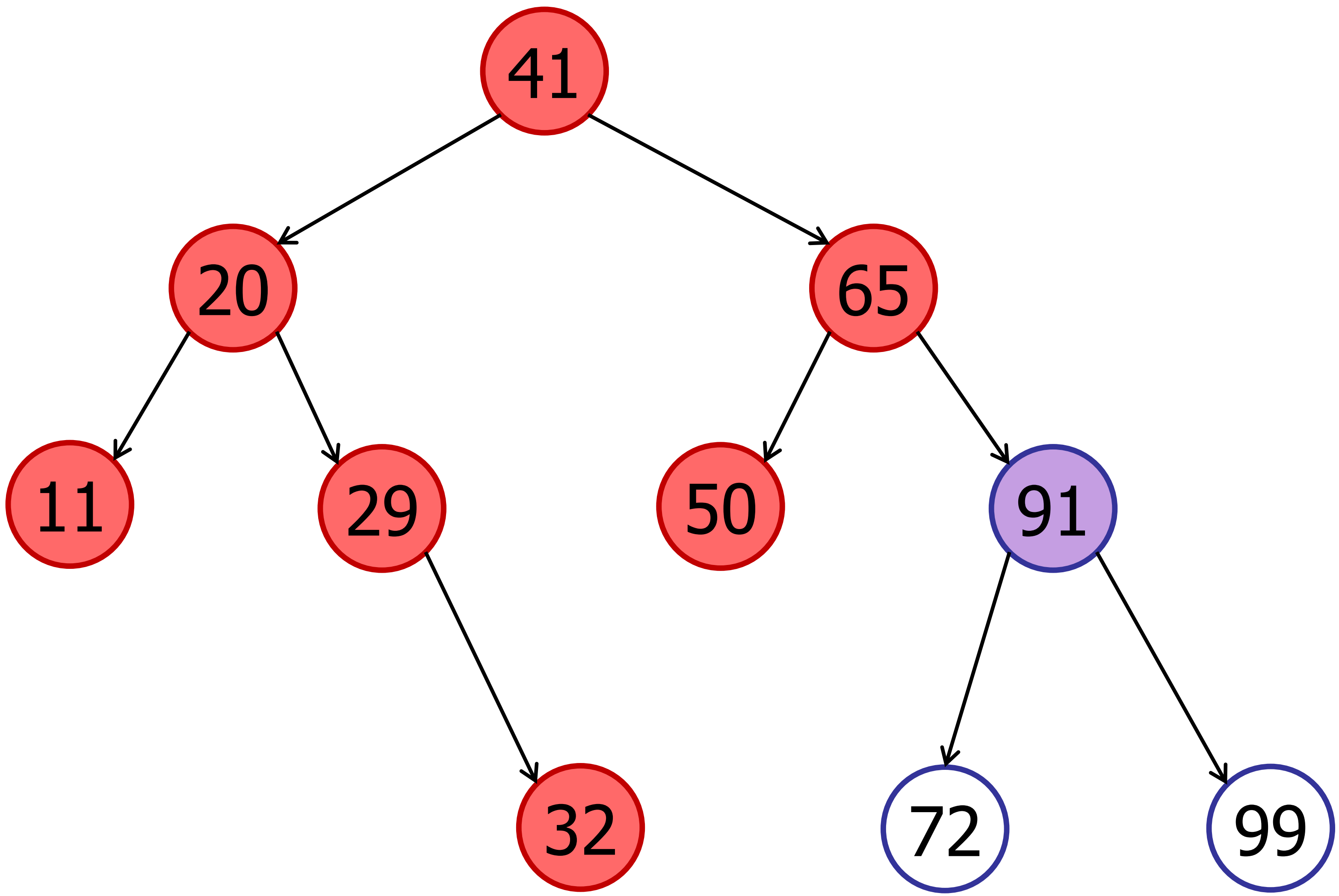
Tree Traversal

in-order-traversal



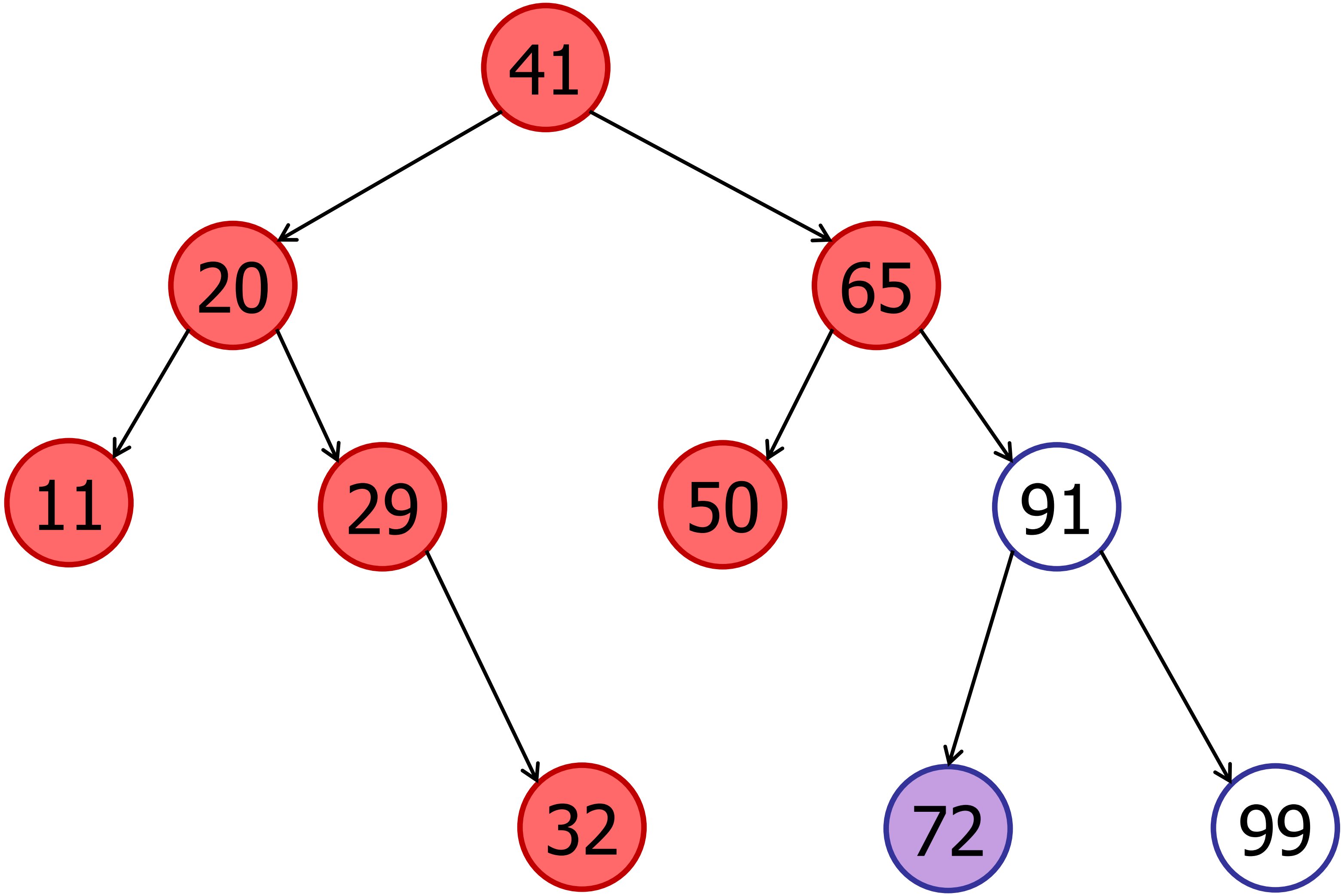
Tree Traversal

in-order-traversal



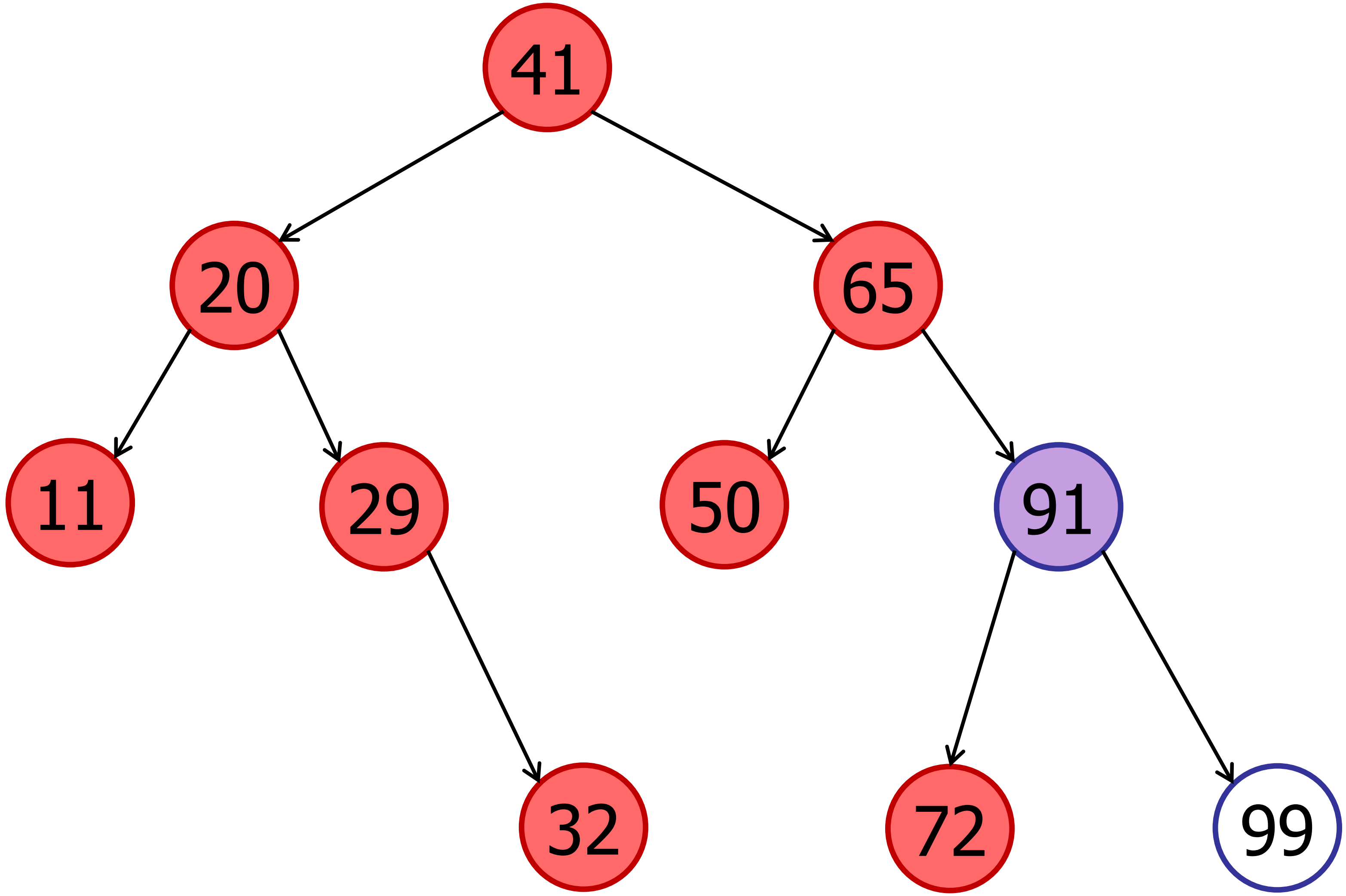
Tree Traversal

in-order-traversal



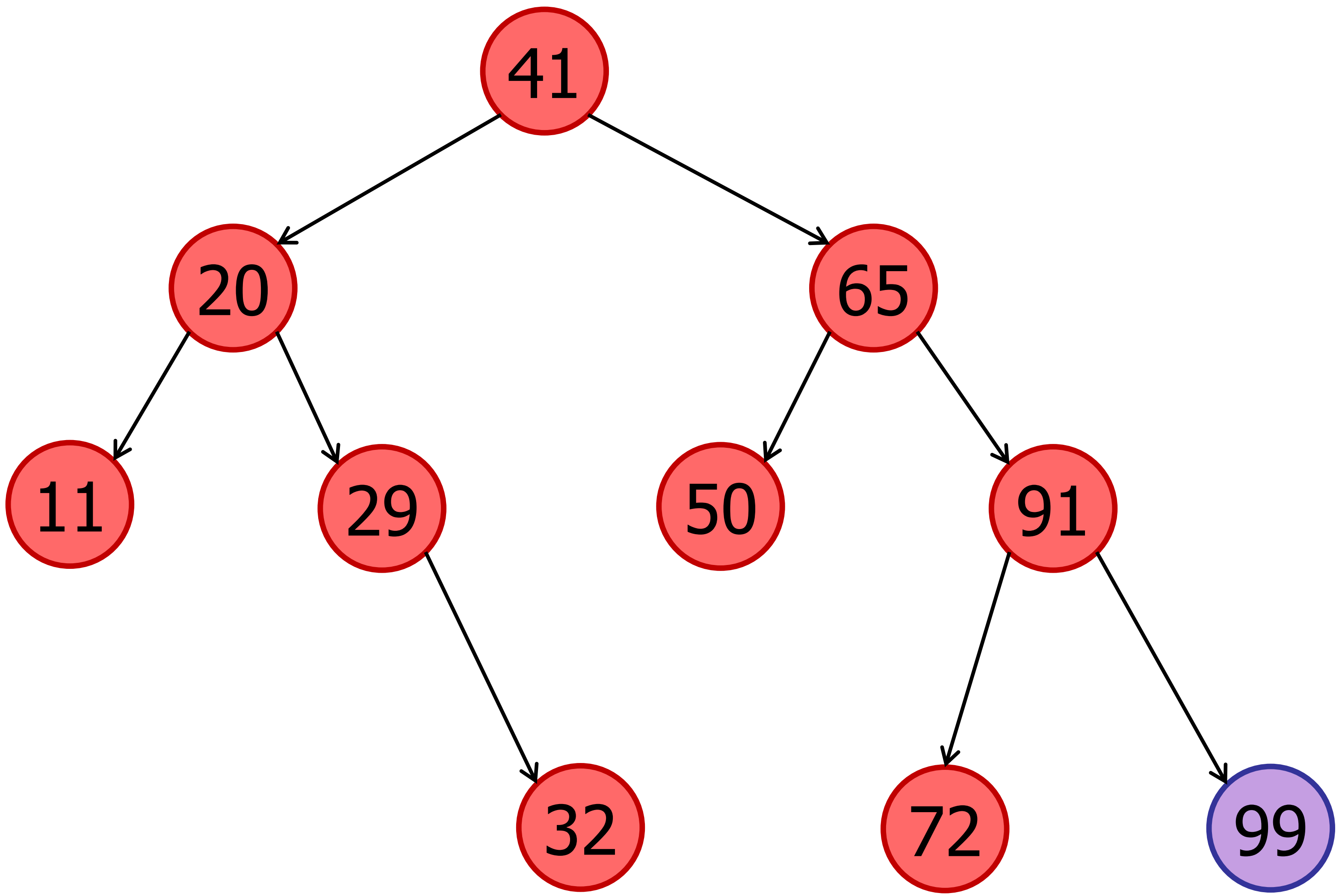
Tree Traversal

in-order-traversal



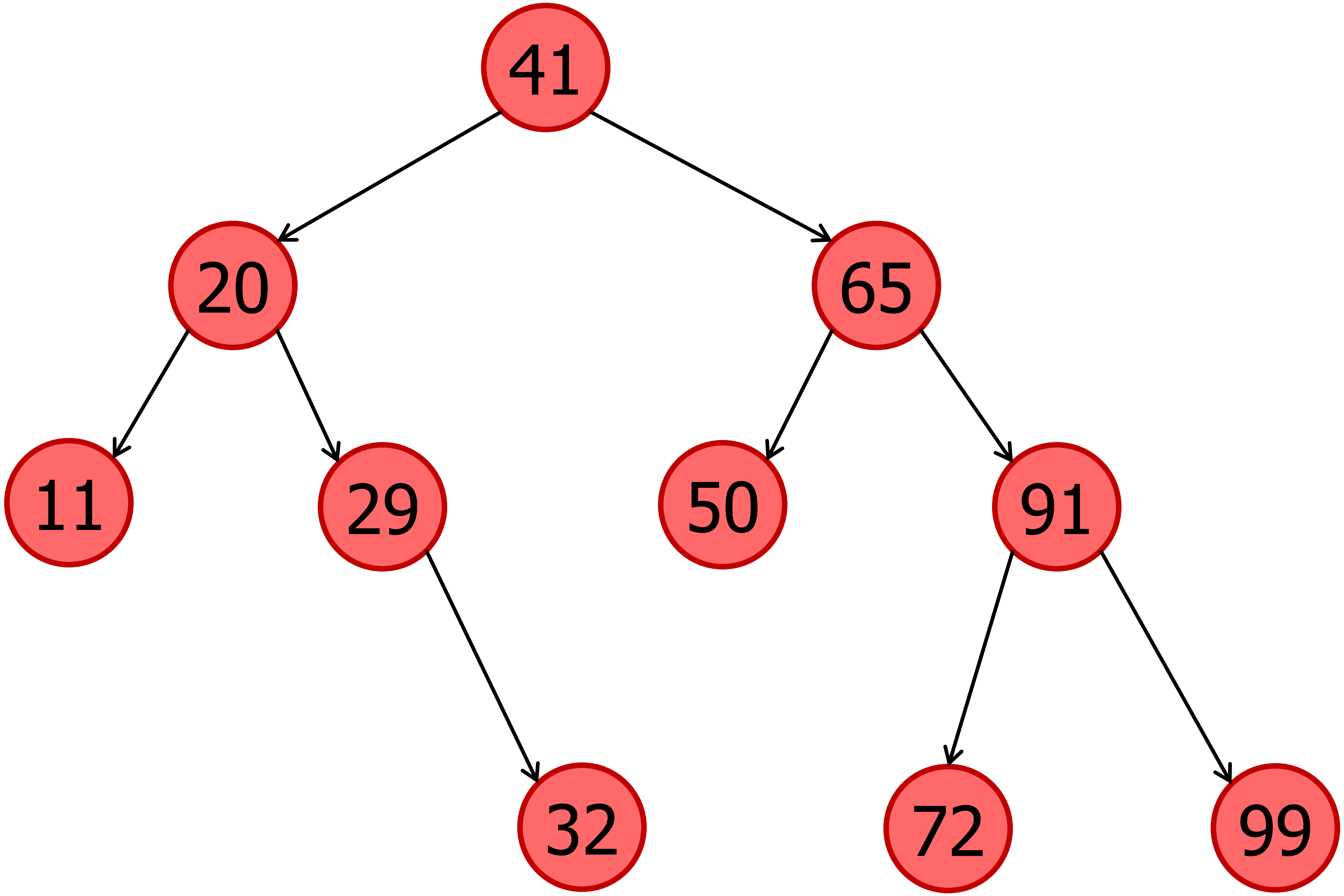
Tree Traversal

in-order-traversal



Tree Traversal

in-order-traversal



Tree Traversal

in-order-traversal(v)

```
public void in-order-traversal(){
       // Traverse left sub-tree
       if (leftTree != null)
               leftTree.in-order-traversal();

       visit(this);

       // Traverse right sub-tree
       if (rightTree != null)
              rightTree.in-order-traversal();
}
```

How long does an in-order-traversal take?

1. $O(1)$
2. $O(\log n)$
3. $O(n)$
4. $O(n \log n)$
5. $O(n^2)$
6. $O(2^n)$

How long does an in-order-traversal take?

