CS2040S
Data Structures and Algorithms
(e-learning edition)

Directed Graphs!

Roadmap

Last time: Graph Basics

- What is a graph?
- Modeling problems as graphs.
- Graph representations (list vs. matrix)
- Searching graphs: BFS

What is a graph?

Graph $G = \langle V, E \rangle$

- V is a set of nodes
  - At least one: $|V| > 0$.
- E is a set of edges:
  - $E \subseteq \{ (v,w) : (v \in V), (w \in V) \}$
  - $e = (v,w)$
  - For all $e_1, e_2 \in E : e_1 \neq e_2$

2 x 2 x 2 Rubik's Cube

Configuration Graph

- Vertex for each possible state
- Edge for each basic move
  - 90 degree turn
  - 180 degree turn

Puzzle: given initial state, find a path to the solved state.

Trade-offs

Adjacency Matrix:

- Fast query: are v and w neighbors?
- Slow query: find me any neighbor of v.
- Slow query: enumerate all neighbors.

Adjacency List:

- Fast query: find me any neighbor.
- Fast query: enumerate all neighbors.
- Slower query: are v and w neighbors?

Searching a Graph

Goal:

- Start at some vertex **s** = start.
- Find some other vertex **f** = finish.

  Or: visit **all** the nodes in the graph;

Two basic techniques:

- Breadth-First Search (BFS)
- Depth-First Search (DFS)

Graph representation:

- Adjacency list

Graph Search

BFS and DFS are the same algorithm:

- BFS: use a queue
  - Every time you visit a node, add all unvisited neighbors to the queue.
- DFS: use a stack
  - Every time you visit a node, add all unvisited neighbors to the stack.

Graph Search

Breadth-first search:

Same algorithm, implemented with a queue:

Add start-node to queue.

Repeat until queue is empty:

- Remove node v from the front of the queue.
- Visit v.
- Explore all outgoing edges of v.
- Add all unvisited neighbors of v to the queue.

Graph Search

Depth-first search:

Same algorithm, implemented with a stack:

Add start-node to stack.

Repeat until stack is empty:

- Pop node v from the front of the stack.
- Visit v.
- Explore all outgoing edges of v.
- Push all unvisited neighbors of v on the front of the stack.

Review: Searching Graphs

BFS and DFS are the same algorithm:

- BFS: use a queue
  - Every time you visit a node, add all unvisited neighbors to the queue.
- DFS: use a stack
  - Every time you visit a node, add all unvisited neighbors to the stack.

Common Mistake

What do BFS and DFS solve?

- They visit every node in the graph?
- They visit every edge in the graph?
- They visit every path in the graph?

Common Mistake

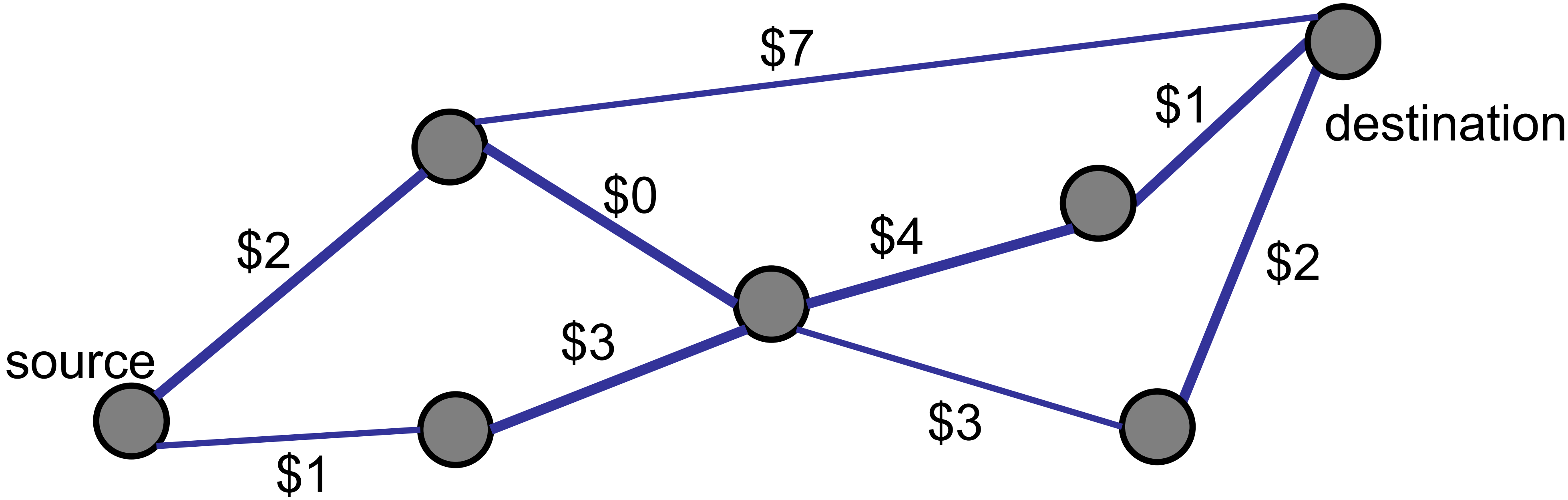
What do BFS and DFS solve?

- They visit every node in the graph? Yes.
- They visit every edge in the graph? Yes.
- ~~They visit every path in the graph?~~

Example

Problem: Make Money

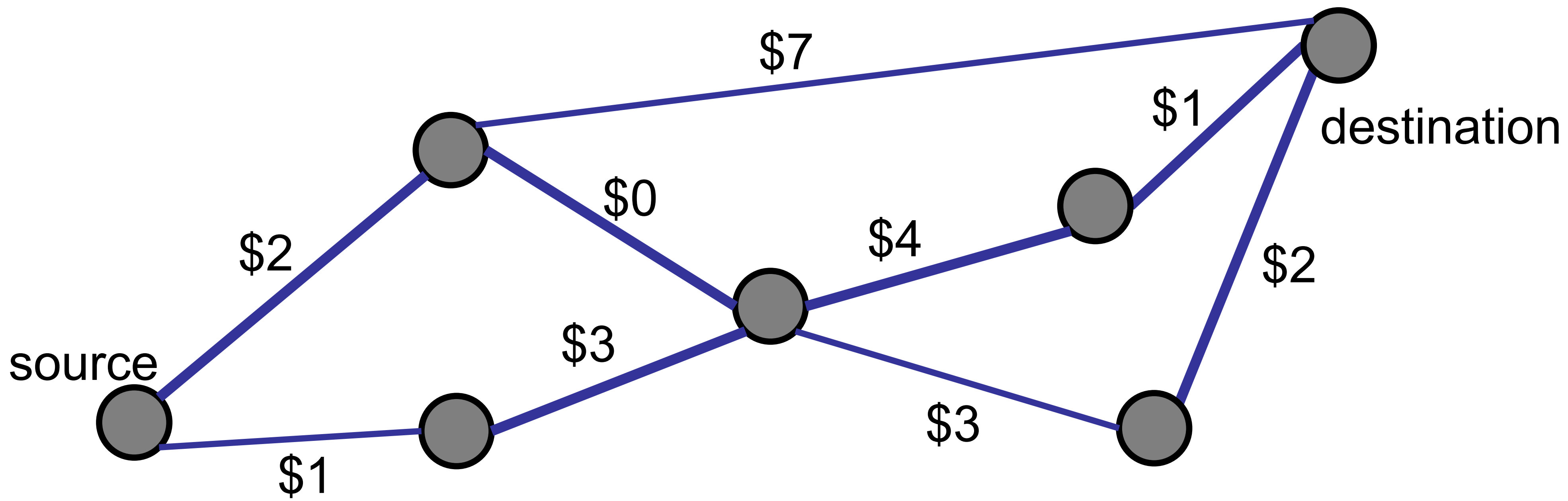
- Start at source s.
- Go to destination d.
- Each edge e earns money m(e).
- Find the path that makes the most money.



Example

NOT a solution:

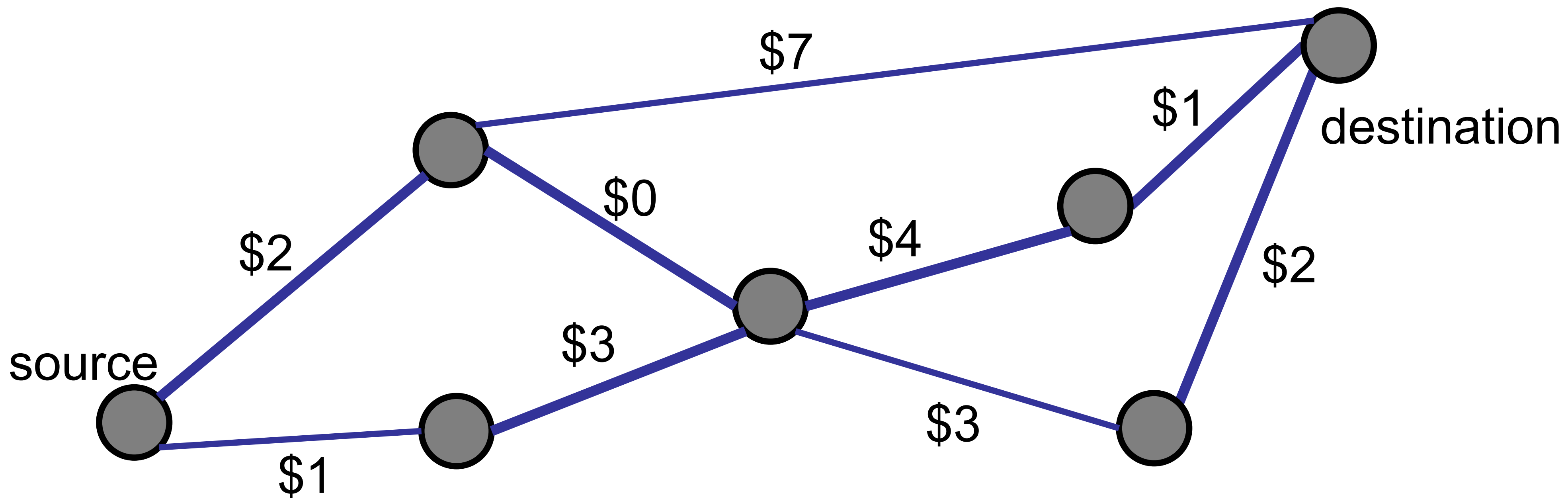
- Start at source s.
- Run BFS or DFS to explore every path.
- Keep track of the best path.



Example

Problem 1: Does not work.

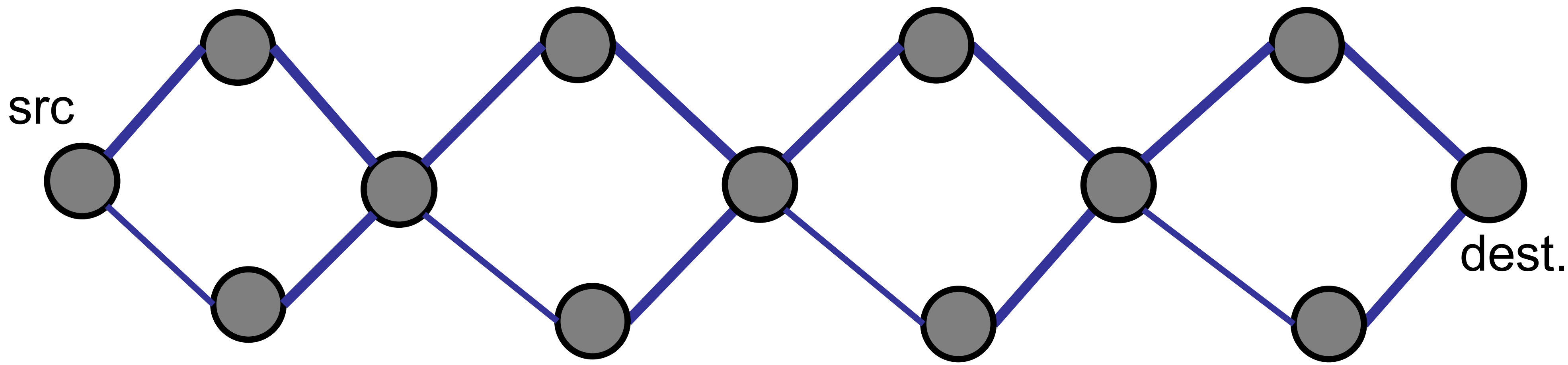
- DFS or BFS do NOT explore every path.
- Once a node is visited, it is never explored again.



Example

Problem 2: Too expensive.

- Some graphs have an exponential number of paths.
- It takes exponential time to explore all paths.



Example: $2^4 > 2^{n/4}$ different s->d paths.

Common Mistake

What do BFS and DFS solve?

- They visit every node in the graph? Yes.
- They visit every edge in the graph? Yes.
- ~~They visit every path in the graph?~~

Roadmap

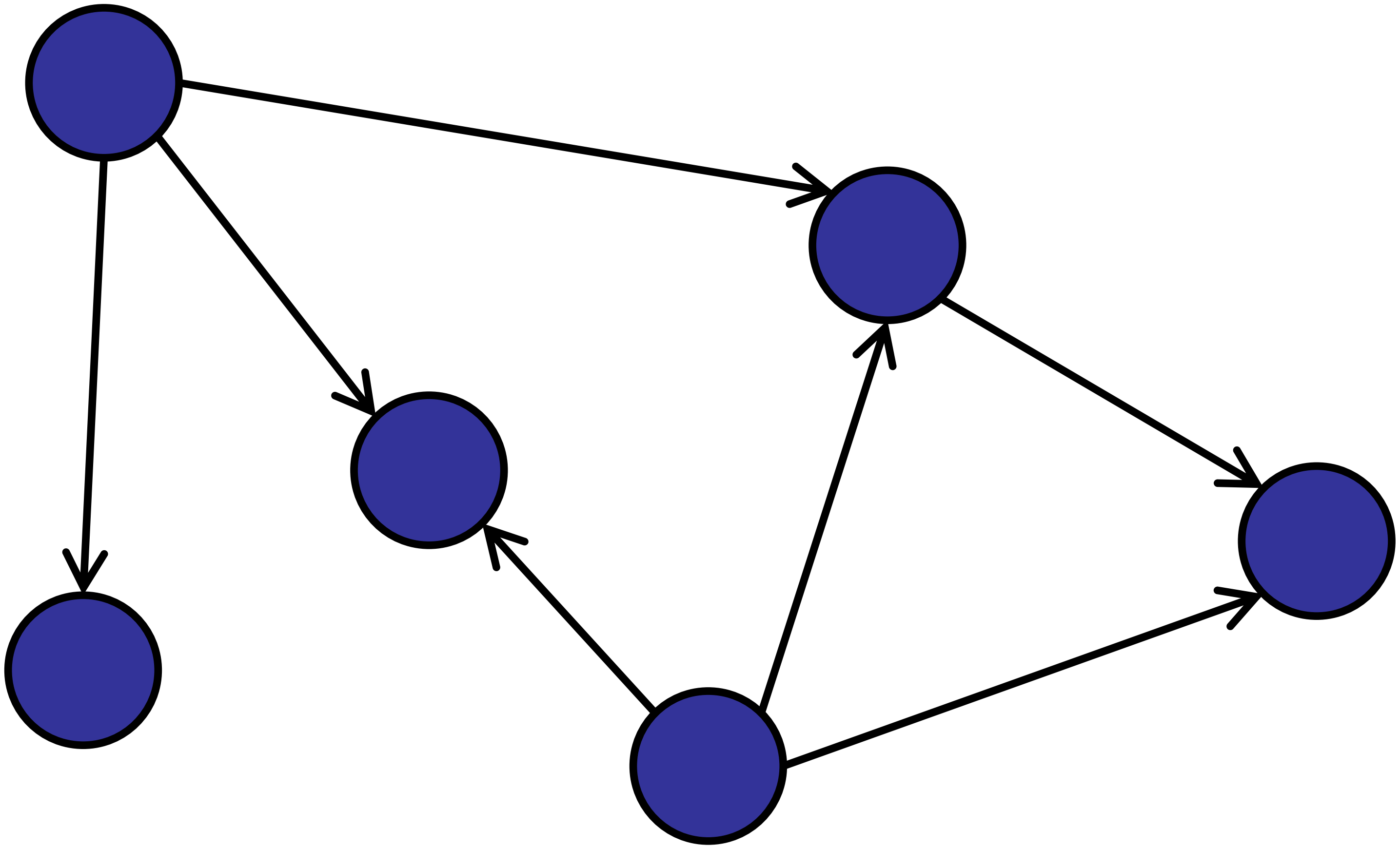
Directed Graphs

- What is a directed graph?
- Searching directed graphs (DFS / BFS)
- Topological Sort
- Connected Components

What is a **directed** graph? (Digraph)

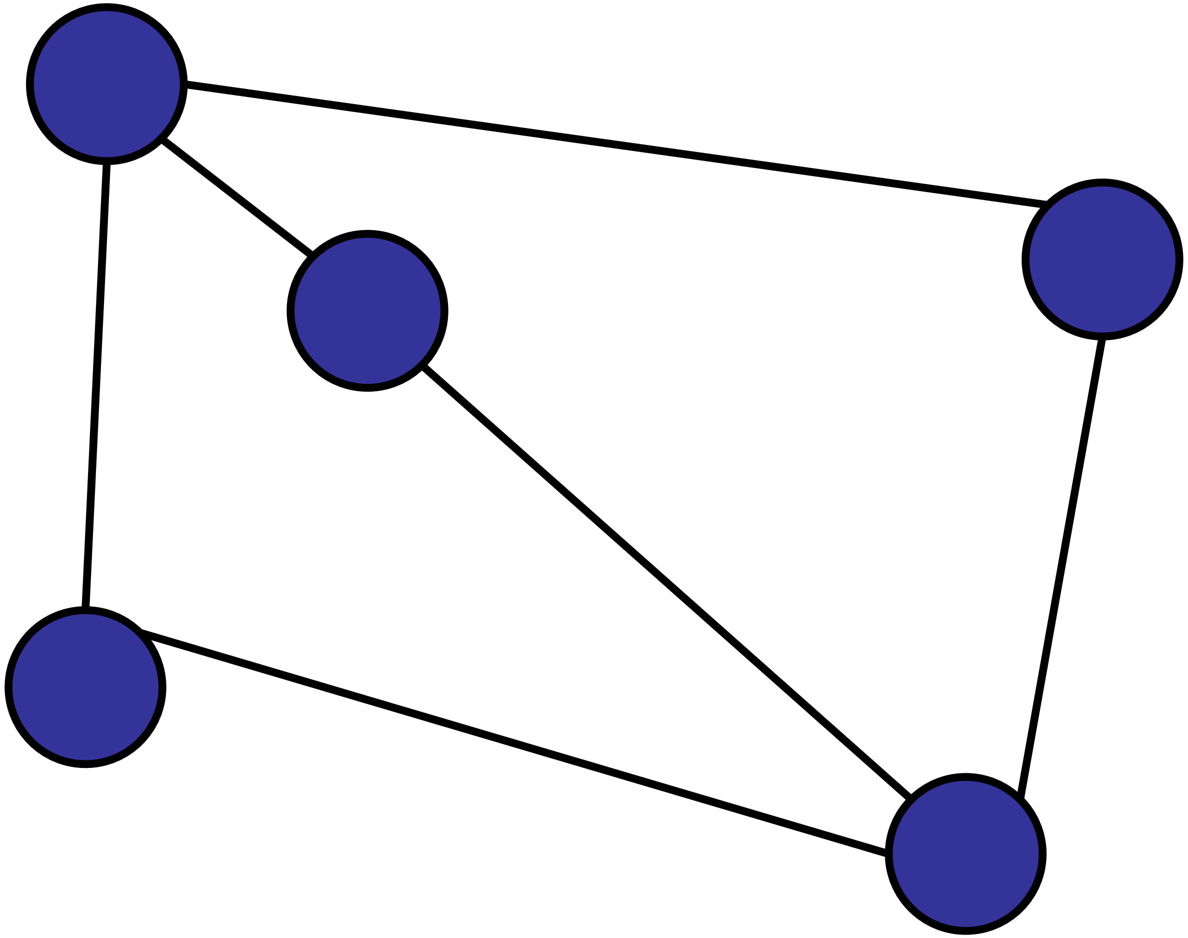
Is it a directed graph?

1. Yes
2. No.



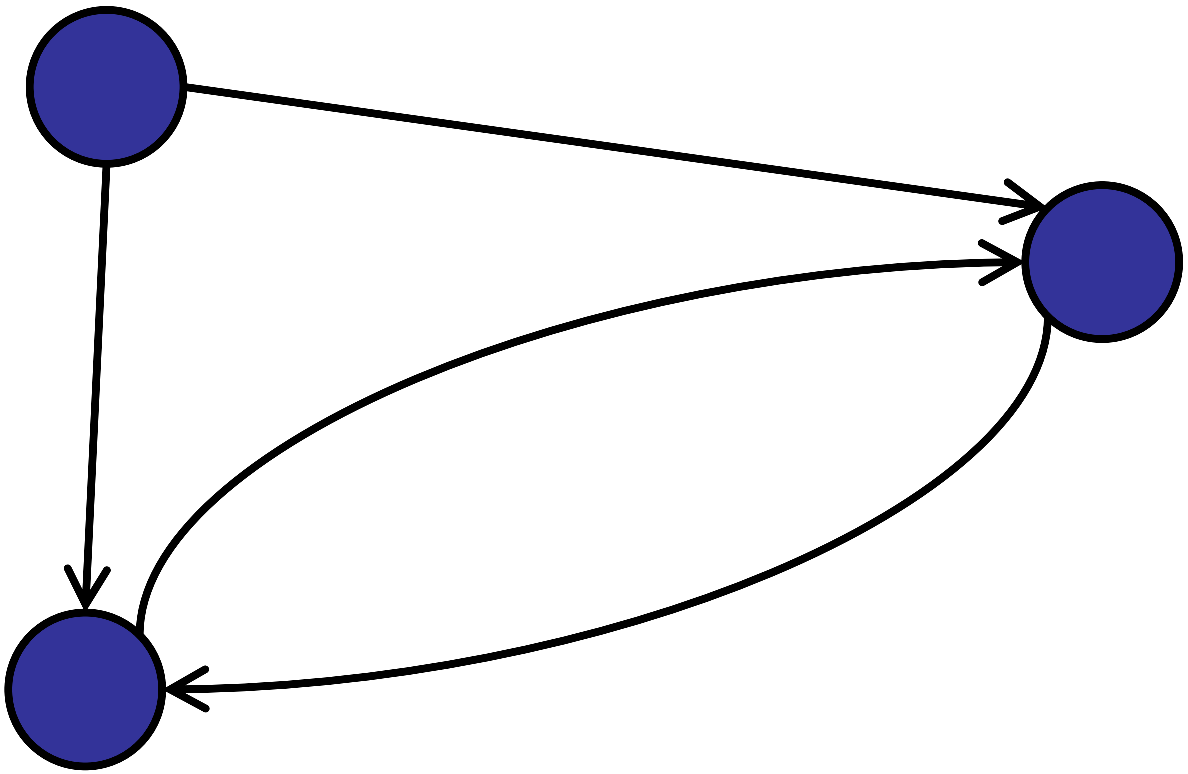
Is it a directed graph?

1. Yes
2. No.



Is it a directed graph?

1. Yes
2. No.



What is a directed graph?

Graph consists of two types of elements:

Nodes (or vertices)

- At least one.

Edges (or arcs)

- Each edge connects two nodes in the graph
- Each edge is unique.
- Each edge is **directed**.

What is a directed graph?

