# CS2040S
# Data Structures and Algorithms
(e-learning edition)

## Hashing II
## (Part 1)

# Today

- Java hashing
- Collision resolution: open addressing
- Table (re)sizing

## Review: Symbol Table Abstract Data Type

Which of the following is *not* typically a symbol table operation?

1. insert(key, data)
2. delete(key)
3. successor(key)
4. search(key)
5. None of the above.

# Hashing in Java

How does your program know which hash function to use?

```
HashMap<MyFoo, Integer> hmap = new ...
MyFoo foo = new MyFoo();

hmap.put(foo, 8);
```

# Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn’t changed.
- If two objects are equal, then they return the same hashCode.

Is it legal for every object to return 32?

# Hashing in Java

How does your program know which hash function to use?

```
HashMap<MyFoo, Integer> hmap = new ...
MyFoo foo = new MyFoo();


hmap.put(foo, 8);


int hash = foo.hashCode();
```

# Java Object

## Every class extends Object

**`public class Object`**

| | | |
|---|---|---|
| `Object` | `clone()` | *creates a copy* |
| **`boolean`** | **`equals(Object obj)`** | ***is obj equal to this?*** |
| `void` | `finalize()` | *used by garbage collector* |
| `Class` | `getClass()` | *returns class* |
| **`int`** | **`hashCode()`** | ***calculates hash code*** |
| `void` | `notify()` | *wakes up a waiting thread* |
| `void` | `notifyAll()` | *wakes up all waiting threads* |
| **`String`** | **`toString()`** | ***returns string representation*** |
| `void` | `wait(…)` | *wait until notified* |

# Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.

Is it legal for every object to return 32?

# Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Default Java implementation:

- hashCode returns the memory location of the object
- Every object hashes to a different location

Must override `hashCode()` for your class.

# Java Library Hashcodes

`Integer`

`Long`

`String`

# Integer

```
public int hashCode() {
   return value;
}
```

# Long

```java
public int hashCode() {
   return (int)(value ^ (value >>> 32));
}
```

$$\text{hash}(\overbrace{0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,1\,0\,0}^{\text{32 bits}}\ \overbrace{0\,0\,1\,0\,1\,0\,0\,0\,0\,1\,1\,0\,0\,1\,0\,0}^{\text{32 bits}})$$

$$\begin{array}{rl} & 0\,1\,1\,0\,0\,1\,0\,1\,1\,0\,0\,1\,1\,1\,0\,0 \\ \text{XOR} & 0\,0\,1\,0\,1\,0\,0\,0\,0\,1\,1\,0\,0\,1\,0\,0 \\ \hline & 0\,1\,0\,0\,1\,1\,0\,1\,1\,1\,1\,1\,1\,0\,0\,0 \end{array}$$

# String

```
public int hashCode() {
  int h = hash; // only calculate hash once
  if (h == 0 && count > 0) {  // empty = 0
       int off = offset;
       char val[] = value;
       int len = count;
       for (int i = 0; i < len; i++) {
            h = 31*h + val[off++];
       }
       hash = h;
  }
  return h;
}
```

# String

HashCode calculation:

```
hash = s[0]*31^(n-1) +
       s[1]*31^(n-2) +
       s[2]*31^(n-3) +
       ... +
       s[n-2]*31 +
       s[n-1]
```

Why did they choose 31?

# Creating a new class

```
public class Pair {
  private int first;
  private int second;

  Pair(int a, int b){
     first = a;
     second = b;
  }
}
```

# Creating a new class

```
public void testPair() {

  HashMap<Pair, Integer> htable =
           new HashMap<Pair, Integer>();

  Pair one = new Pair(20, 20);
  htable.put(one, 7);

  Pair two = new Pair(20, 20);
  int question = htable.get(two);
}
```

htable.get(new Pair(20, 20)) == ?

1. 1
2. 7
3. 11
4. null

# Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);

one.hashCode() != two.hashCode()
```

# Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);
htable.put(one, “first item”);

htable.get(one) → “first item”

htable.get(two) → null
```

# Creating a new class

```
public class Pair {
  private int first;
  private int second;

  Pair(int a, int b){
     first = a;
     second = b;
  }

  int hashCode(){
     return (first ^ second);
  }
}
```

# Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);
htable.put(one, “first item”);

htable.get(one) → “first item”
htable.get(two) → null

one.equals(two) → false
```

# Java Hash Functions

Every object supports the method:

```
int hashCode()
```

Rules:

- Always returns the same value, if the object hasn't changed.
- If two objects are equal, then they return the same hashCode.
- **Must redefine .equals to be consistent with hashCode.**

# Creating a new class

```
Pair one = new Pair(20, 20);
Pair two = new Pair(20, 20);
htable.put(one, “first item”);

htable.get(one) => “first item”

htable.get(two) => null
```

# Java Hash Functions

Every object supports the method:

```
boolean equals(Object o)
```

Rules:

- **Reflexive**: x.equals(x) ➔ true
- **Symmetric**: x.equals(y) == y.equals(x)
- **Transitive**: x.equals(y), y.equals(z) ➔ x.equals(z)
- **Consistent**: always returns the same answer
- **Null is null**: x.equals(null) ➔ false

# Java Hash Functions

Every object supports the method:

```
boolean equals(Object o)
```

```
boolean equals(Object p){
  if (p == null) return false;
  if (p == this) return true;

  if (!(p instanceOf Pair)) return false;
  Pair pair = (Pair)p;

  if (pair.first != first) return false;
  if (pair.second != second) return false;
  return true;
}
```

# Java HashMap

```
public V get(Object key) {
   if (key == null) return getForNullKey();
   int hash = hash(key.hashCode());
   for (Entry<K,V> e = table[indexFor(hash,table.length)];
        e != null;
        e = e.next)
   {
      Object k;
      if (e.hash==hash &&((k=e.key)==key)||key.equals(k)))
         return e.value;
   }
   return null;
}
```

Java checks if the key is equal to the item in the hash table before returning it!

# Java HashMap

```java
// This function ensures that hashCodes that differ only
// by constant multiples at each bit position have a
// bounded number of collisions (approximately 8 at
// default load factor).

static int hash(int h) {
   h ^= (h >>> 20) ^ (h >>> 12);
   return h ^ (h >>> 7) ^ (h >>> 4);
}
```

# Java HashMap