1. $O(1)$
2. $O(\log n)$
3. $\boxed{O(n)}$
4. $O(n \log n)$
5. $O(n^2)$
6. $O(2^n)$

Tree Traversal

in-order-traversal(v)

```
public void in-order-traversal(){
      // Traverse left sub-tree
      if (leftTree != null)
              leftTree.in-order-traversal();

      visit(this);

      // Traverse right sub-tree
      if (rightTree != null)
             rightTree.in-order-traversal();
}
```

Running time: O(n)

– visits each node at most once

Tree Traversal

in-order-traversal(v)

- left-subtree
- SELF
- right-subtree

pre-order-traversal(v)

- SELF
- left-subtree
- right-subtree

post-order-traversal(v)

- left-subtree
- right-subtree
- SELF

Tree Traversals

pre-order-traversal(v)

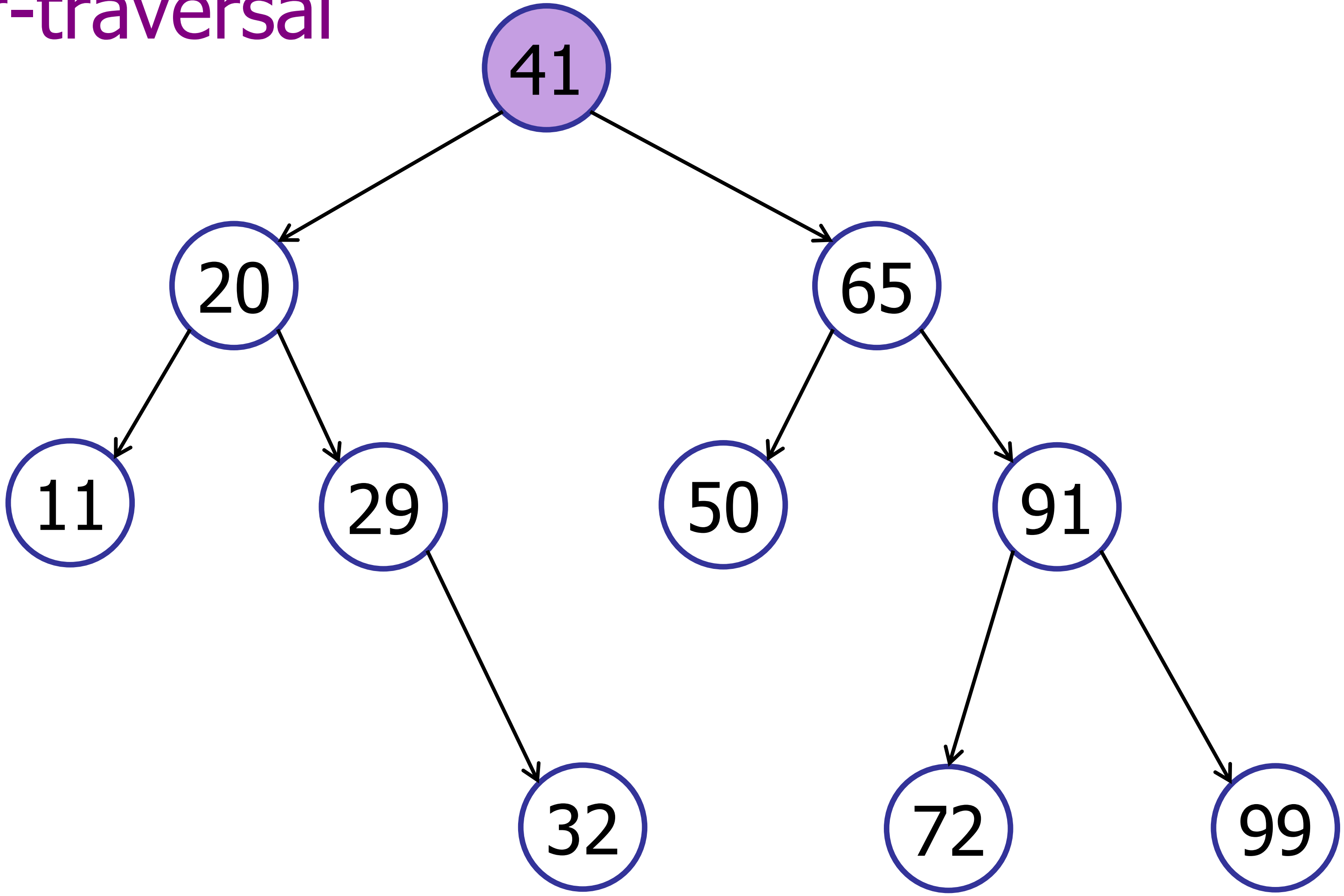
```
public void pre-order-traversal(){
        visit(this);

        // Traverse left sub-tree
        if (leftTree != null)
                leftTree.in-order-traversal();

        // Traverse right sub-tree
        if (rightTree != null)
                rightTree.in-order-traversal();
}
```