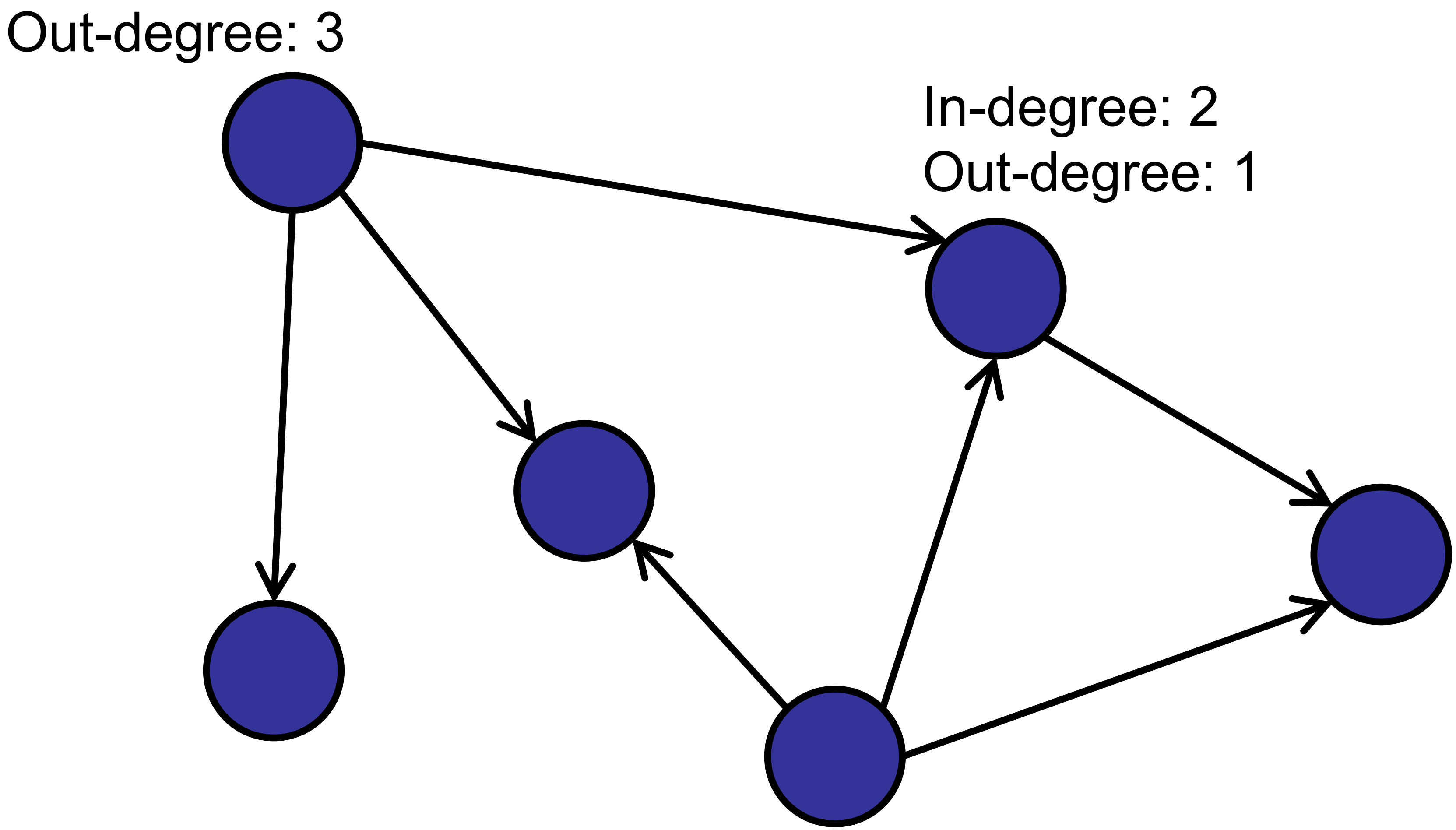
Graph $G = \langle V, E \rangle$

- V is a set of nodes
  - At least one: $|V| > 0$.
- E is a set of edges:
  - $E \subseteq \{ (v,w) : (v \in V), (w \in V) \}$
  - $e = (v,w)$ Order matters!
  - For all $e_1, e_2 \in E$: $e_1 \neq e_2$

What is a directed graph?

In-degree: number of incoming edges

Out-degree: number of outgoing edges



Representing a (Directed) Graph

Adjacency List:

- Array of nodes
- Each node maintains a list of neighbors
- Space: $O(V + E)$

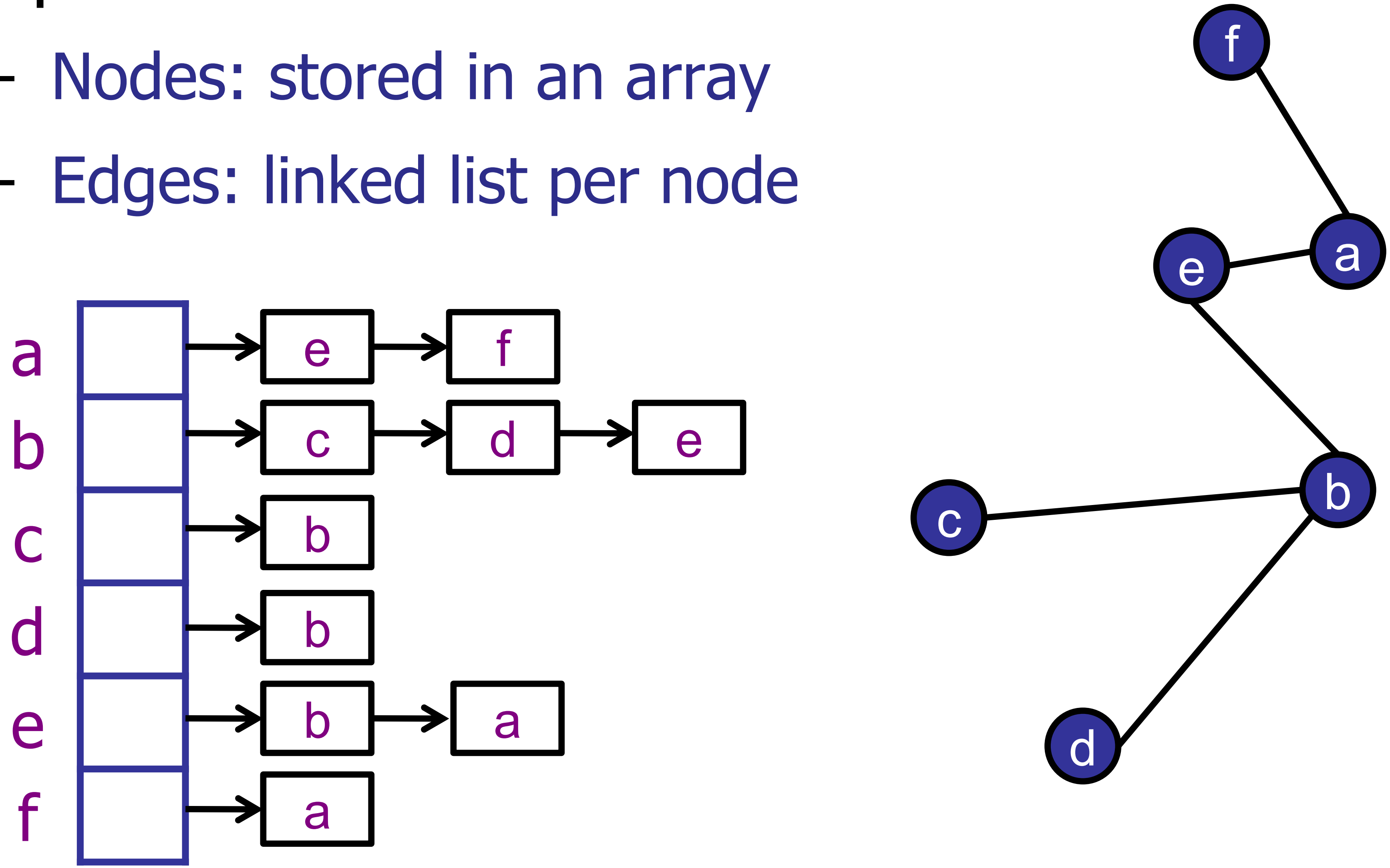
Adjacency Matrix:

- Matrix $A[v,w]$ represents edge (v,w)
- Space: $O(V^2)$

Adjacency List

Graph consists of:

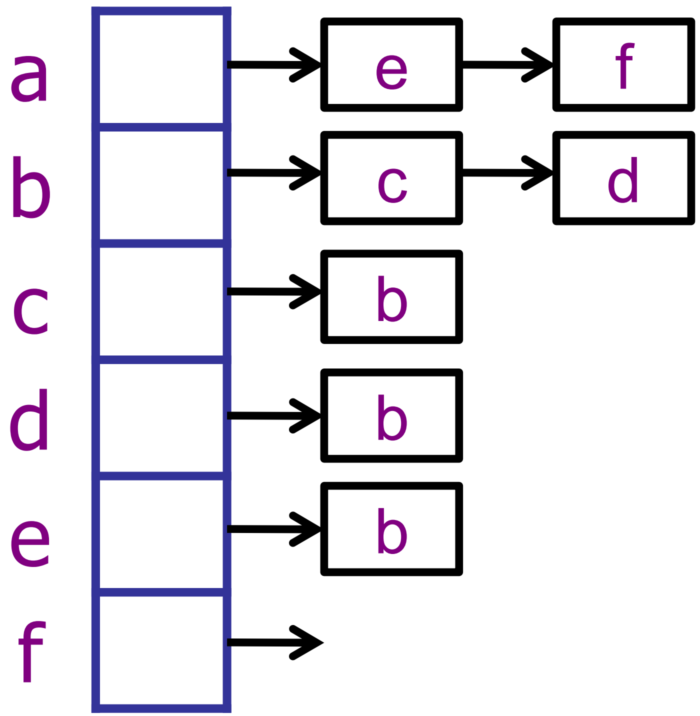
- Nodes: stored in an array
- Edges: linked list per node

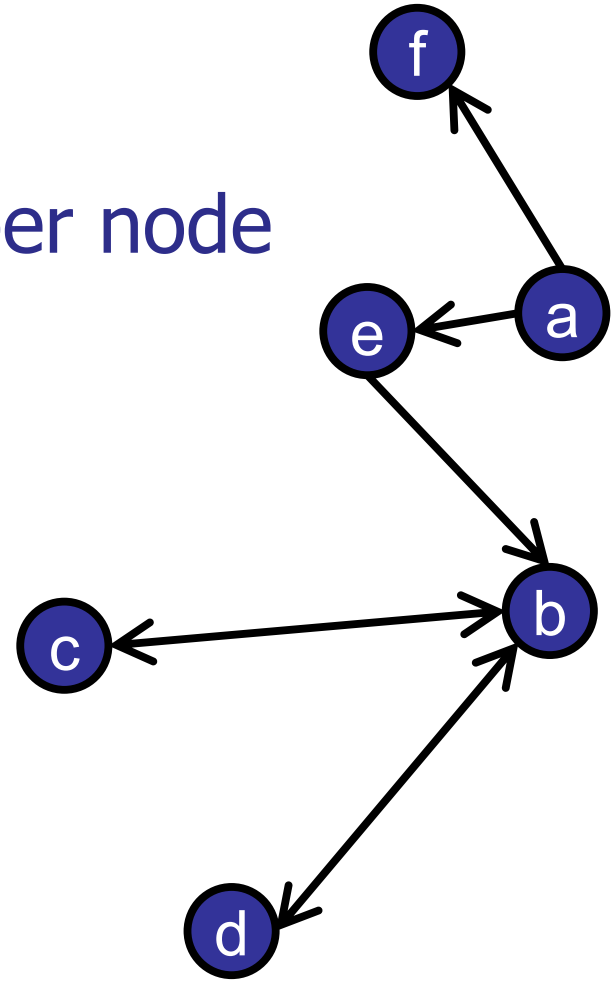


Adjacency List

Directed Graph consists of:

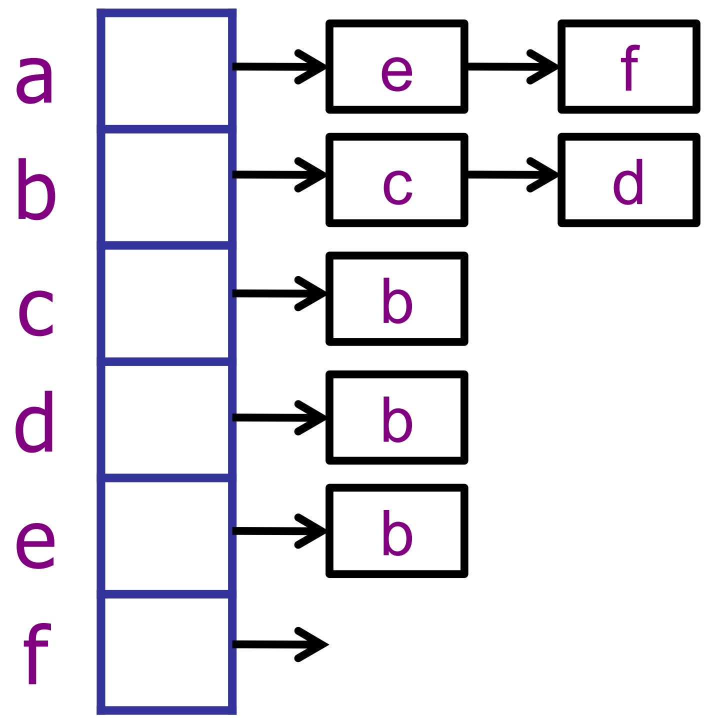
- Nodes: stored in an array
- **Outgoing** Edges: linked list per node





Adjacency List in Java

```
class NeighborList extends ArrayList<Integer> {
}

class Node {
  int key;
  NeighborList nbrs;
}

class Graph {
  Node[] nodeList;


}
```



Representing a (Directed) Graph

Adjacency List:

- Array of nodes
- Each node maintains a list of neighbors
- Space: $O(V + E)$

Adjacency Matrix:

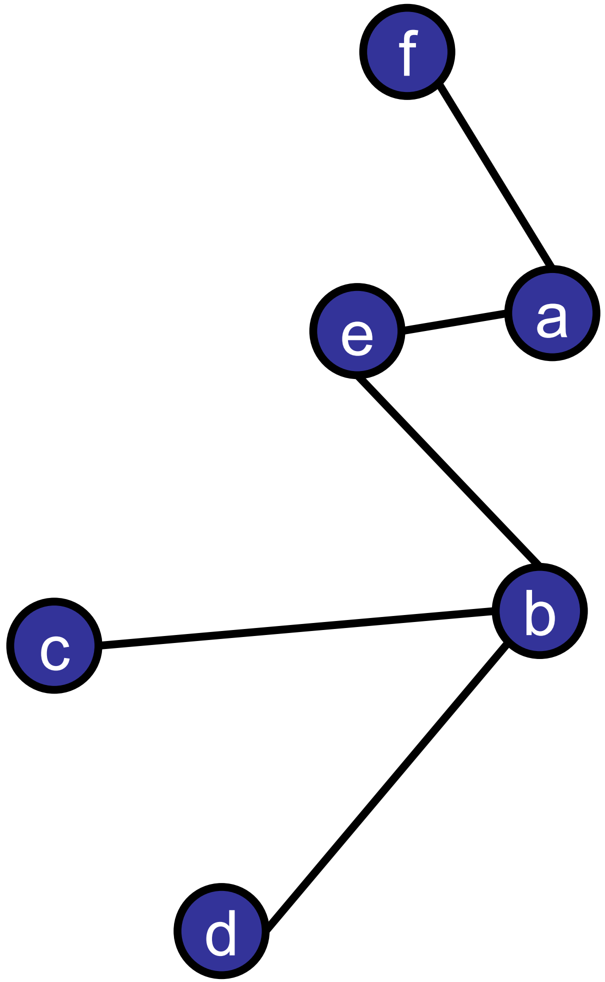
- Matrix $A[v,w]$ represents edge (v,w)
- Space: $O(V^2)$

Adjacency Matrix

Graph consists of:

- Nodes
- Edges = pairs of nodes

	a	b	c	d	e	f
a	0	0	0	0	1	1
b	0	0	1	1	1	0
c	0	1	0	0	0	0
d	0	1	0	0	0	0
e	1	1	0	0	0	0
f	1	0	0	0	0	0

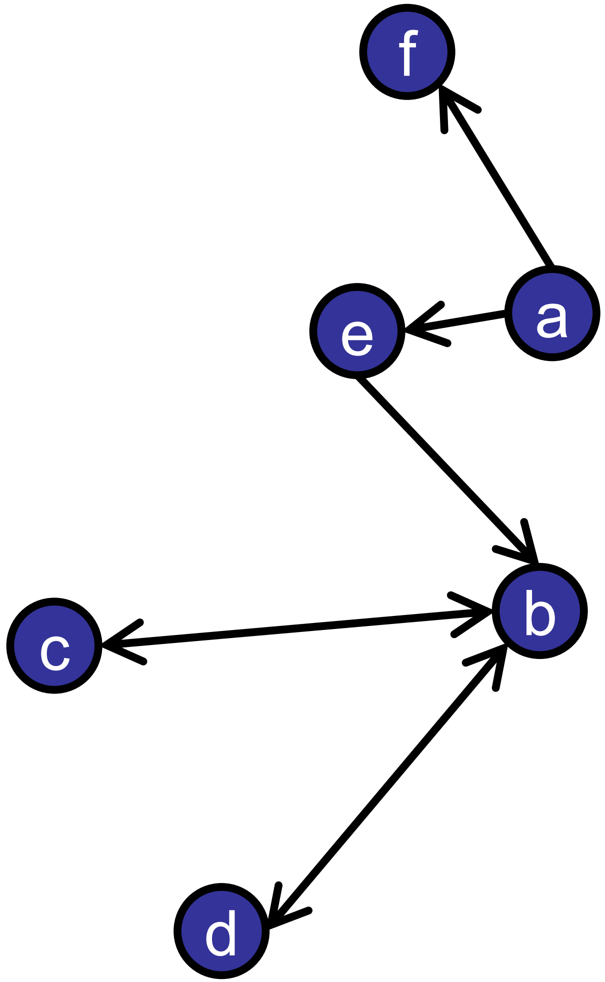


Adjacency Matrix

Directed Graph consists of:

- Nodes
- Edges = pairs of nodes

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**0**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**0**	**1**	0	0	0	0
f	**0**	0	0	0	0	0



Adjacency Matrix

Graph represented as:

$A[v][w] = 1$ iff $(v,w) \in E$

	a	b	c	d	e	f
a	0	0	0	0	**1**	**1**
b	0	0	**1**	**1**	**0**	0
c	0	**1**	0	0	0	0
d	0	**1**	0	0	0	0
e	**0**	**1**	0	0	0	0
f	**0**	0	0	0	0	0

Searching a (Directed) Graph

Breadth-First Search:

- Search level-by-level
- Follow outgoing edges
- Ignore incoming edges

Depth-First Search:

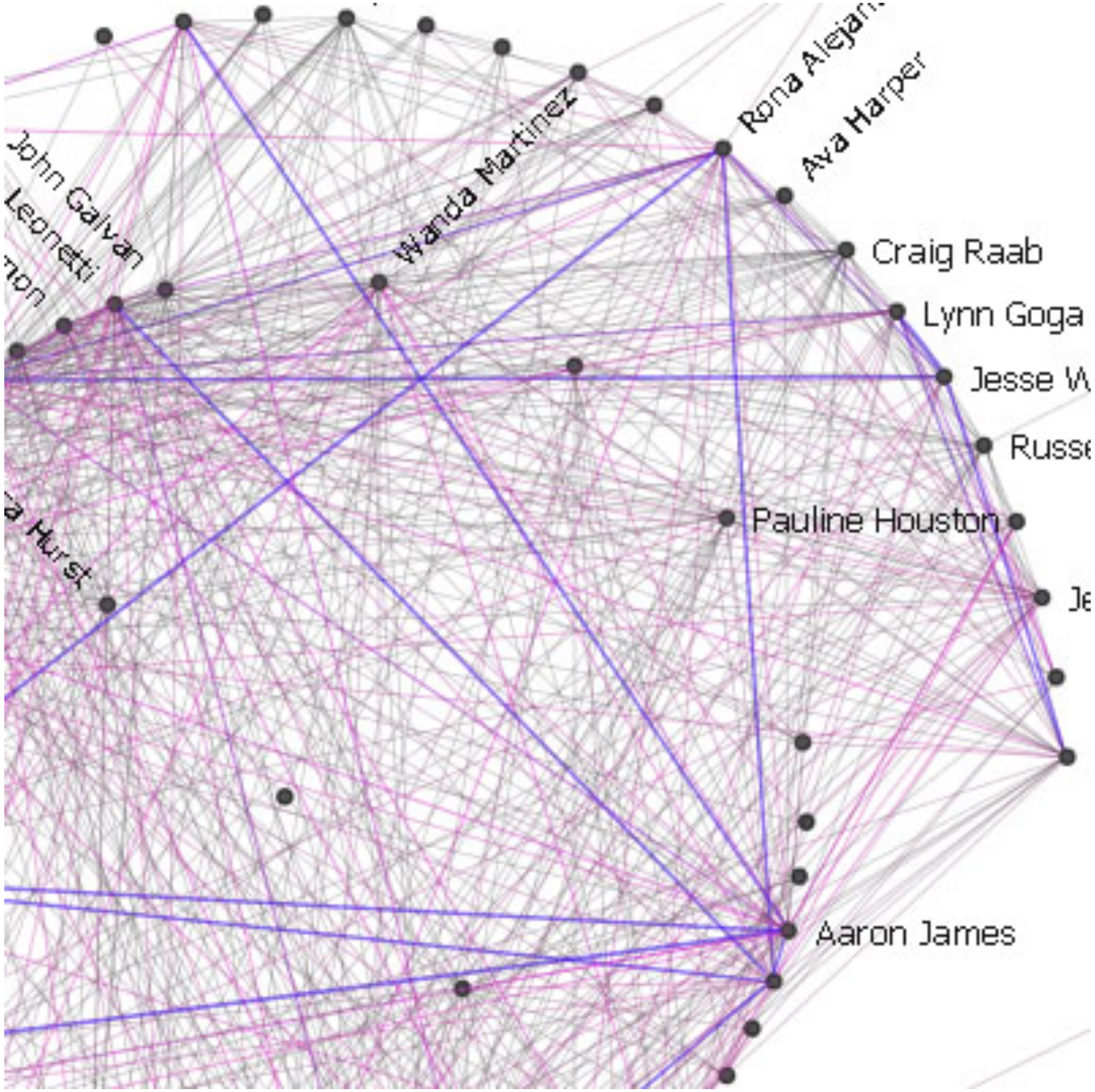
- Search recursively
- Follow outgoing edges
- Backtrack (through incoming edges)

Example of directed graphs

Directed Graphs

Is friendship always bidirectional?:

- Nodes are people
- Edge = friendship



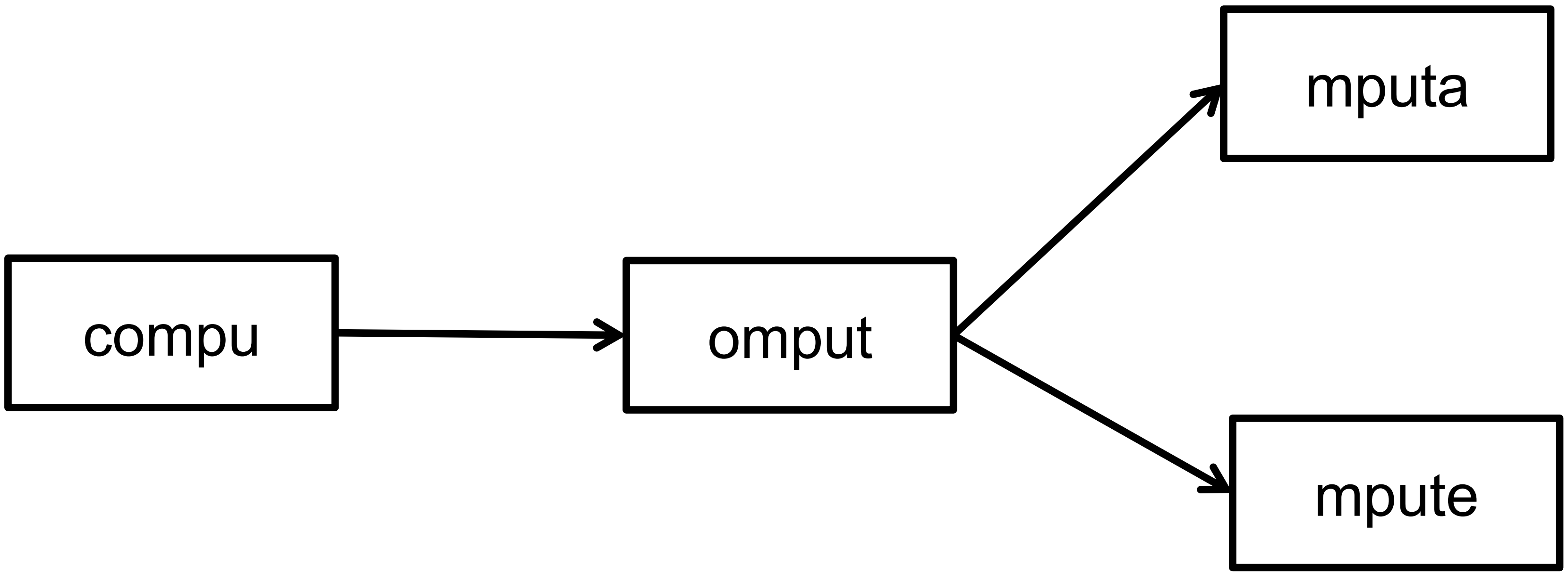
Facebook: yes

Google+: no

Directed Graphs

Markov text generation:

- Nodes are kgrams
- Edge = one kgram follows another



Scheduling

Set of tasks for baking cookies:

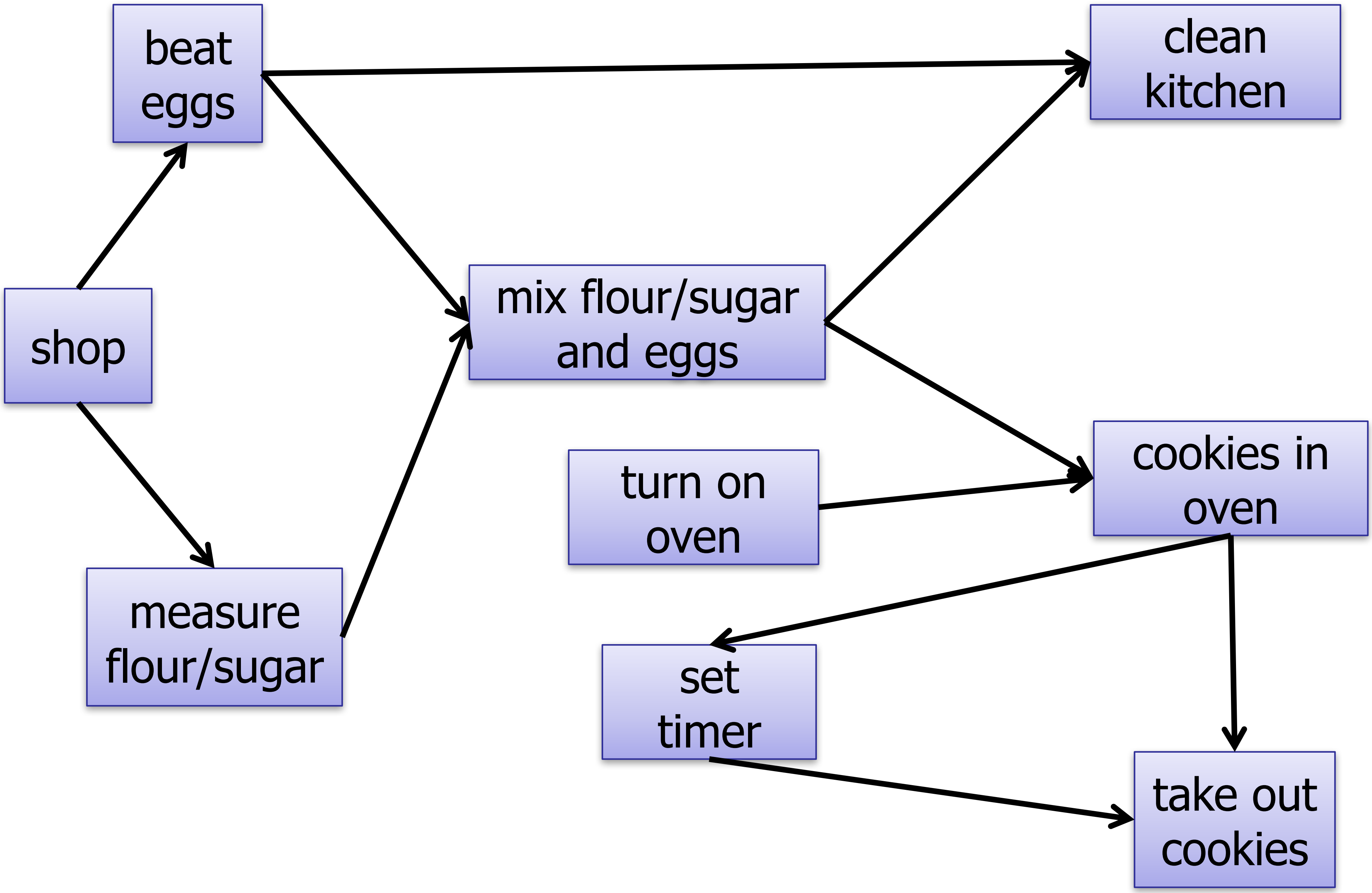
- Shop for groceries
- Put the cookies in the oven
- Clean the kitchen
- Beat the eggs in a bowl
- Measure the flour and sugar in a bowl
- Mix the eggs with the flour and sugar
- Turn on the oven
- Set the timer
- Take out the cookies

Scheduling

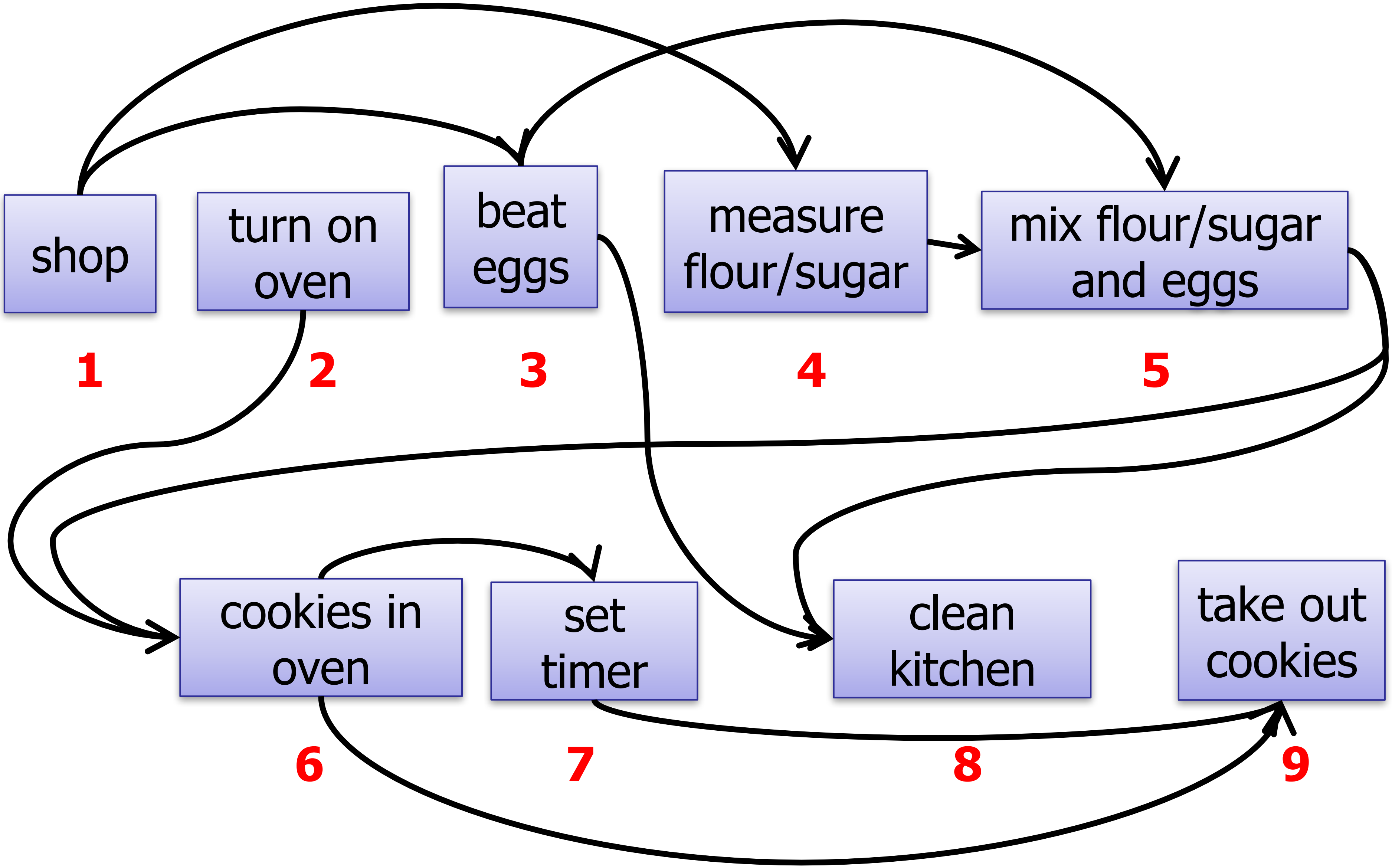
Ordering:

- Shop for groceries before beat the eggs
- Shop for groceries before measure the flour
- Turn on the oven before put the cookies in the oven
- Beat the eggs before mix the eggs with the flour
- Measure the flour before mix the eggs with the flour
- Put the cookies in the oven before set the timer
- Measure the flour before clean the kitchen
- Beat the eggs before clean the kitchen
- Mix the flour and the eggs before clean the kitchen

Scheduling


beat eggs
clean kitchen
shop
mix flour/sugar and eggs
turn on oven
cookies in oven
measure flour/sugar
set timer
take out cookies

Topological Ordering



Topological Order

Properties:

1. Sequential total ordering of all nodes

1. shop | 2. turn on oven | 3. measure flour/sugar | 4. eggs

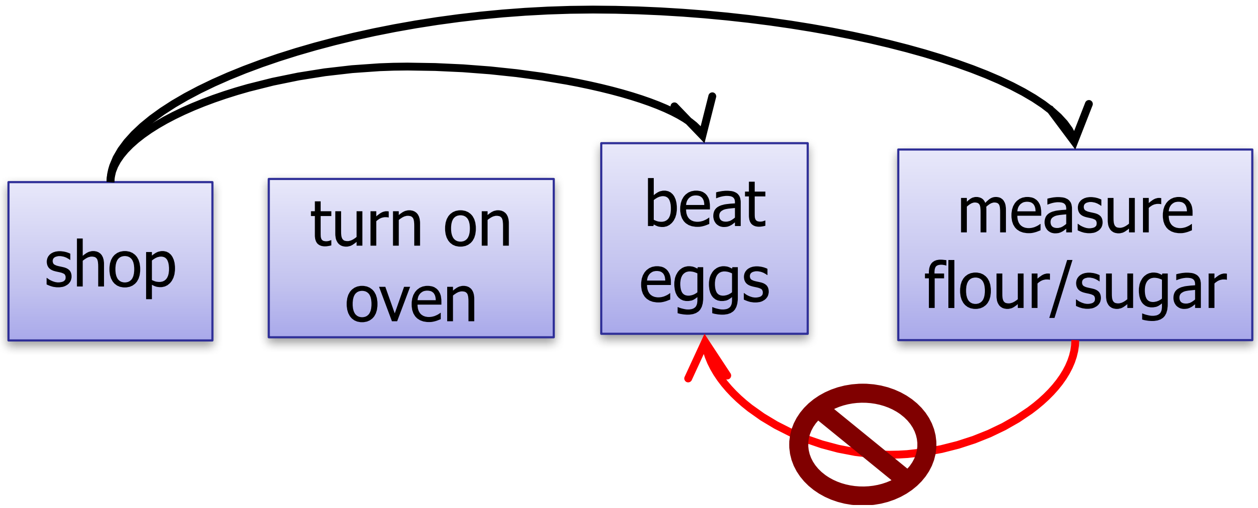
Topological Order

Properties:

1. Sequential total ordering of all nodes

1. shop | 2. turn on oven | 3. measure flour/sugar | 4. eggs

2. Edges only point forward

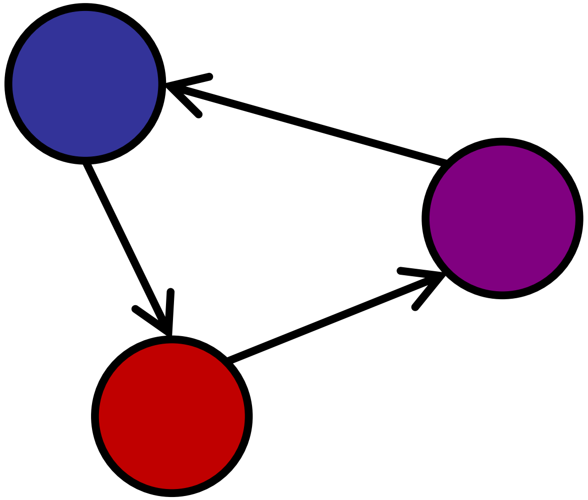


Does every directed graph have a topological ordering?

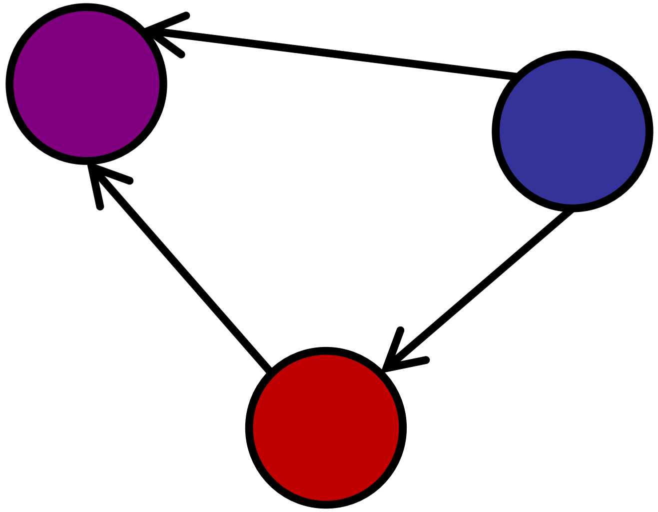
1. Yes
2. No
3. Only if the adjacency matrix has small second eigenvalue.

Directed Acyclic Graphs

Cyclic

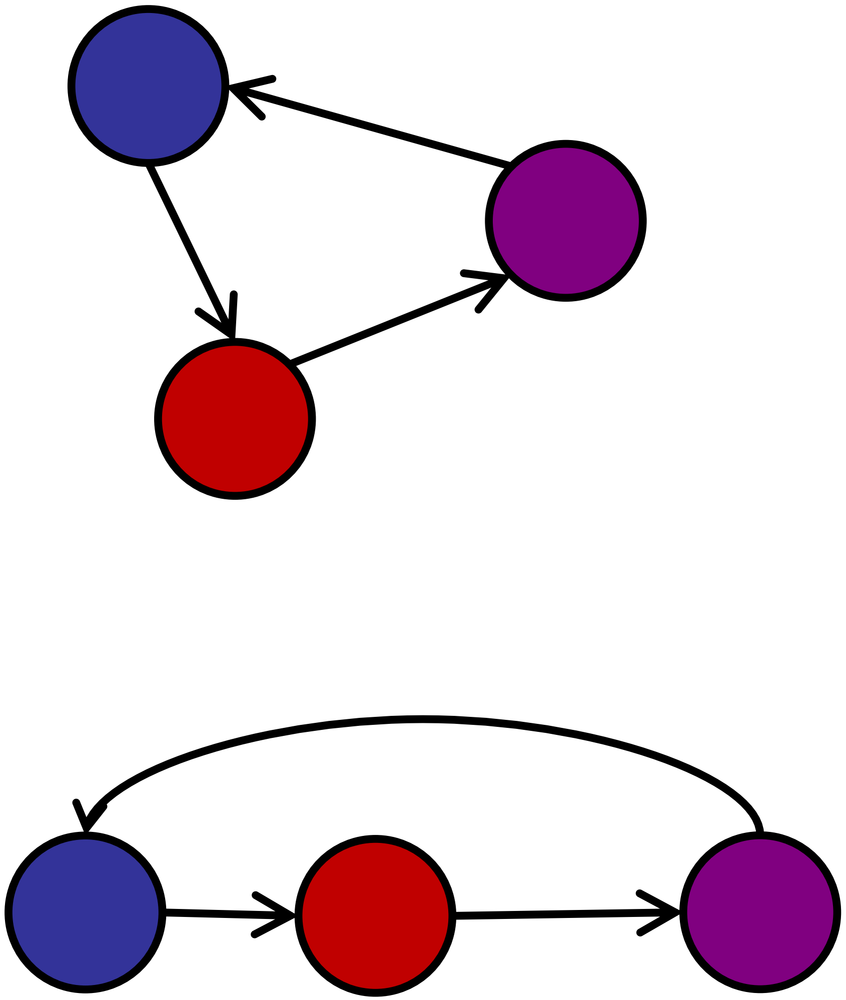


Acyclic

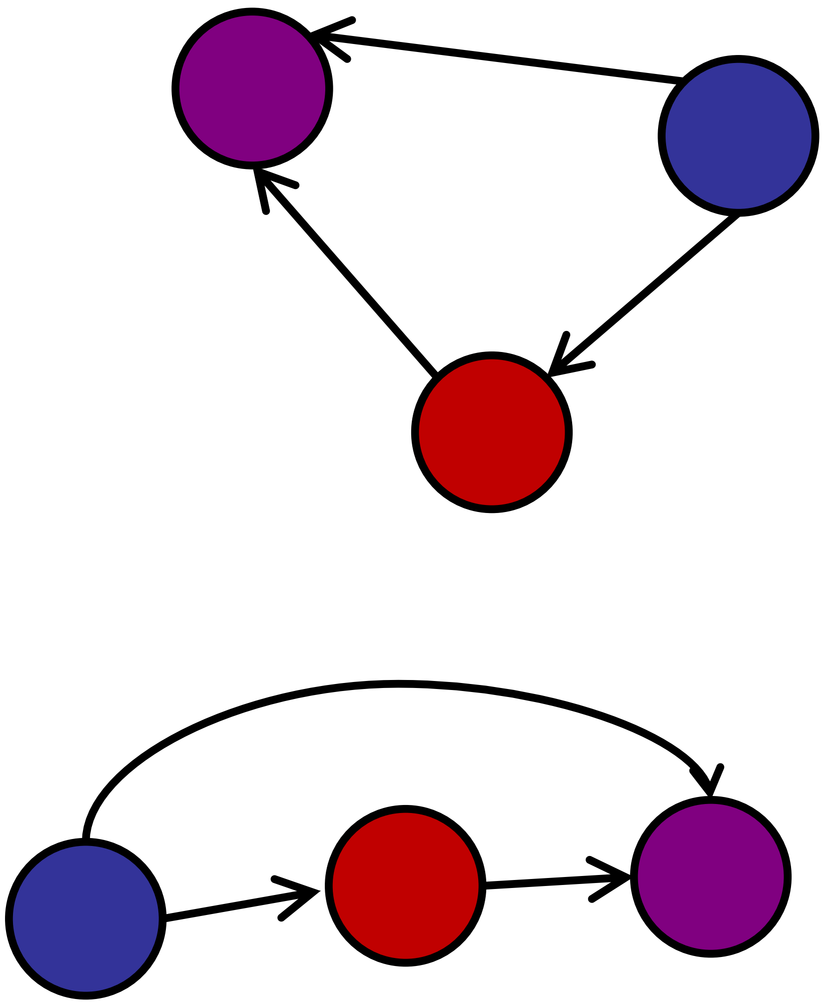


Directed Acyclic Graphs

Cyclic

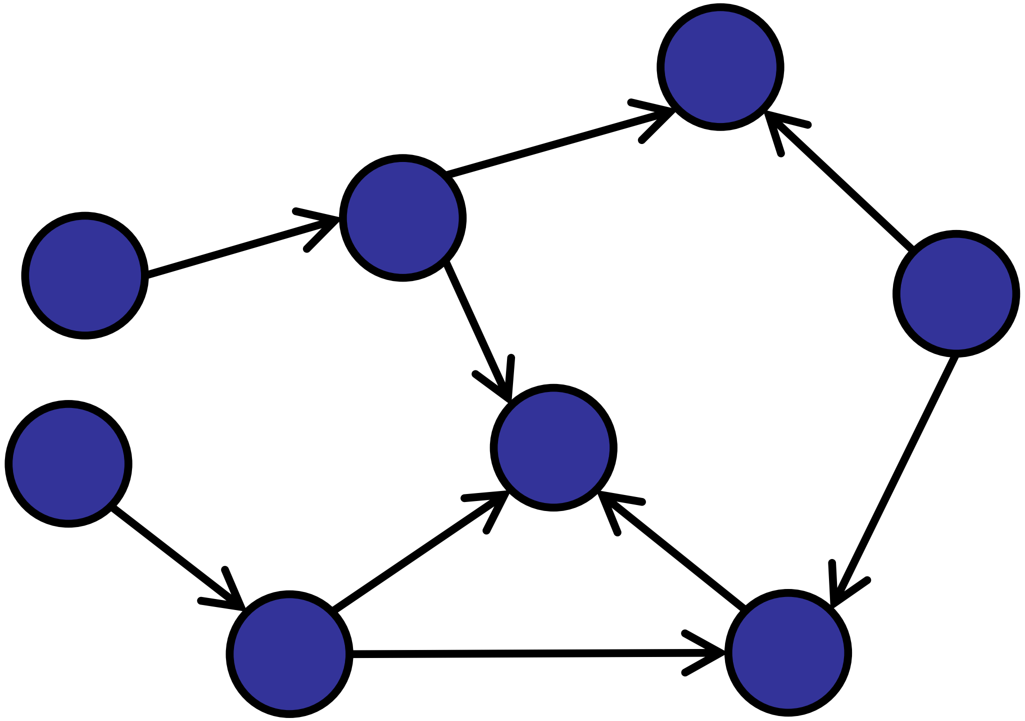


Acyclic



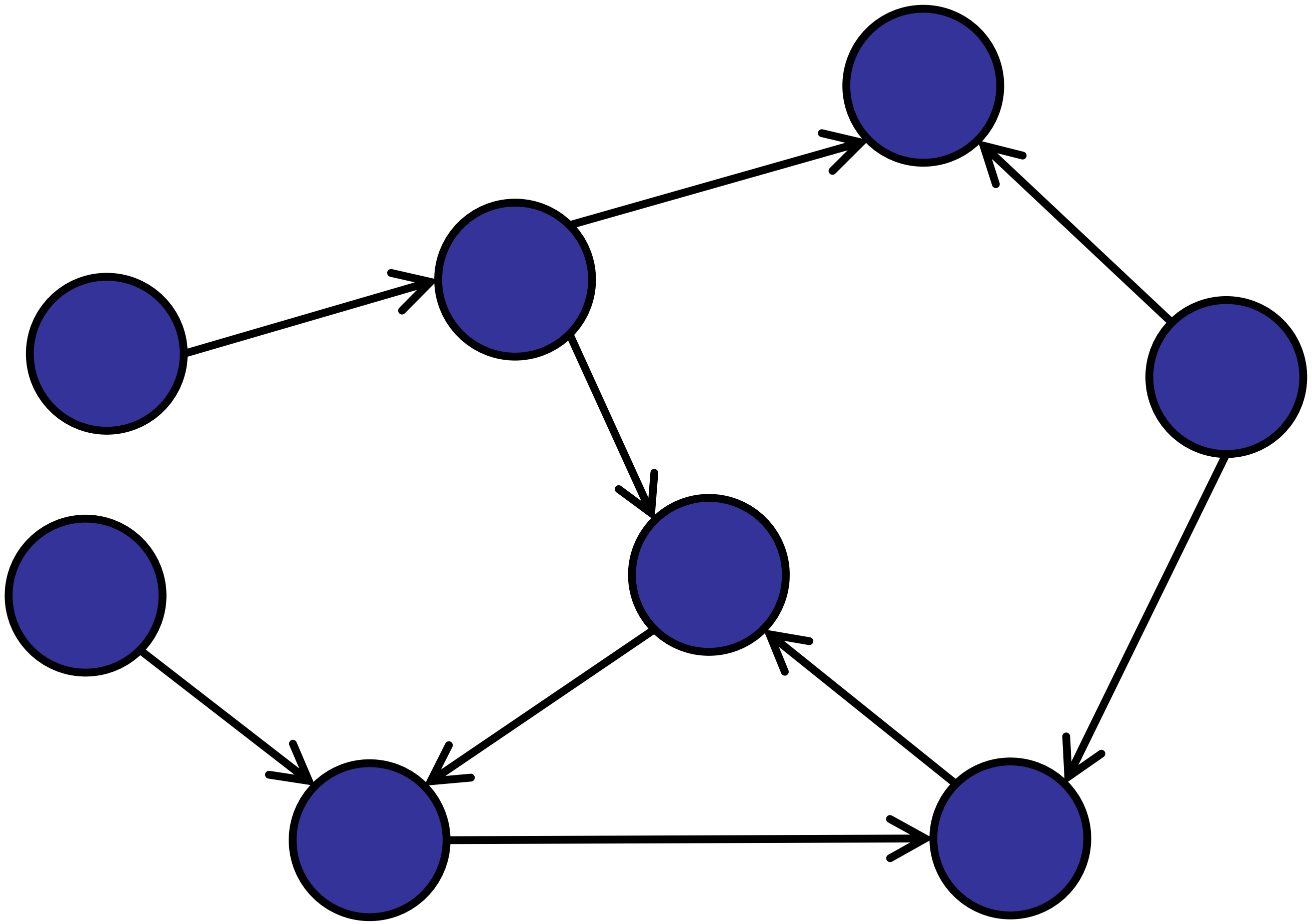
Is this graph:

1. Cyclic
2. Acyclic
3. Transcendental

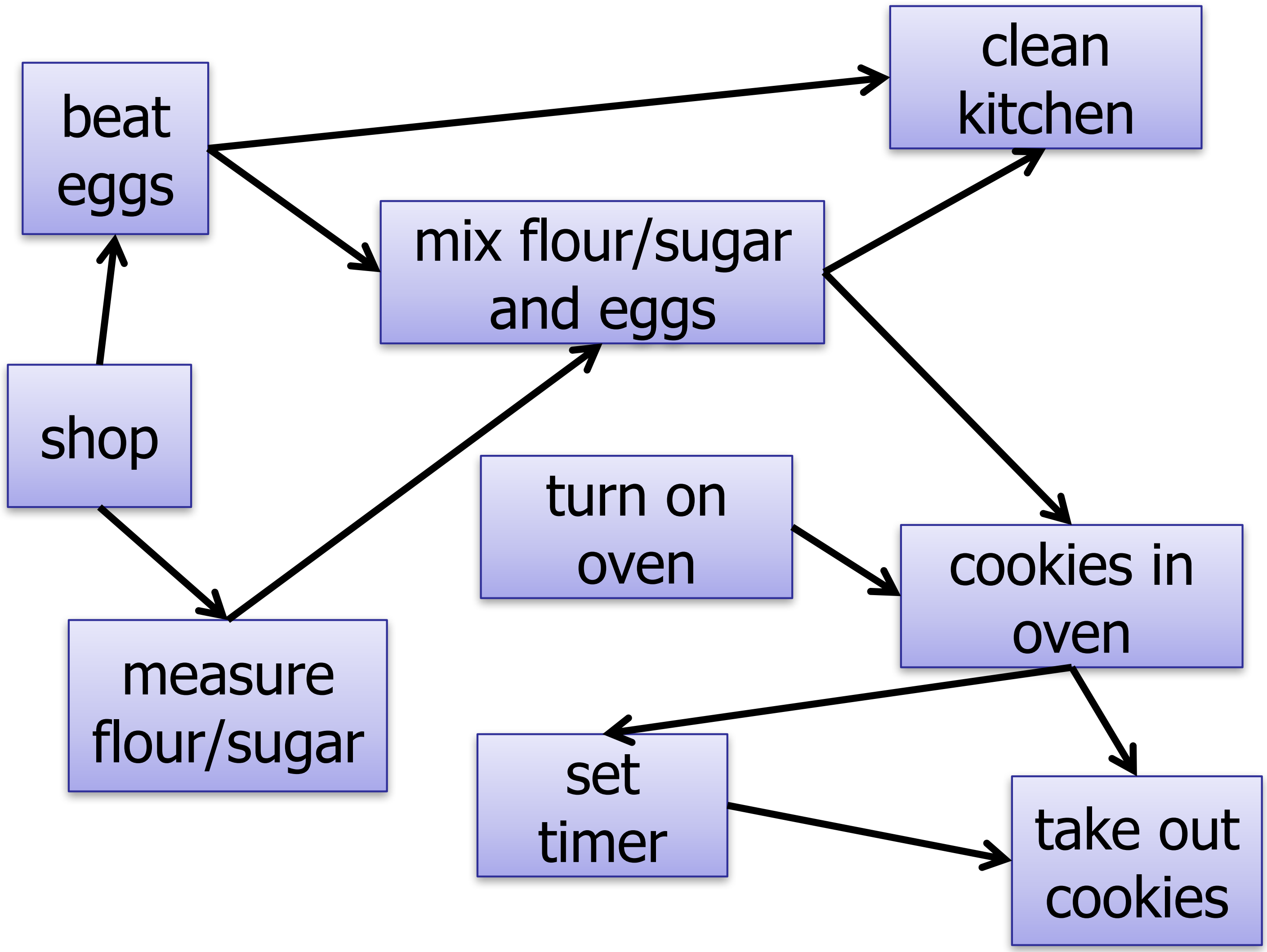


Directed Acyclic Graphs

Cyclic or Acyclic?



Directed Acyclic Graph



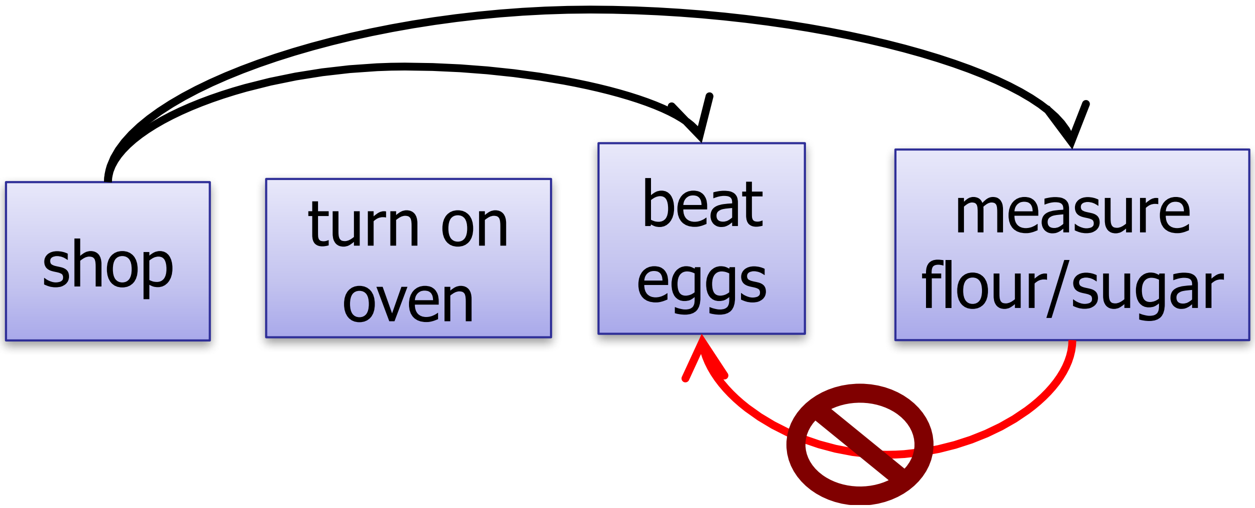
Topological Order

Properties:

1. Sequential total ordering of all nodes

1. shop | 2. turn on oven | 3. measure flour/sugar | 4. eggs

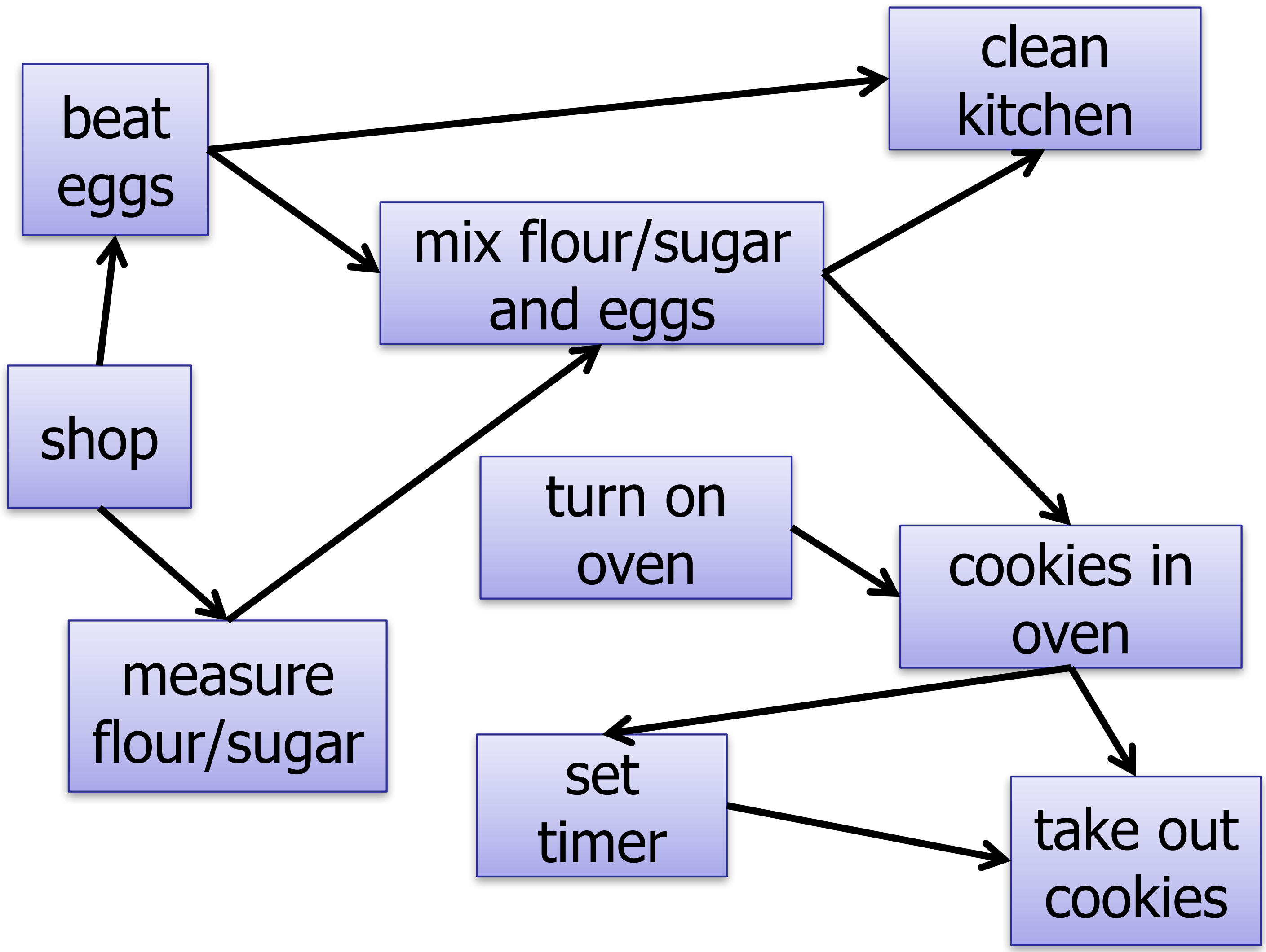
2. Edges only point forward



Which algorithm is best for finding a Topological Ordering in a DAG?

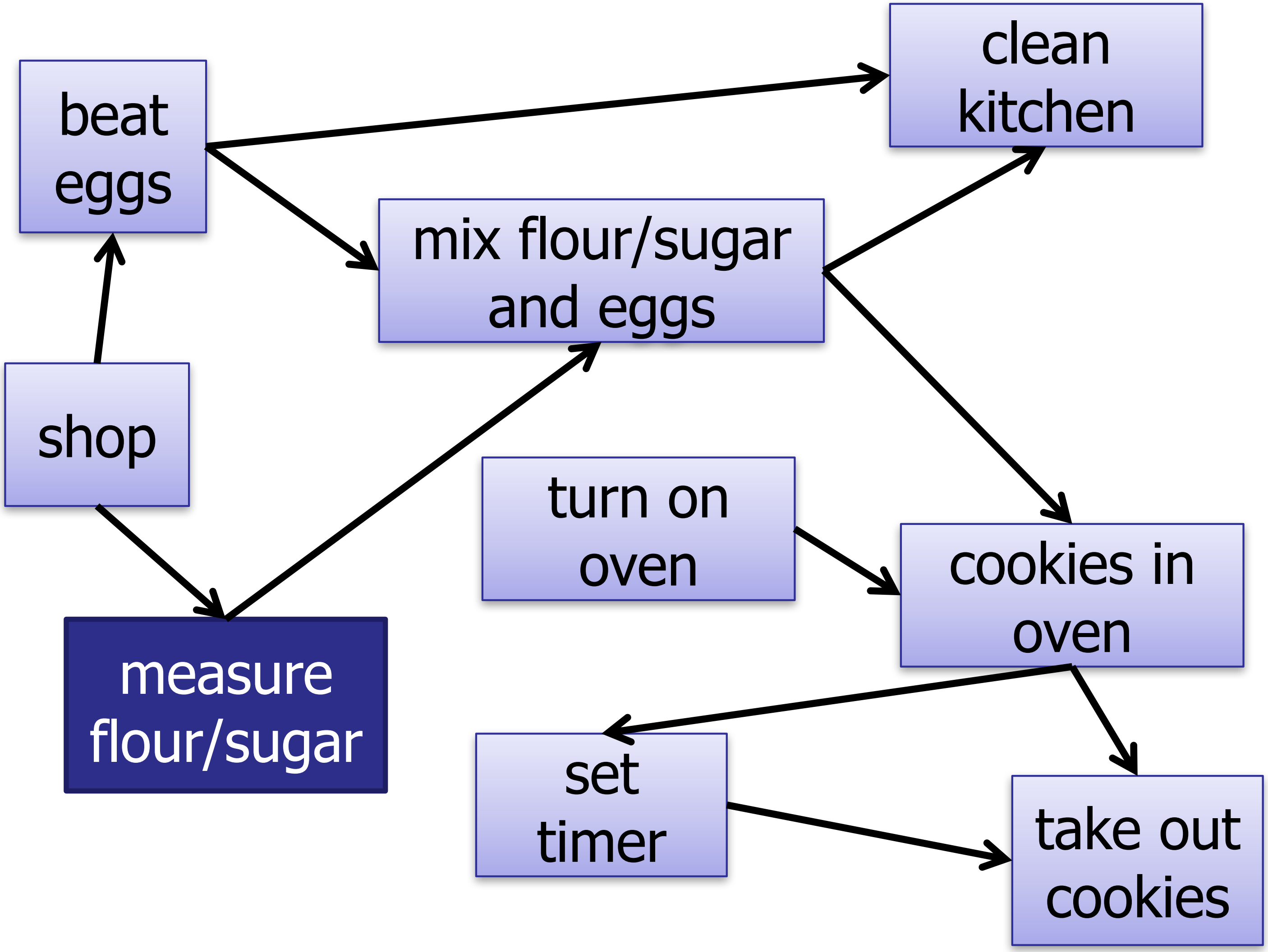
1. Breadth-first search
2. ✓ Depth-first search
3. Bellman-Ford
4. Prim’s
5. Something else