```
public V get(Object key) {
   if (key == null) return getForNullKey();
   int hash = hash(key.hashCode());
   for (Entry<K,V> e = table[indexFor(hash,table.length)];
        e != null;
        e = e.next)
   {
      Object k;
      if (e.hash==hash &&((k=e.key)==key)||key.equals(k)))
         return e.value;
   }
   return null;
}
```

# Today

- Java hashing
- Collision resolution: open addressing
- Table (re)sizing

# CS2040S
# Data Structures and Algorithms
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## Hashing II
## (Part 2)

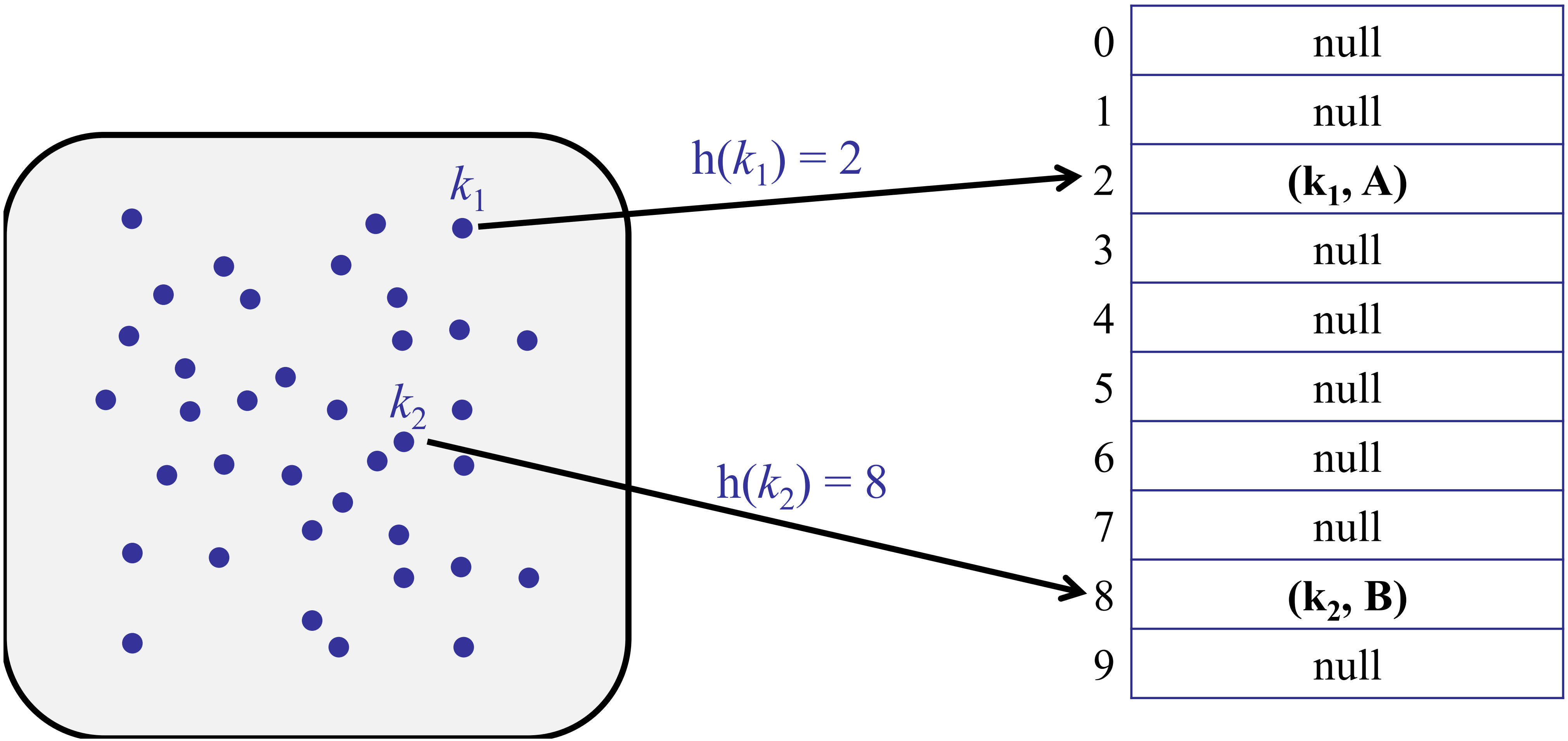
# Today

- Java hashing
- Collision resolution: open addressing
- Table (re)sizing

# Review

## Hash Tables

- Store each item from the symbol table in a table.
- Use hash function to map each key to a bucket.

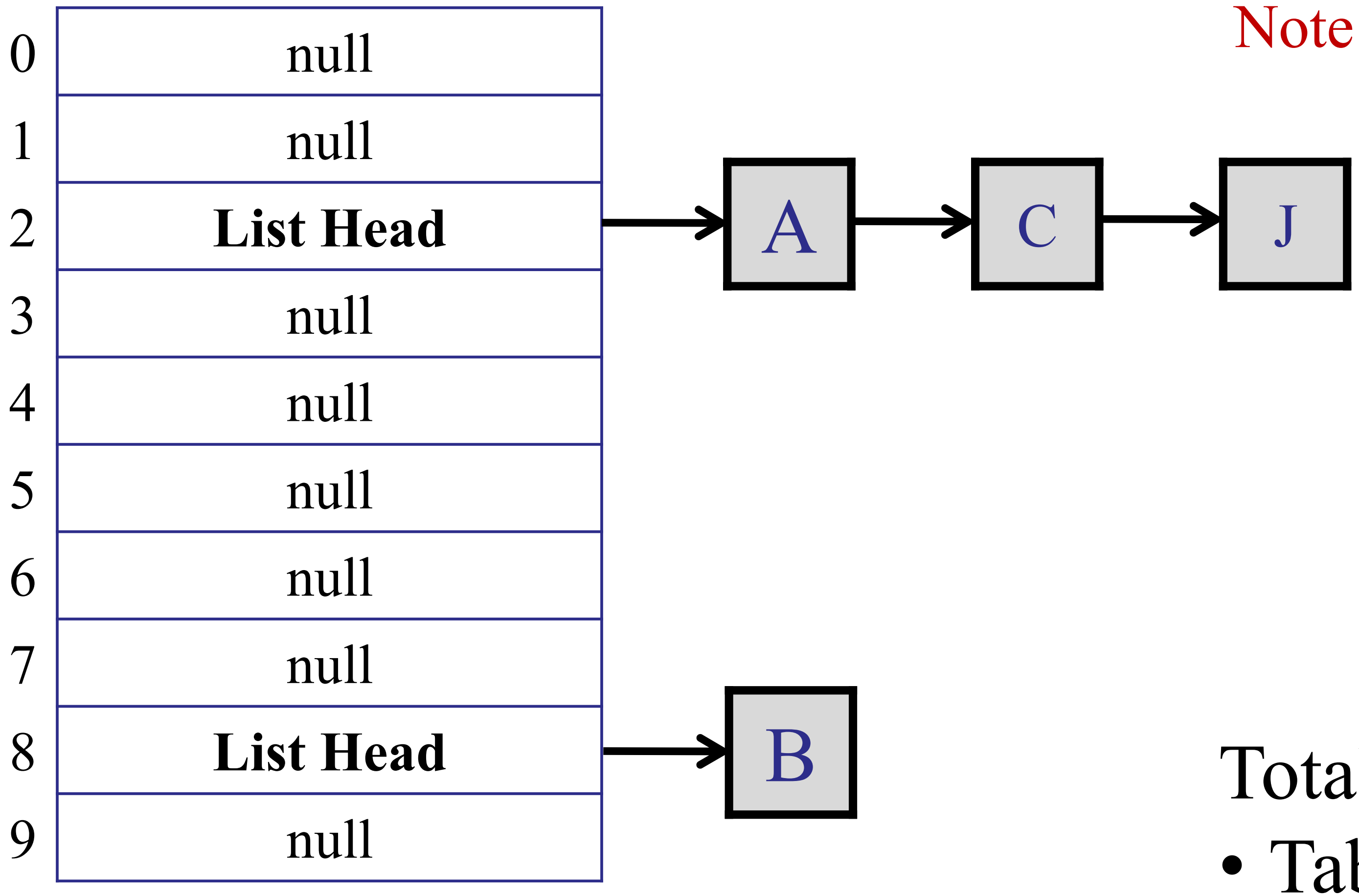


# Resolving Collisions

- Basic problem:
  - What to do when two items hash to the same bucket?
- Solution 1: Chaining
  - Insert item into a linked list.
- Solution 2: Open Addressing
  - Find another free bucket.

# Review: Chaining

Each bucket contains a linked list of items.




Note: h(A) == h(C) == h(J)

Total space: $O(m + n)$
- Table size: $m$
- Linked list size: $n$

# Review

The Simple Uniform Hashing Assumption

- Every key is equally likely to map to every bucket.

Load of a Hash Table:

- # elements: n
- # buckets: m
- Define: load(hash table) = $n/m$

  = average #items / bucket.
- Expected search time = $1 + n/m$

# Open Addressing

Advantages:

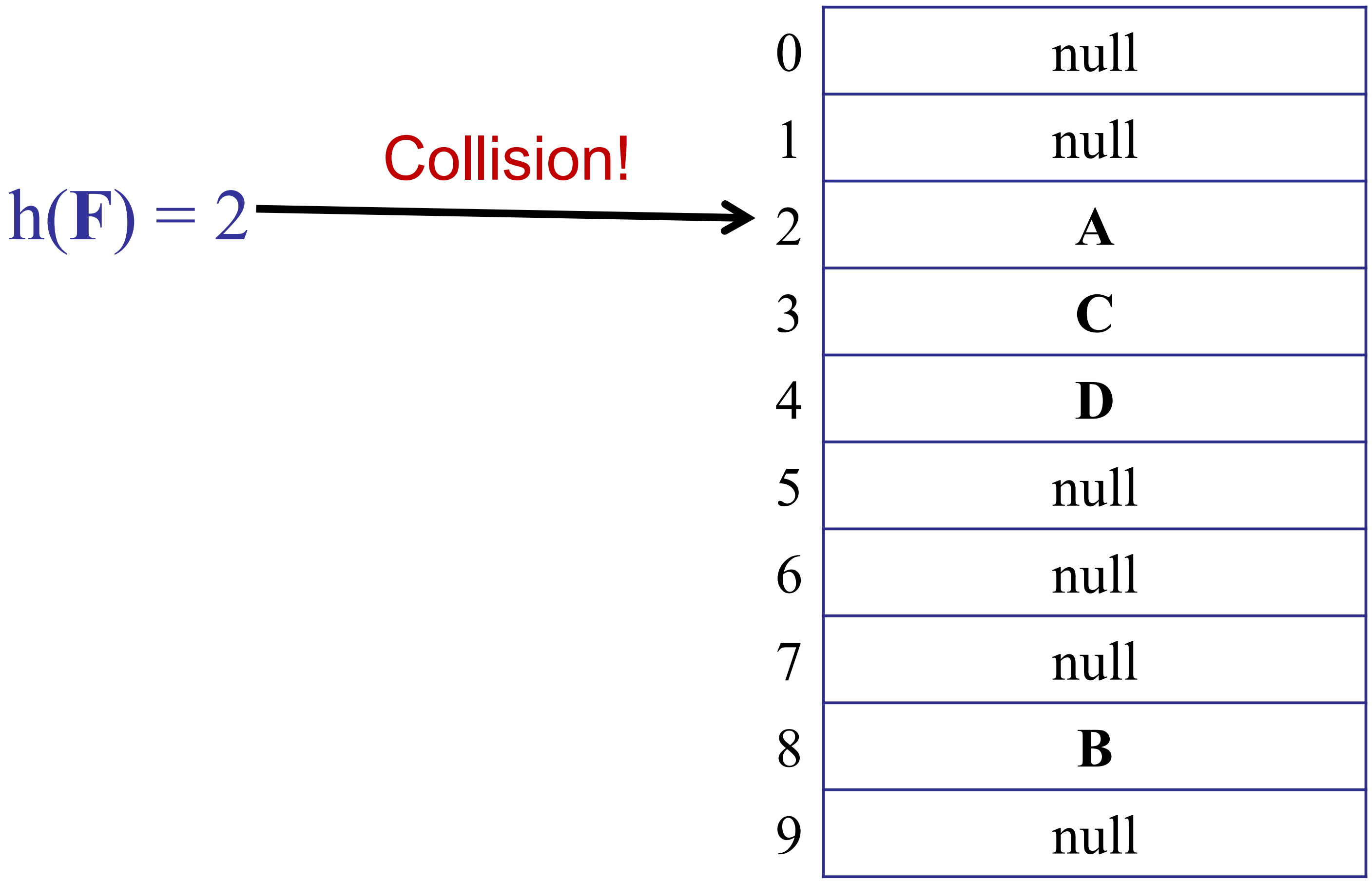
- No linked lists!
- All data directly stored in the table.