Tree Traversals

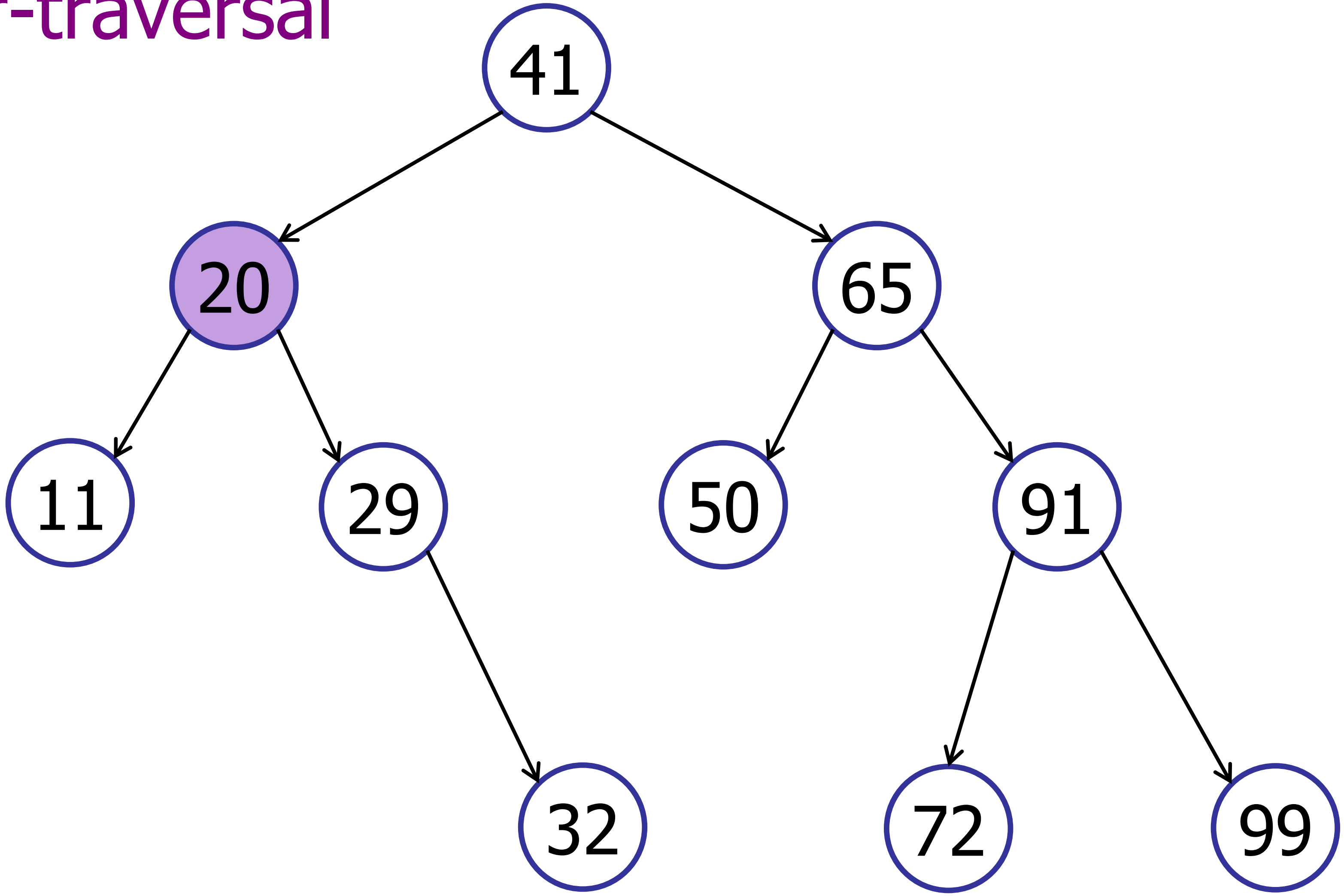
pre-order-traversal



41

Tree Traversals

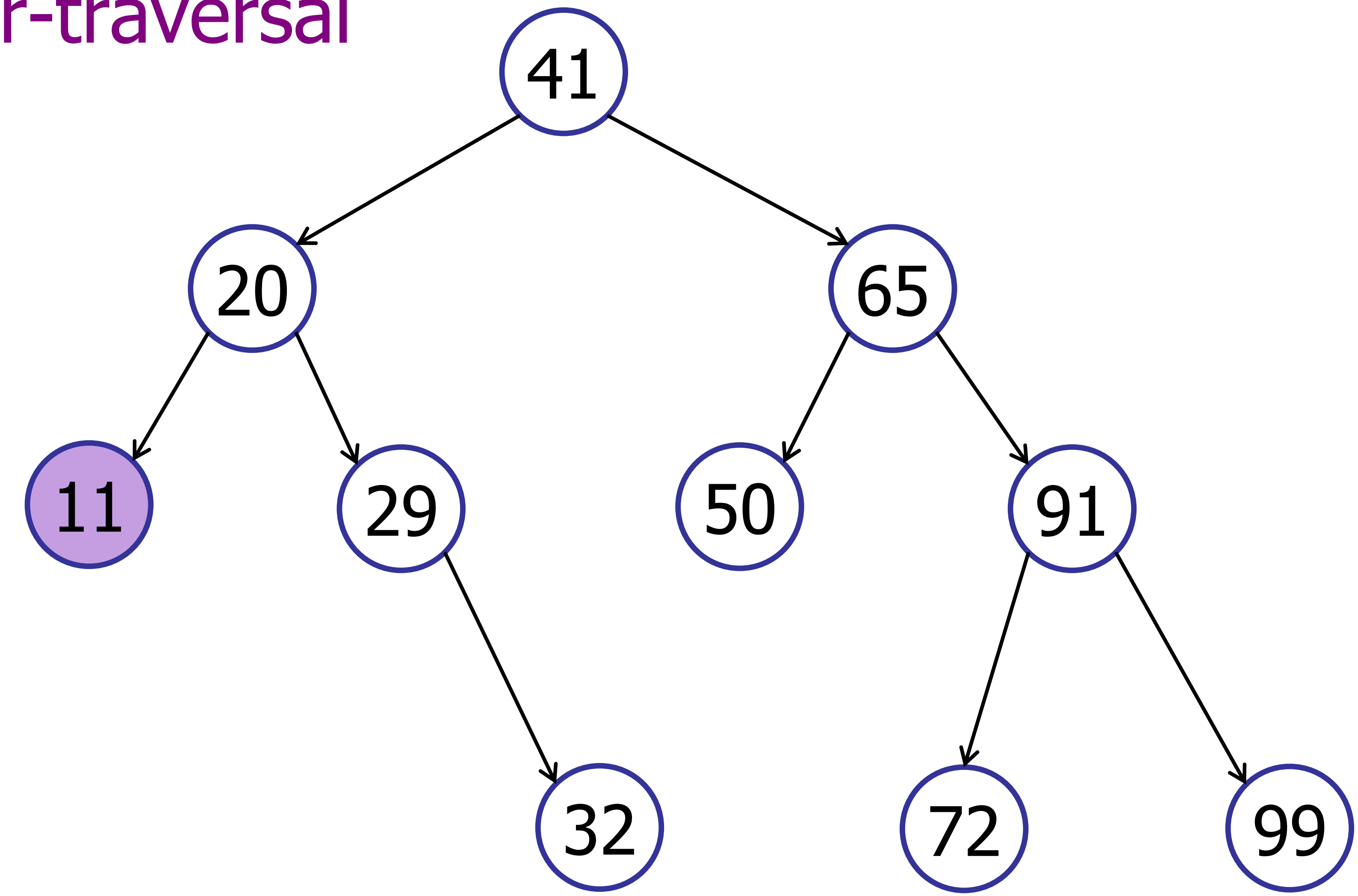
pre-order-traversal



41 20

Tree Traversals

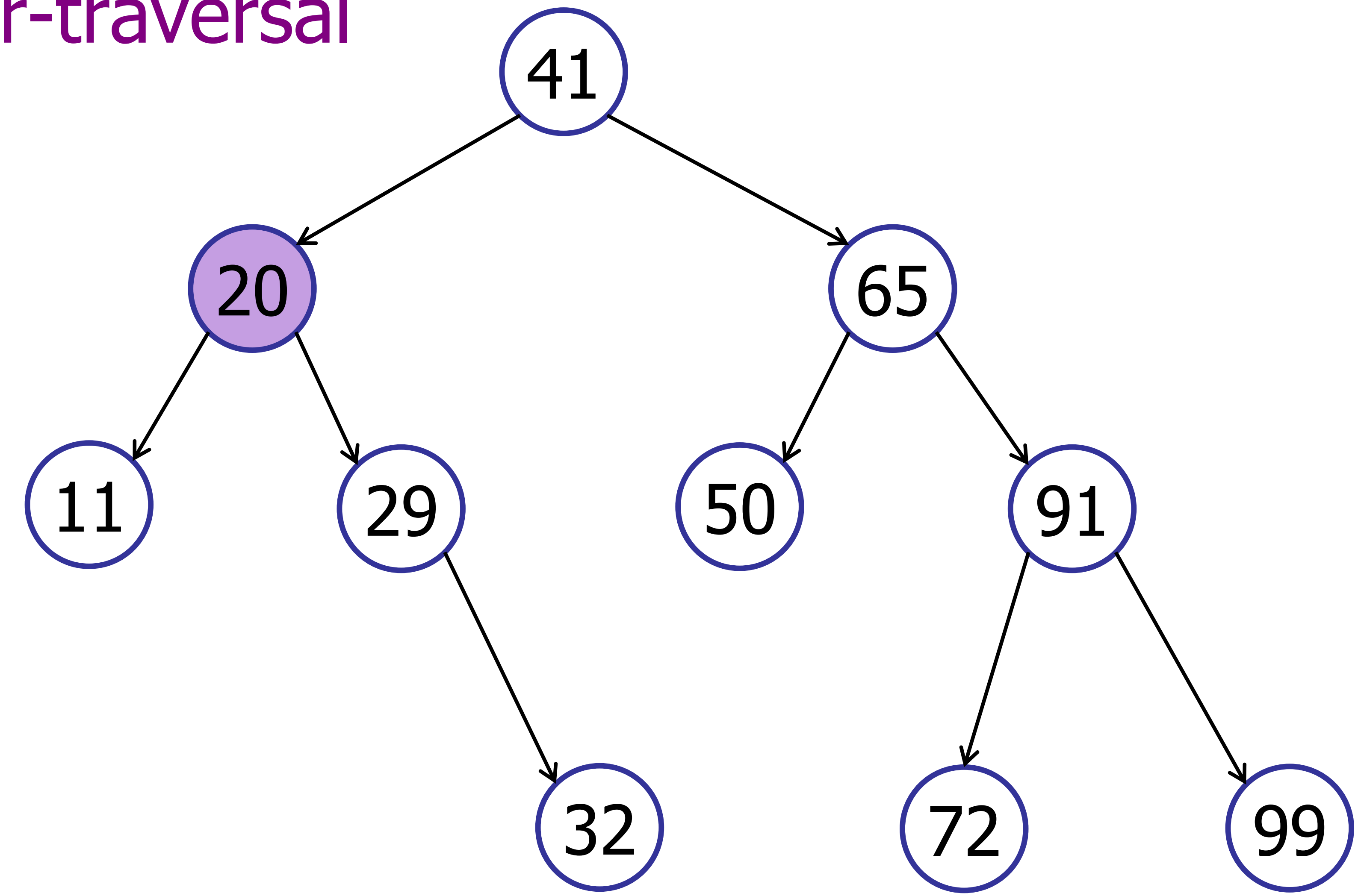
pre-order-traversal



41 20 11

Tree Traversals

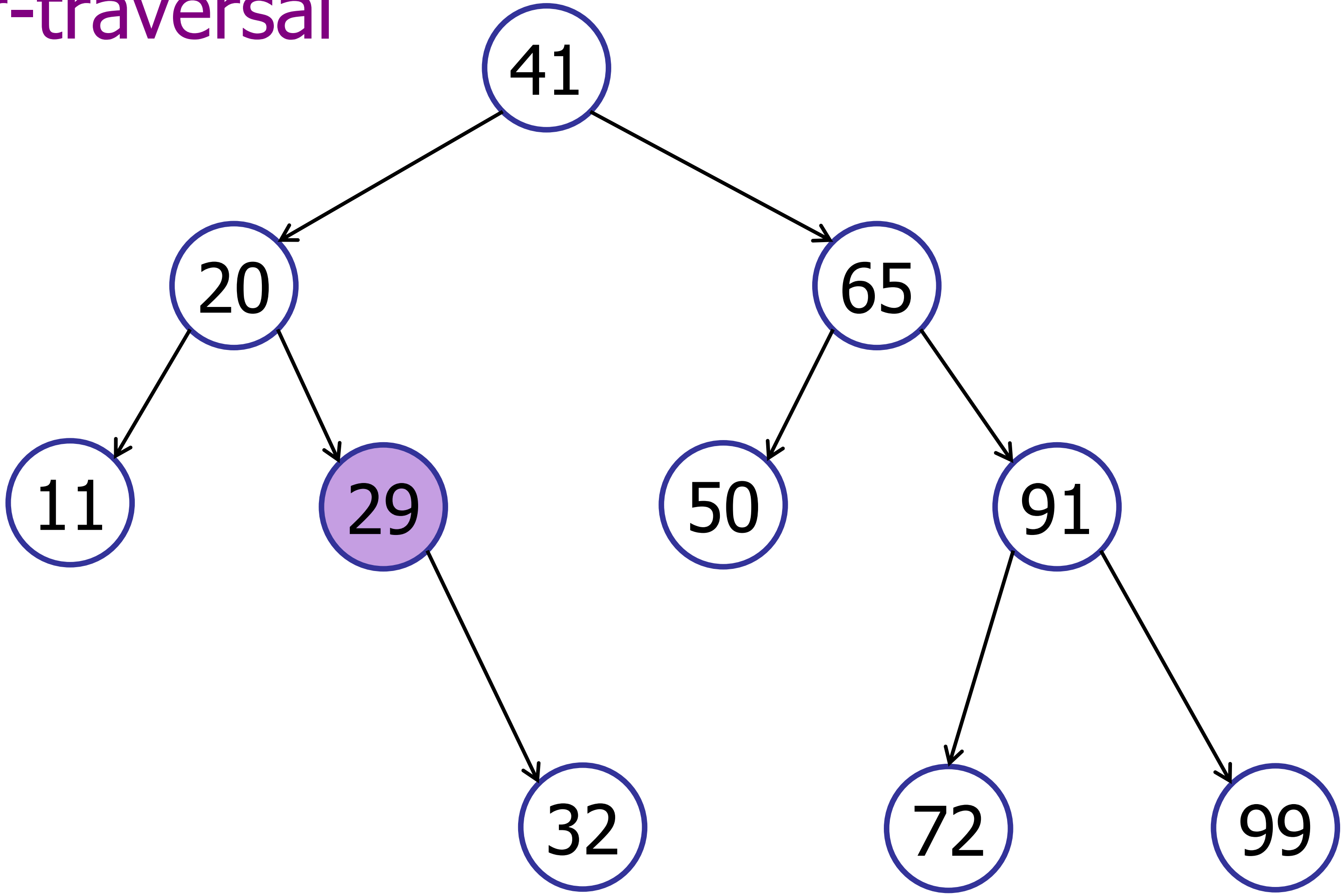
pre-order-traversal



41 20 11

Tree Traversals

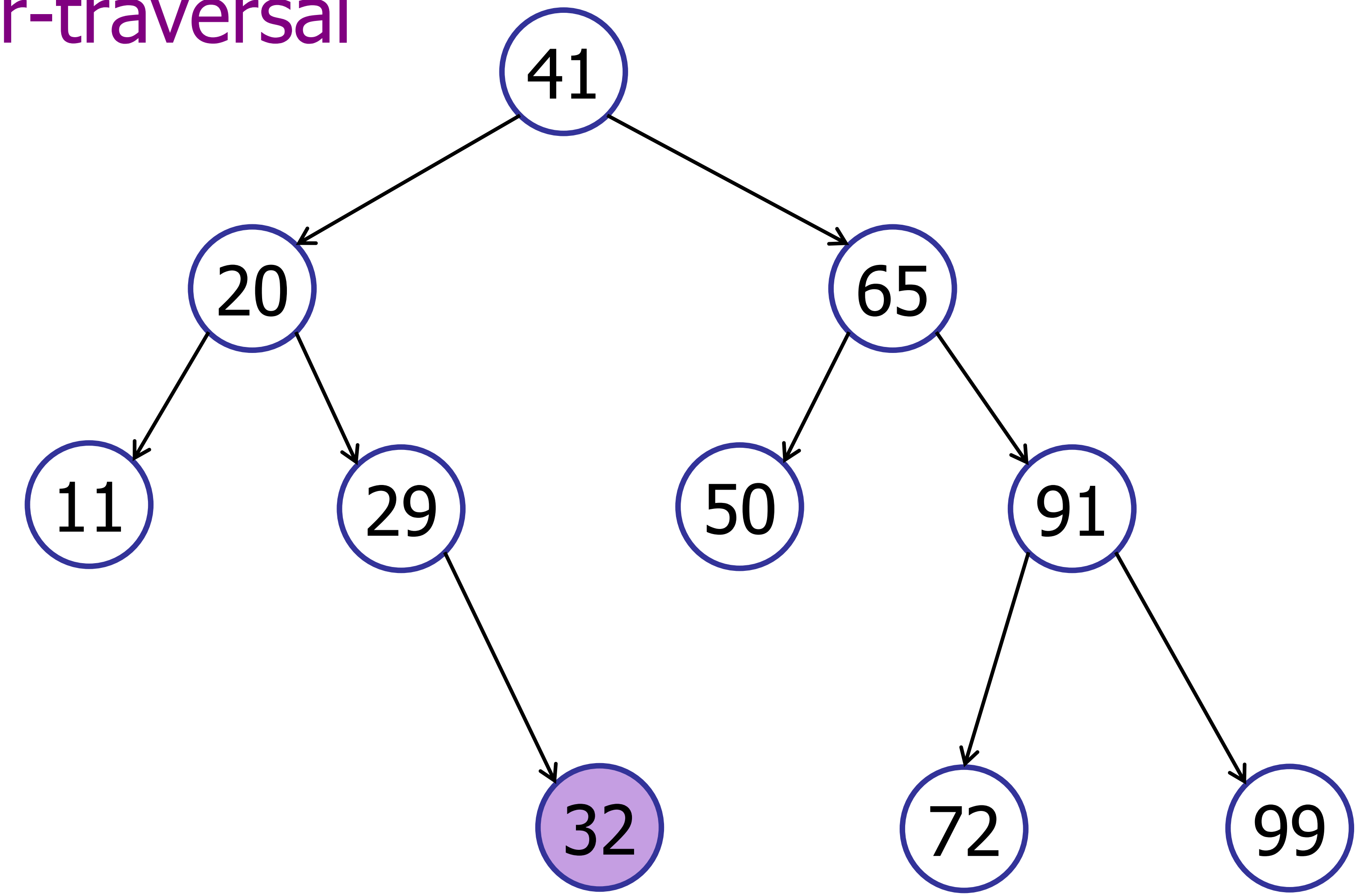
pre-order-traversal



41 20 11 29

Tree Traversals

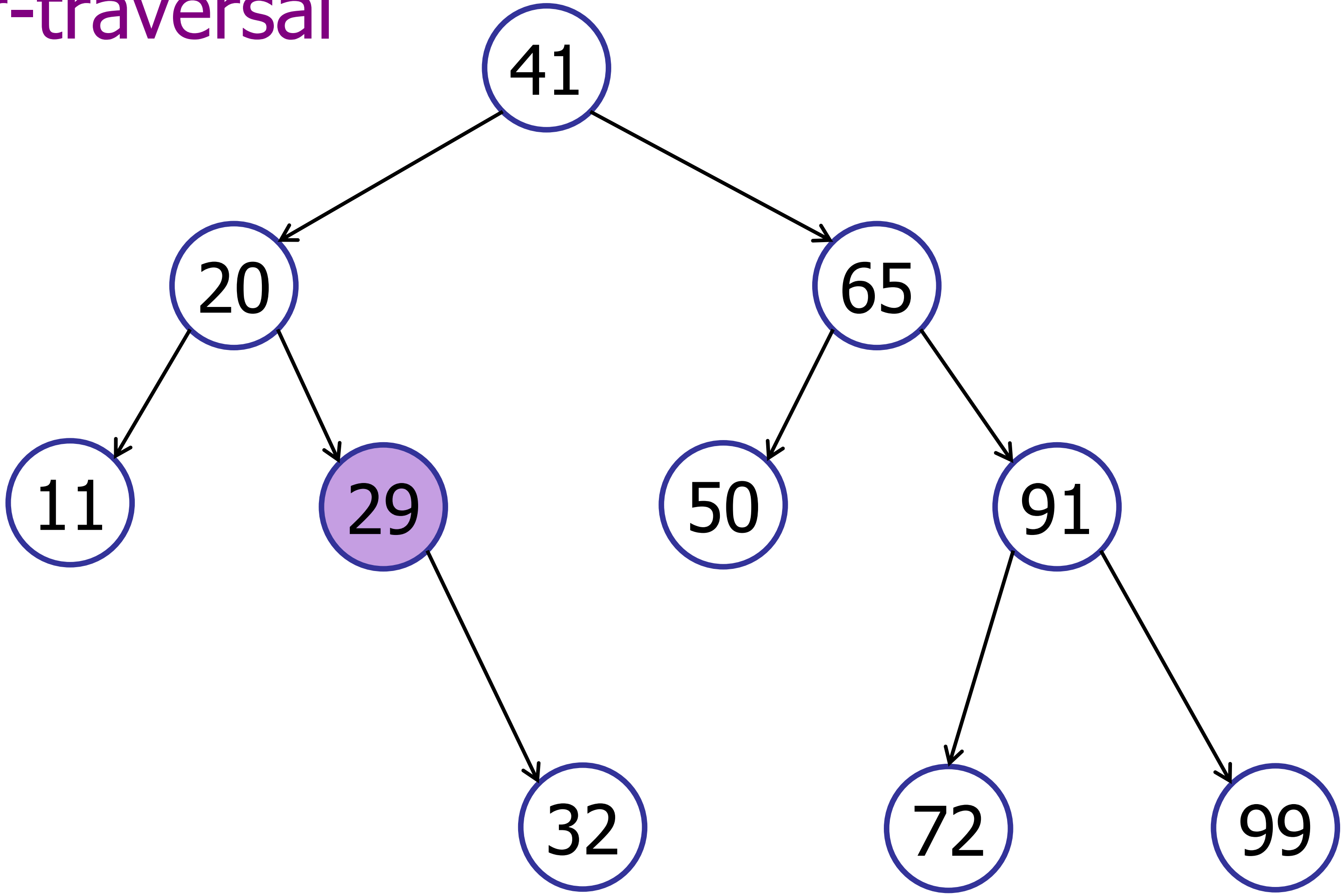
pre-order-traversal



41 20 11 29 32

Tree Traversals

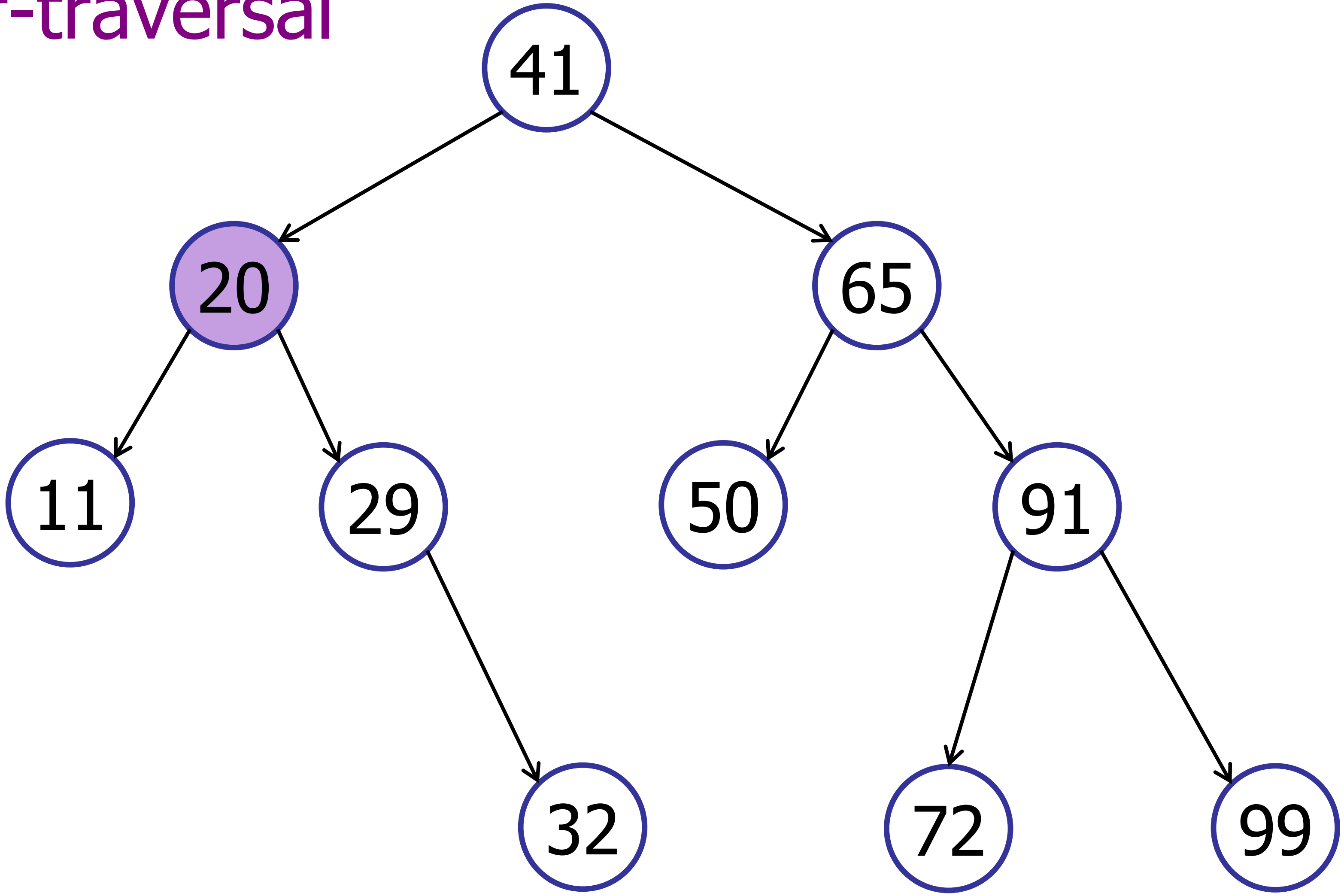
pre-order-traversal



41 20 11 29 32

Tree Traversals

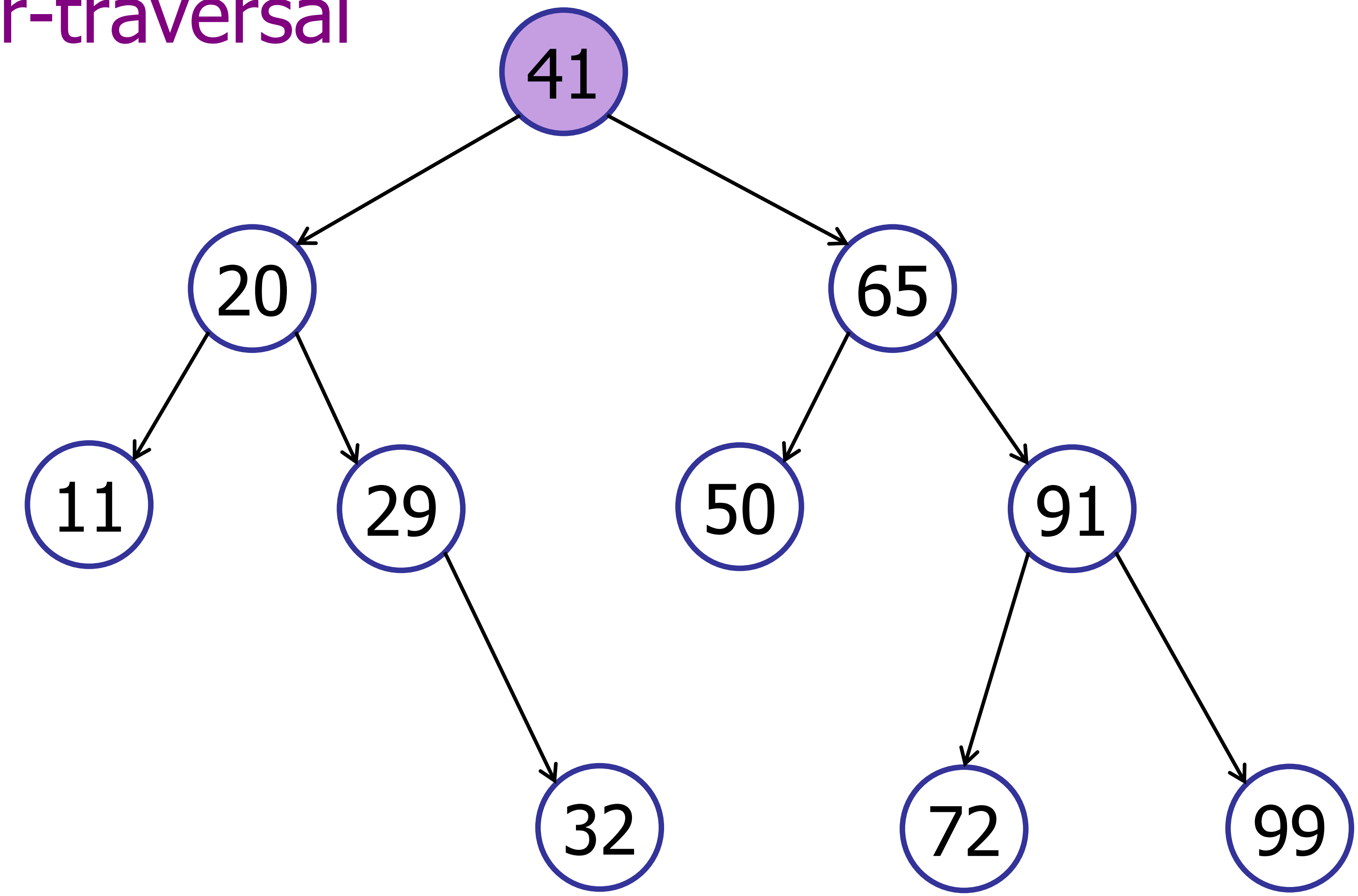
pre-order-traversal



41 20 11 29 32

Tree Traversals

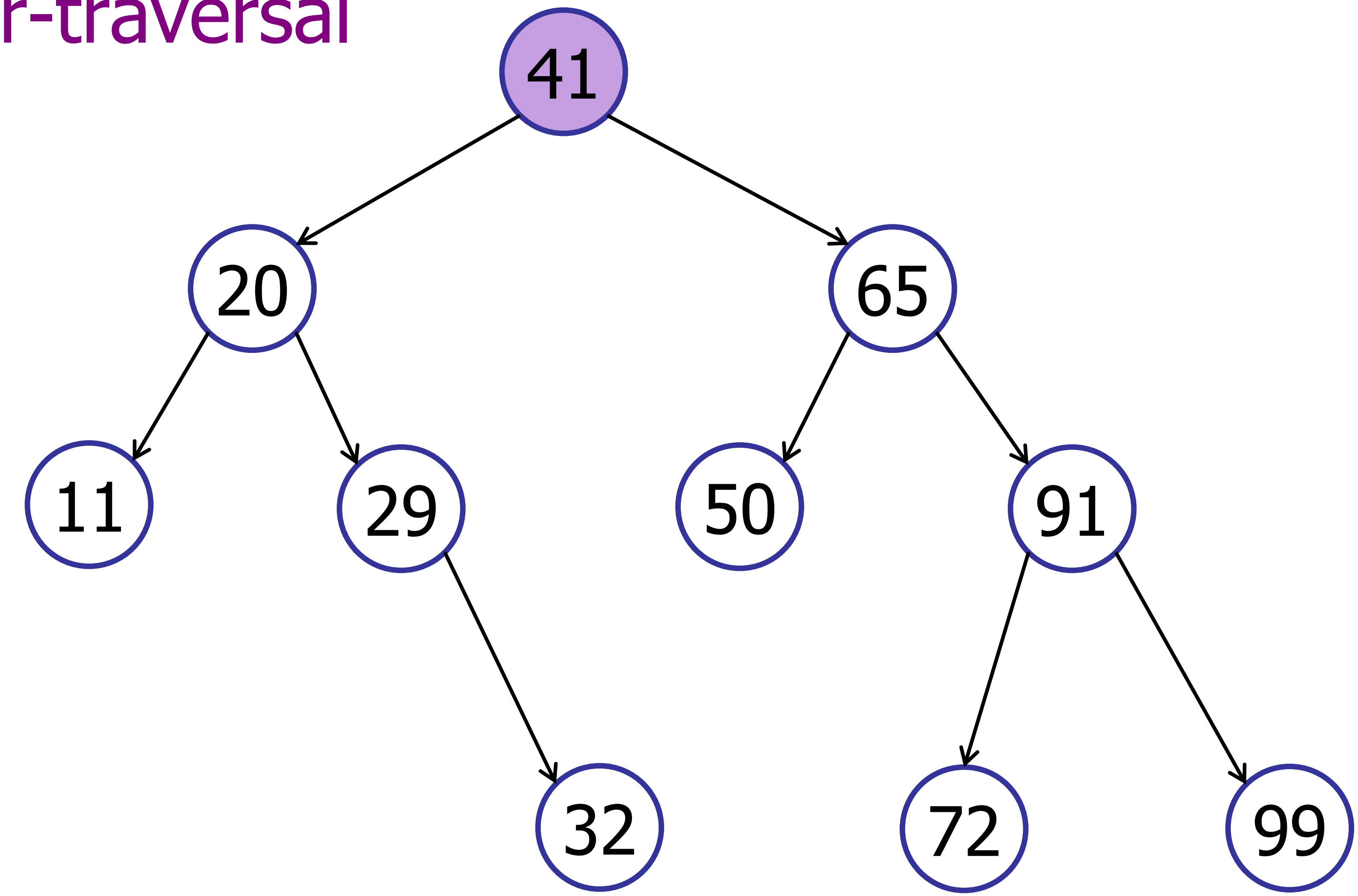
pre-order-traversal



41 20 11 29 32

Tree Traversals

pre-order-traversal



41 20 11 29 32 65 50 91 72 99

Tree Traversals

post-order-traversal(v)

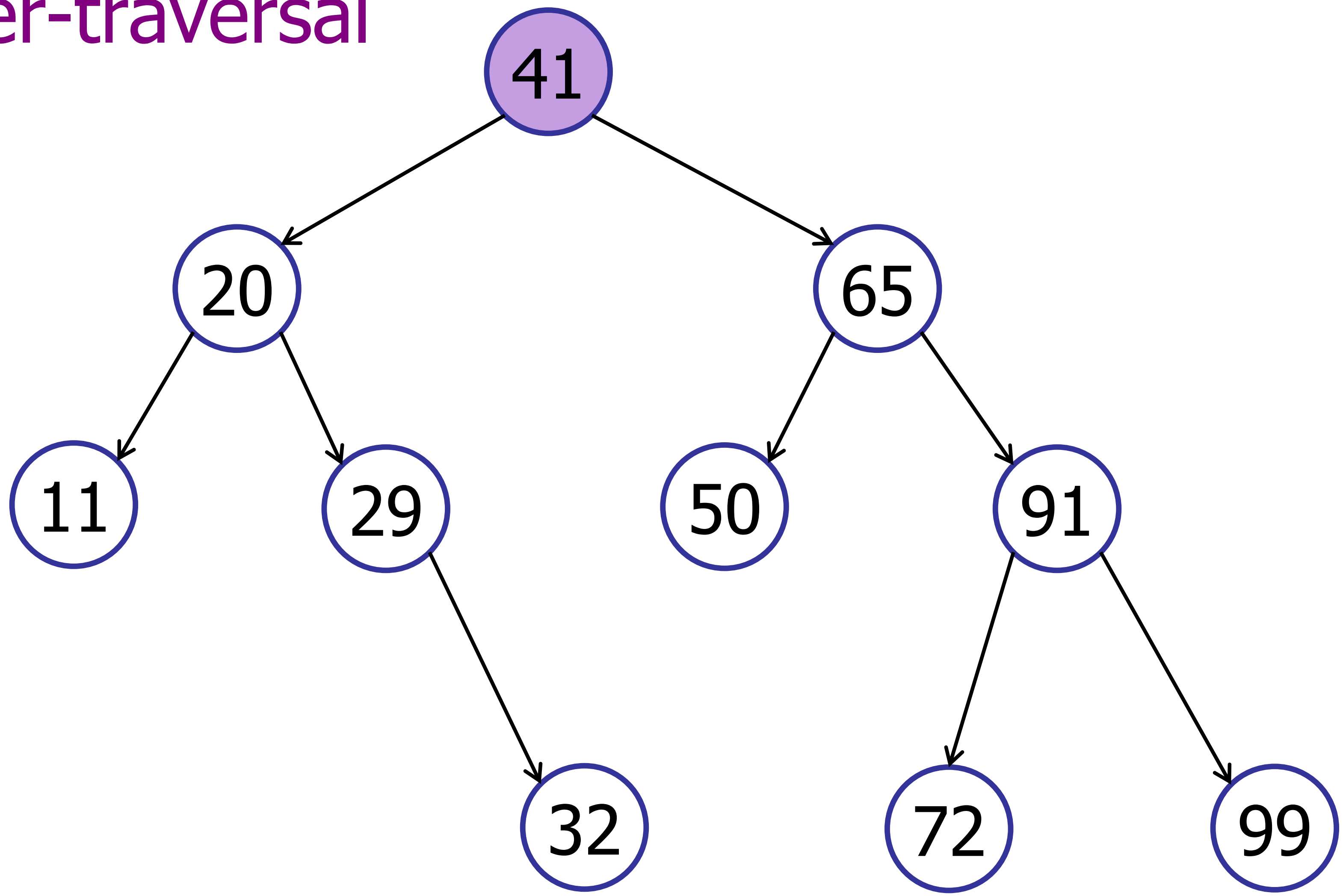
```
public void post-order-traversal(){
        // Traverse left sub-tree
        if (leftTree != null)
                leftTree.in-order-traversal();

        // Traverse right sub-tree
        if (rightTree != null)
                rightTree.in-order-traversal();

        visit(this);
}
```