Depth-First Search



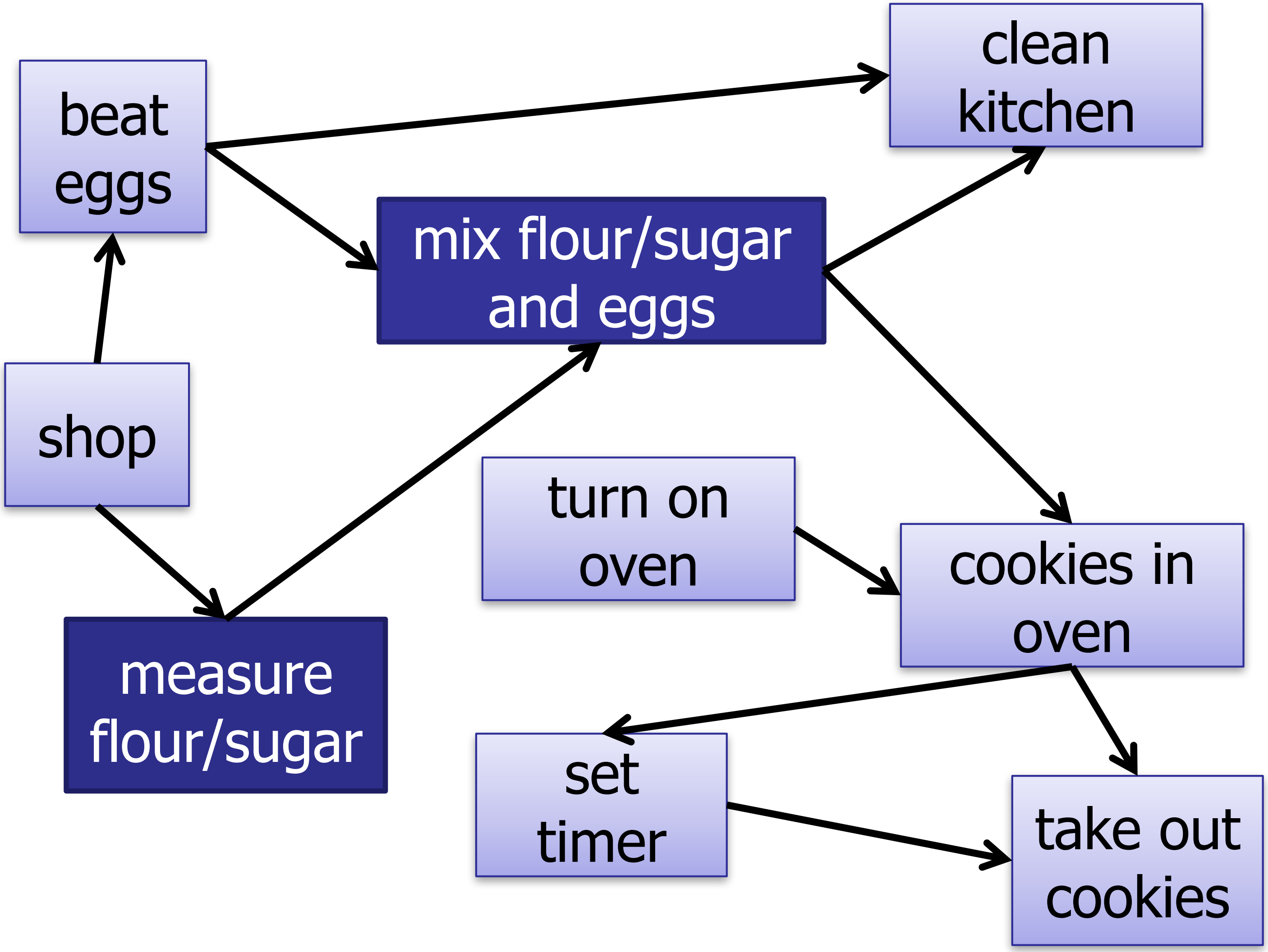
Depth-First Search

1. measure



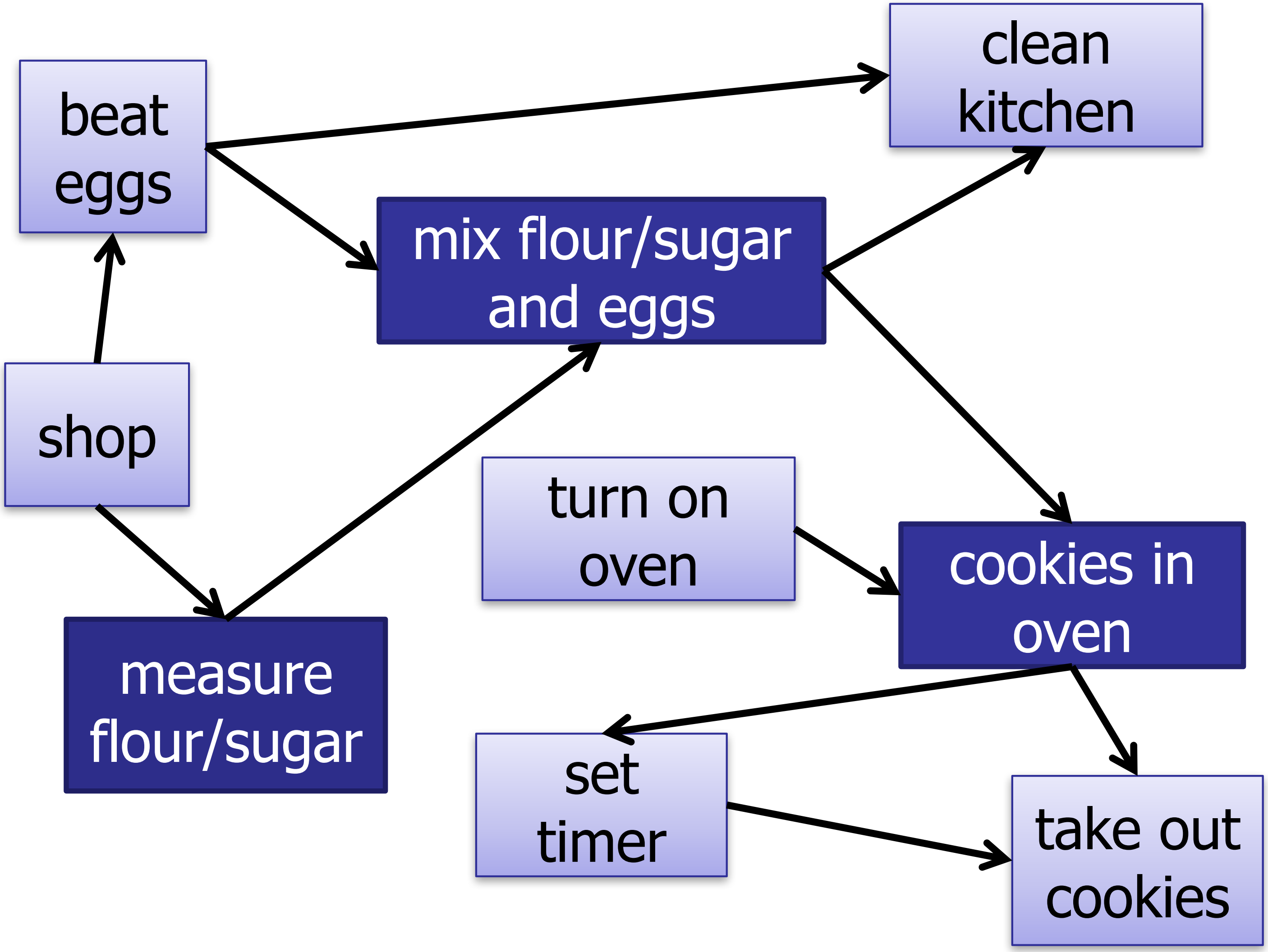
Depth-First Search

1. measure
2. mix



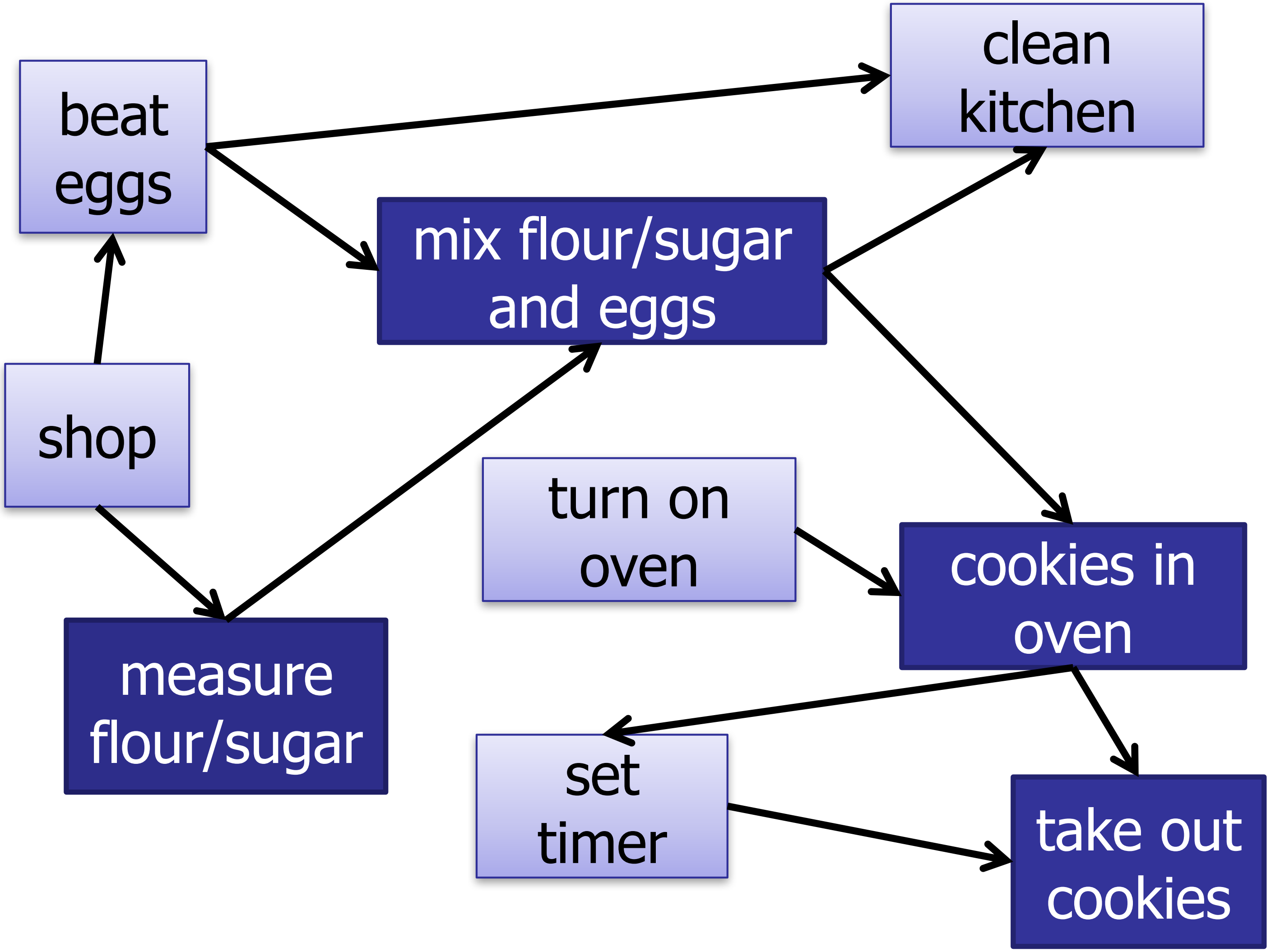
Depth-First Search

1. measure
2. mix
3. in oven



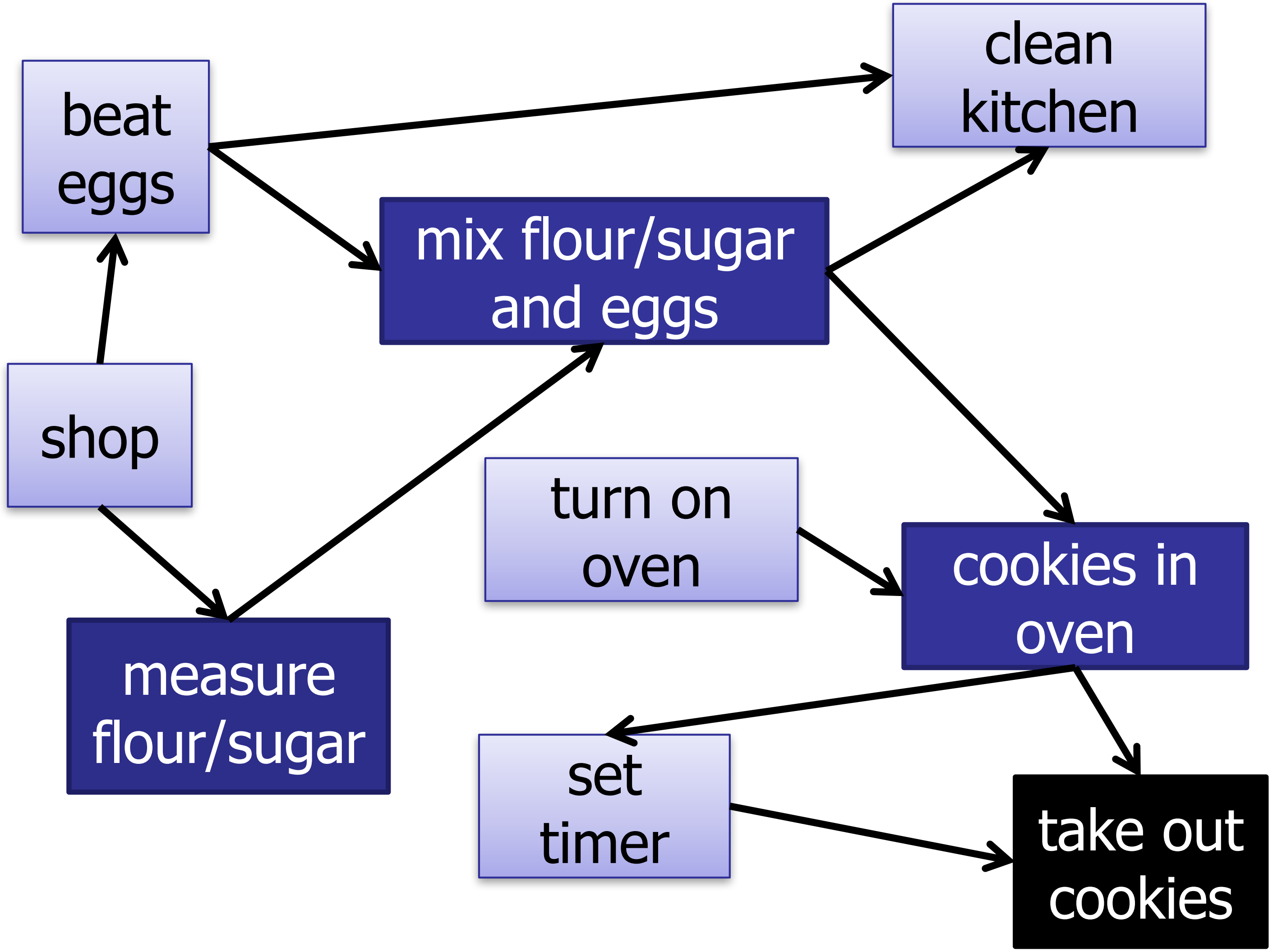
Depth-First Search

1. measure
2. mix
3. in oven
4. take out



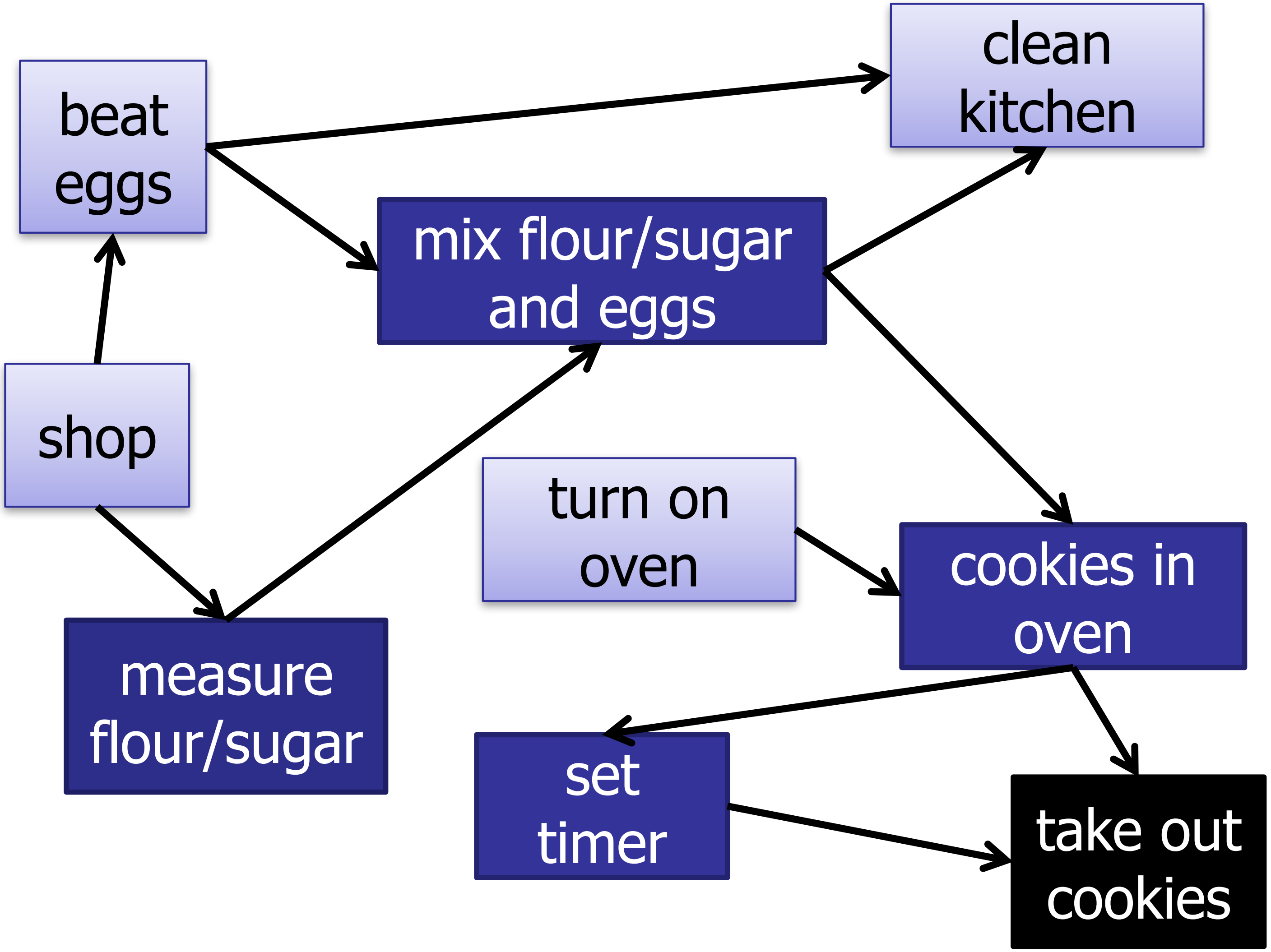
Depth-First Search

1. measure
2. mix
3. in oven
4. take out



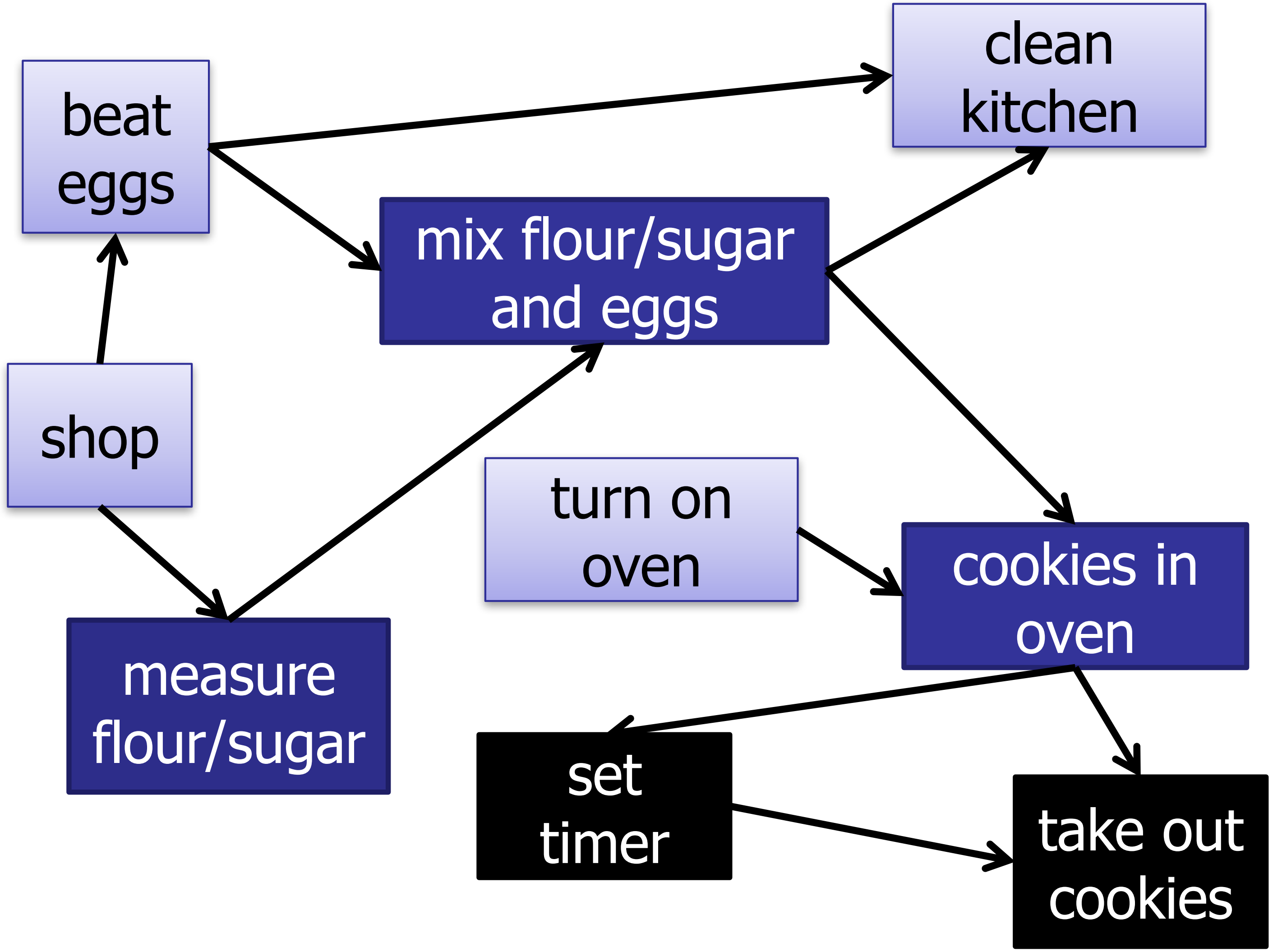
Depth-First Search

1. measure
2. mix
3. in oven
4. take out
5. set timer



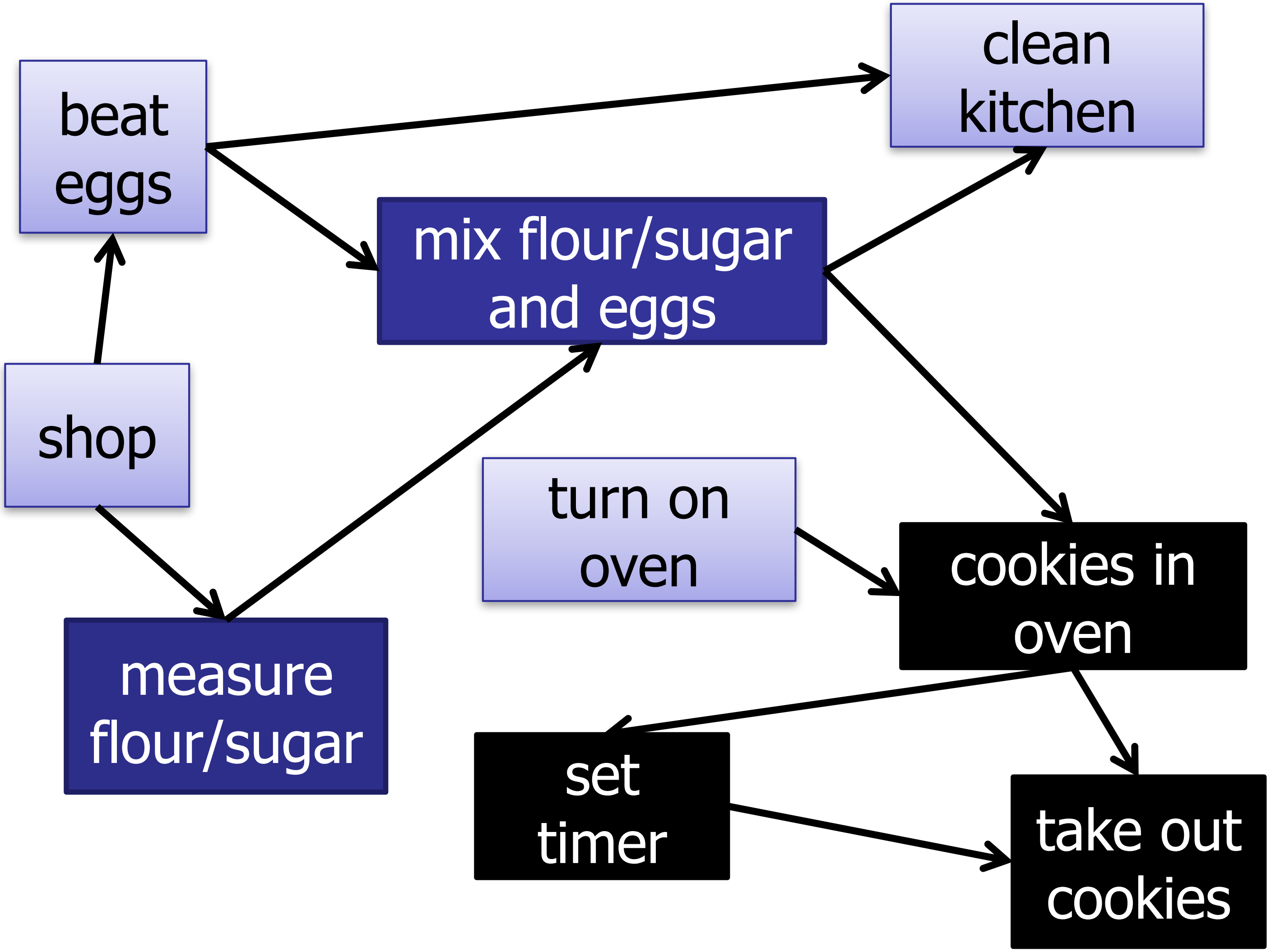
Depth-First Search

1. measure
2. mix
3. in oven
4. take out
5. set timer



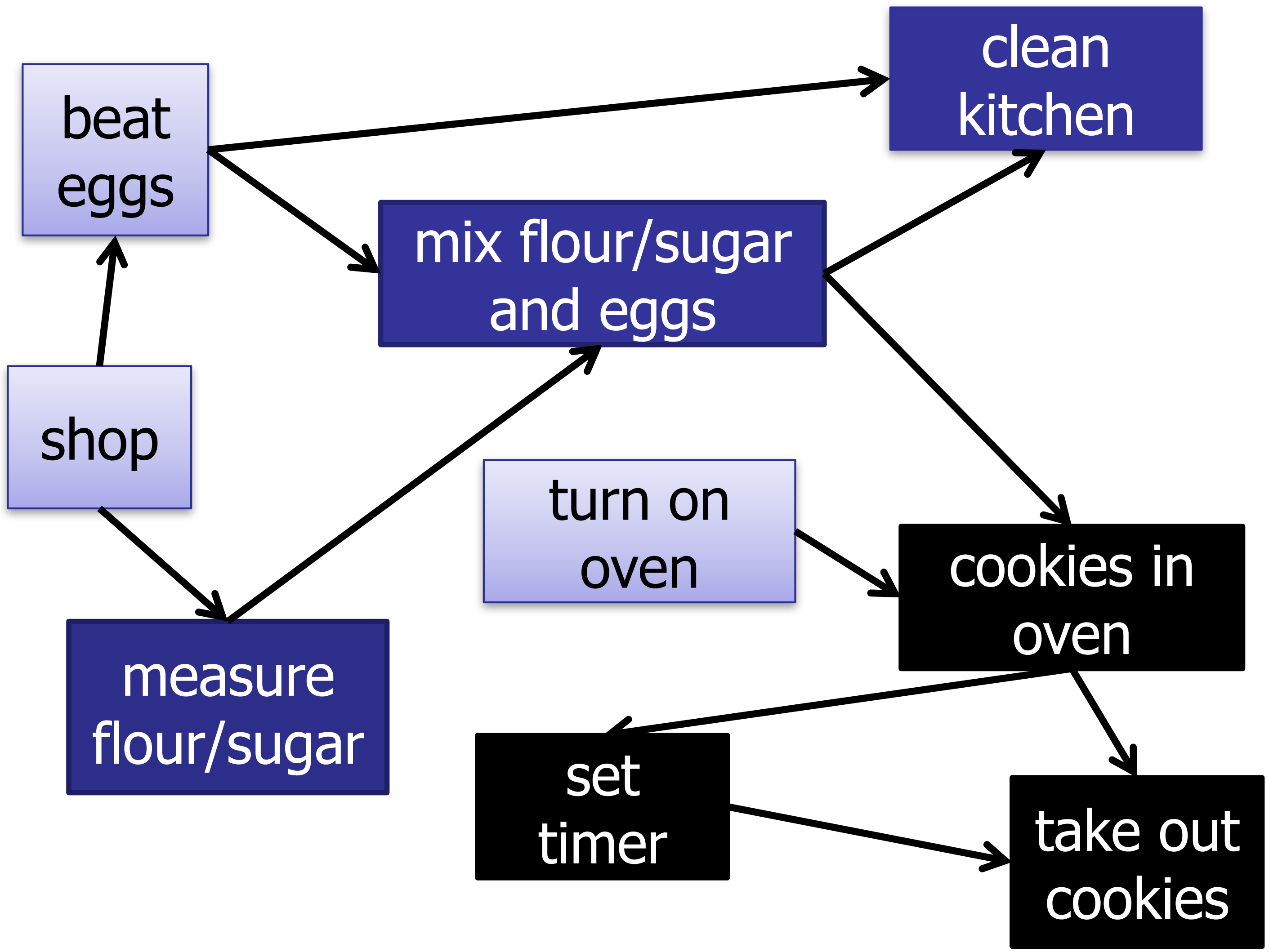
Depth-First Search

1. measure
2. mix
3. in oven
4. take out
5. set timer



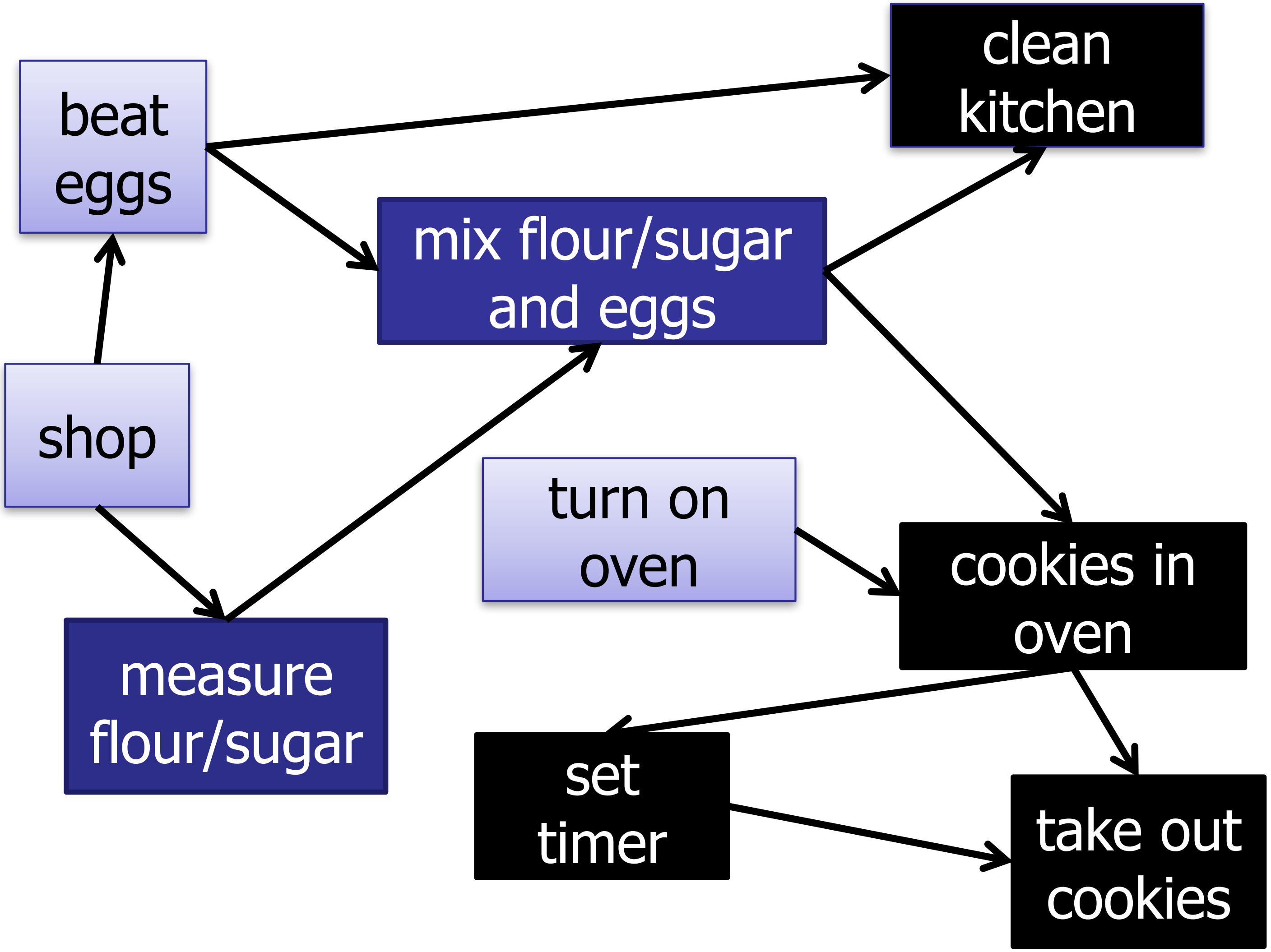
Depth-First Search

1. measure
2. mix
3. in oven
4. take out
5. set timer
6. clean



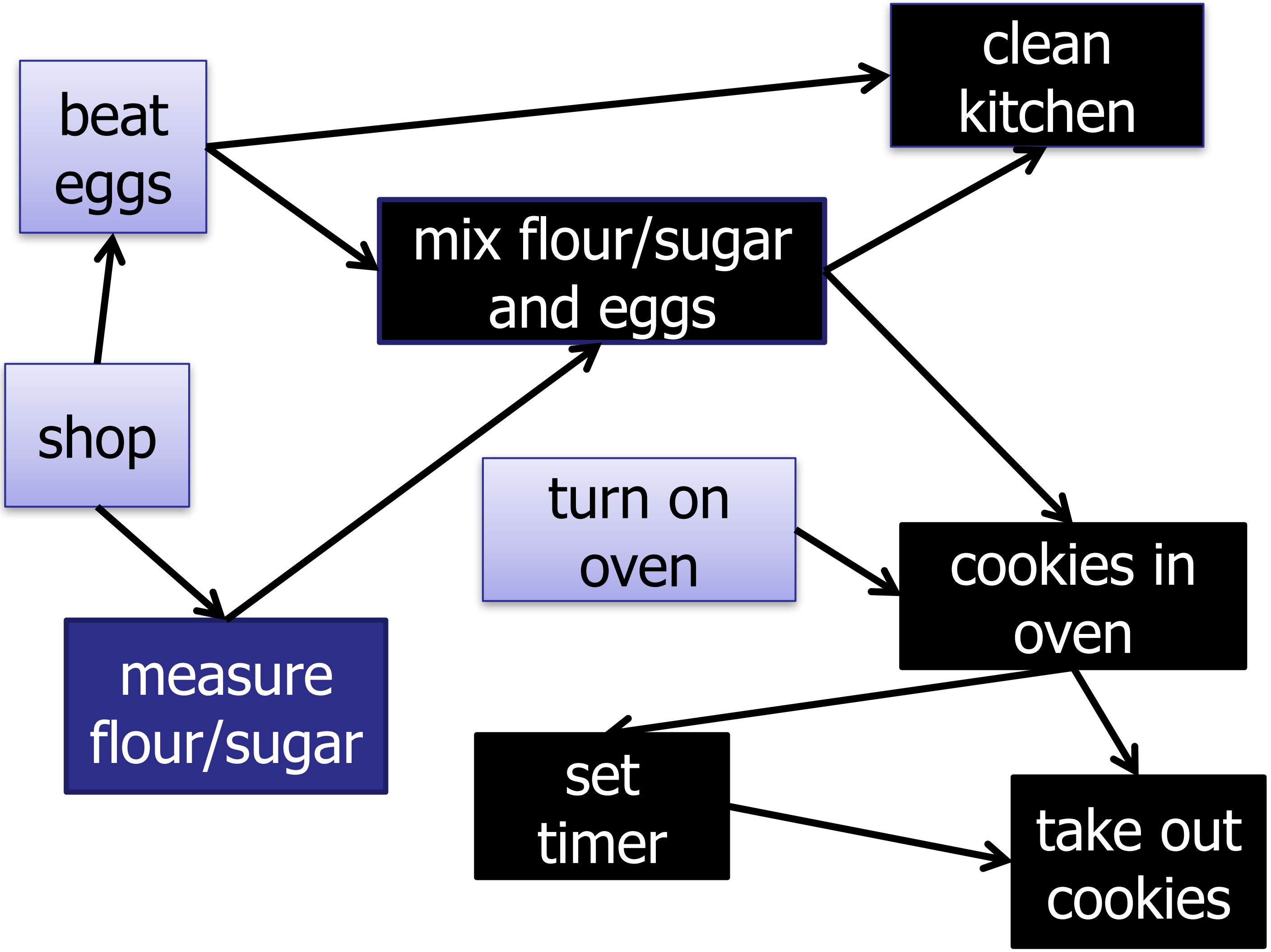
Depth-First Search

1. measure
2. mix
3. in oven
4. take out
5. set timer
6. clean



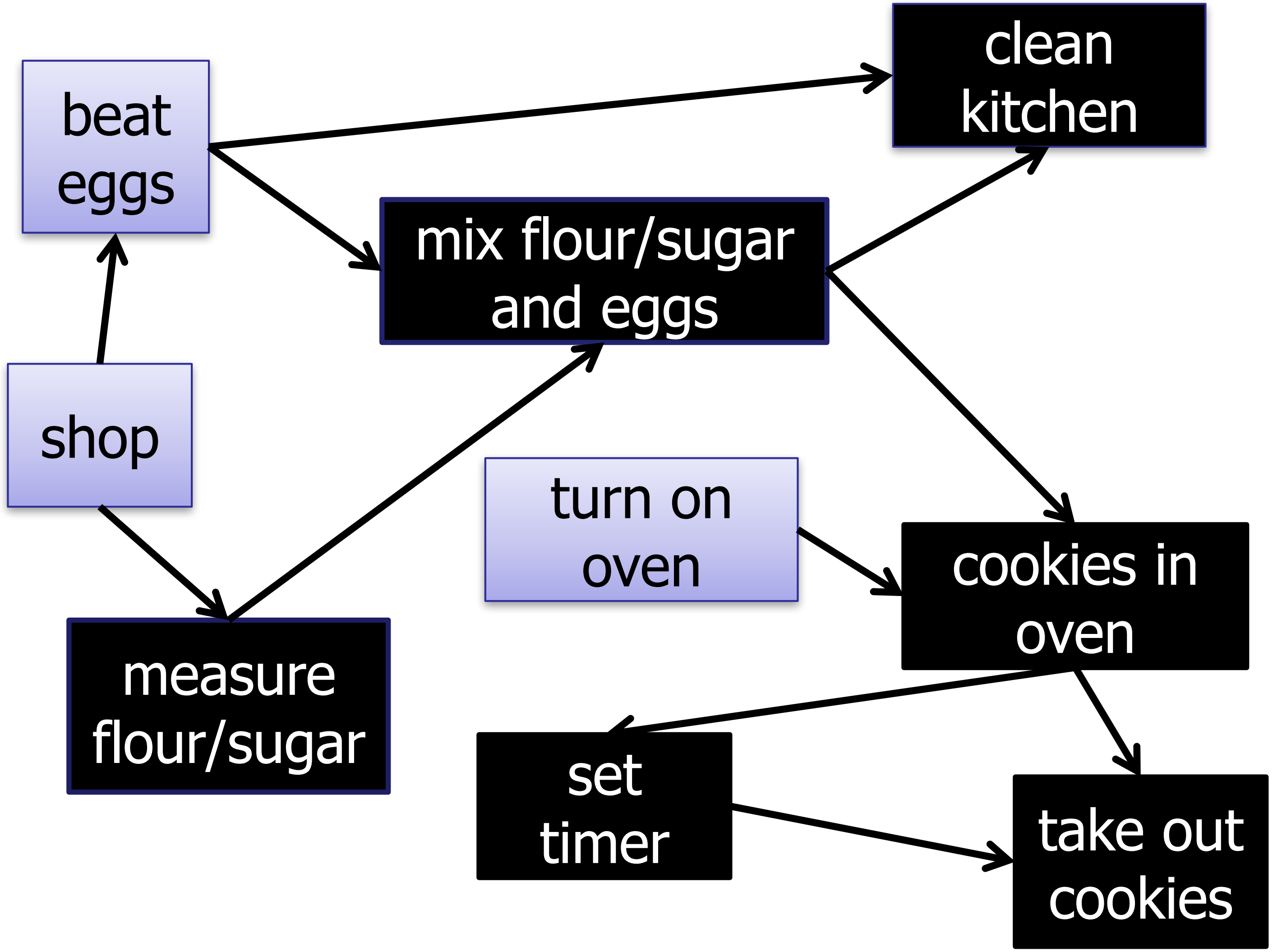
Depth-First Search

1. measure
2. mix
3. in oven
4. take out
5. set timer
6. clean