- One item per slot.

| | |
|---|---|
| 0 | null |
| 1 | null |
| 2 | **A** |
| 3 | null |
| 4 | null |
| 5 | null |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

On collision:

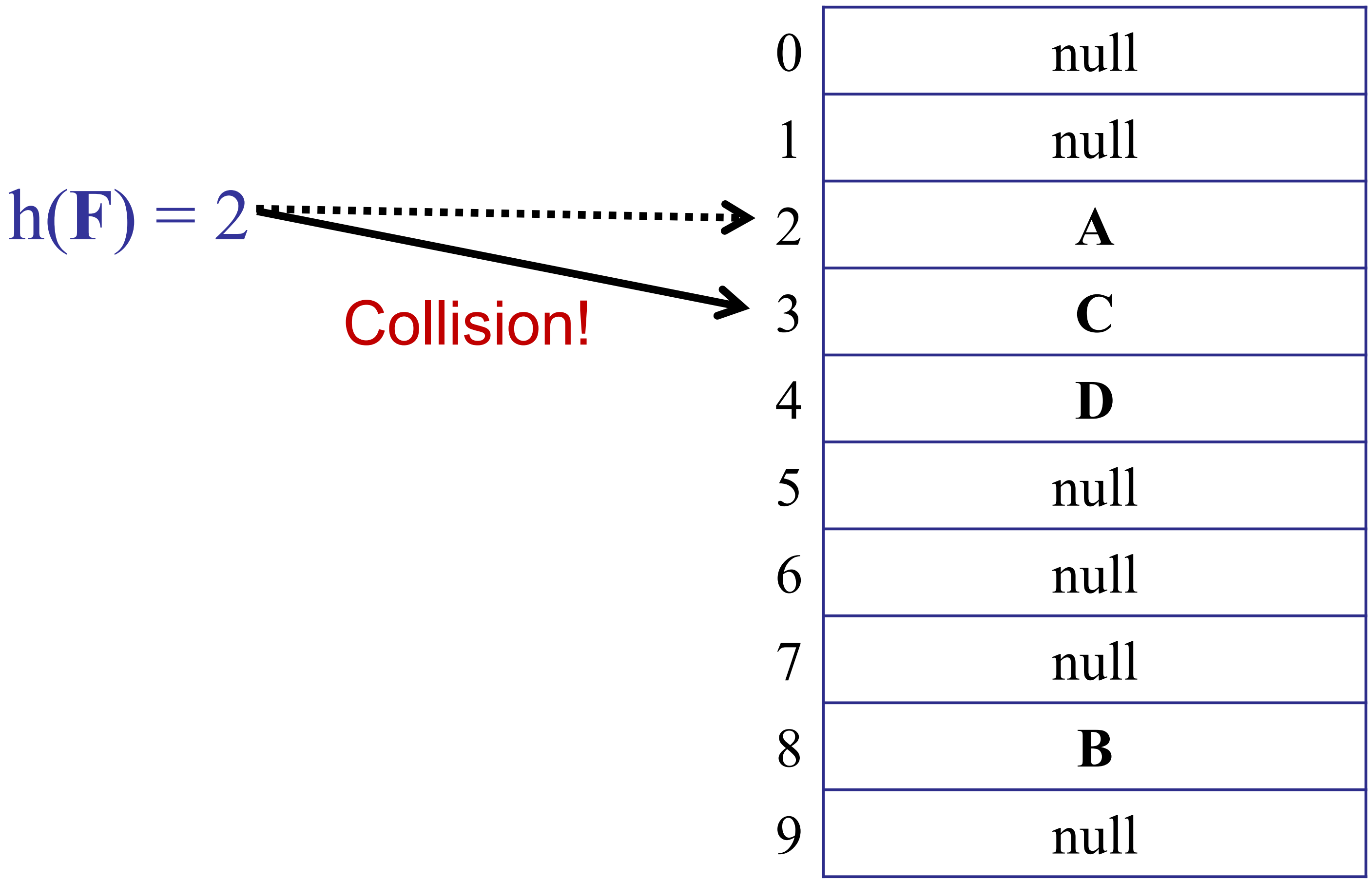
Probe a sequence of buckets until you find an empty one.



# Open Addressing

On collision:

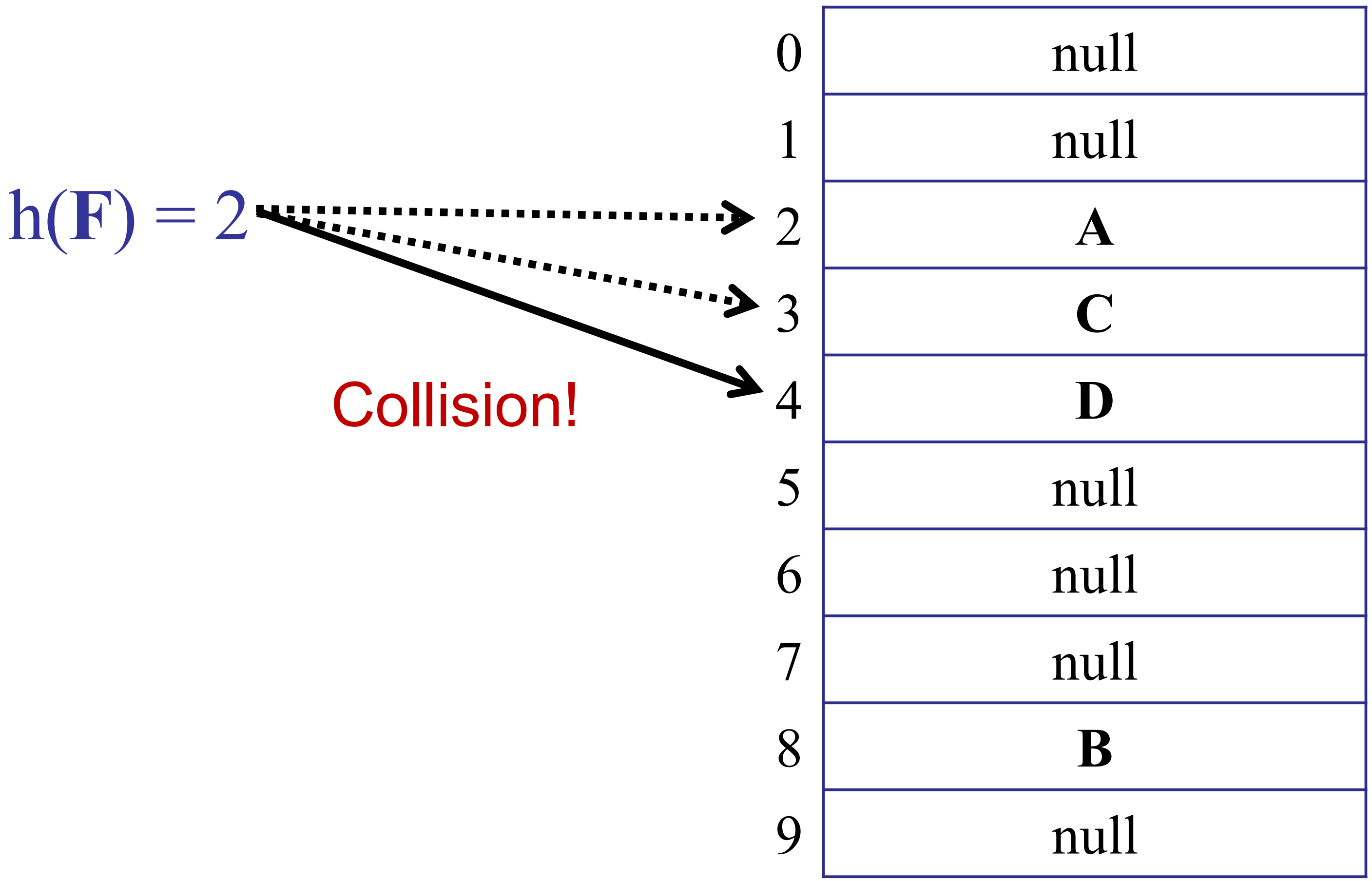
Probe a sequence of buckets until you find an empty one.



# Open Addressing

On collision:

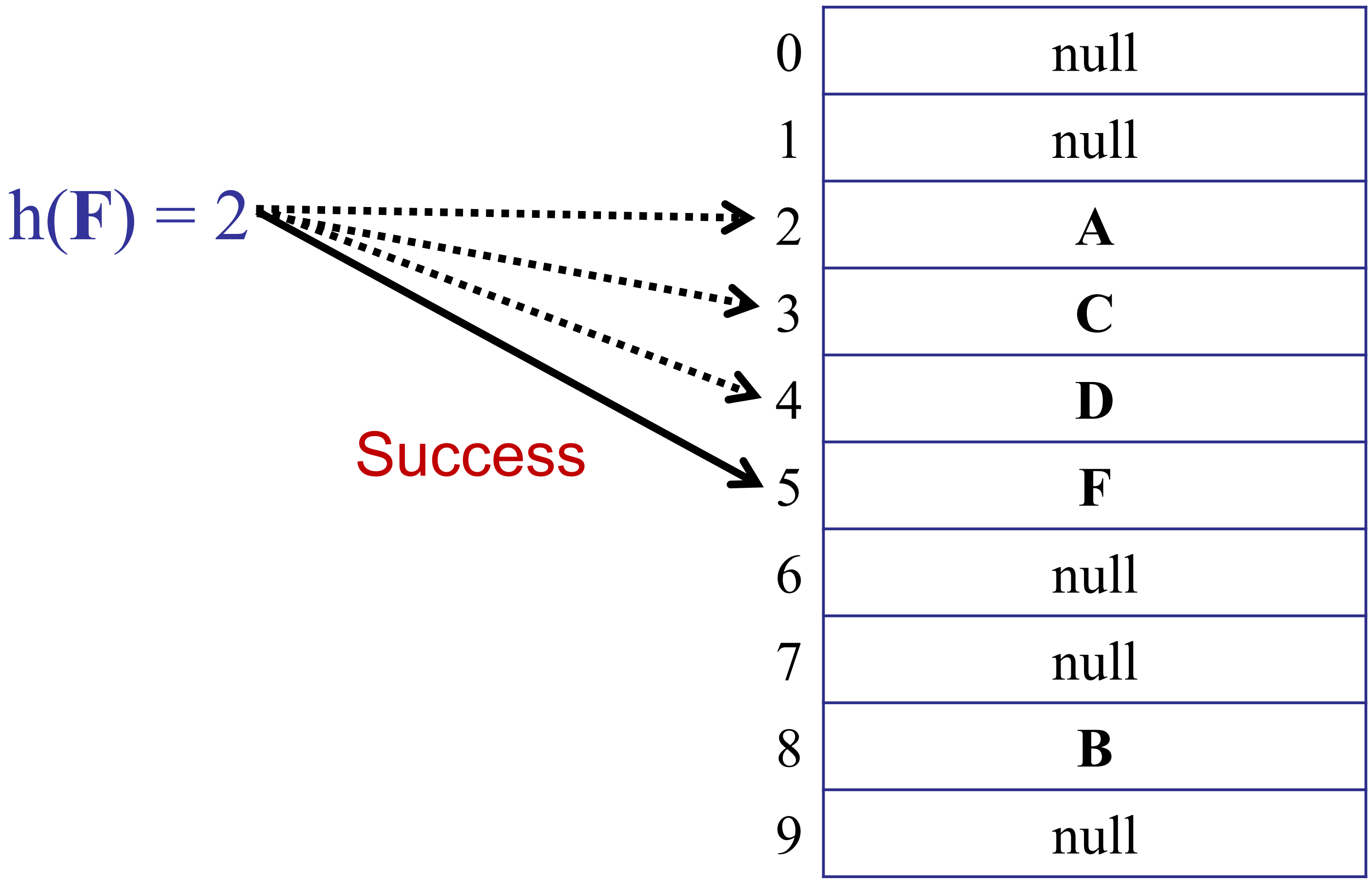
Probe a sequence of buckets until you find an empty one.



# Open Addressing

On collision:

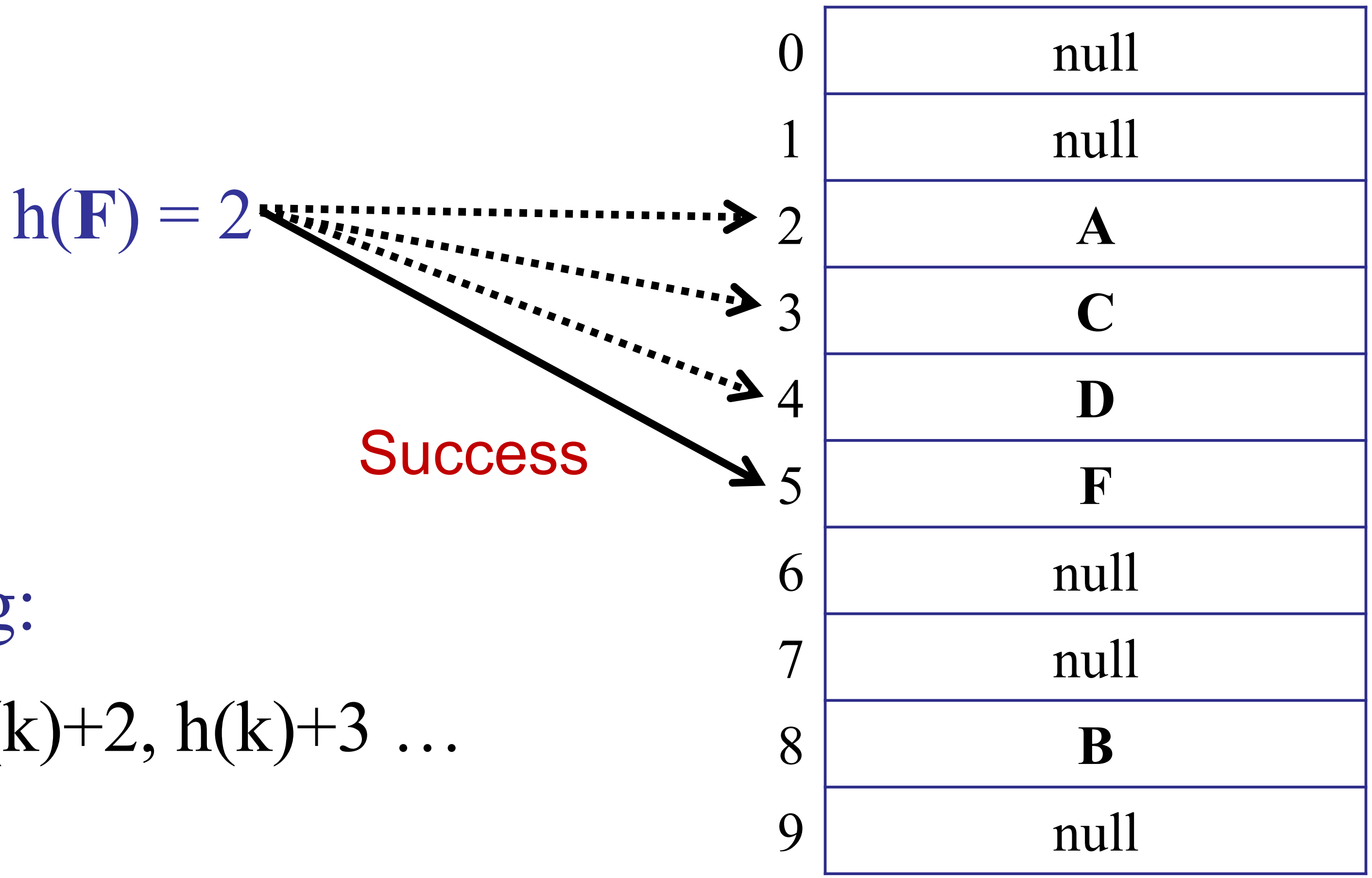
Probe a sequence of buckets until you find an empty one.



# Open Addressing

On collision:

Probe a sequence of buckets until you find an empty one.




Linear Probing:

- h(k)+1, h(k)+2, h(k)+3 …

# Open Addressing

Hash Function re-defined:

$$h(key, i) : U \rightarrow \{1..m\}$$

Two parameters:

- key : the thing to map
- i : number of collisions

# Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

Example: Linear Probing

- $h(k, 1)$ = hash of key k
- $h(k, 2) = h(k, 1) + 1$
- $h(k, 3) = h(k, 1) + 2$
- $h(k, 4) = h(k, 1) + 3$
- …
- $h(k, i) = h(k, 1) + i \bmod m$

| | |
|---|---|
| 0 | null |
| 1 | null |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | **F** |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

Example: Weird Probing

- $h(k, 1) = 4$
- $h(k, 2) = 1$
- $h(k, 3) = 8$
- $h(k, 4) = 5$

| | |
|---|---|
| 0 | null |
| 1 | **G** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | null |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