Tree Traversals

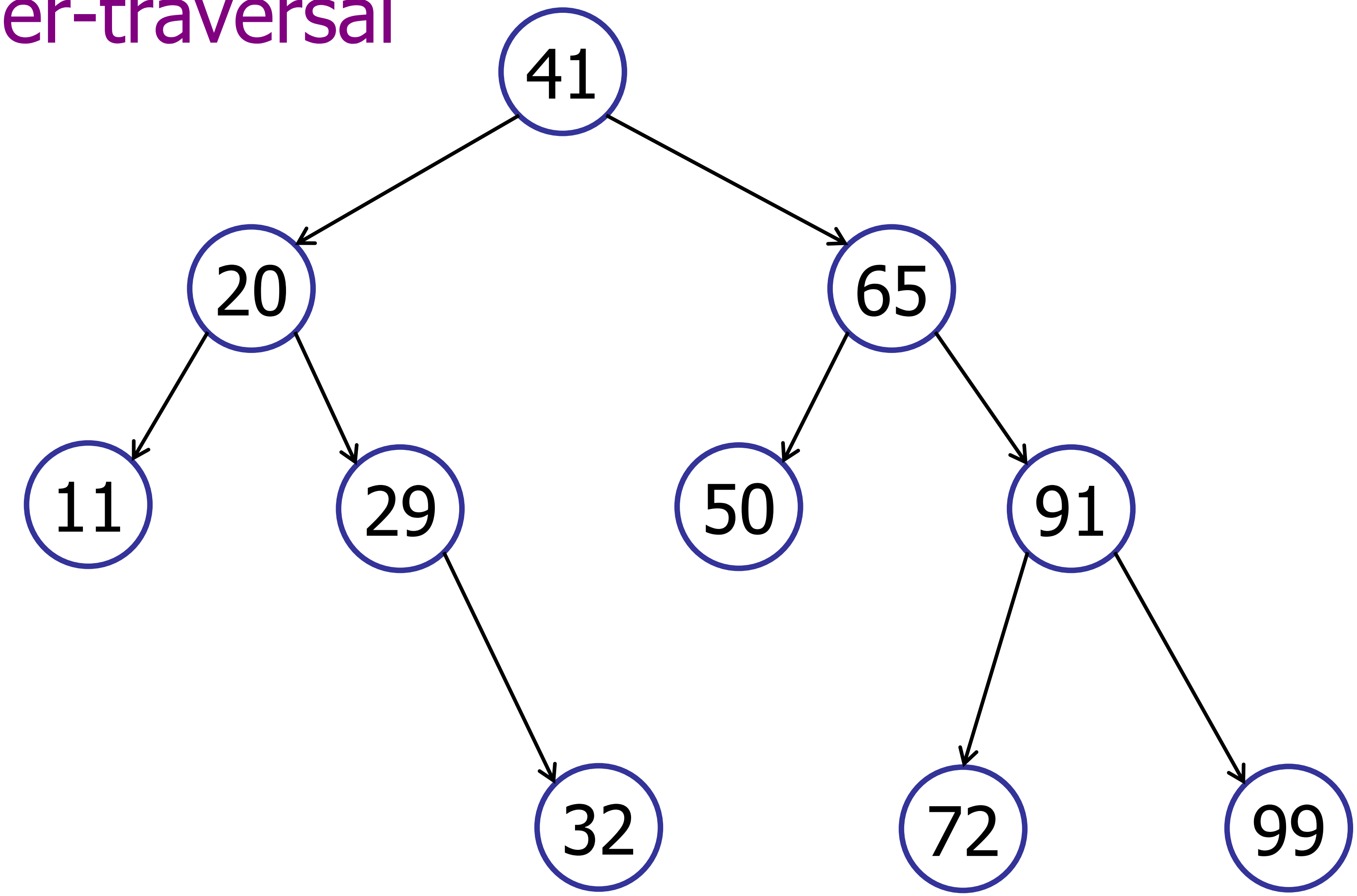
post-order-traversal



11 32 29 20 50 72 99 91 65 41

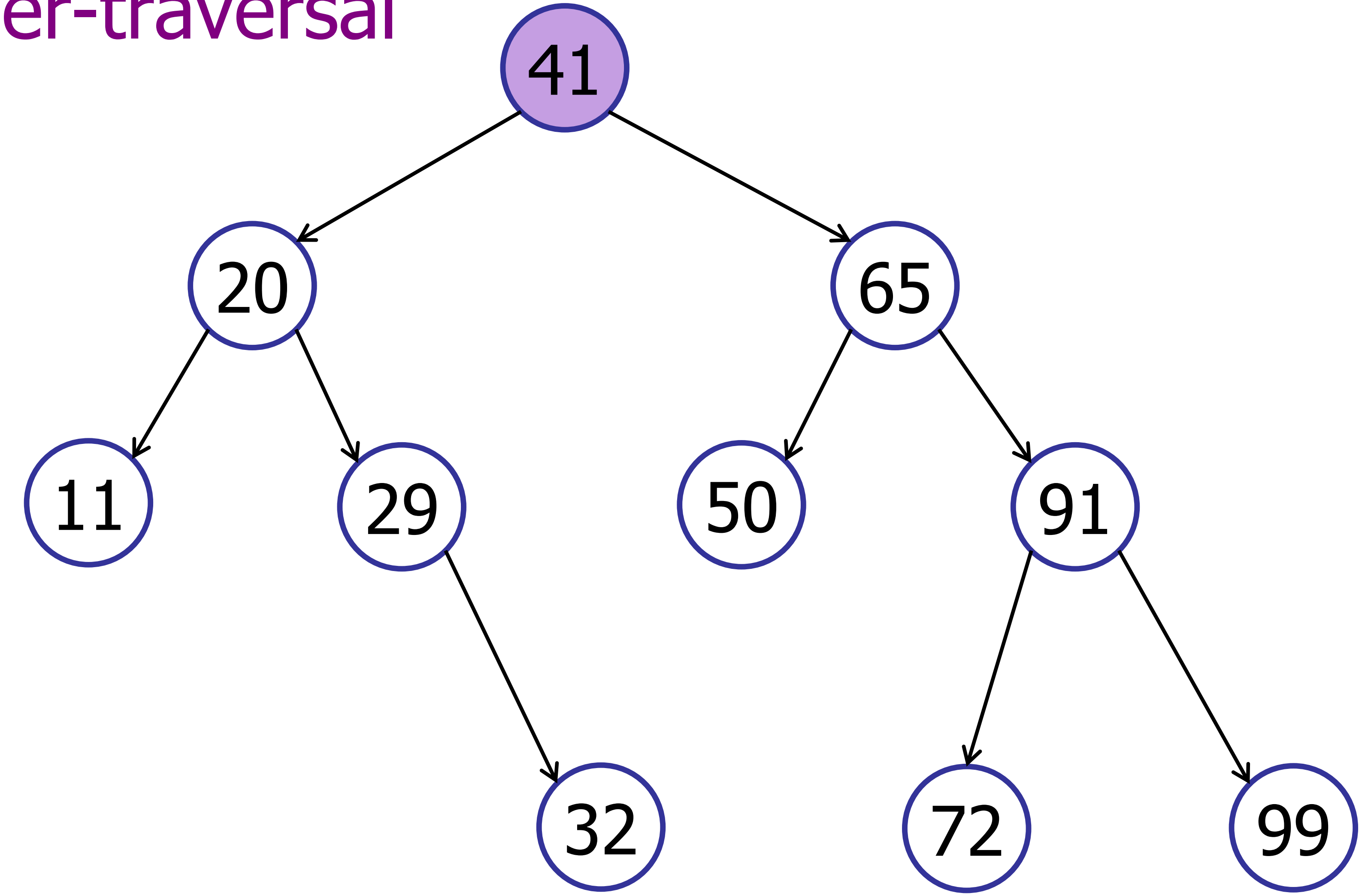
Tree Traversals

level-order-traversal



Tree Traversals

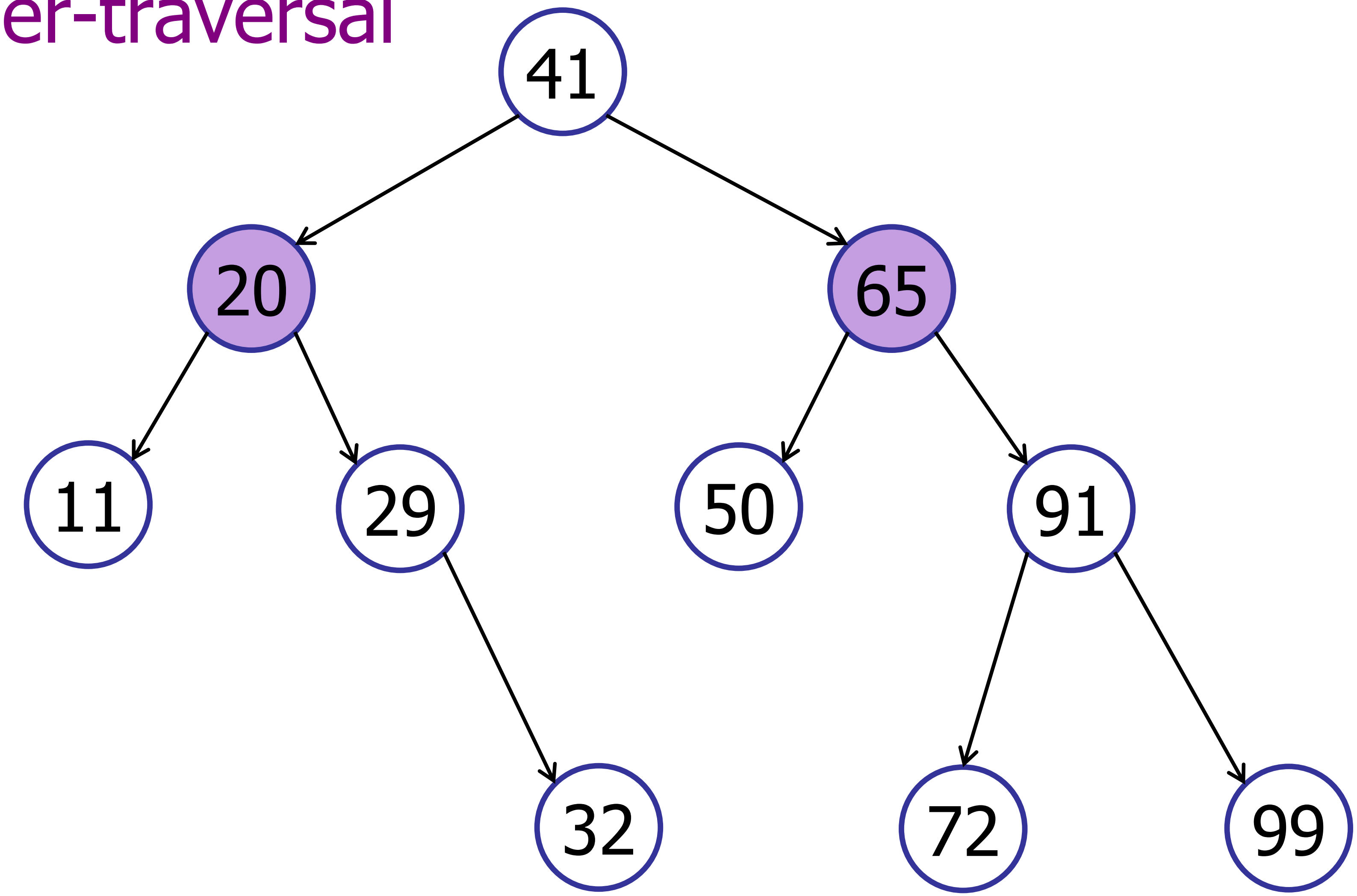
level-order-traversal



41

Tree Traversals

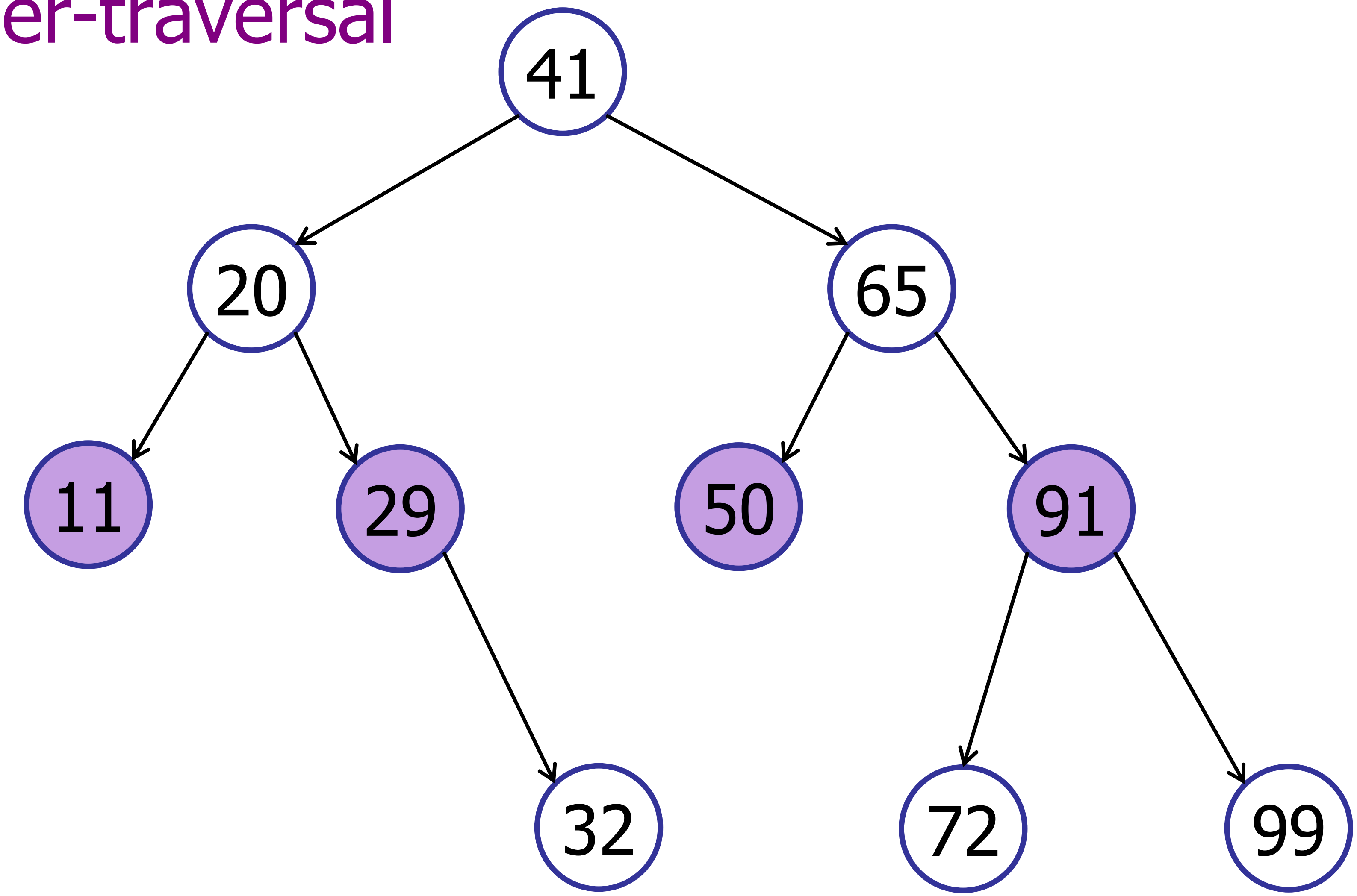
level-order-traversal



41 20 65

Tree Traversals

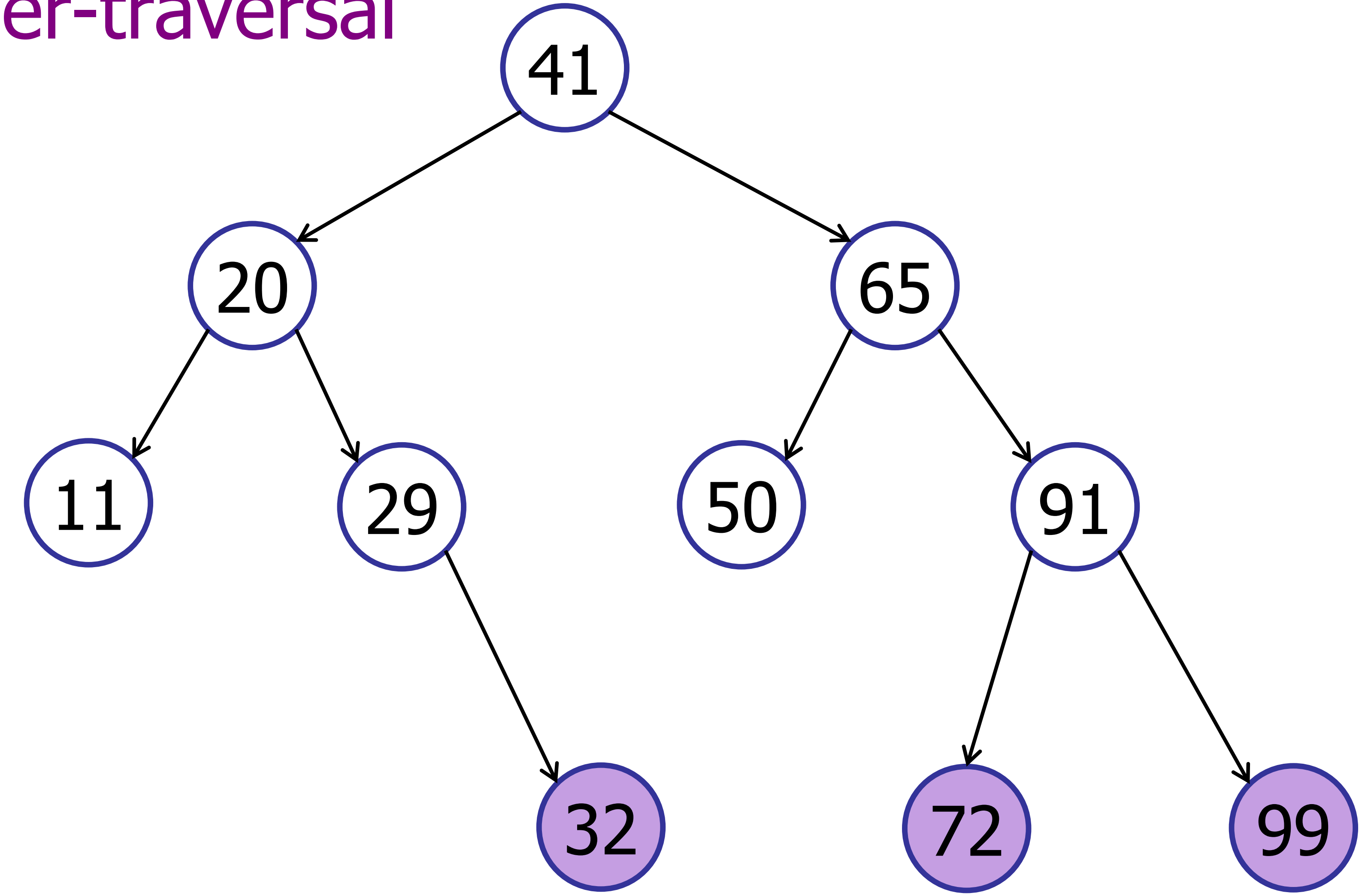
level-order-traversal



41 20 65 11 29 50 91

Tree Traversals

level-order-traversal



41 20 65 11 29 50 91 32 72 99

Tree Traversals

Several varieties:

- pre-order
- in-order
- post-order
- level-order

Binary Search Trees

1. Terminology and Definitions

2. Basic operations:
   - height
   - searchMin, searchMax
   - search, insert

3. Traversals
   - in-order, pre-order, post-order

4. Other operations

Airport Scheduling

Dictionary

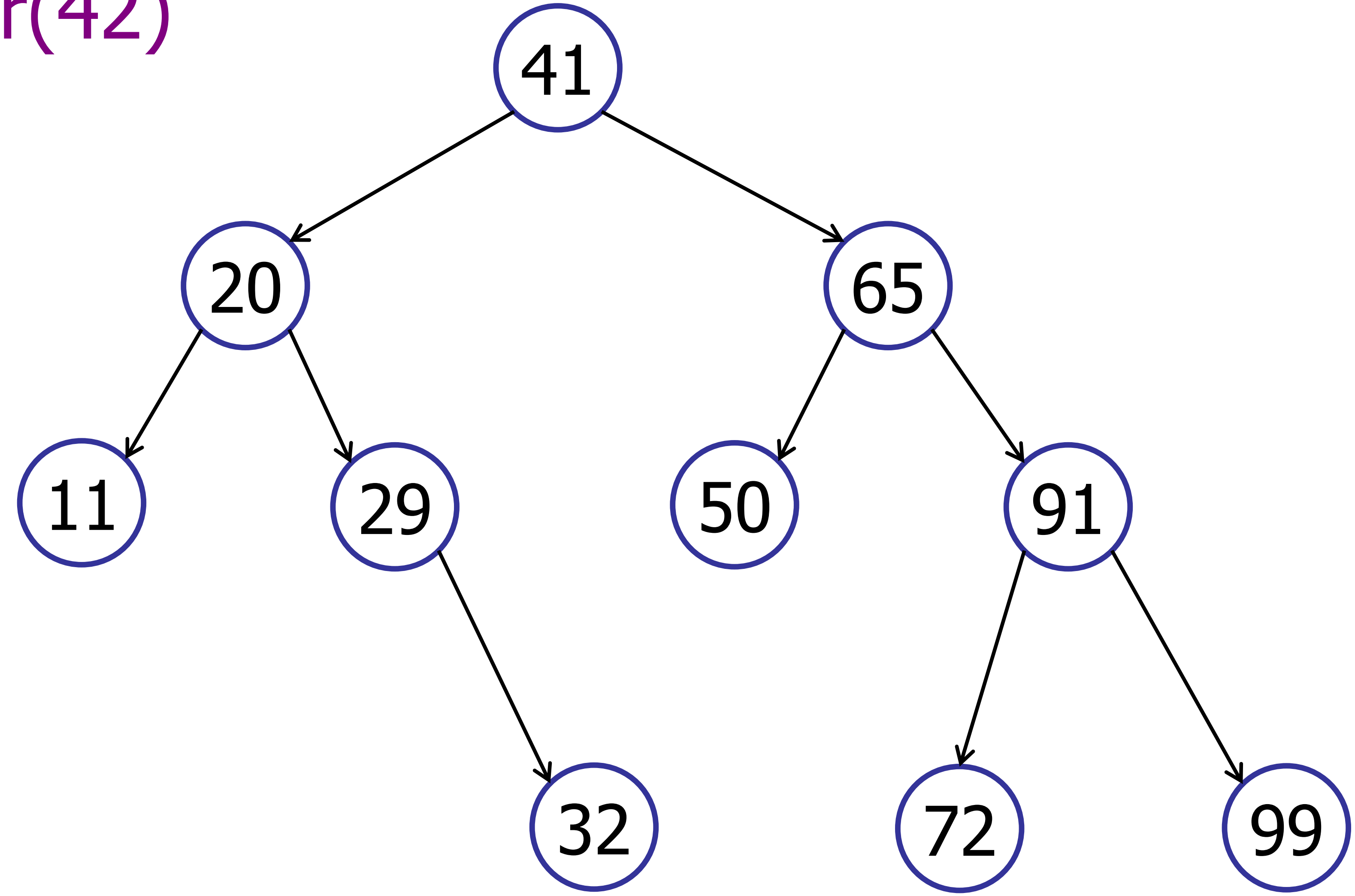
6:35	**7:00**	**7:19**	**8:21**	**12:21**	**14:23**	**14:42**			

– successor(8:24) = 12:21

How do we implement this?