Depth-First Search

1. measure
2. mix
3. in oven
4. take out
5. set timer
6. clean



Searching a (Directed) Graph

Pre-Order Depth-First Search:

– Process each node when it is *first* visited.

Searching a (Directed) Graph

Pre-Order Depth-First Search:

– Process each node when it is *first* visited.

Post-Order Depth-First Search:

– Process each node when it is *last* visited.

Depth-First Search

```
DFS-visit(Node[] nodeList, boolean[] visited, int startId){
  for (Integer v : nodeList[startId].nbrList) {
     if (!visited[v]){
           visited[v] = true;
           ProcessNode(v);
           DFS-visit(nodeList, visited, v);
     }
  }
}
```

Depth-First Search

```
DFS-visit(Node[] nodeList, boolean[] visited, int startId){
  for (Integer v : nodeList[startId].nbrList) {
     if (!visited[v]){
           visited[v] = true;
           DFS-visit(nodeList, visited, v);
           ProcessNode(v);
     }
  }
}
```

Searching a (Directed) Graph

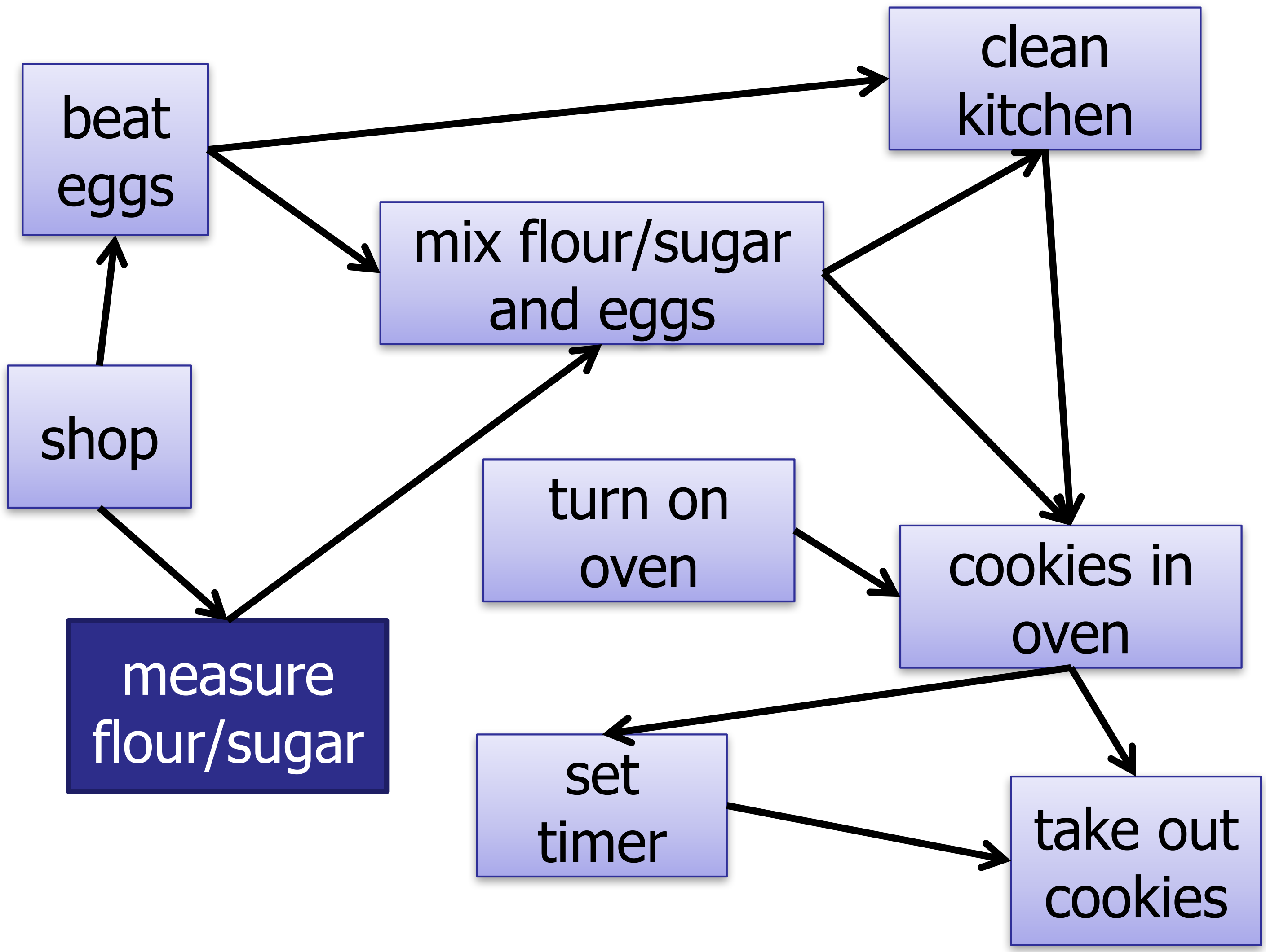
Pre-Order Depth-First Search:

– Process each node when it is *first* visited.

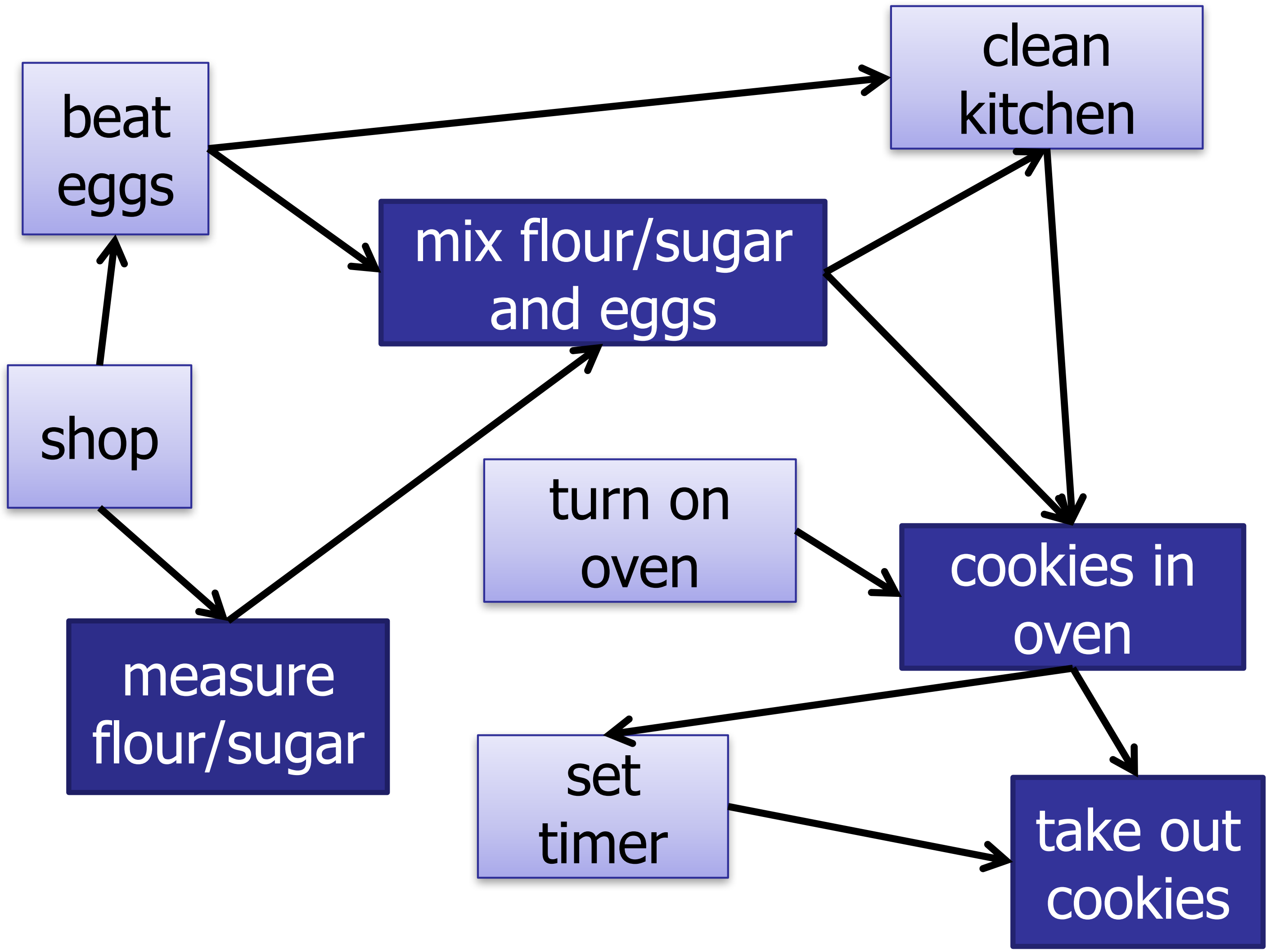
Post-Order Depth-First Search:

– Process each node when it is *last* visited.