```
hash-insert(key, data)
int i = 1;
while (i <= m) {                          // Try every bucket
      int bucket = h(key, i);
      if (T[bucket] == null){             // Found an empty bucket
            T[bucket] = {key, data};      // Insert key/data
            return success;               // Return
      }
      i++;
}
throw new TableFullException();           // Table full!
```

# Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

search(key)

- $h(key, 1) = 4$
- $h(key, 2) = 1$
- $h(key, 3) = 8$
- $h(key, 4) = 5$

| | |
|---|---|
| 0 | null |
| 1 | **G** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | key |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

```
hash-search(key)
int i = 1;
while (i <= m) {
    int bucket = h(key, i);
    if (T[bucket] == null)  // Empty bucket!
        return key-not-found;
    if (T[bucket].key == key)  // Full bucket.
            return T[bucket].data;
    i++;
}
return key-not-found;  // Exhausted entire table.
```

# Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

search(key)

- $h(key, 1) = 4$
- $h(key, 2) = 1$
- $h(key, 3) = 8$
- $h(key, 4) = 5$

| | |
|---|---|
| 0 | null |
| 1 | **G** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | null |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

delete(key)

- Find key to delete
- Remove it from table.
- Set bucket to null.

| | |
|---|---|
| 0 | null |
| 1 | **G** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | **NULL** |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# What is wrong with delete?

✓ 1. Search may fail to find an element.
2. The table will have gaps in it.
3. Space is used inefficiently.
4. If the key is inserted again, it may end up in a different bucket.

# Open Addressing

insert(key)

Probe sequence:

3

1

5

| | |
|---|---|
| 0 | null |
| 1 | **G** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | key |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

insert(key)

delete(G)

| | |
|---|---|
| 0 | null |
| 1 | **G → NULL** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | key |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

insert(key)

delete(G)

search(key)

| | |
|---|---|
| 0 | null |
| 1 | **NULL** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | key |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

insert(key)

delete(G)

search(key)

Probe sequence:

3

1

5

| | |
|---|---|
| 0 | null |
| 1 | **NULL** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | key |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

insert(key)

delete(G)

search(key)

Probe sequence:

3

1

Not found!

| | |
|---|---|
| 0 | null |
| 1 | **NULL** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | key |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

Hash Function re-defined:

$h(key, i) : U \rightarrow \{1..m\}$

delete(key)

- Find key to delete
- Remove it from table.
- Set bucket to DELETED.

(Tombstone value.)

| | |
|---|---|
| 0 | null |
| 1 | **G** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | **DELETED** |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

# Open Addressing

insert(key)

delete(G)

search(key)

Probe sequence:

3

1

5

| | |
|---|---|
| 0 | null |
| 1 | **DELETED** |
| 2 | **A** |
| 3 | **C** |
| 4 | **D** |
| 5 | key |
| 6 | null |
| 7 | null |
| 8 | **B** |
| 9 | null |

What happens when an insert finds a DELETED cell?

1. Overwrite the deleted cell.
2. Continue probing.
3. Fail.

# Hash Functions

Two properties of a good hash function:

1. h($key$, $i$) enumerates all possible buckets.
   - For every bucket $j$, there is some $i$ such that:

     $$h(key, i) = j$$

   - The hash function is permutation of $\{1..m\}$.
   - For linear probing: true!

## What goes wrong if the sequence is not a permutation?

1. Search incorrectly returns key-not-found.
2. Delete fails.
3. Insert puts a key in the wrong place
4. Returns table-full even when there is still space left.

# Hash Functions

Two properties of a good hash function:

2. Simple Uniform Hashing Assumption

   Every key is equally likely to be mapped to every bucket, independently of every other key.

For h(*key*, 1)?

For every h(*key*, *i*)?

# Hash Functions

Two properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

   ***n!*** permutations for probe sequence: e.g.,

   - 1 2 3 4
   - 1 2 4 3
   - 1 4 2 3
   - 1 4 3 2
   - …

# Hash Functions

Two properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

***n!*** permutations for probe sequence: e.g.,

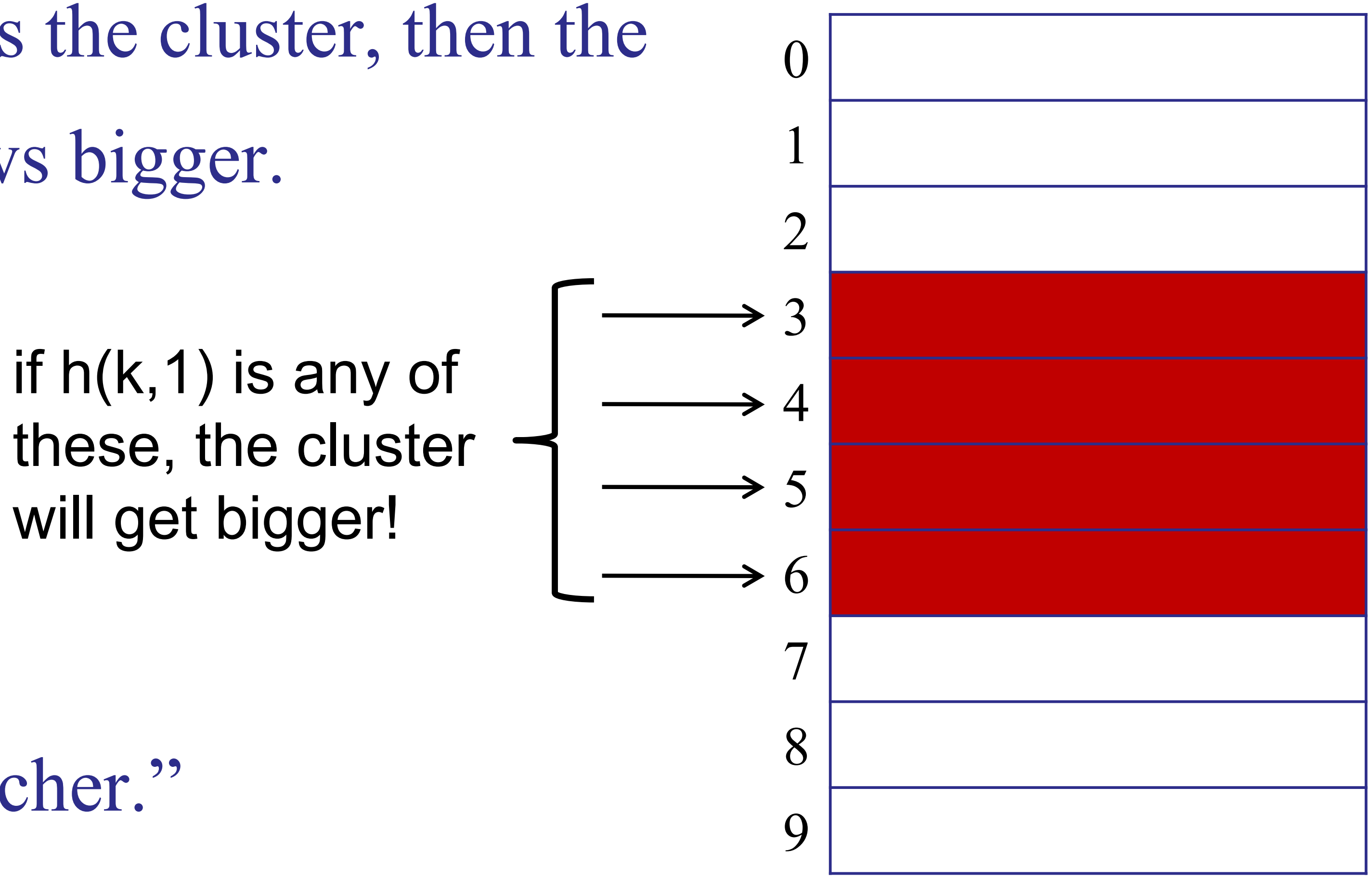
- 1 2 3 4 Pr(1/m)
- 1 2 4 3 Pr(0)
- 1 4 2 3 Pr(0)
- 1 4 3 2 Pr(0)
- …

NOT Linear Probing

# Linear Probing

Problem with linear probing: ***clusters***

- If there is a cluster, then there is a higher probability that the next h(k) will hit the cluster.
- If h(k,1) hits the cluster, then the cluster grows bigger.




- "Rich get richer."

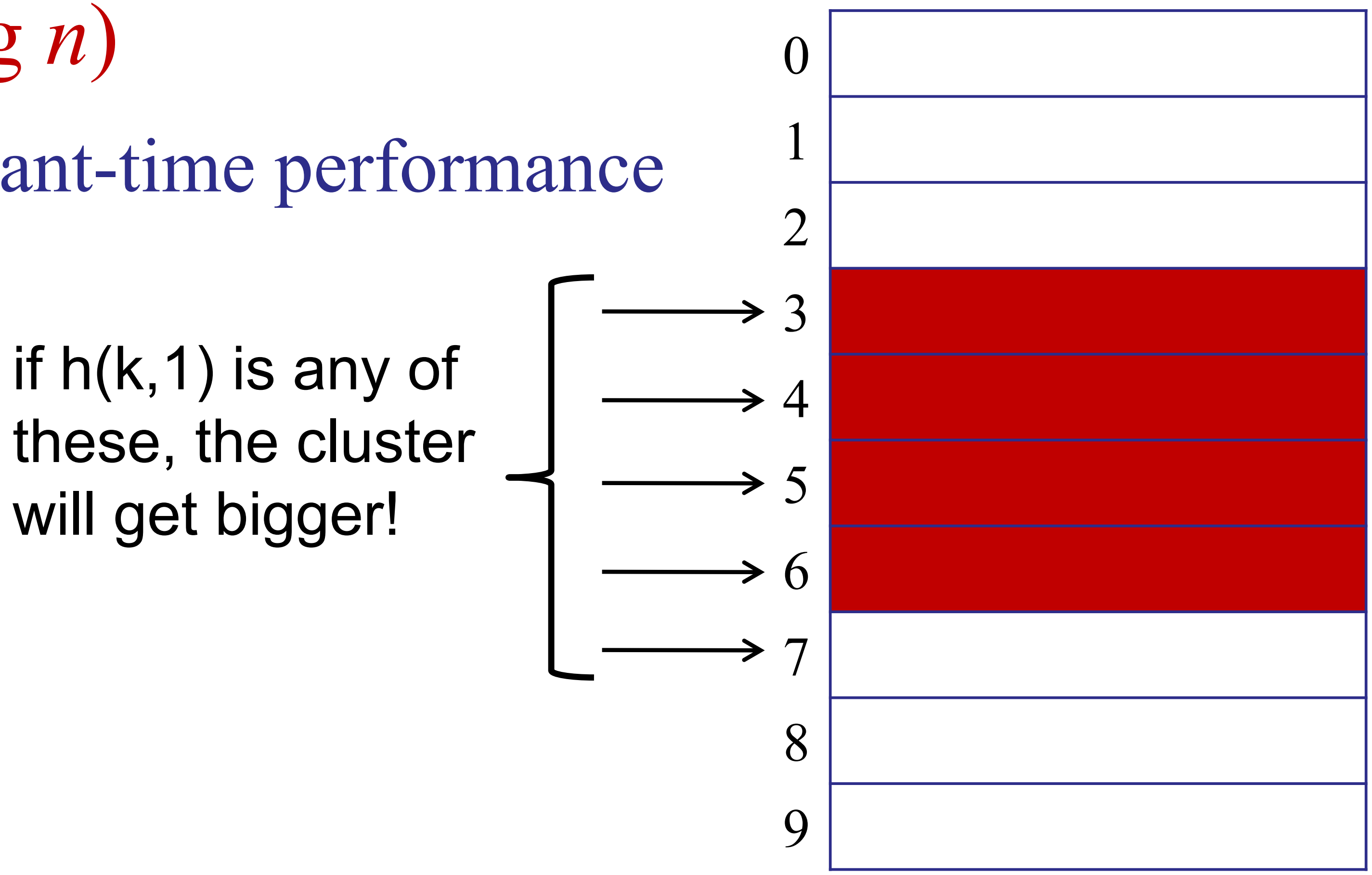
# Linear Probing

Problem with linear probing: ***clusters***