Successor Queries

successor(42)

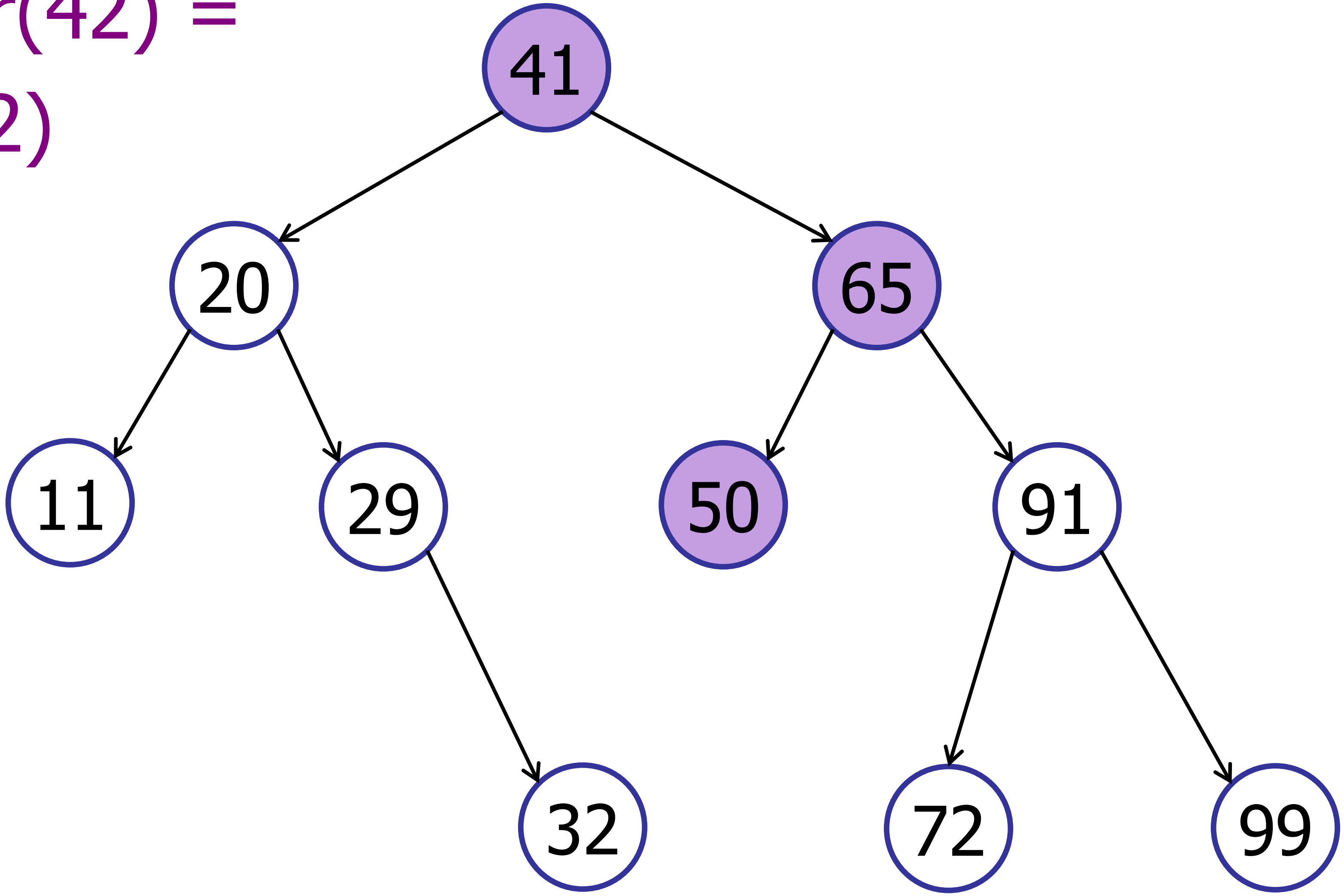


Key 42 is not in the tree

Successor Queries

successor(42) =

search(42)

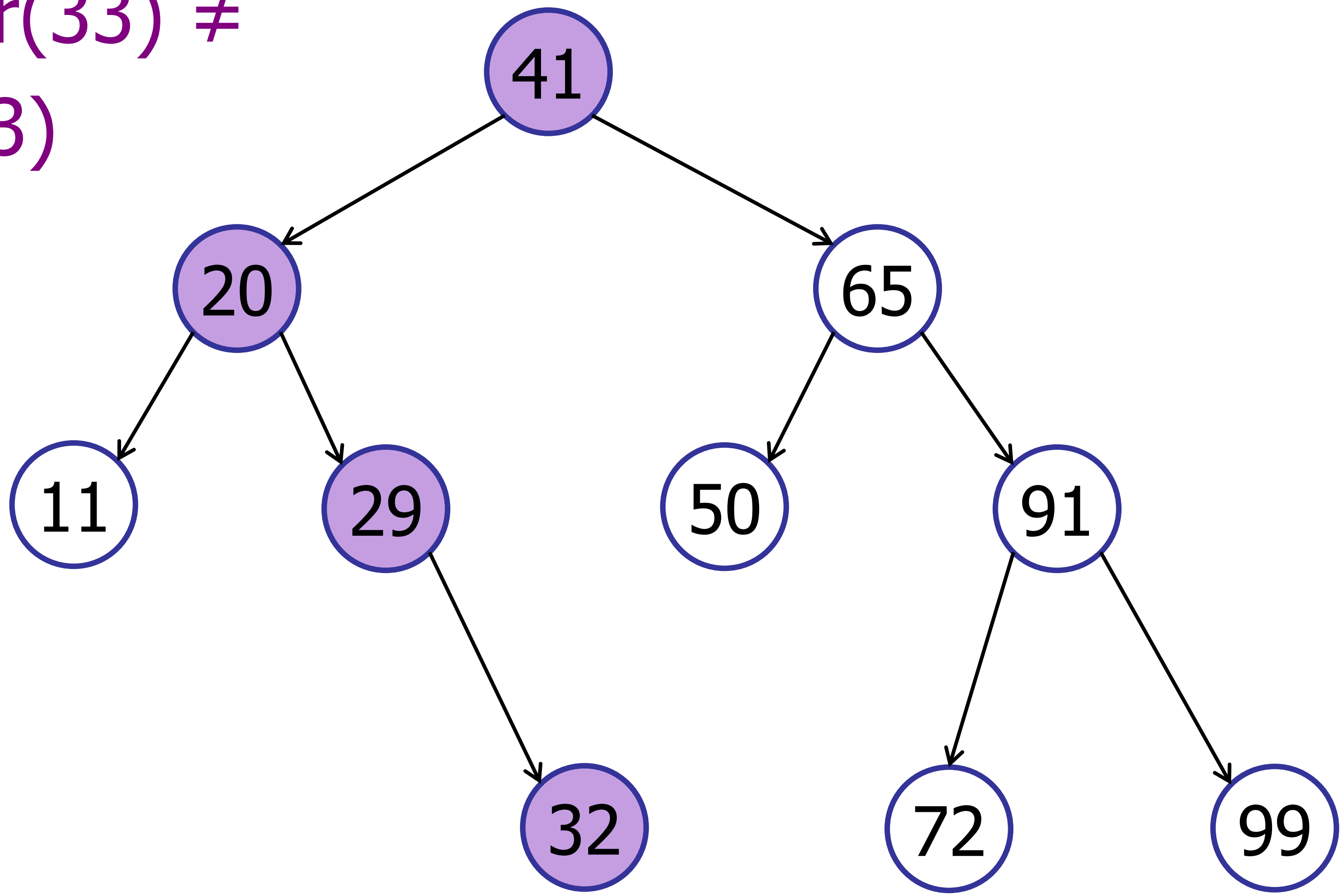


Key 42 is not in the tree

Successor Queries

successor(33) ≠

search(33)



Key 33 is not in the tree

Successor Queries

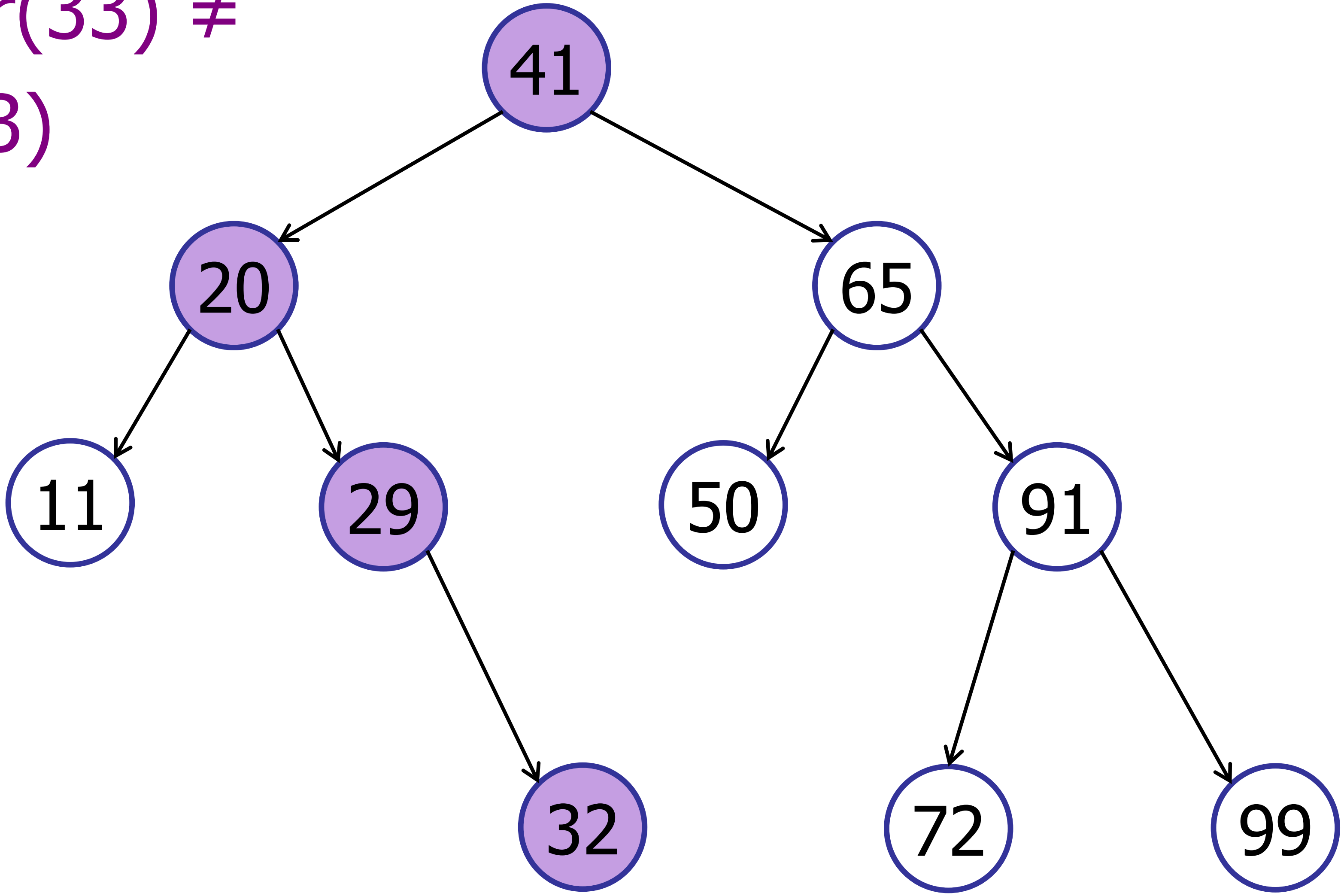
Basic strategy: successor(key)

1. Search for key in the tree.
2. If (result > key), then return result.
3. If (result <= key), then search for successor of result.

Successor Queries

successor(33) ≠

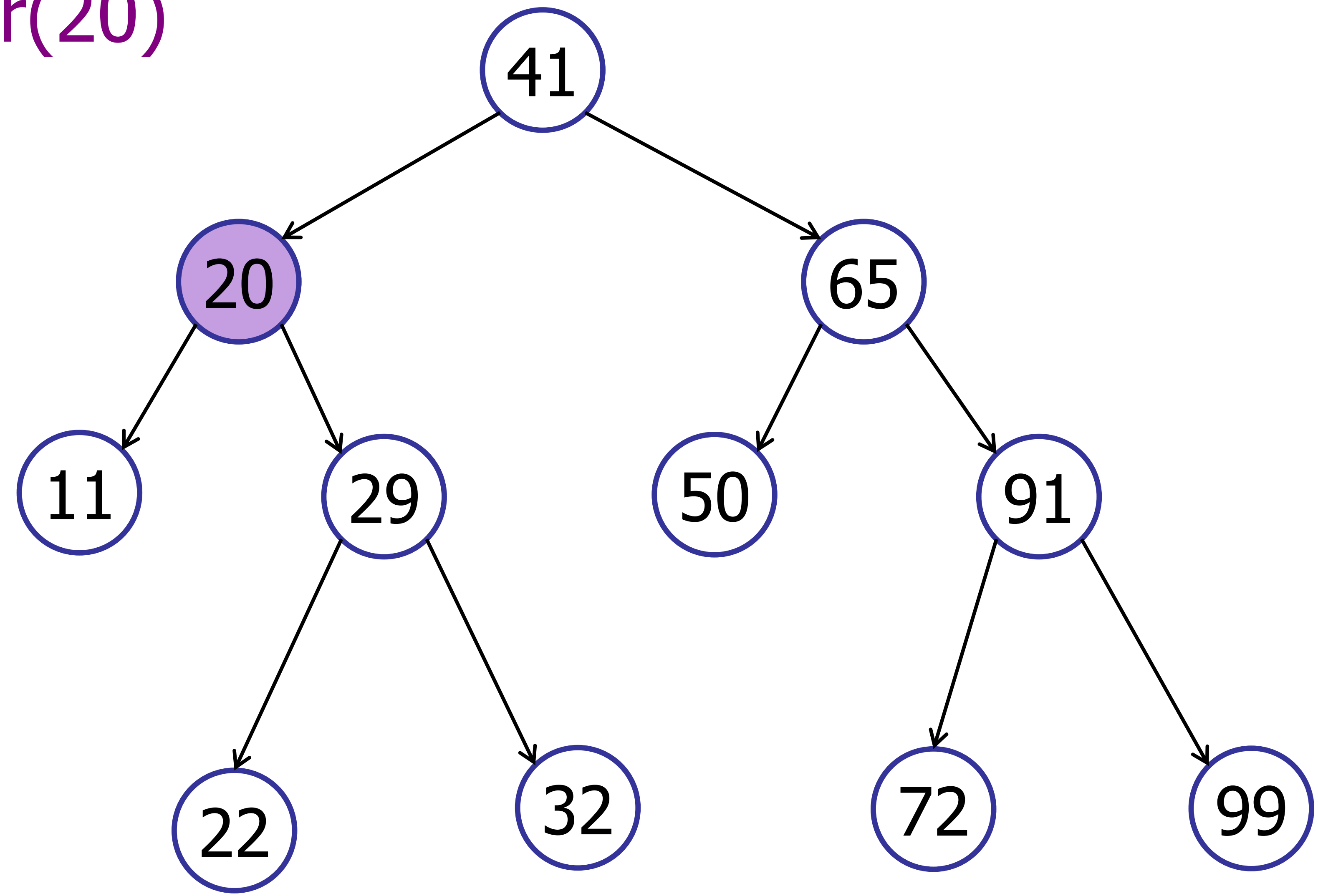
search(33)



Key 33 is not in the tree

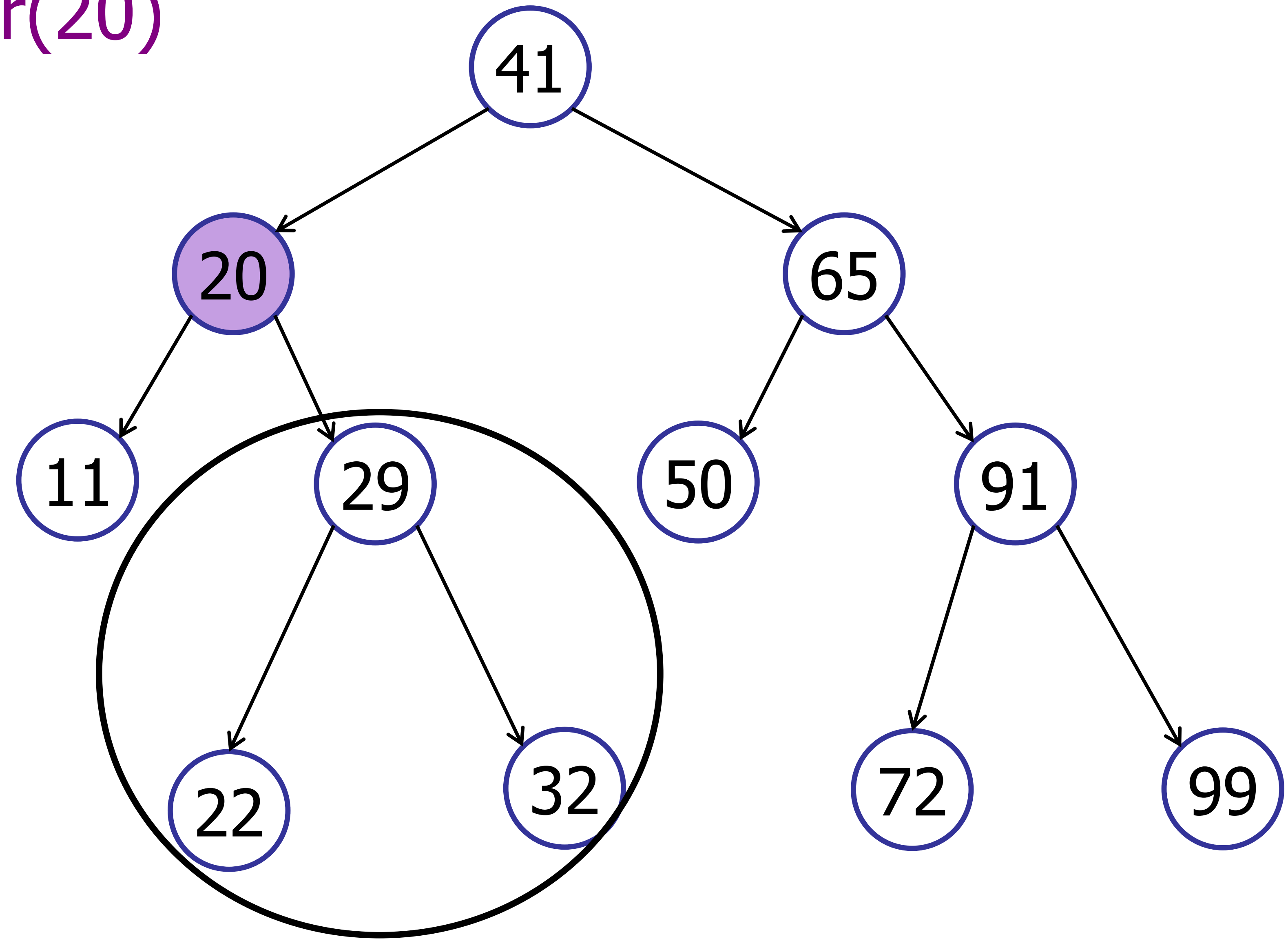
Successor Queries

successor(20)



Successor Queries

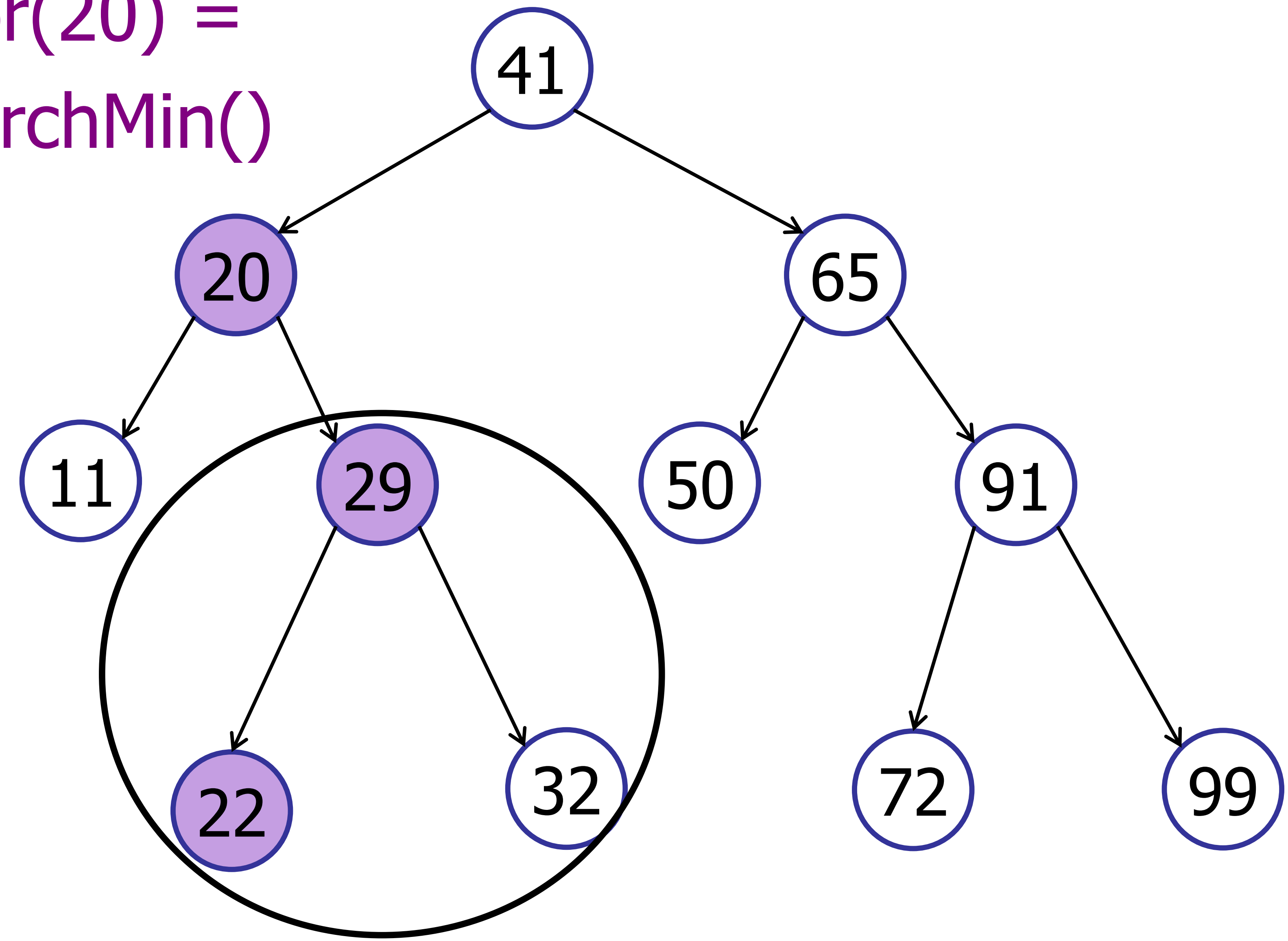
successor(20)



Case 1: node has a right child.