Post-Order Depth-First Search

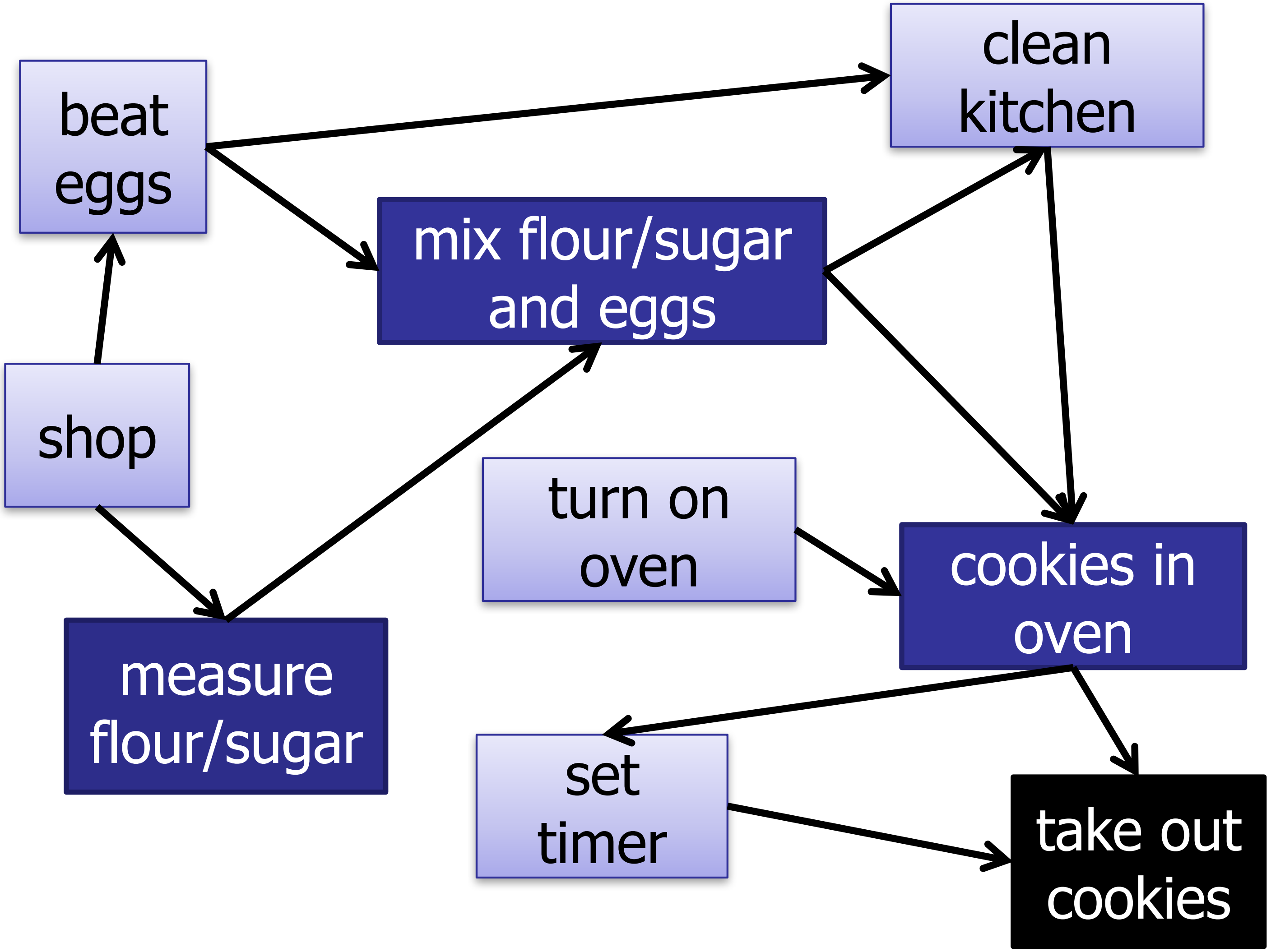


Post-Order Depth-First Search



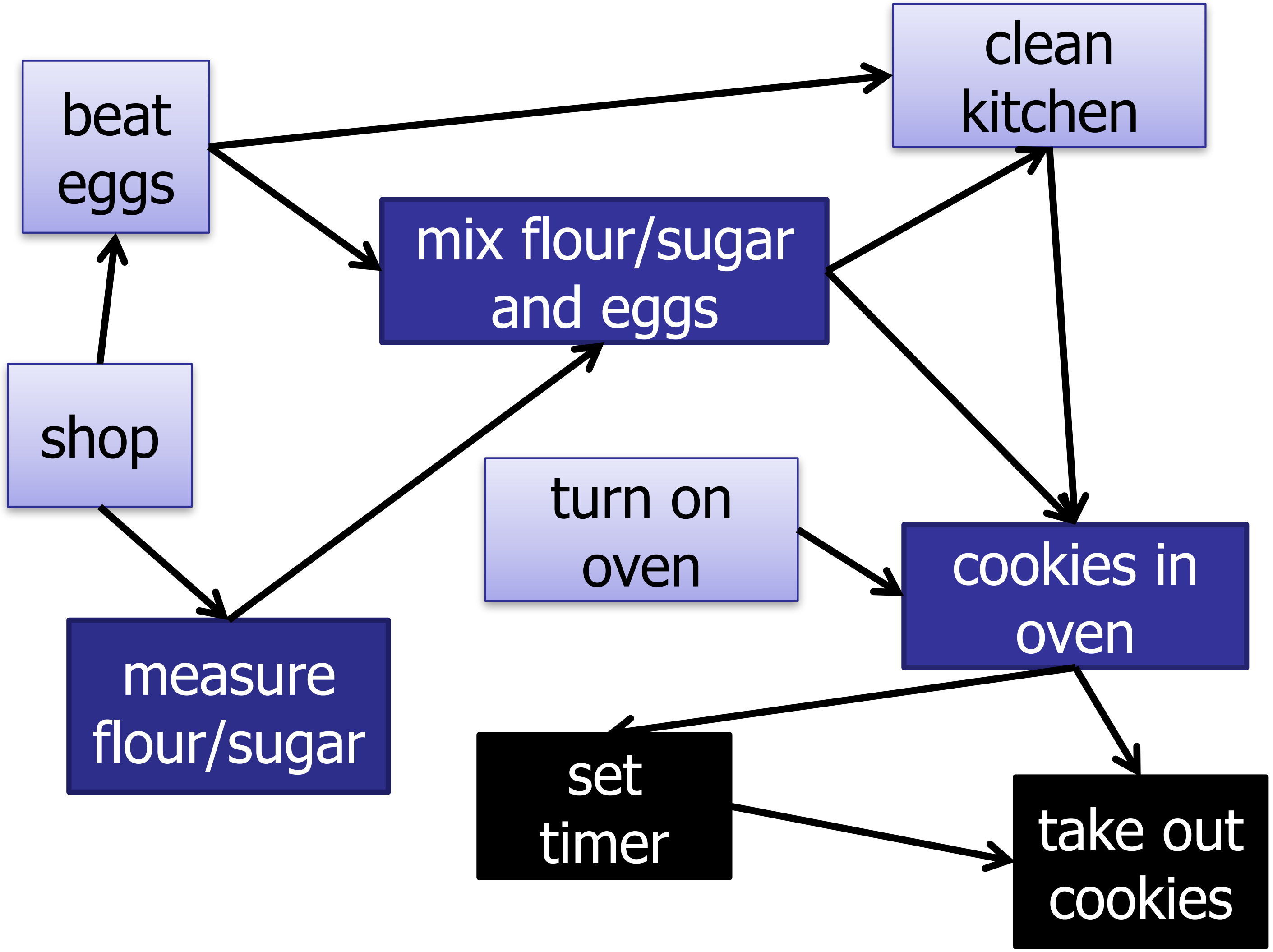
Post-Order Depth-First Search

1.
2.
3.
4.
5.
6.
7.
8.
9. take out



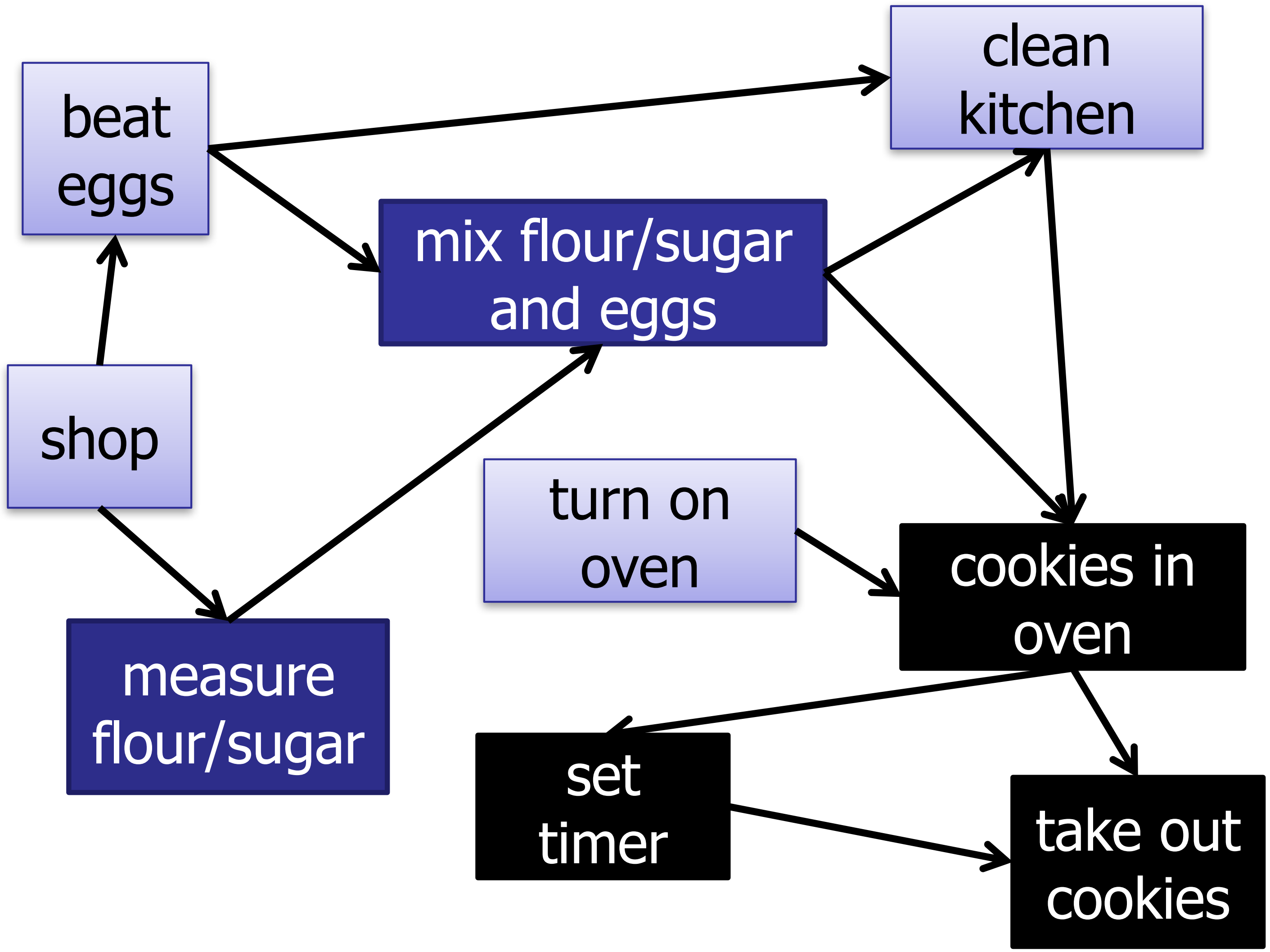
Post-Order Depth-First Search

1.
2.
3.
4.
5.
6.
7.
8. set timer
9. take out



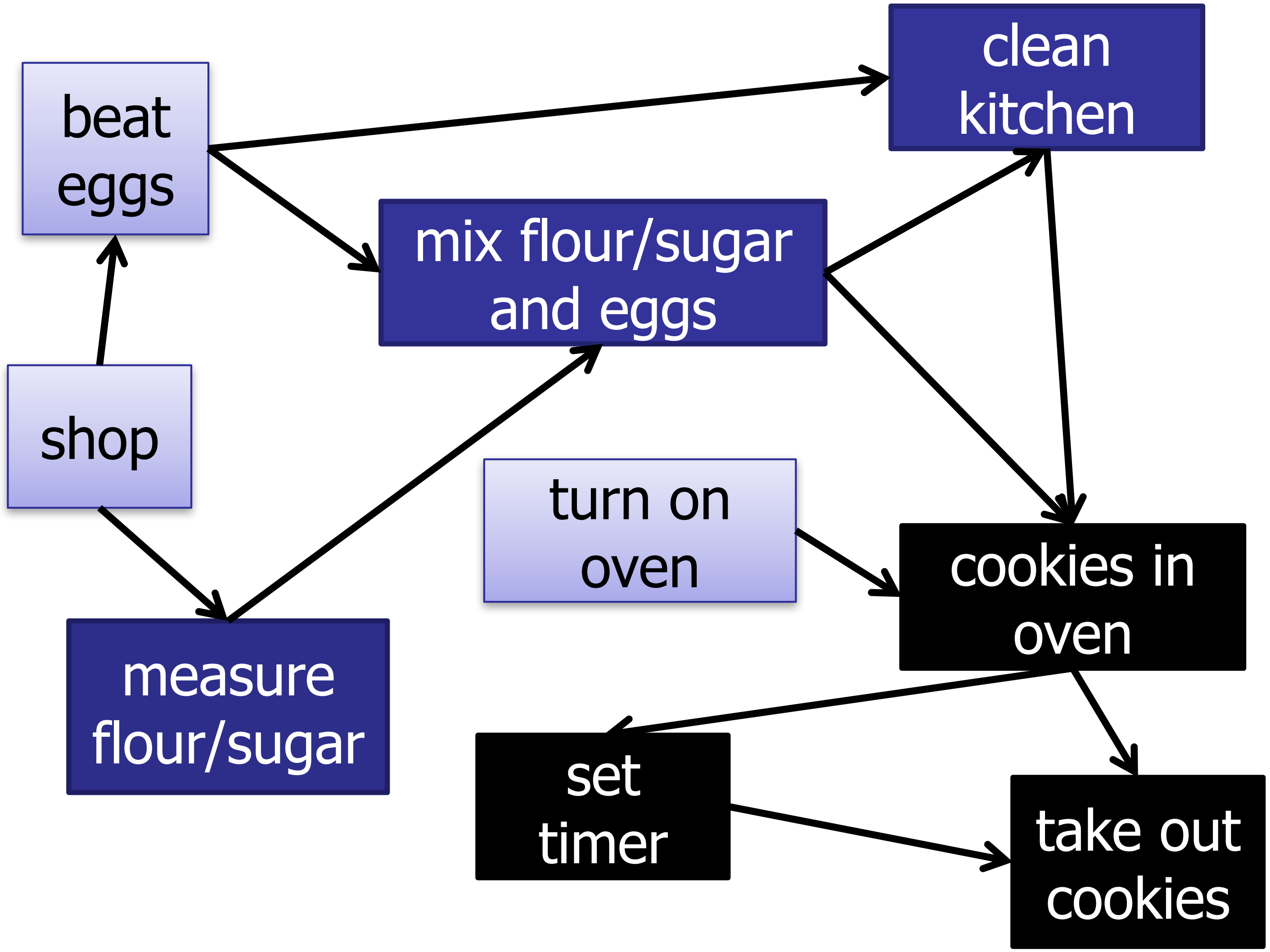
Post-Order Depth-First Search

1.
2.
3.
4.
5.
6.
7. in oven
8. set timer
9. take out



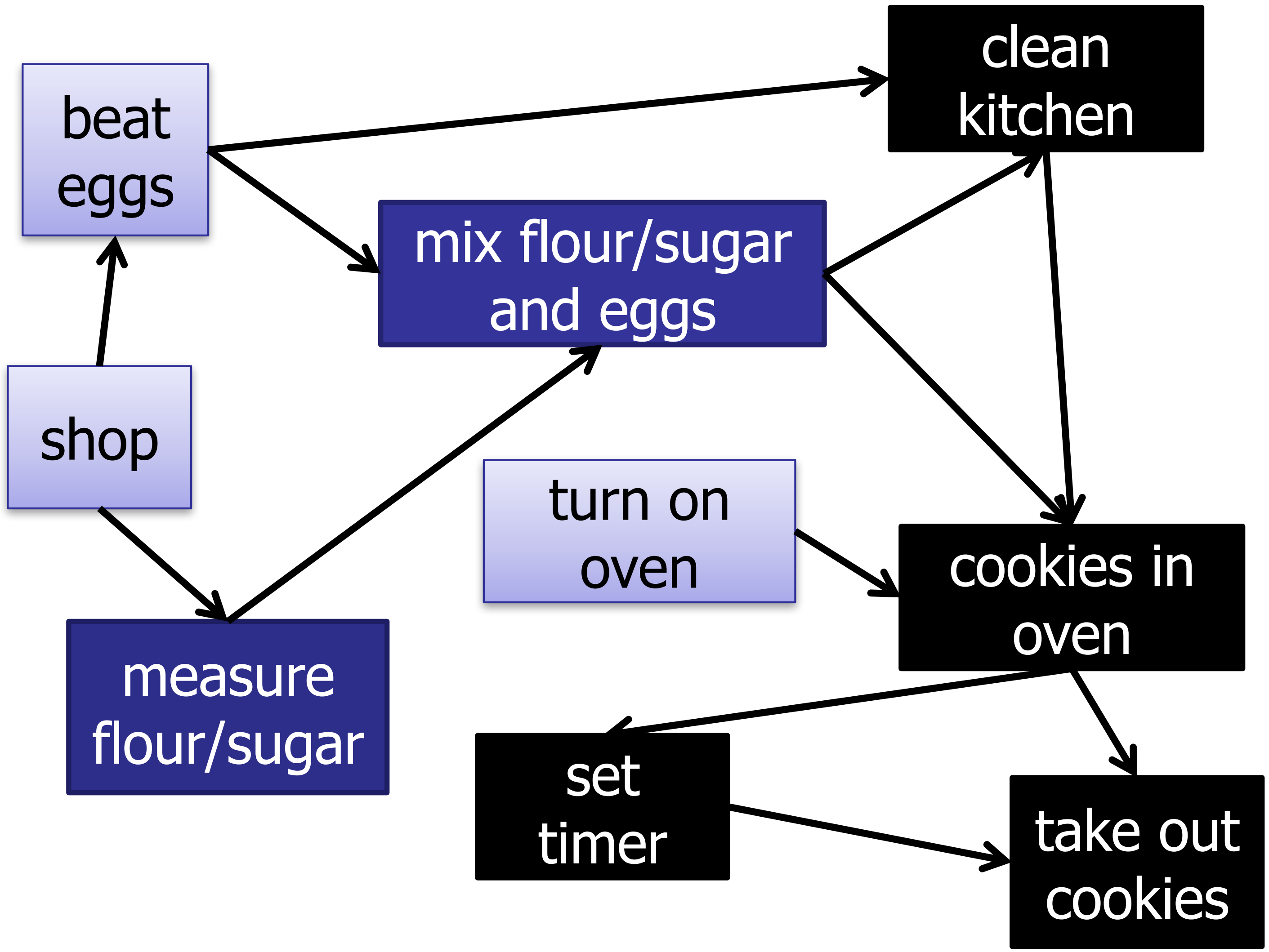
Post-Order Depth-First Search

1.
2.
3.
4.
5.
6.
7. in oven
8. set timer
9. take out



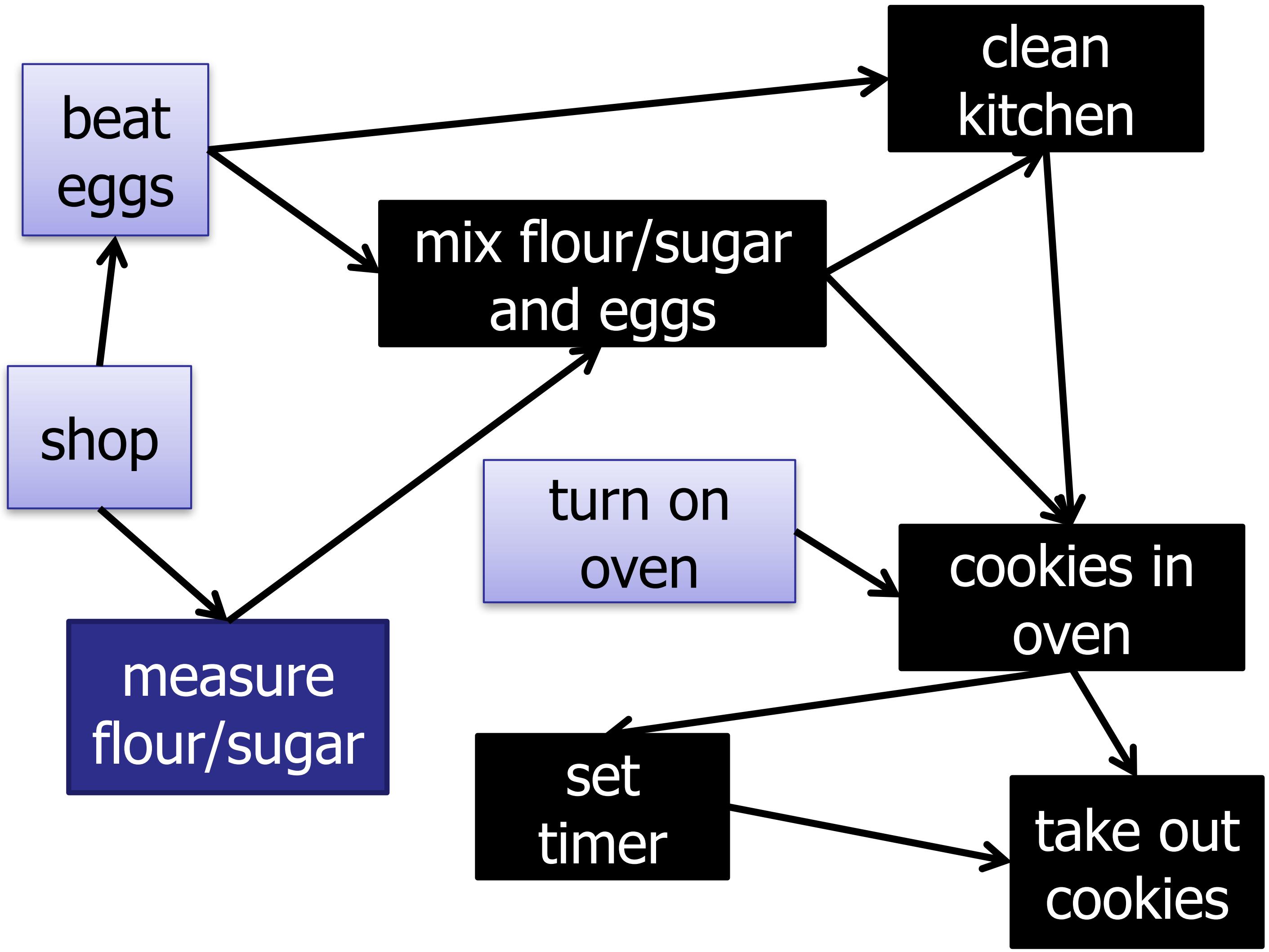
Post-Order Depth-First Search

1.
2.
3.
4.
5.
6. clean
7. in oven
8. set timer
9. take out



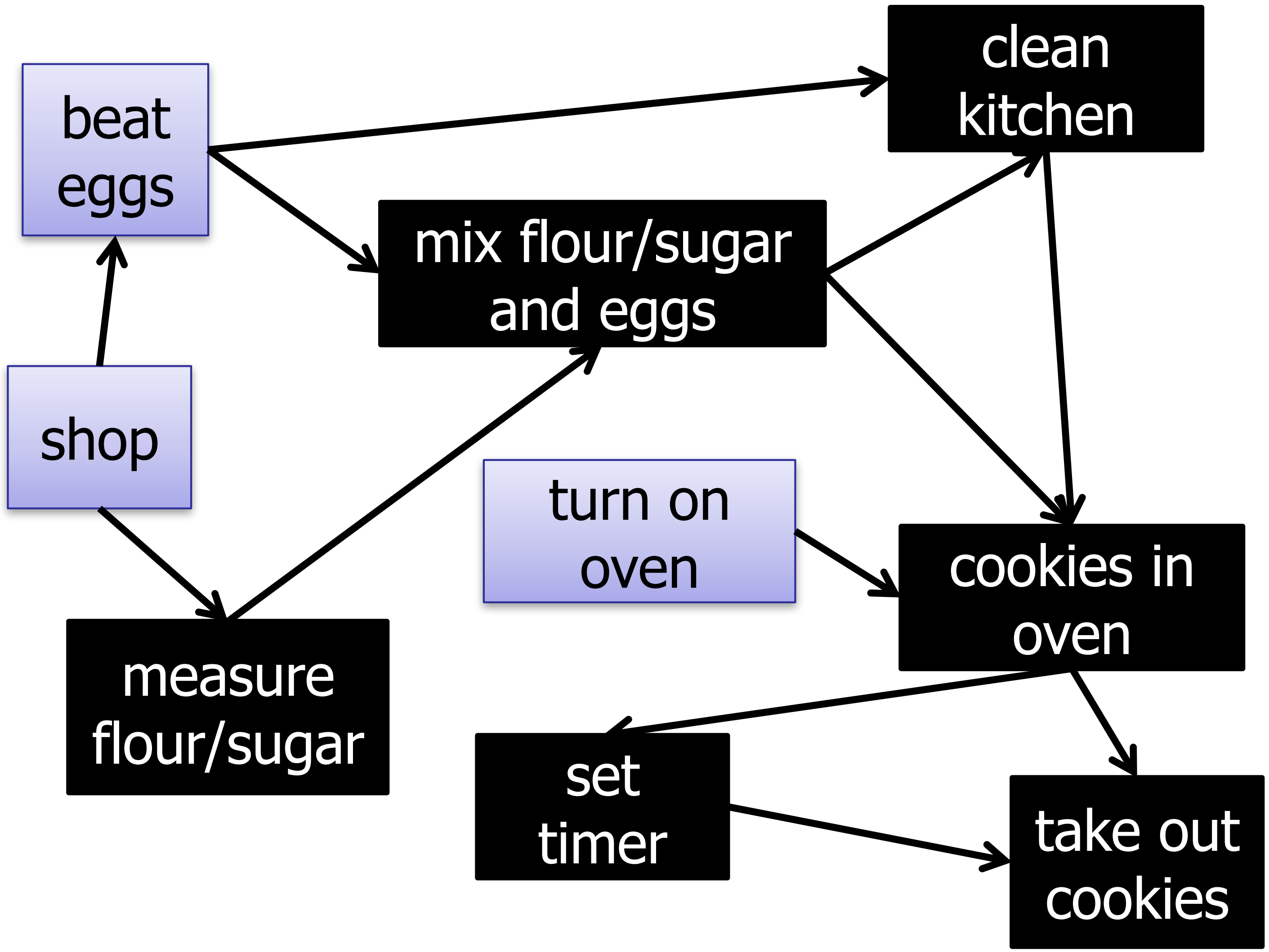
Post-Order Depth-First Search

1.
2.
3.
4.
5. mix
6. clean
7. in oven
8. set timer
9. take out



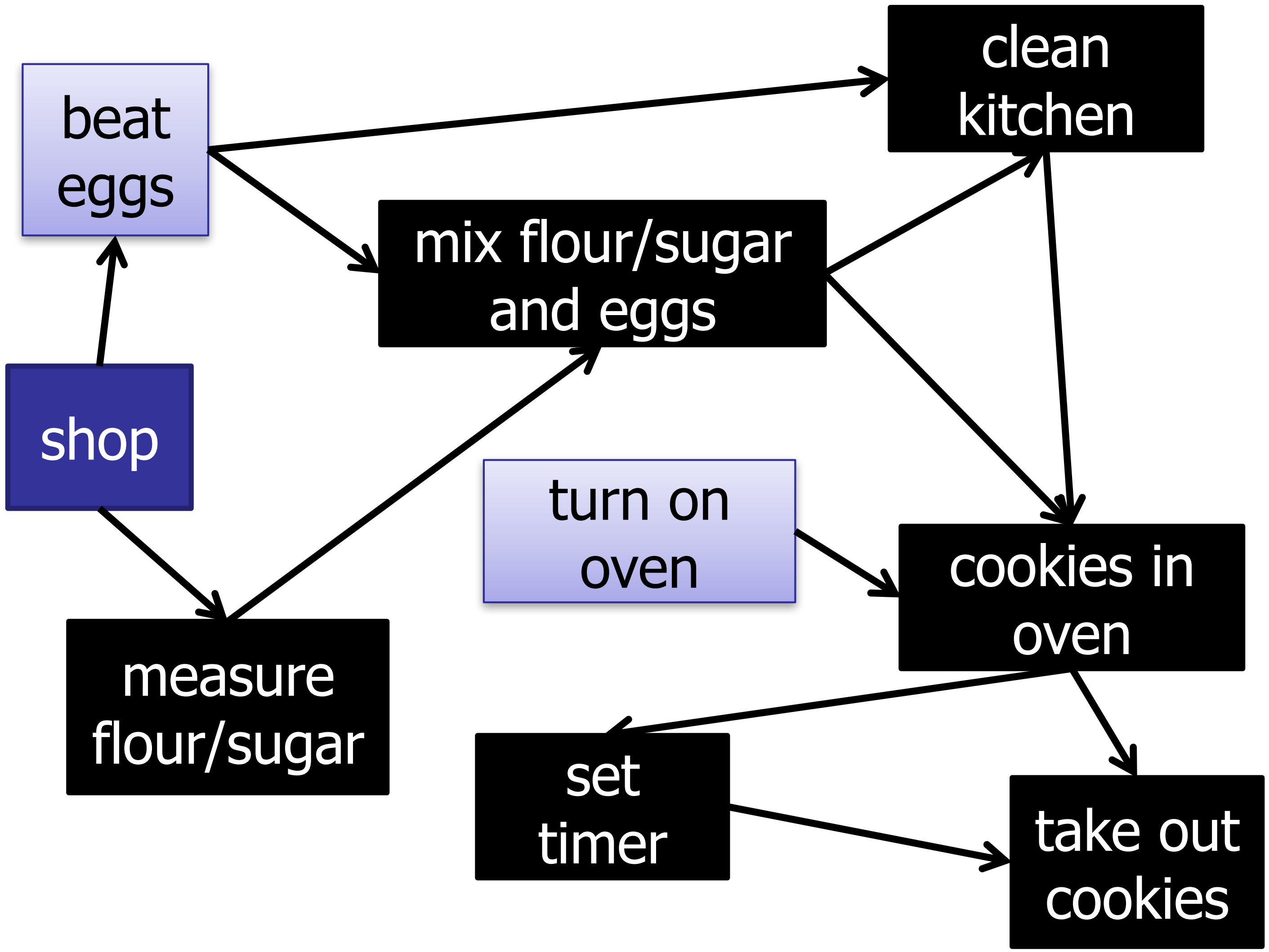
Post-Order Depth-First Search

1.
2.
3.
4. measure
5. mix
6. clean
7. in oven
8. set timer
9. take out



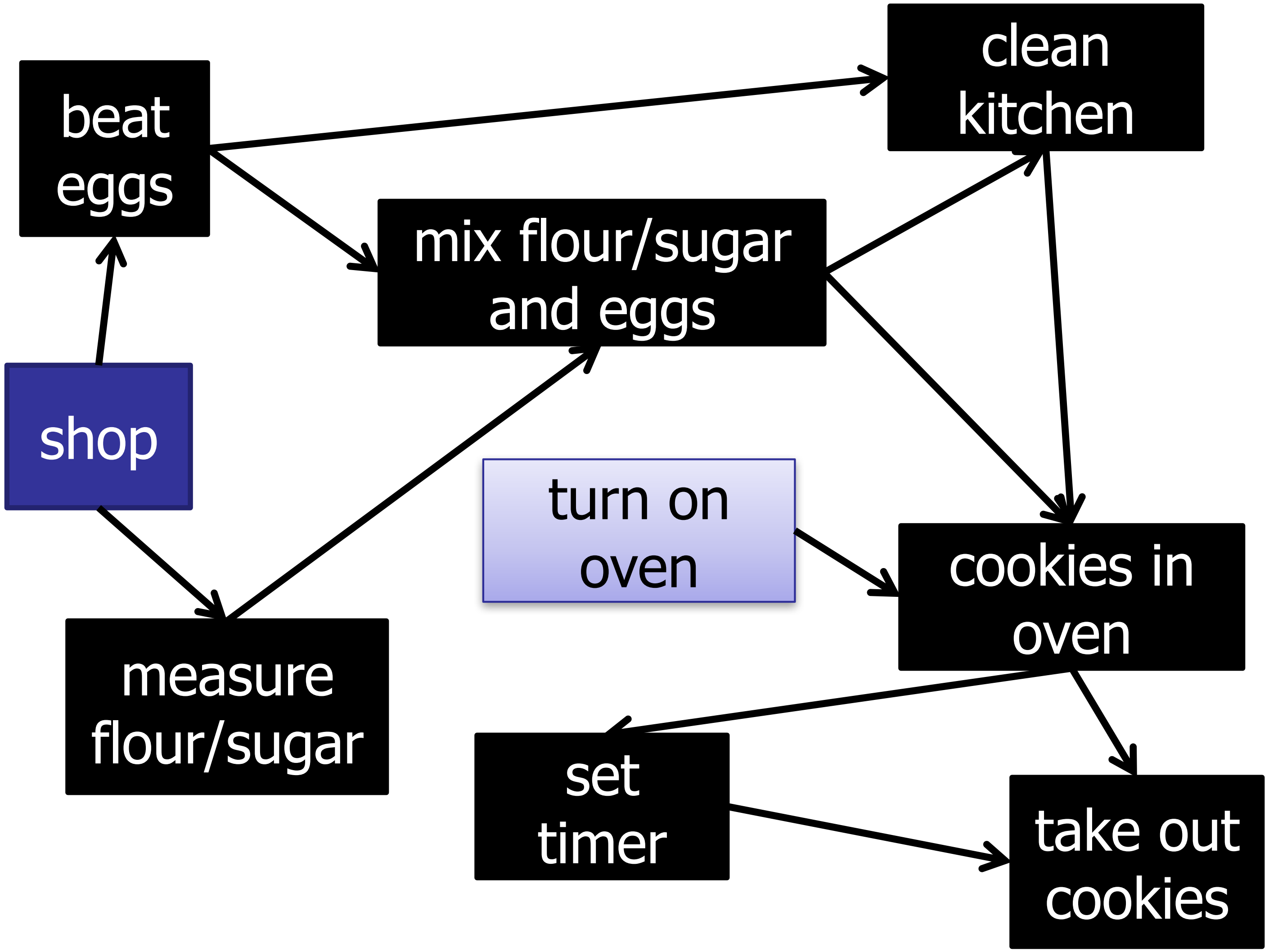
Post-Order Depth-First Search

1.
2.
3.
4. measure
5. mix
6. clean
7. in oven
8. set timer
9. take out



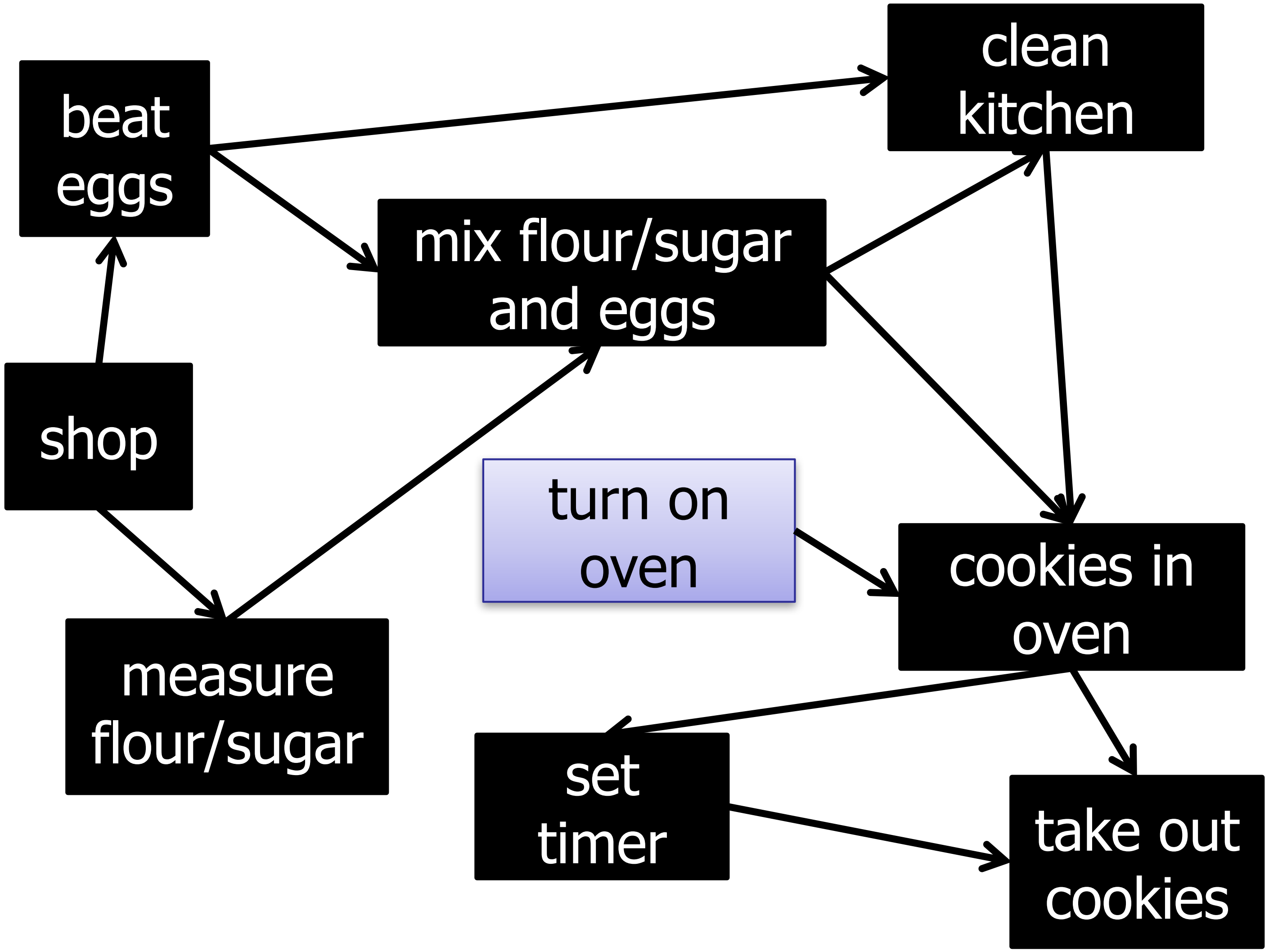