- If the table is 1/4 full, then there will be clusters of size:

$$\theta(\log n)$$

- Ruins constant-time performance



# Linear probing

In practice, linear probing is very fast!

- Why? Caching!
- It is *cheap* to access nearby array cells.
  - Example: access T[17]
  - Cache loads: T[10..50]
  - Almost 0 cost to access T[18], T[19], T[20], …
- If the table is 1/4 full, then there will be clusters of size: $\theta(\log n)$
  - Cache may hold entire cluster!
  - No worse than wacky probe sequence.

# Open Addressing

Properties of a good hash function:

2. Uniform Hashing Assumption

   Every key is equally likely to be mapped to every ***permutation***, independent of every other key.

***n!*** permutations for probe sequence: e.g.,

- 1 2 3 4
- 1 2 4 3
- 1 4 2 3
- 1 4 3 2
- …

# Double Hashing

- Start with two ordinary hash functions:

$$\mathrm{f}(k),\ \mathrm{g}(k)$$

- Define new hash function:

$$\mathrm{h}(k, i) = \mathrm{f}(k) + i \cdot \mathrm{g}(k) \mod m$$

- Note:
  - Since f(k) is good, f(k, 1) is "almost" random.
  - Since g(k) is good, the probe sequence is "almost" random.

# Double Hashing

Hash function

$$h(k, i) \;=\; f(k) + i \cdot g(k) \quad \text{mod } m$$

**Claim**: if g(k) is relatively prime to $m$, then $h(k, i)$ hits all buckets.

- Assume not: then for some distinct $i, j < m$:

$$f(k) + i \cdot g(k) = f(k) + j \cdot g(k) \quad \text{mod } m$$

$\Rightarrow$ $i \cdot g(k) = j \cdot g(k) \quad \text{mod } m$

$\Rightarrow$ $(i - j) \cdot g(k) = 0 \quad \text{mod } m$

$\Rightarrow$ g(k) not relatively prime to $m$, since $(i,j < m)$

# Double Hashing

Hash function

$$h(k, i) = f(k) + i \cdot g(k) \mod m$$

**Claim**: if $g(k)$ is relatively prime to $m$, then $h(k, i)$ hits all buckets.

Example: if $(m = 2^r)$, then choose $g(k)$ odd.

# Performance of Open Addressing

If (m==n), what is the expected insert time, under uniform hashing assumption?

1. O(1)
2. O(log n)
3. O(n)
4. O($n^2$)
5. None of the above.

# Performance of Open Addressing

- Chaining:
  - When (m==n), we can still add new items to the hash table.
  - We can still search efficiently.

- Open addressing:
  - When (m==n), the table is full.
  - We cannot insert any more items.
  - We cannot search efficiently.

# Performance of Open Addressing

Define:

- Load $\alpha = n / m$ — Average # items / bucket
- Assume $\alpha < 1$.

# Performance of Open Addressing

Define:

- Load $\alpha = n / m$ ← Average # items / bucket
- Assume $\alpha < 1$.

**Claim:**

For $n$ items, in a table of size $m$, assuming *uniform hashing*, the expected cost of an operation is:

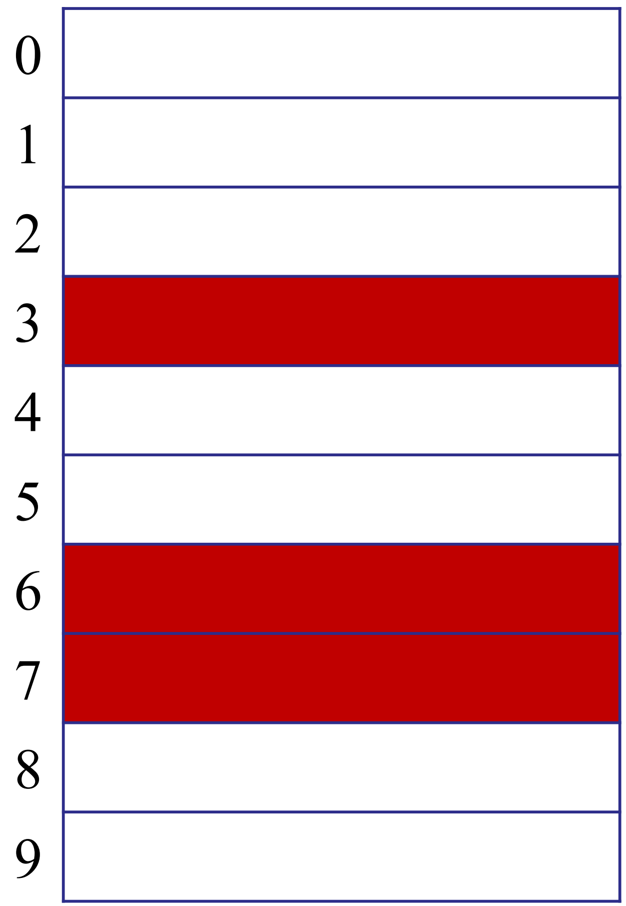
$$\leq \frac{1}{1-\alpha}$$

Example: if ($\alpha$=90%), then E[# probes] = 10

# Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: $n/m$

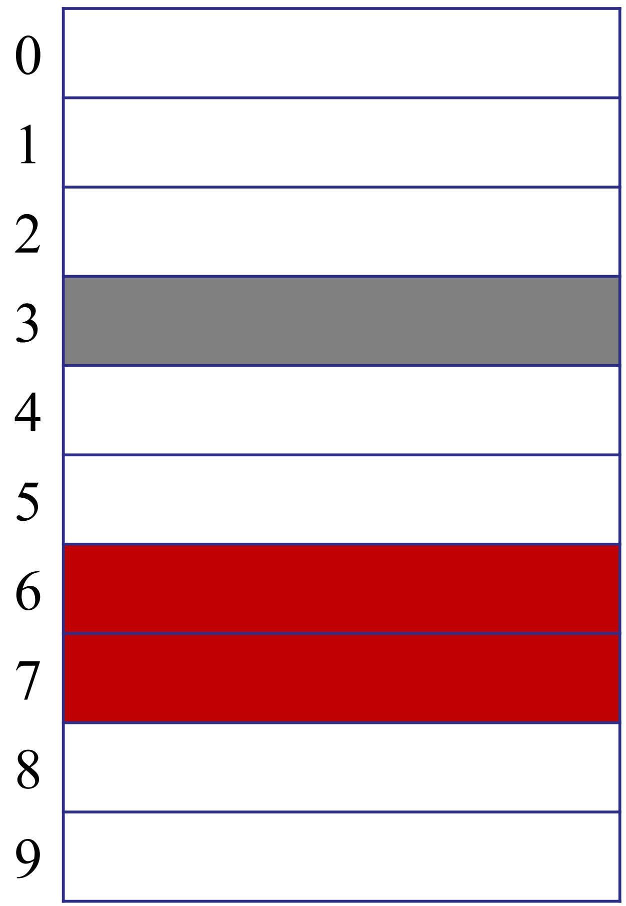


# Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: $n/m$

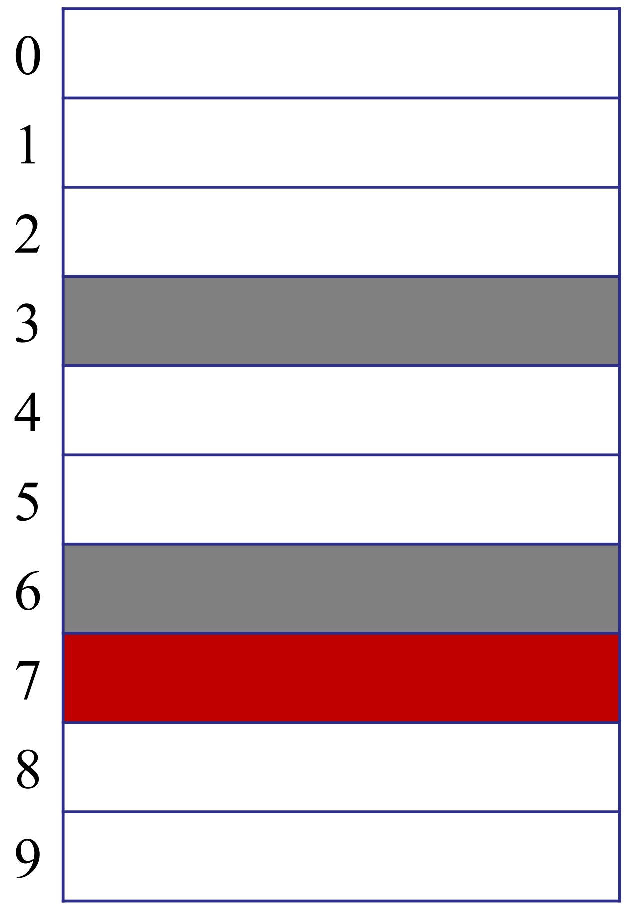
- Second probe: if first bucket is full, then the probability that the second bucket is also full: $(n - 1) / (m - 1)$



# Performance of Open Addressing

Proof of Claim:

- First probe: probability that first bucket is full is: $n/m$
- Second probe: if first bucket is full, then the probability that the second bucket is also full: $(n - 1) / (m - 1)$
- Third probe: probability is full: $(n - 2) / (m - 2)$



# Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(\text{Expected cost of remaining probes}\right)$$

First probe

Probability of collision on first probe

# Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(\text{Expected cost of remaining probes}\right)\right)$$

First probe

Probability of collision on first probe

Probability of collision on second probe

# Performance of Open Addressing

Proof of Claim:

- Expected cost:

$$\underbrace{1}_{\text{First probe}}+\frac{n}{m}\left(\underbrace{1}_{\text{Second probe}}+\frac{n-1}{m-1}\left(\underbrace{1}_{\text{Third probe}}+\frac{n-2}{m-2}\left(\quad\cdots\cdots\cdots\quad\right)\right)\right)$$

# Performance of Open Addressing

Proof of Claim:

- Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\quad \cdots\cdots\cdots \quad\right)\right)\right)$$

- Note:

$$\frac{n-i}{m-i} \le \frac{n}{m} \le \alpha$$

# Performance of Open Addressing

Proof of Claim:

- Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\quad\cdots\cdots\cdots\quad\right)\right)\right)$$

$$\leq 1+\alpha\left(1+\alpha\left(1+\alpha\left(\cdots\cdots\right)\right)\right)$$

# Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\quad\cdots\cdots\cdots\quad\right)\right)\right)$$

$$\leq 1+\alpha\left(1+\alpha\left(1+\alpha\left(\cdots\cdots\right)\right)\right)$$

$$\leq 1+\alpha+\alpha^{2}+\alpha^{3}+\cdots$$

# Performance of Open Addressing

Proof of Claim:

– Expected cost:

$$1+\frac{n}{m}\left(1+\frac{n-1}{m-1}\left(1+\frac{n-2}{m-2}\left(\quad \cdots\cdots\cdots \quad\right)\right)\right)$$

$$\leq 1+\alpha\left(1+\alpha\left(1+\alpha\left(\cdots\cdots\right)\right)\right)$$

$$\leq 1+\alpha+\alpha^{2}+\alpha^{3}+\cdots$$