Successor Queries

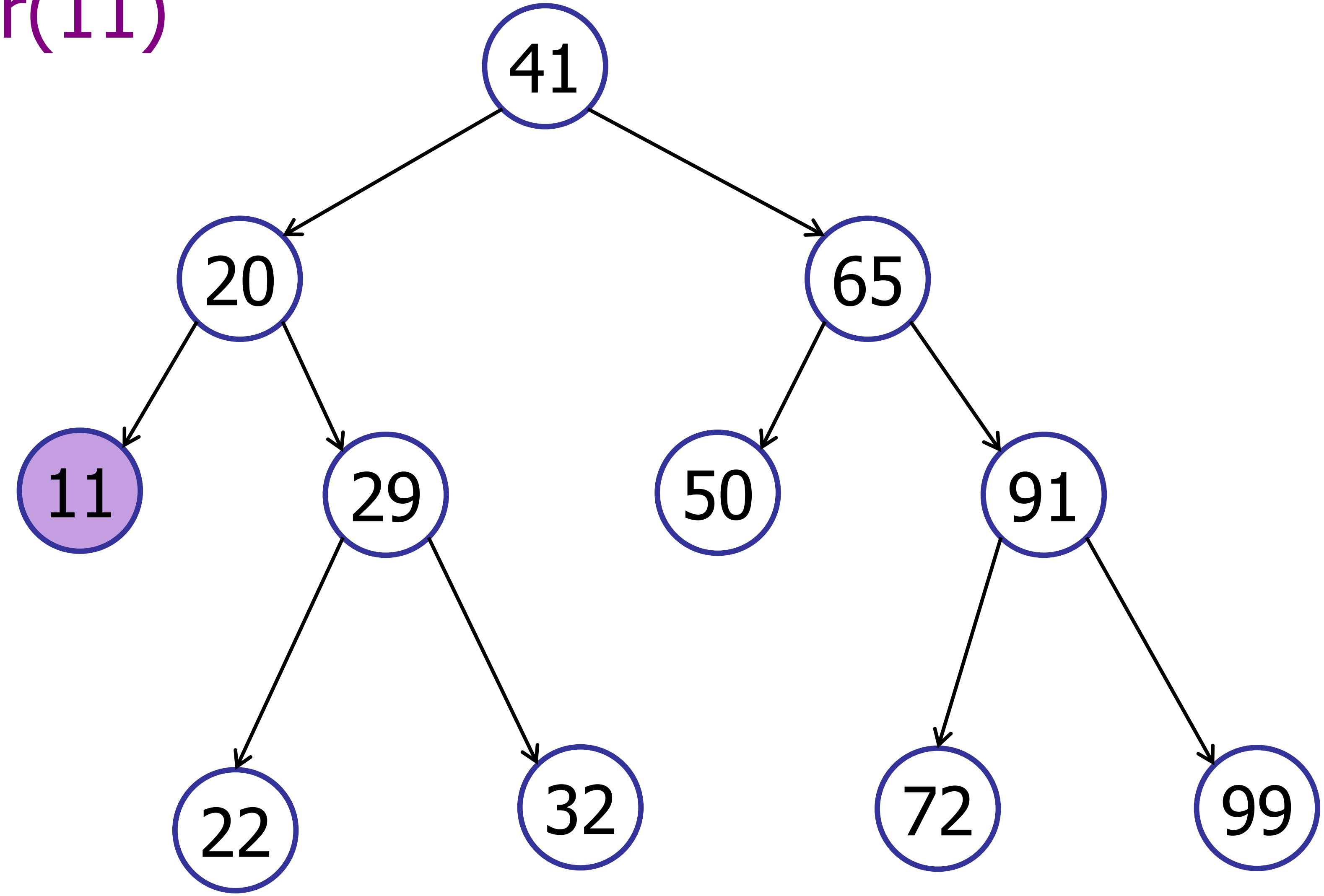
successor(20) =
right.searchMin()



Case 1: node has a right child.

Successor Queries

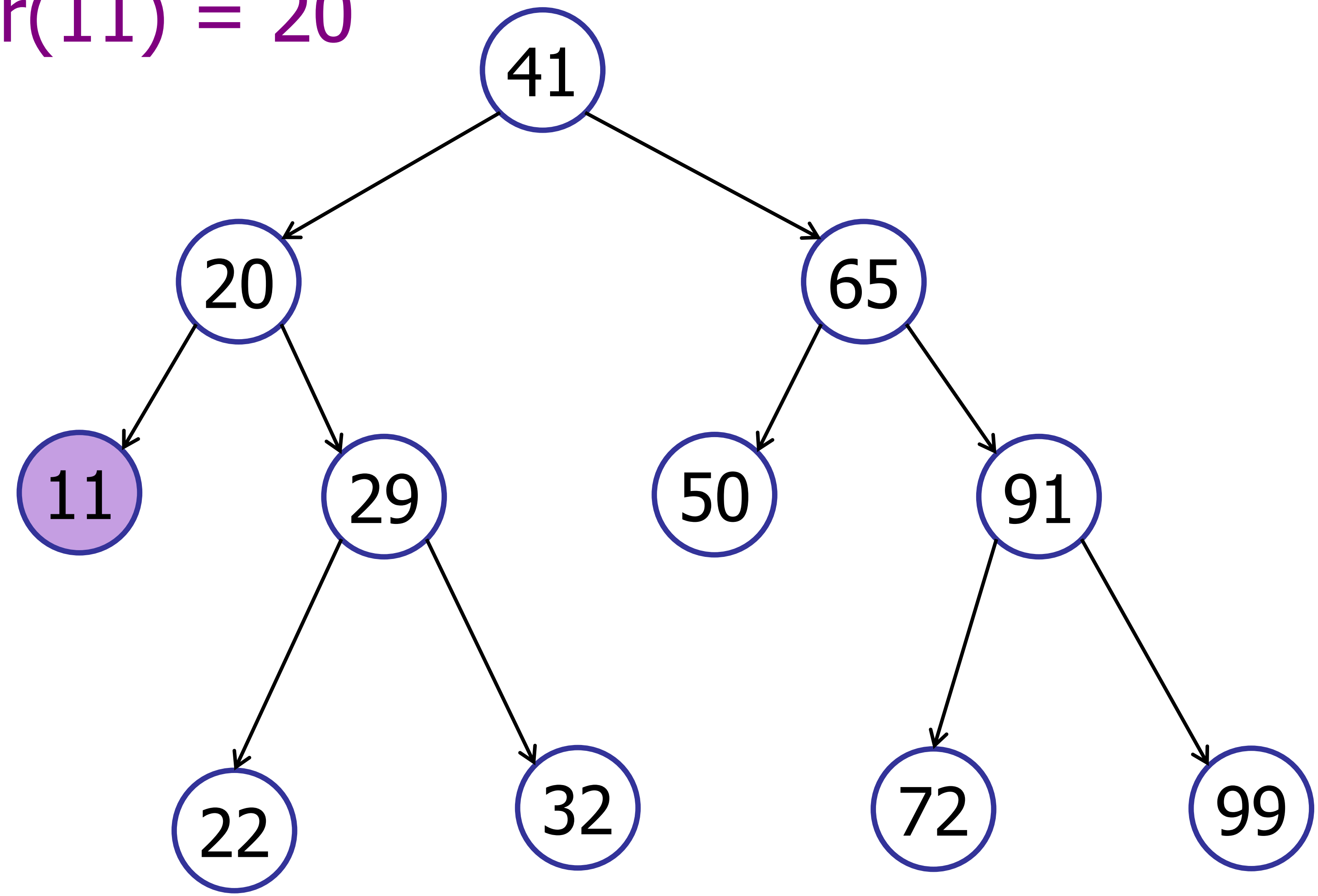
successor(11)



Case 2: node has no right child.

Successor Queries

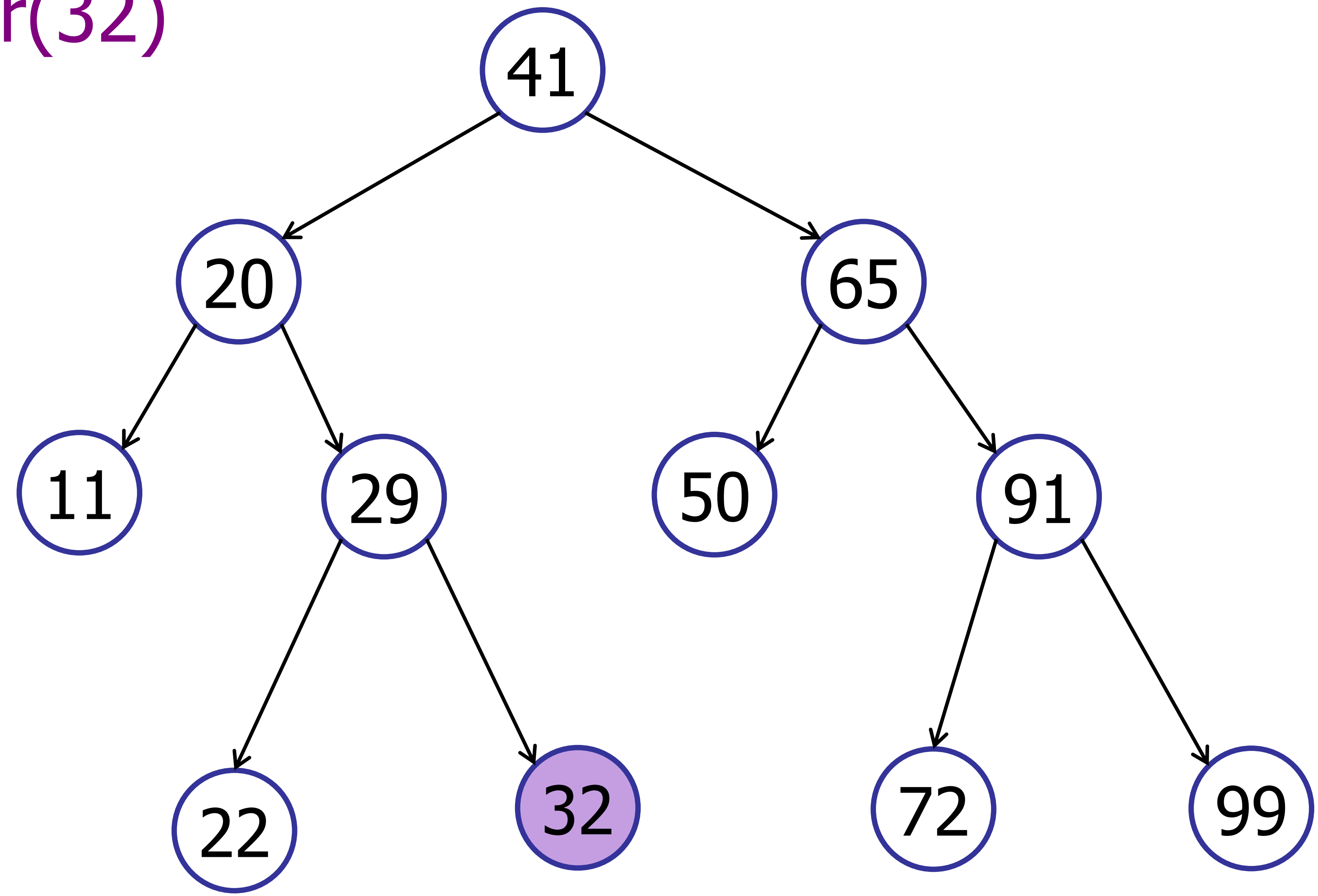
successor(11) = 20



Case 2: node has no right child.

Successor Queries

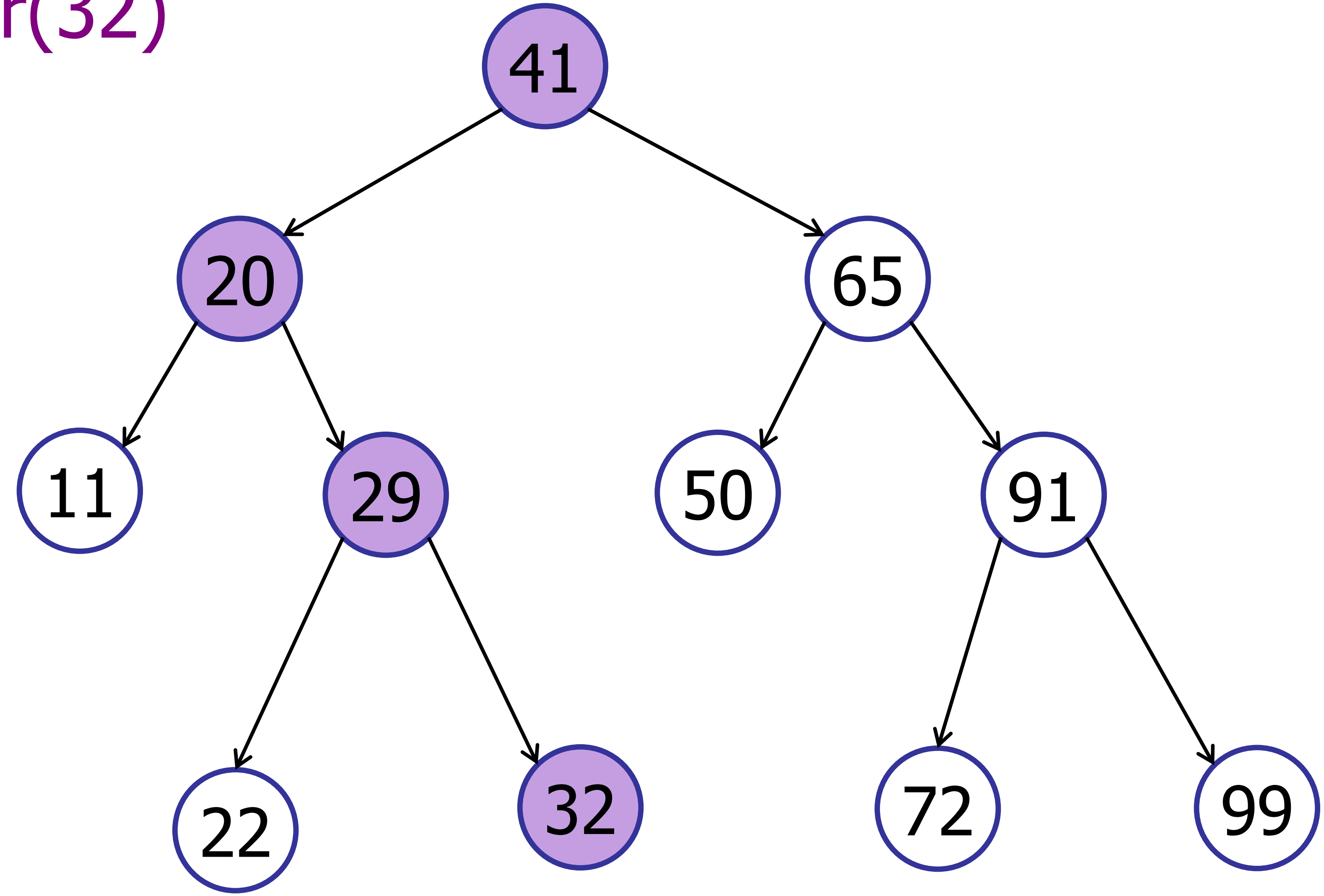
successor(32)



Case 2: node has no right child.

Successor Queries

successor(32)



Case 2: node has no right child.

Successor Queries

Find the next TreeNode:

```
public TreeNode successor(){
        if (rightTree != null)
                return rightTree.searchMin();

        TreeNode parent = parentTree;
        TreeNode child = this;
        while ((parent != null) && (child = parent.rightTree))
                child = parent;
                parent = child.parentTree;
        }
        return parent;
}
```

Binary Search Trees

1. Terminology and Definitions

2. Basic operations:
   - height
   - searchMin, searchMax
   - search, insert

3. Traversals
   - in-order, pre-order, post-order

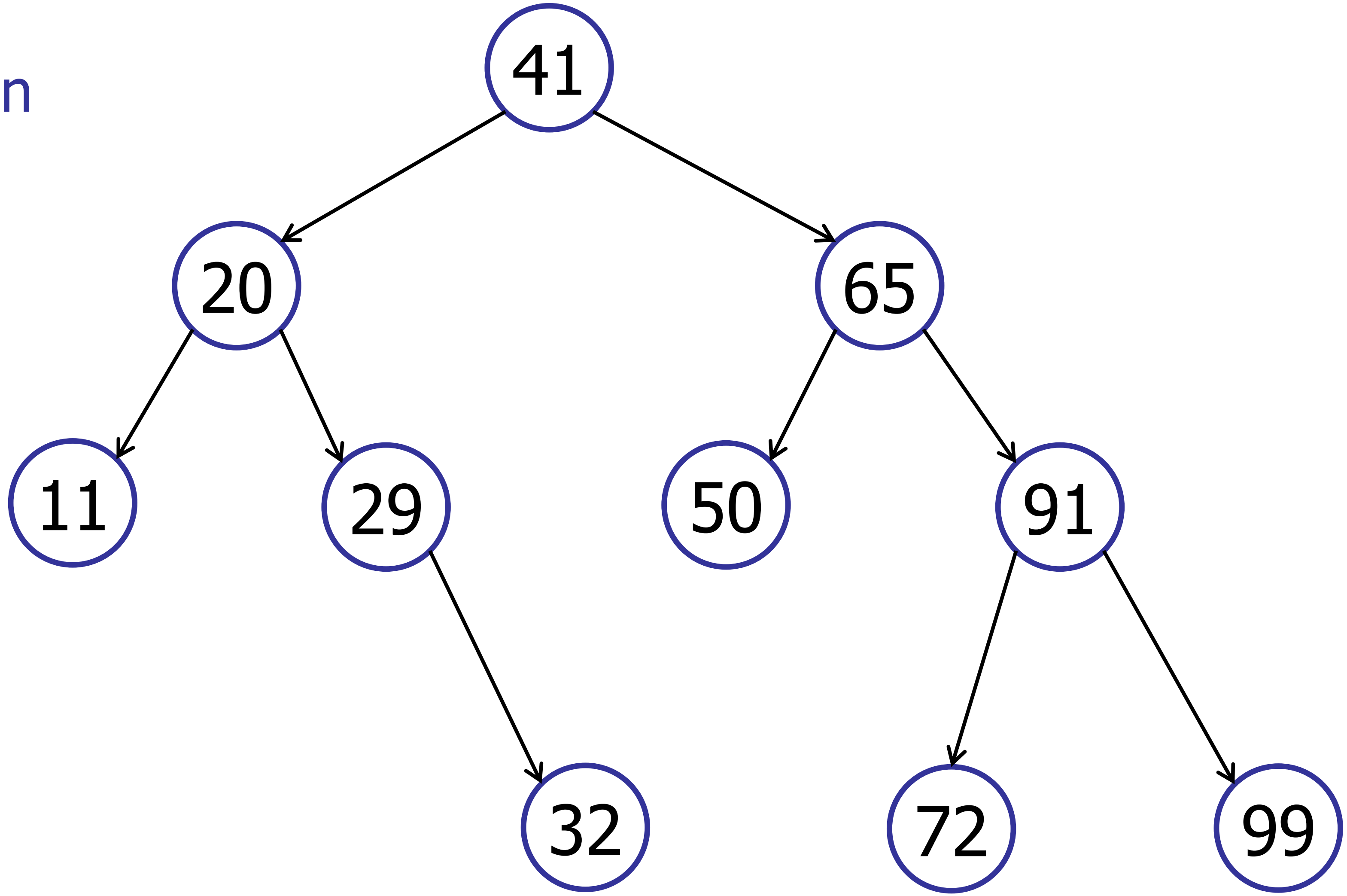
4. Other operations

Binary Search Tree

delete(v)