$$\leq \frac{1}{1-\alpha}$$

# Performance of Open Addressing

Define:

- Load $\alpha = n / m$ ← Average # items / bucket
- Assume $\alpha < 1$.

**Claim:**

For $n$ items, in a table of size $m$, assuming *uniform hashing*, the expected cost of an operation is:

$$\leq \frac{1}{1-\alpha}$$

Example: if ($\alpha$=90%), then E[# probes] = 10

# Advantages…

Open addressing:

- Saves space
  - Empty slots vs. linked lists.
- Rarely allocate memory
  - No new list-node allocations.
- Better cache performance
  - Table all in one place in memory
  - Fewer accesses to bring table into cache.
  - Linked lists can wander all over the memory.

# Disadvantages…

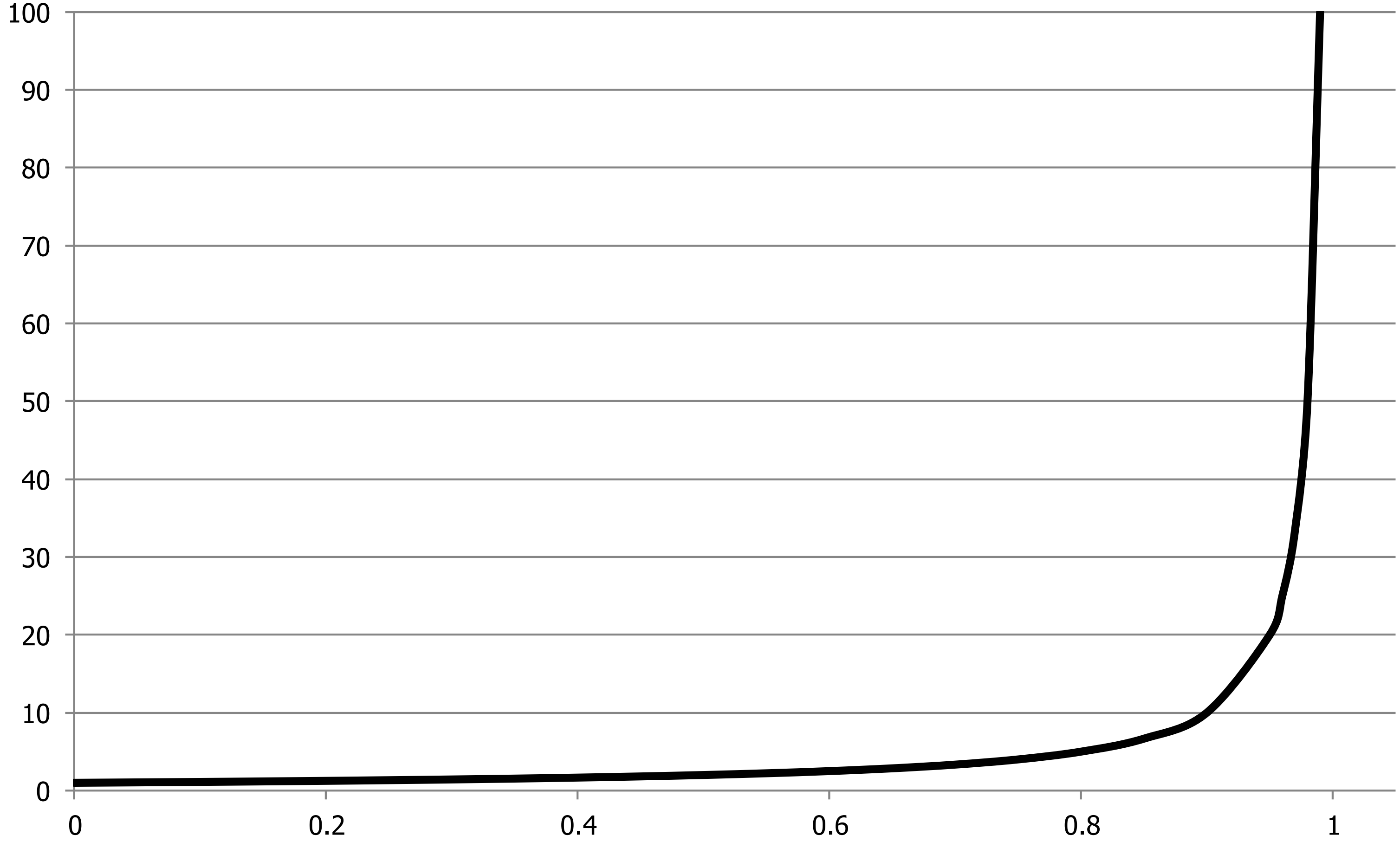
Open addressing:

- More sensitive to choice of hash functions.
  - Clustering is a common problem.
  - See issues with linear probing.
- More sensitive to load.
  - Performance degrades badly as $\alpha \rightarrow 1$.

# Disadvantages…

## Open addressing:

– Performance degrades badly as $\alpha \rightarrow 1$.



# Today

- Java hashing
- Collision resolution: open addressing
- Table (re)sizing

# CS2040S
# Data Structures and Algorithms
(e-learning edition)

## Hashing II
## (Part 3)

# Today

- Java hashing
- Collision resolution: open addressing
- Table (re)sizing

# Table Size

How large should the table be?

- Assume: Hashing with Chaining
- Assume: Simple Uniform Hashing
- Expected search time: $O(1 + n/m)$
- Optimal size: $m = \Theta(n)$

  - if $(m < 2n)$ : too many collisions.
  - if $(m > 10n)$ : too much wasted space.

- Problem: we don't know $n$ in advance.

# Table Size

Idea:

- Start with small (constant) table size.
- Grow (and shrink) table as necessary.

Example:

- Initially, $m = 10$.
- After inserting 6 items, table too small! Grow…
- After deleting $n$-1 items, table too big! Shrink…

# Table Size

How to grow the table:

1. Choose new table size $m$.
2. Choose new hash function h.
   - Hash function depends on table size!
   - Remember: $h : U \rightarrow \{1..m\}$
3. For each item in the old hash table:
   - Compute new hash function.
   - Copy item to new bucket.

# Table Size

Time complexity of growing the table:

- Assume:
  - Let $m_1$ be the size of the old hash table.
  - Let $m_2$ be the size of the new hash table.
  - Let $n$ be the number of elements in the hash table.
- Costs:
  - Scanning old hash table: $O(m_1)$
  - Inserting each element in new hash table: $O(1)$
  - Total: $O(m_1 + n)$

# Table Size

Time complexity of growing the table:

- Assume:
  - Size $m_1 < n$.
  - Size $m_2 > 2\mathrm{n}$
- Costs:
  - Total: $O(m_1 + n)$ .

    $= O(n)$

# Table Size

Time complexity of growing the table:

Wait! What is the cost of initializing the new table?

- Initializing a table of size x takes x time!

- Costs:

  Total: $O(m_1 + m_2 + n)$

# Table Size

Time complexity of growing the table:

- Assume:
  - Let $m_1$ be the size of the old hash table.
  - Let $m_2$ be the size of the new hash table.
  - Let $n$ be the number of elements in the hash table.
- Costs:
  - Scanning old hash table: $O(m_1)$
  - Creating new hash table: $O(m_2)$
  - Inserting each element in new hash table: $O(1)$
  - Total: $O(m_1 + m_2 + n)$

# How fast to grow?

Idea 1: Increment table size by 1

- if ($n == m$): $m = m+1$

- Cost of resize:
  - Size $m_1 = n$.
  - Size $m_2 = n+1$.
  - Total: O($n$)

Initially: $m = 8$
What is the cost of inserting $n$ items?

1. O(n)
2. O(n log n)
3. O($n^2$)
4. O($n^3$)
5. None of the above.

# How fast to grow?

Idea 1: Increment table size by 1

- When ($n == m$): $m = m+1$
- Cost of each resize: O($n$)

| **Table size** | 8 | 8 | 9 | 10 | 11 | 12 | … | n+1 |
|---|---|---|---|---|---|---|---|---|
| **Number of items** | 0 | 7 | 8 | 9 | 10 | 11 | … | n |
| **Number of inserts** | | 7 | 1 | 1 | 1 | 1 | … | 1 |
| **Cost** | | 7 | 8 | 9 | 10 | 11 | | n |

- Total cost: $(7 + 8 + 9 + 10 + 11 + \ldots + n) = O(n^2)$

# How fast to grow?

## Idea 2: Double table size

- if $(n == m)$: $m = 2m$

- Cost of resize:
  - Size $m_1 = n$.
  - Size $m_2 = 2n$.
  - Total: $O(n)$

# How fast to grow?

Idea 2: Double table size

- When $(n == m)$: $m = 2m$
- Cost of each resize: $O(n)$

| **Table size** | 8 | 8 | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 16 | 32 | 32 | 32 | … | 2n |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **# of items** | 0 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | … | n |
| **# of inserts** | | 7 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | … | 1 |
| **Cost** | | 7 | 8 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 16 | 1 | 1 | | n |

- Total cost: $(7 + 15 + 31 + \ldots + n) = O(n)$

# How fast to grow

## Idea 2: Double table size

### Cost of Resizing:

| Table size | Total Resizing Cost |
|---|---|
| 8 | 8 |
| 16 | (8 + 16) |
| 32 | (8 + 16 + 32) |
| 64 | (8 + 16 + 32 + 64) |
| 128 | (8 + 16 + 32 + 64 + 128) |
| … | … |
| m | $<(1+2+4+8+\ldots+m) \leq O(m)$ |

# How fast to grow?

Idea 2: Double table size

- if $(n == m)$: $m = 2m$

  - Cost of resize: $O(n)$
  - Cost of inserting $n$ items + resizing: $O(n)$

- Most insertions: $O(1)$
- Some insertions: linear cost (expensive)
- Average cost: $O(1)$

# How fast to grow?

Idea 3: Square table size

– When ($n == m$): $m = m^2$

| Table size | Total Resizing Cost |
|---|---|
| 8 | ? |
| 64 | ? |
| 4,096 | ? |
| 16,777,216 | ? |
| … | … |
| m | ? |

Assume: square table size
What is the cost of inserting $n$ items?

1. $O(\log n)$
2. $O(\sqrt{n})$
3. $O(n)$
4. $O(n \log n)$
5. $O(n^2)$
6. $O(2^n)$
7. None of the above.

# How fast to grow?

Idea 3: Square table size

- if $(n == m)$: $m = m^2$

- Cost of resize:
  - Size $m_1 = n$.
  - Size $m_2 = n^2$.
  - Total: $O(m_1 + m_2 + n)$

    $= O(n + n^2 + n)$

    $= O(n^2)$

# How fast to grow?

Idea 3: Square table size

– When ($n == m$): $m = m^2$

| # Items | Total Resizing Cost |
|---|---|
| 8 | 64 |
| 64 | (64 + 4,096) |
| 4,096 | (64 + 4,096 + …) |
| … | … |
| $n$ | $> n^2$ |
| | $= O(n^2)$ |

# How fast to grow?

Idea 3: Square table size

– When ($n == m$): $m = m^2$

| # Items | Resizing Cost | Insert Cost |
|---|---|---|
| 8 | 64 | 8 |
| 64 | (64 + 4,096) | 64 |
| 4,096 | (64 + 4,096 + …) | 4,096 |
| … | … | … |
| $n$ | > $n^2$ | n |
| | < $O(n^2)$ | O(n) |

# How fast to grow?

Idea 3: Square table size

- if ($n == m$): $m = m^2$

- Cost of resize:
  - Total: $O(n^2)$

- Cost of inserts:
  - Total: $O(n)$

# Why else is squaring the table size bad?

1. Resize takes too long to find items to copy.
2. Inefficient space usage.
3. Searching is more expensive in a big table.
4. Inserting is more expensive in big table.
5. Deleting is more expensive in a big table.

# Deleting Elements

Basic procedure: (chained hash tables)

Delete(*key*)

1. Calculate hash of *key*.
2. Let $L$ be the linked list in the specified bucket.
3. Search for item in linked list $L$.
4. Delete item from linked list $L$.

Cost:

– Total: $O(1 + n/m)$

# Deleting Elements

What happens if too many items are deleted?

- Table is too big!
- Shrink the table…

- Try 1:
  - If $(n == m)$, then $m = 2m$.
  - If $(n < m/2)$ then $m = m/2$.

# Deleting Elements

Rules for shrinking and growing:

- Try 1:
  - If $(n == m)$, then $m = 2m$.
  - If $(n < m/2)$ then $m = m/2$.

- Example problem:
  - Start: $n$=100, $m$=200
  - Delete: $n$=99, $m$=200 → shrink to $m$=100
  - Insert: $n$=100, $m$=100 → grow to $m$=200
  - Repeat…

# Deleting Elements

Example execution:

- Start: $n$=100, $m$=200
- cost=100 • Delete: $n$=99, $m$=200 → shrink to $m$=100
- cost=100 • Insert: $n$=100, $m$=100 → grow to $m$=200
- cost=100 • Delete: $n$=99, $m$=200 → shrink to $m$=100
- cost=100 • Insert: $n$=100, $m$=100 → grow to $m$=200
- cost=100 • Delete: $n$=99, $m$=200 → shrink to $m$=100
- cost=100 • Insert: $n$=100, $m$=100 → grow to $m$=200
- Repeat…

# Deleting Elements

Rules for shrinking and growing:

- Try 2:
  - If $(n == m)$, then $m = 2m$.
  - If $(n < m/4)$, then $m = m/2$.
- Claim:
  - Every time you double a table of size $m$, at least $m/2$ new items were added.
  - Every time you shrink a table of size $m$, at least $m/4$ items were deleted.

# Amortized Analysis

Technique for analyzing "average" cost:

- Common in data structure analysis
- Like paying rent:
  - You don't pay rent every day!
  - Pay \$900/month = \$30/day.

Definition:

- Operation has <u>amortized cost</u> $\mathrm{T}(n)$ if for every integer $k$, the cost of $k$ operations is $\leq k\,\mathrm{T}(n)$

# Amortized Analysis

Definition:

– Operation has amortized cost $T(n)$ if for every integer $k$, the cost of $k$ operations is $\leq k\,T(n)$

Example: amortized cost = 7

1. insert: 5
2. insert: 5
3. insert: 5
4. insert: 13
5. insert: 7

5 <= 7

5+5 <= 2*7 = 14

5+5+5 <= 3*7 = 21

5+5+5+13 <= 4*7 = 28

5+5+5+13+7 <= 5*7 = 35

# Amortized Analysis

"amortized" is NOT "average"

Definition:

– Operation has <u>amortized cost</u> T($n$) if for every integer $k$, the cost of $k$ operations is $\leq k$ T($n$)

Example: amortized cost **NOT** 7

1. insert: 13
2. insert: 5
3. insert: 5
4. insert: 5
5. insert: 7

13 > 7

13+5 > 2*7 = 14

13+5+5 > 3*7 = 21

13+5+5+5 <= 4*7 = 28

5+5+5+13+7 <= 5*7 = 35

# Amortized Analysis

## Definition:

– Operation has amortized cost $T(n)$ if for every integer $k$, the cost of $k$ operations is $\leq k\,T(n)$

## Example: (Hash Tables)

– Inserting $k$ elements into a hash table takes time $O(k)$.

– Conclusion:

The insert operation has amortized cost $O(1)$.

# Amortized Analysis

## Accounting Method (paying rent)

- Imagine a bank account **B**.
- Each operation adds money to the bank account.
- Every step of the algorithm spends money:
  - Immediate money: to perform the operation.
  - Deferred money: from the bank account.
- Total cost execution = total money
  - Average time / operation = money / num. ops

# Amortized Analysis

## Accounting Method Example (Hash Table)

- Each table has a bank account.