Three cases:

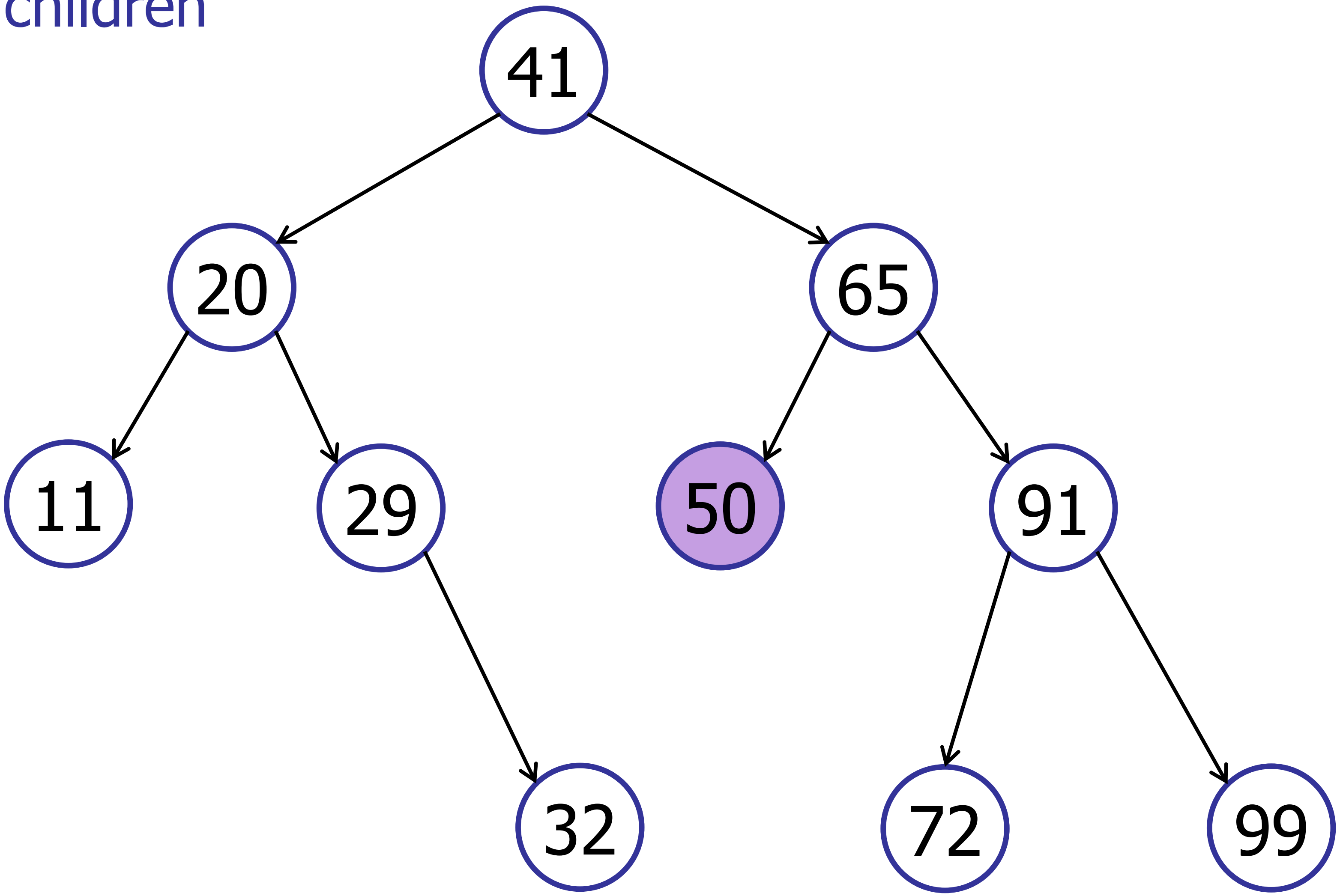
1. No children
2. 1 child
3. 2 children



Binary Search Tree

delete(50)

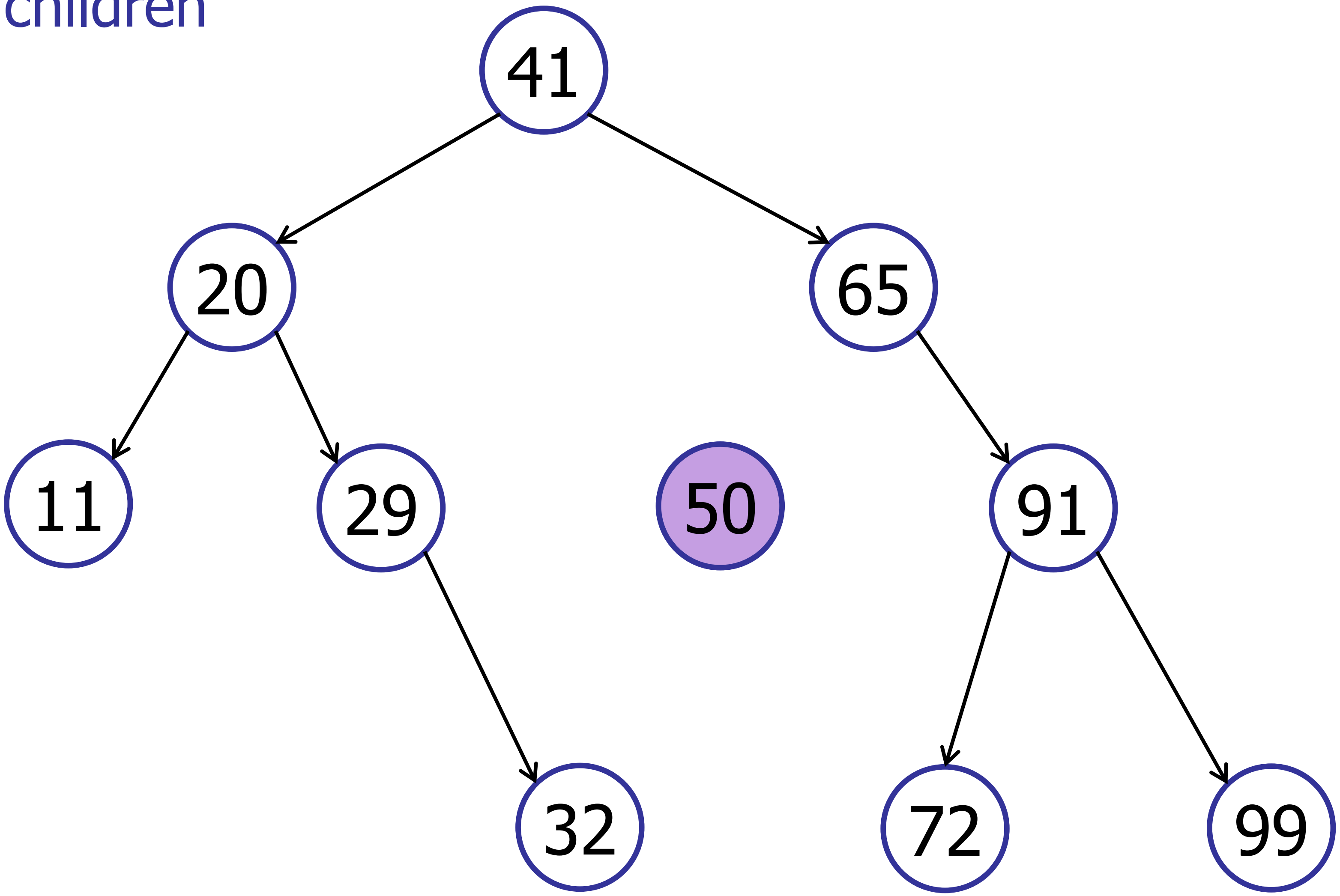
Case 1: No children



Binary Search Tree

delete(50)

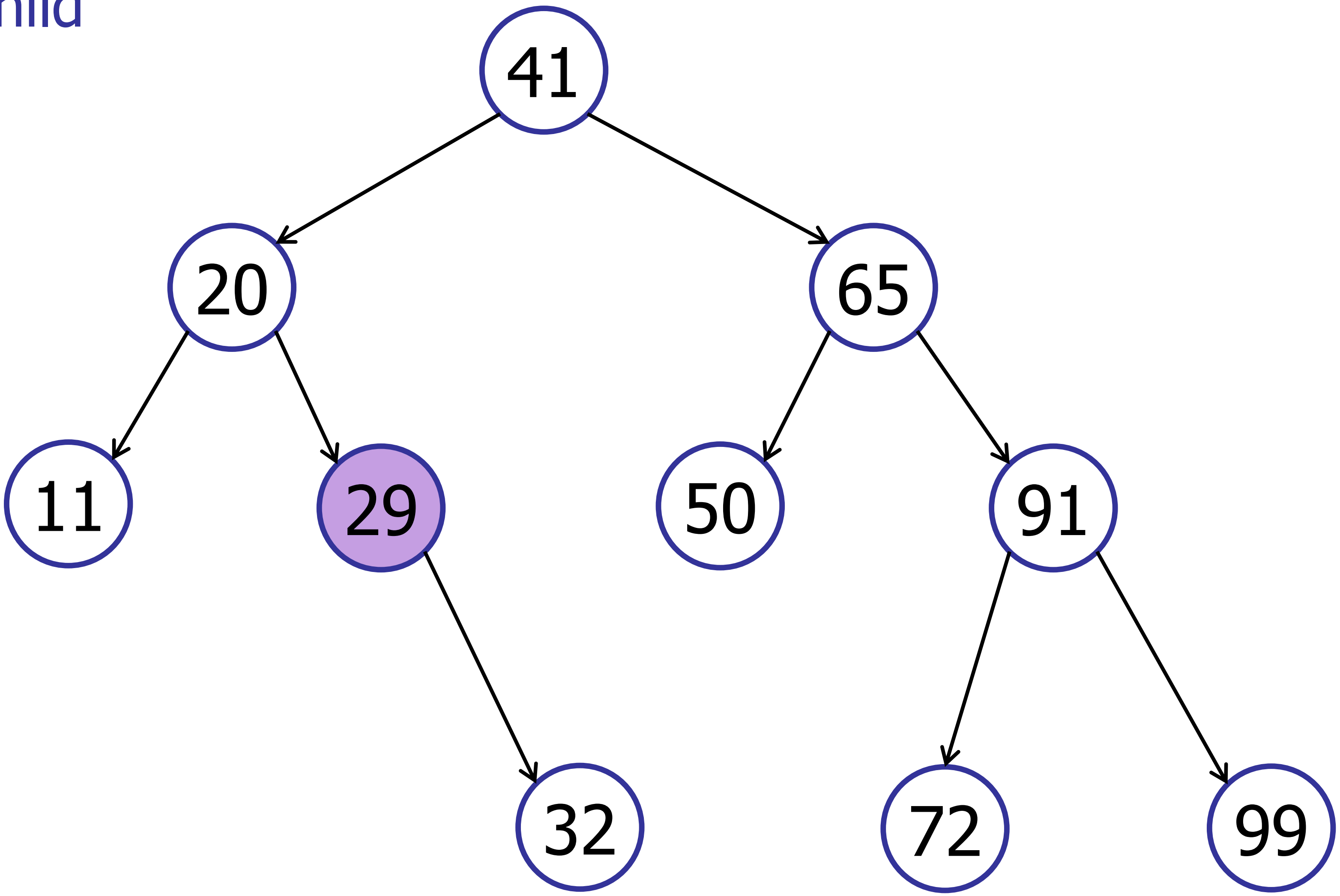
Case 1: No children



Binary Search Tree

delete(29)

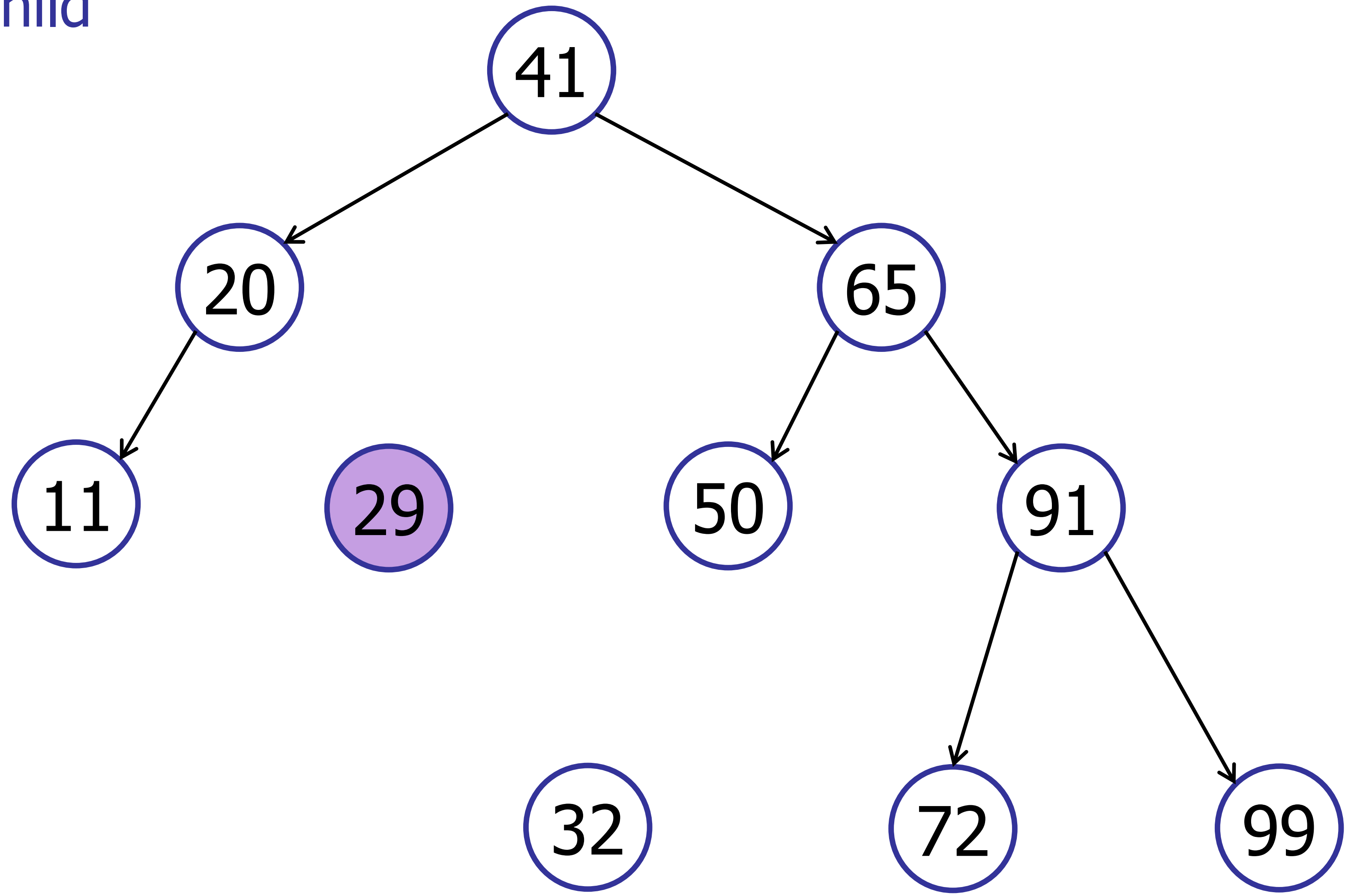
Case 2: 1 child



Binary Search Tree

delete(29)

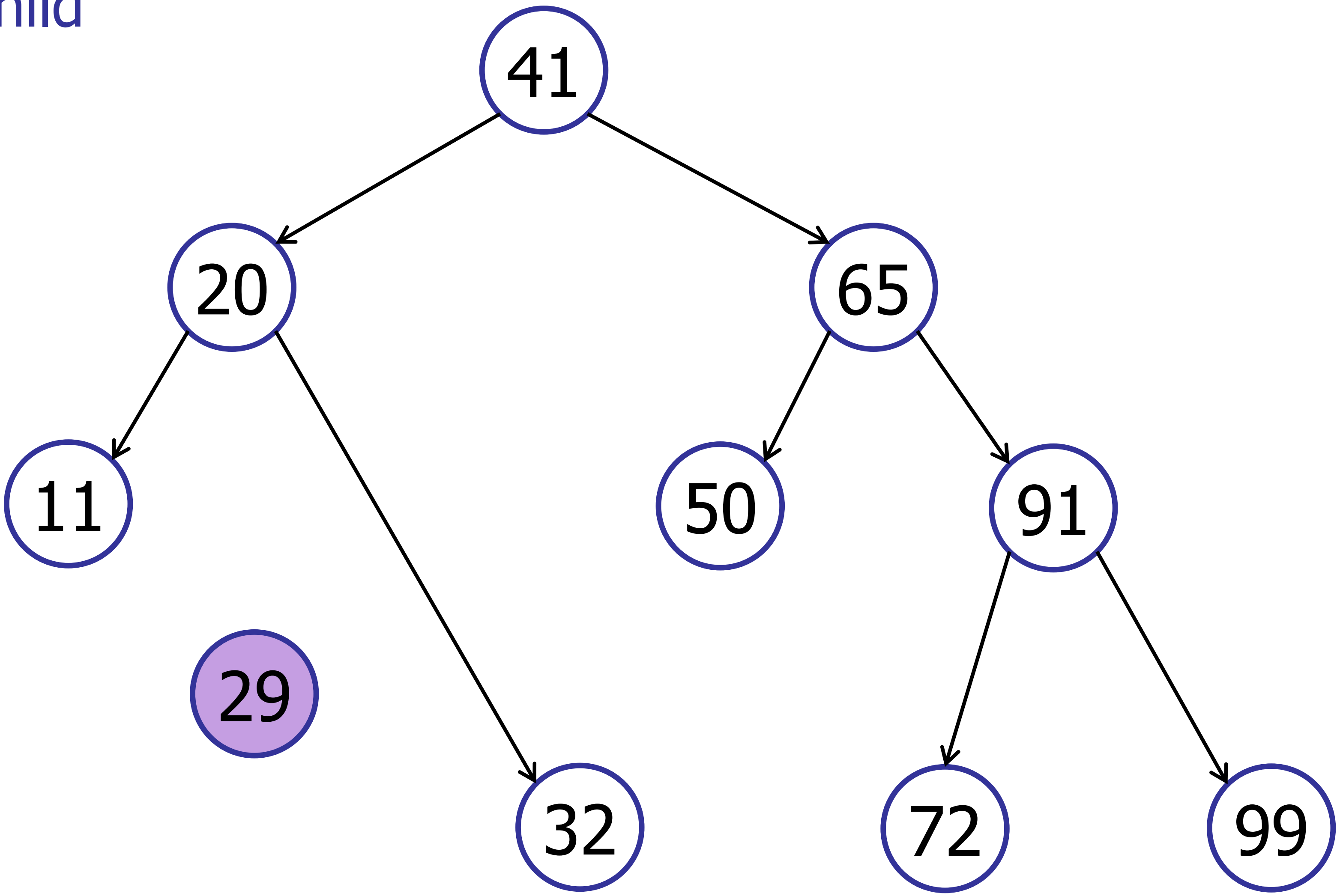
Case 2: 1 child



Binary Search Tree

delete(29)

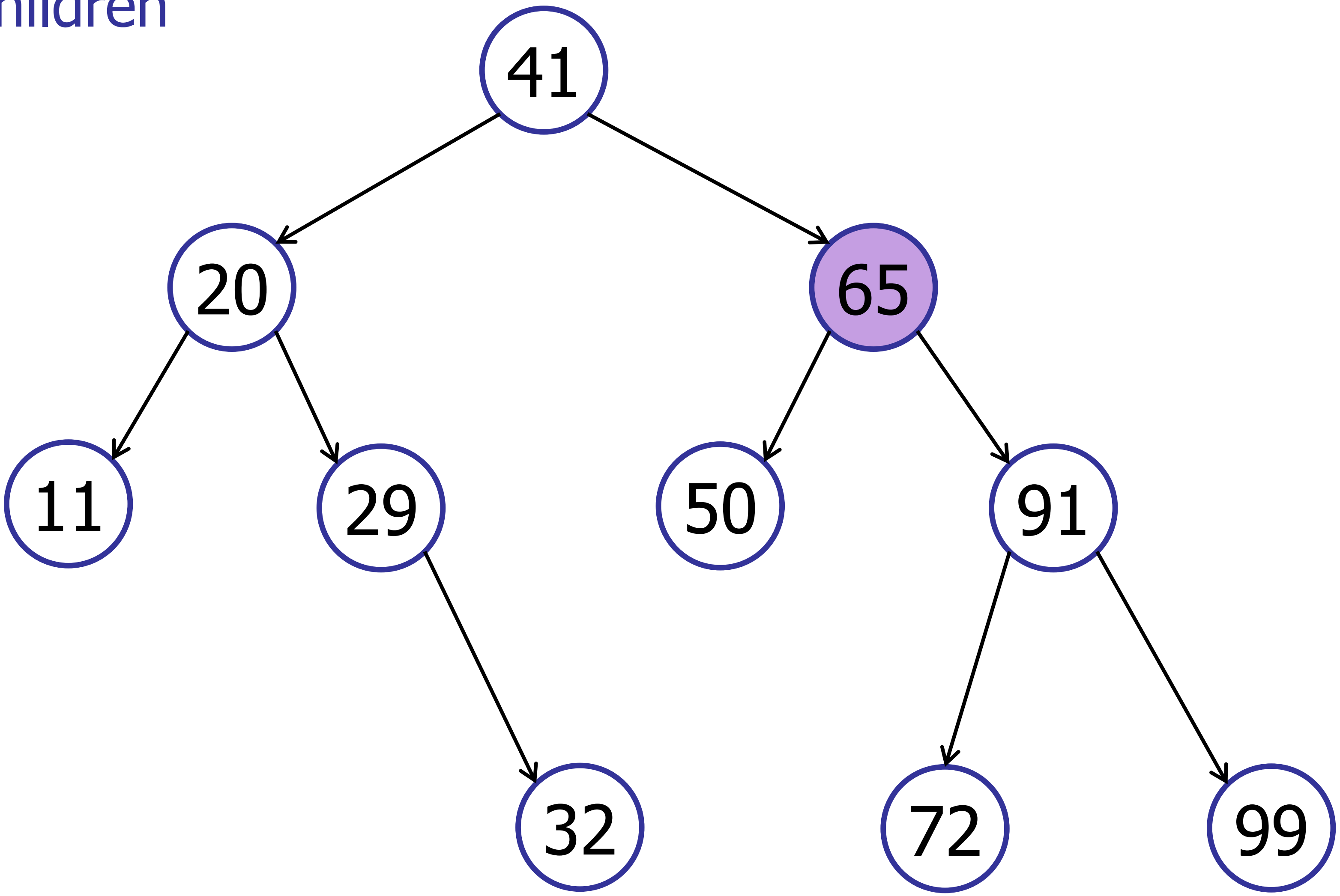
Case 2: 1 child



Binary Search Tree

delete(65)

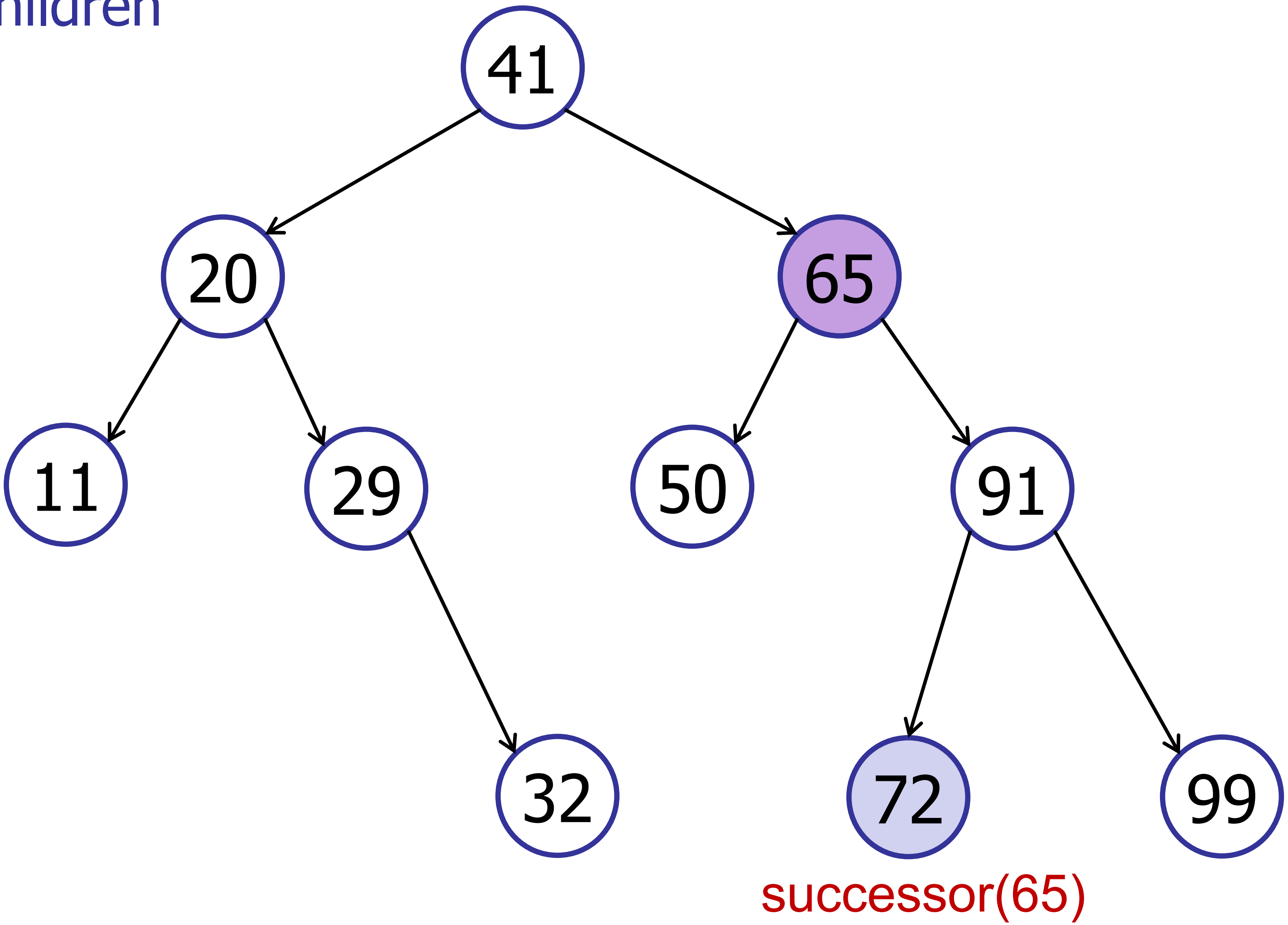
Case 3: 2 children



Binary Search Tree

delete(65)

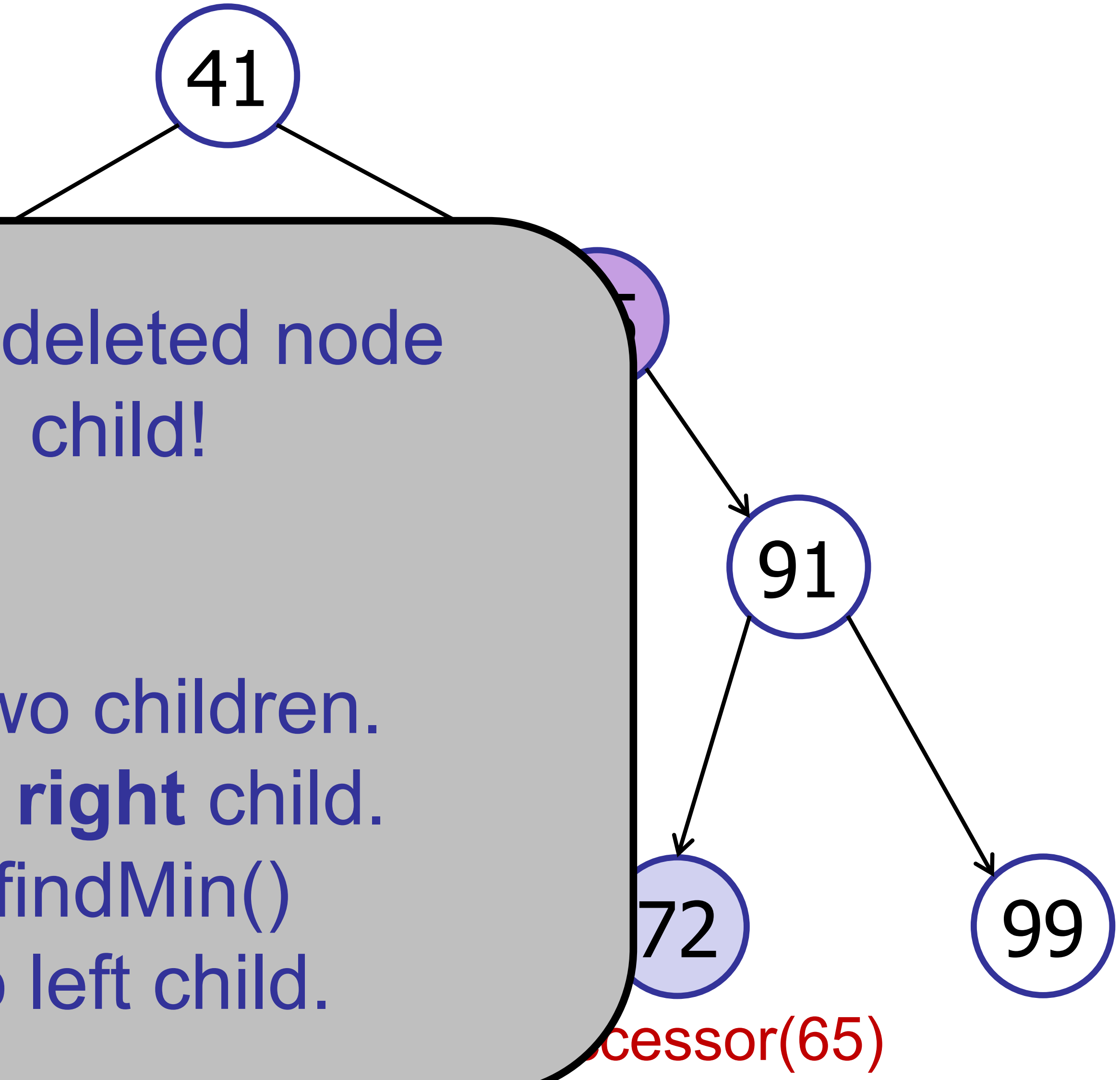
Case 3: 2 children



Binary Search Tree

delete(65)

Case 3: 2 children



Claim: successor of deleted node has at most 1 child!

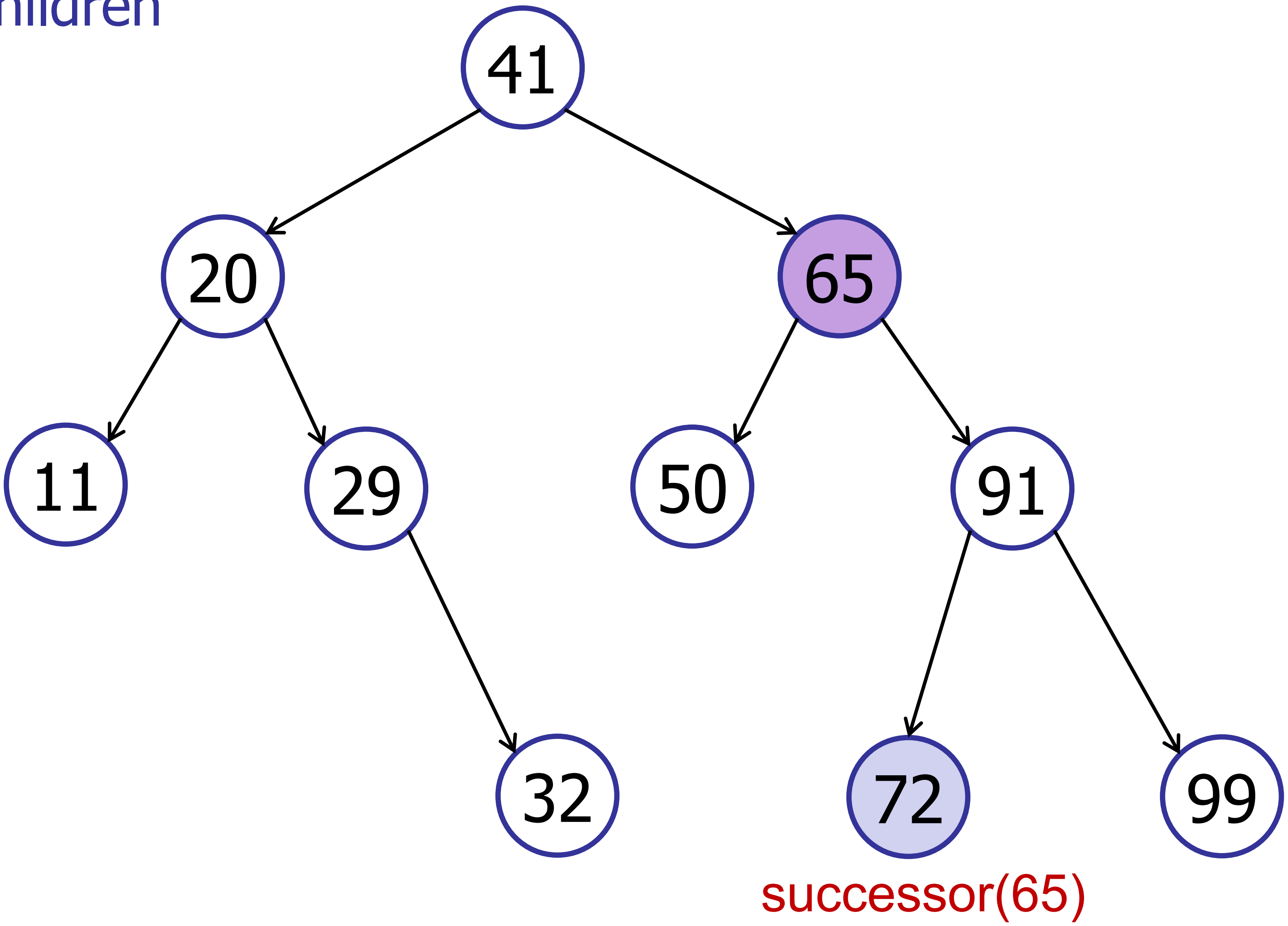
Proof:

- DeletedNode has two children.
- DeletedNode has a **right** child.
- successor() = right.findMin()
- min element has no left child.

Binary Search Tree

delete(65)

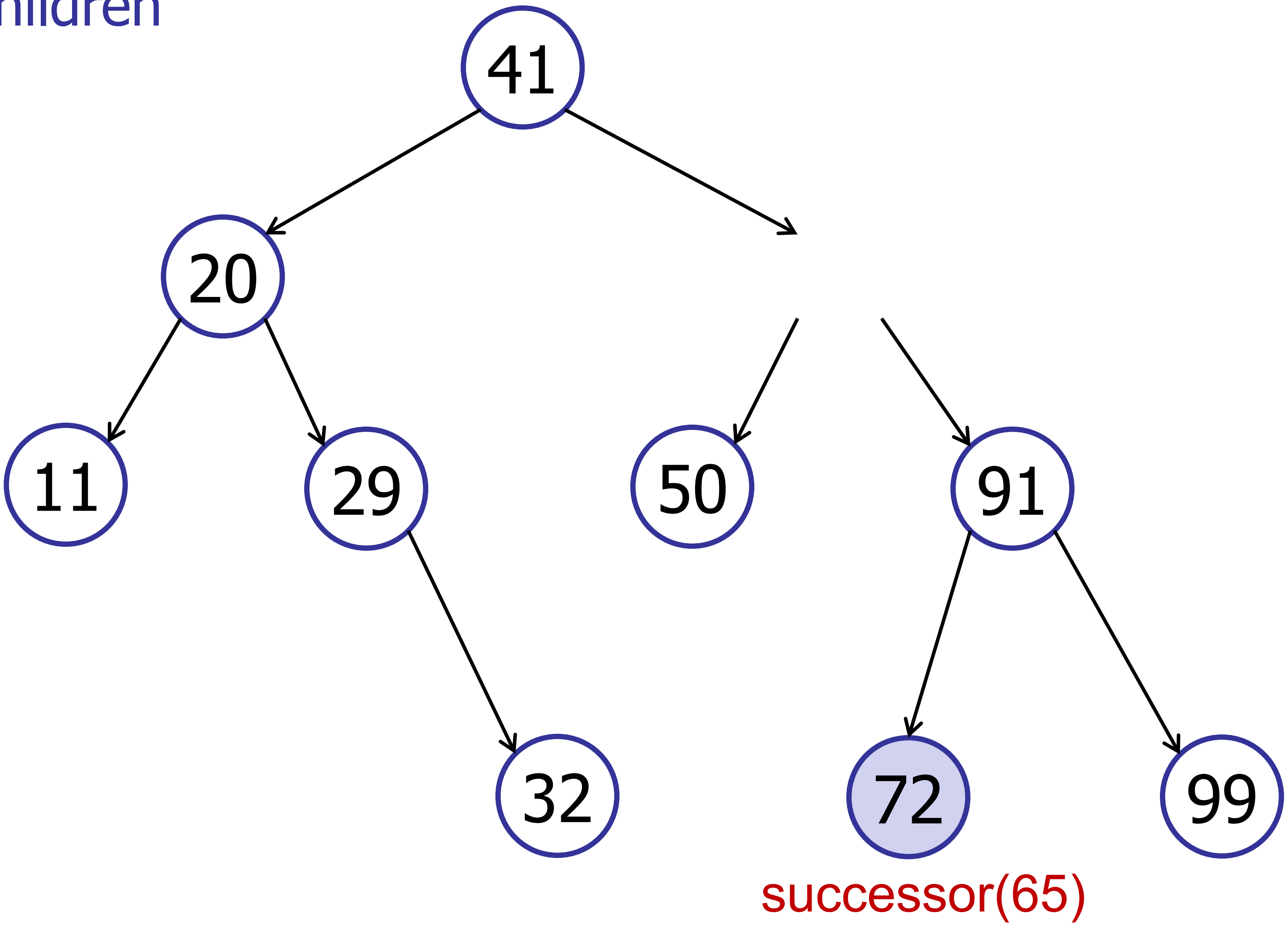
Case 3: 2 children



Binary Search Tree

delete(65)

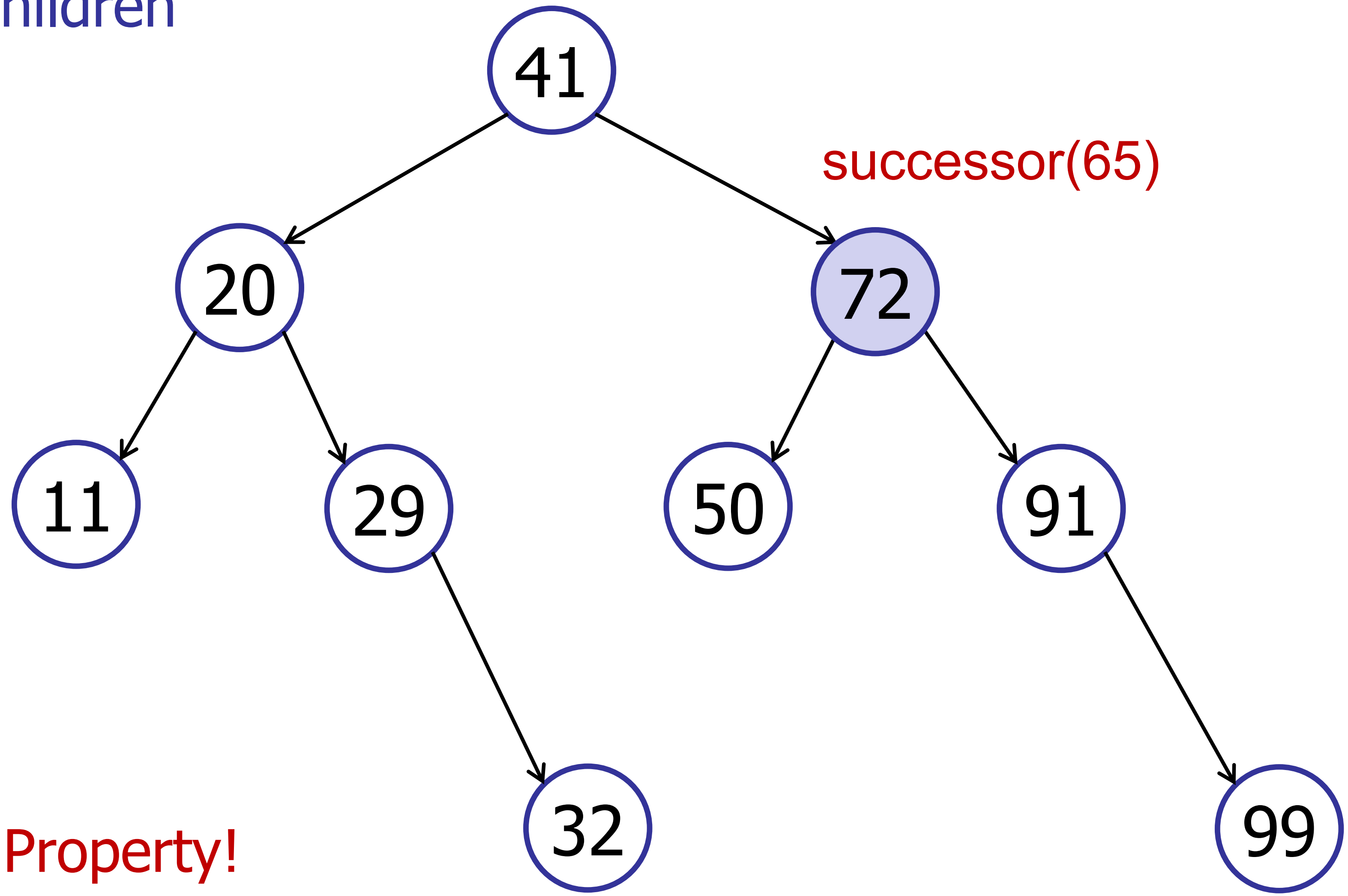
Case 3: 2 children



Binary Search Tree

delete(65)

Case 3: 2 children



Check BST Property!

Binary Search Tree

delete(v)

Running time: O(h)

Three cases:

1. No children:
   - remove v
2. 1 child:
   - remove v
   - connect child(v) to parent(v)
3. 2 children
   - x = successor(v)
   - delete(x)
   - remove v
   - connect x to left(v), right(v), parent(v)

Binary Search Tree

Modifying Operations

- insert: O(h)
- delete: O(h)

Query Operations:

- search: O(h)
- predecessor, successor: O(h)
- findMax, findMin: O(h)
- in-order-traversal: O(n)

Plan of the Day

Trees

- Terminology
- Traversals
- Operations

Balanced Trees

- Height-balanced binary search trees
- AVL trees
- Rotations