- Each time an element is added to the table, it adds O(1) dollars to the bank account, uses O(1) dollars to insert element.
- A table with *k* new elements since last resize has *k* dollars in bank.

| Bank account |
| --- |
| $2 dollars |

| | |
| --- | --- |
| 0 | null |
| 1 | null |
| 2 | **($k_1$, A)** |
| 3 | null |
| 4 | null |
| 5 | null |
| 6 | null |
| 7 | null |
| 8 | **($k_2$, B)** |
| 9 | null |

# Amortized Analysis

## Accounting Method Example (Hash Table)

- Each table has a bank account.
- Each time an element is added to the table, it adds O(1) dollars to the bank account.

- Claim:
  - Resizing a table of size $m$ takes $O(m)$ time.
  - If you resize a table of size $m$, then:
    - at least $m/2$ new elements since last resize
    - bank account has $\Theta(m)$ dollars.

# Amortized Analysis

## Accounting Method Example (Hash Table)

- Each table has a bank account.
- Each time an element is added to the table, it adds O(1) dollars to the bank account.
- Pay for resizing from the bank account!
- Strategy:
  - Analyze inserts ignoring cost of resizing.
  - Ensure that bank account always is big enough to pay for resizing.

# Amortized Analysis

Total cost: Inserting $k$ elements costs:

- Deferred dollars: O($k$)    (to pay for resizing)
- Immediate dollars: O($k$) for inserting elements in table
- Total (Deferred + Immediate): O($k$)

# Amortized Analysis

Total cost: Inserting $k$ elements costs:

- Deferred dollars: $O(k)$ (to pay for resizing)
- Immediate dollars: $O(k)$ for inserting elements in table
- Total (Deferred + Immediate): $O(k)$

Cost per operation:

- Deferred dollars: $O(1)$
- Immediate dollars: $O(1)$
- Total: $O(1)$ / per operation

# Example: Binary Counter

Counter ADT:

- increment()
- read()

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Example: Binary Counter

Counter ADT:

- increment()
- read()

increment()

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

# Example: Binary Counter

Counter ADT:

- increment()
- read()

increment(), increment()

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

# Example: Binary Counter

Counter ADT:

- increment()
- read()

increment(), increment(), increment()

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 |

What is the worst-case cost of incrementing a counter with max-value n?

1. O(1)
2. O(log n)
3. O(n)
4. O($n^2$)
5. I have no idea.

# Example: Binary Counter

Counter ADT:

- increment()
- read()

Some increments are expensive…

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

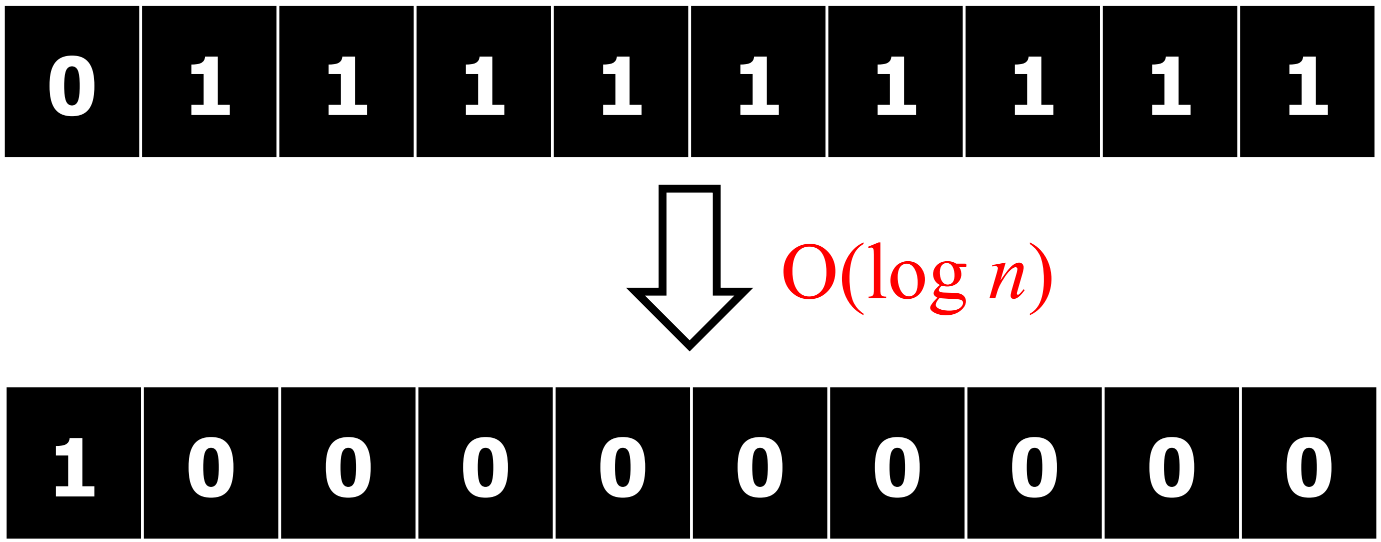
$\Downarrow$ O(log $n$)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Example: Binary Counter

Question: If we increment the counter to $n$, what is the amortized cost per operation?

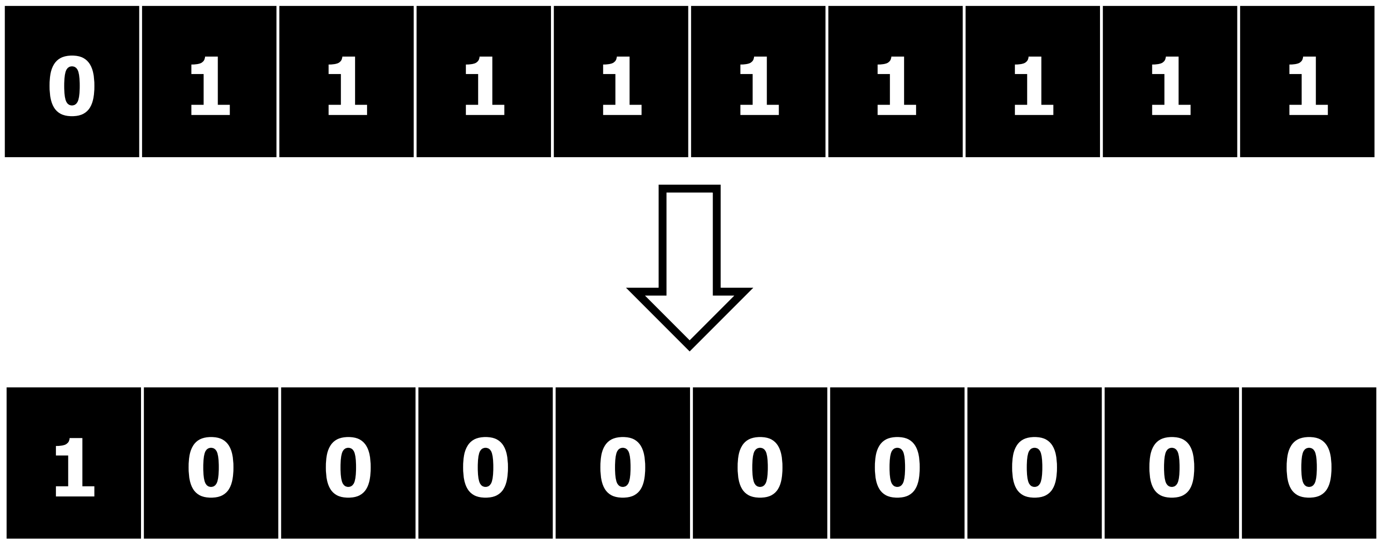
- Easy answer: $O(\log n)$
- More careful analysis….



# Example: Binary Counter

Observation:

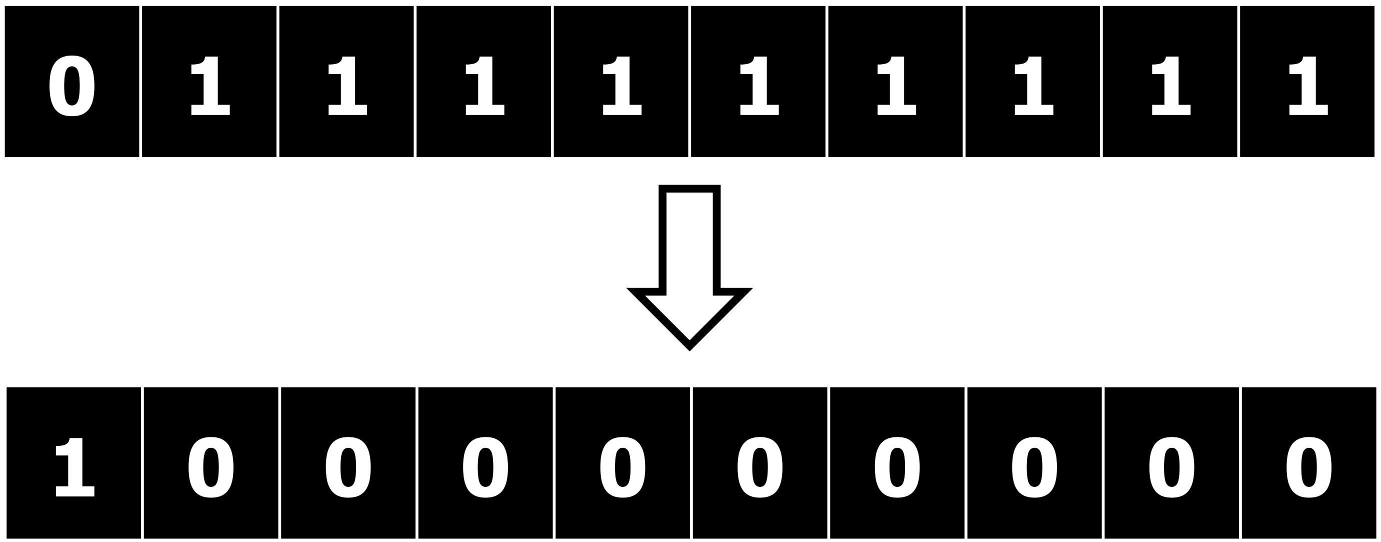
During each increment, only one bit is changed from: 0 → 1



# Example: Binary Counter

Observation:

During each increment, <u>many</u> bits may be changed from: 1 → 0

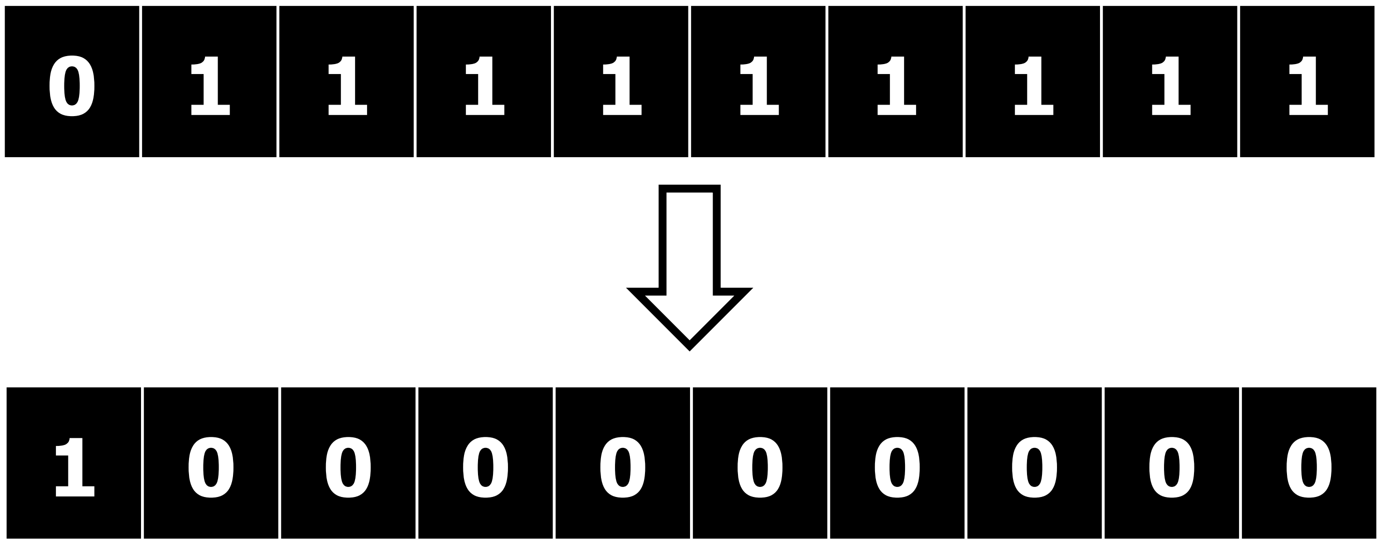


# Example: Binary Counter

Observation:

Accounting method: each bit has a bank account.

Whenever you change it from 0→1, add one dollar.



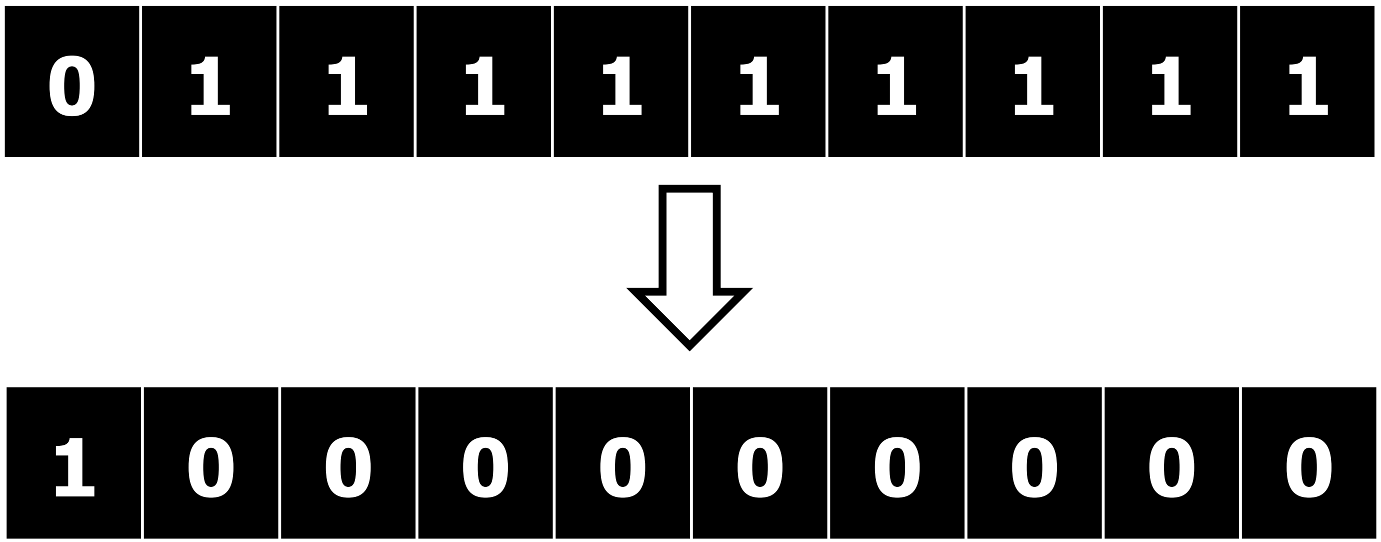
# Example: Binary Counter

Observation:

Accounting method: each bit has a bank account.

Whenever you change it from 0→1, add one dollar.

Whenever you change it from 1→0, pay one dollar.



# Example: Binary Counter

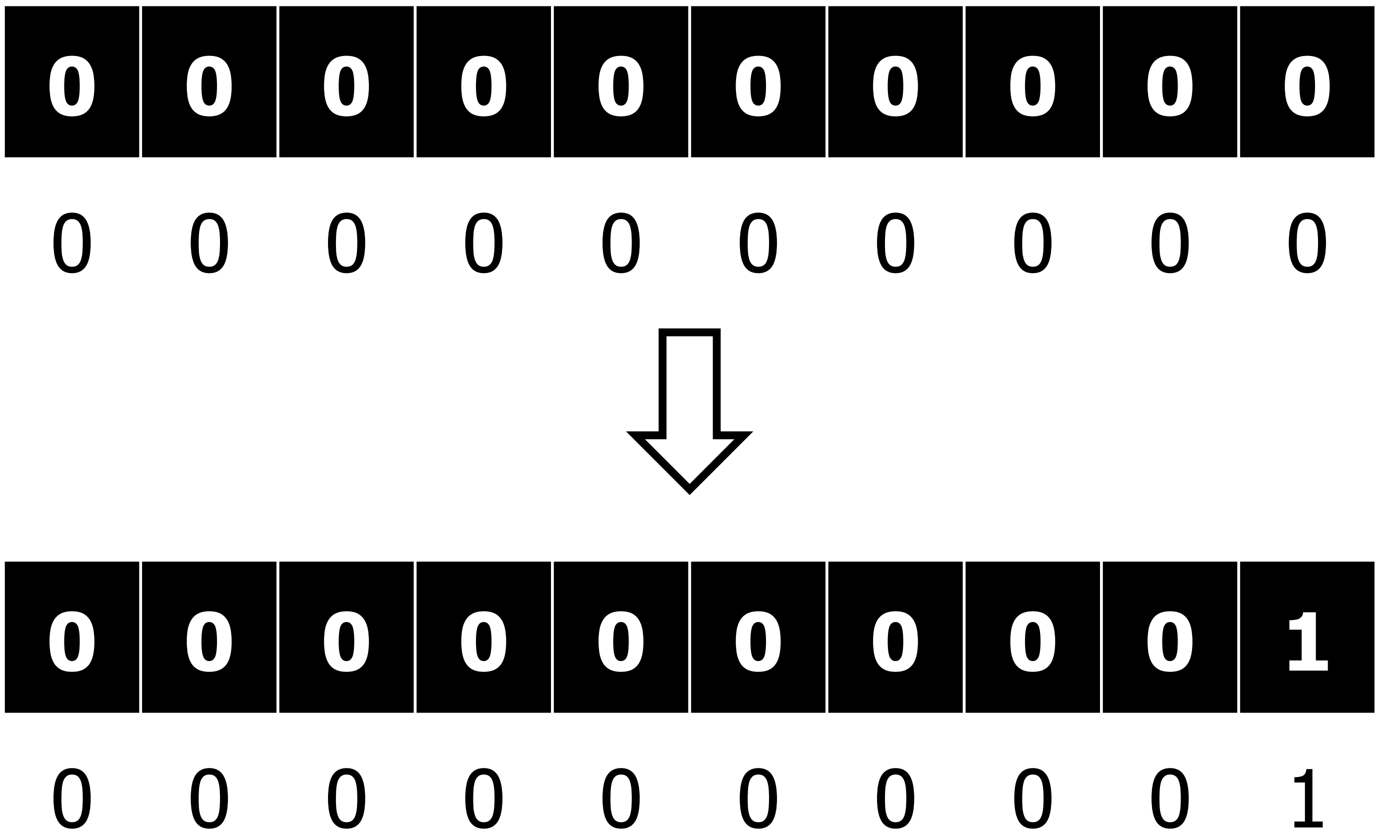
Counter ADT

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Example: Binary Counter

Counter ADT

increment()



# Example: Binary Counter

Counter ADT

increment(), increment()

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **1** |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |

⇩

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **1** | **0** |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |

# Example: Binary Counter

Counter ADT

increment(), increment(), increment()

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **1** | **0** |

0 0 0 0 0 0 0 0 1 0

$\Downarrow$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **1** | **1** |

0 0 0 0 0 0 0 0 1 1

# Example: Binary Counter

Counter ADT

increment()

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **0** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** |

0 1 1 1 1 1 1 1 1 1

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| **1** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** | **0** |

1 0 0 0 0 0 0 0 0 0

# Example: Binary Counter

Observation:

Amortized cost of increment: 2

- One operation to switch <u>one</u> 0→1
- One dollar (for bank account of switched bit).

(All switches from 1→0 paid for by bank account.)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

⇩

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Today

- Java hashing
- Resolving collisions: open addressing
- Table (re)sizing