Post-Order Depth-First Search

1.
2.
3. beat
4. measure
5. mix
6. clean
7. in oven
8. set timer
9. take out



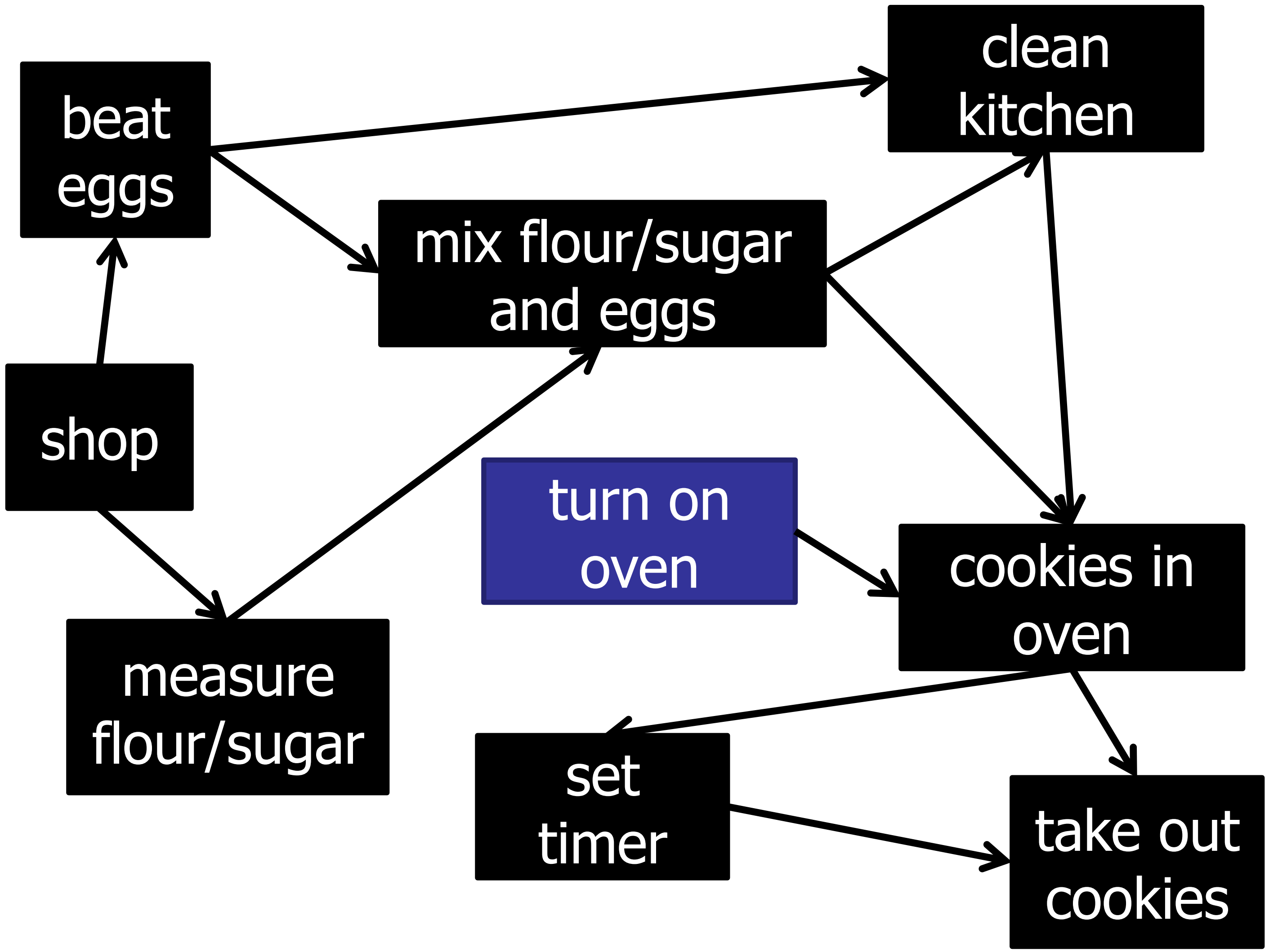
Post-Order Depth-First Search

1.
2. shop
3. beat
4. measure
5. mix
6. clean
7. in oven
8. set timer
9. take out



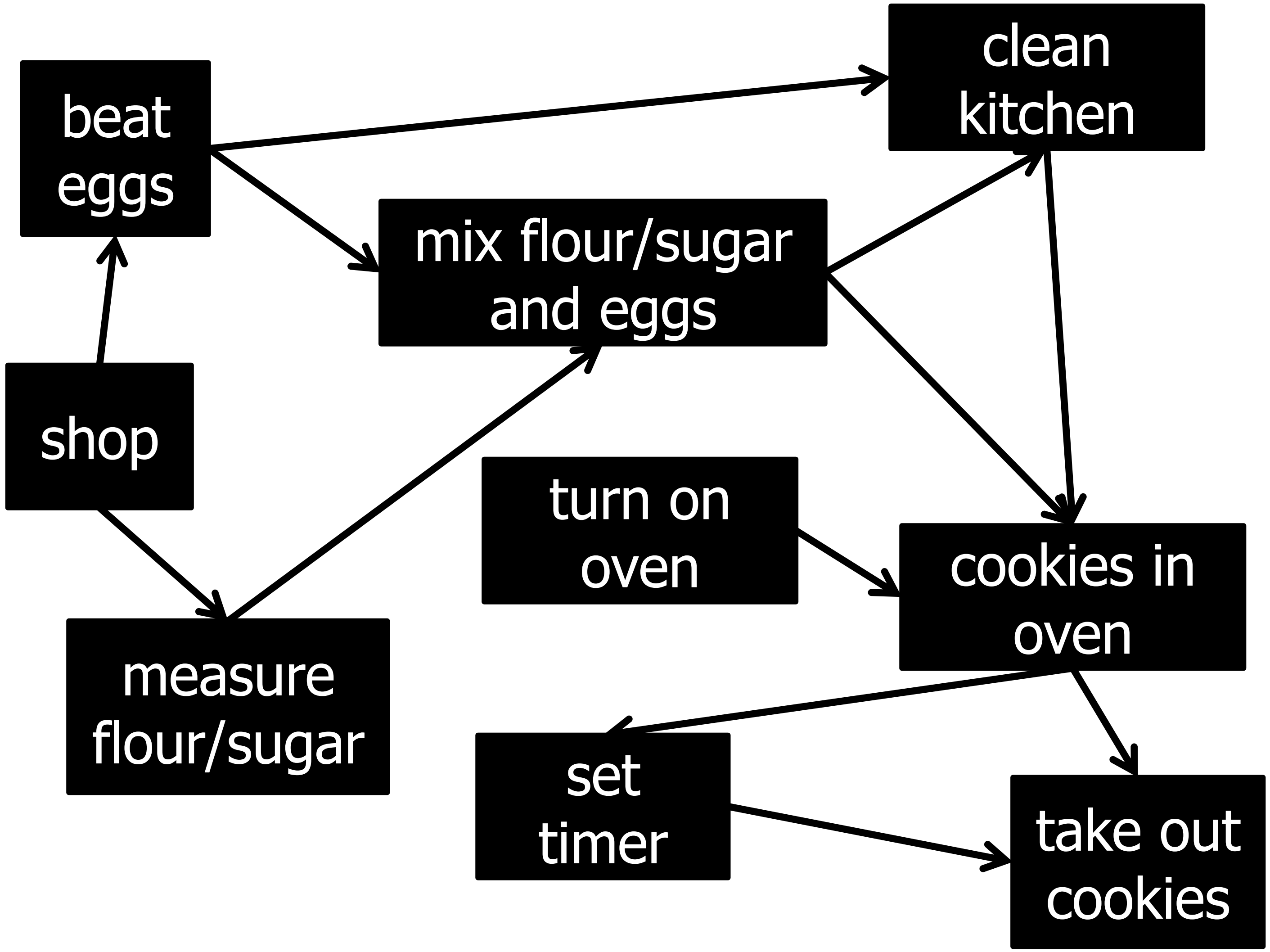
Post-Order Depth-First Search

1.
2. shop
3. beat
4. measure
5. mix
6. clean
7. in oven
8. set timer
9. take out



Post-Order Depth-First Search

1. on oven
2. shop
3. beat
4. measure
5. mix
6. clean
7. in oven
8. set timer
9. take out



Topological Sort

What is the time complexity of topological sort?

DFS: $O(V+E)$

Depth-First Search

```
DFS-visit(Node[] nodeList, boolean[] visited, int startId){
  for (Integer v : nodeList[startId].nbrList) {
     if (!visited[v]){
           visited[v] = true;
           DFS-visit(nodeList, visited, v);
           schedule.prepend(v);
     }
  }
}
```

Depth-First Search

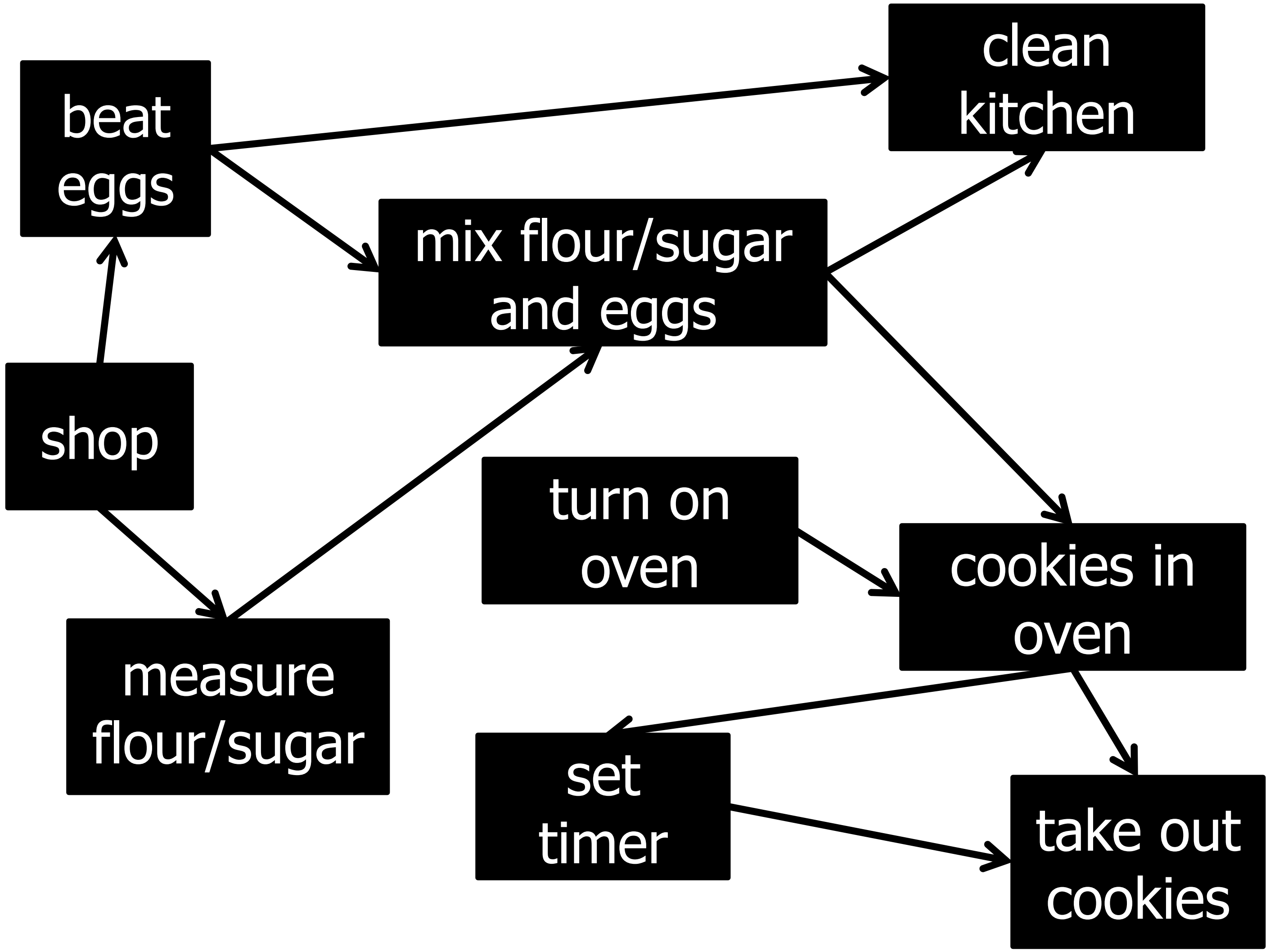
```
DFS(Node[] nodeList){
boolean[] visited = new boolean[nodeList.length];
Arrays.fill(visited, false);

  for (start = i; start<nodeList.length; start++) {
     if (!visited[start]){
           visited[start] = true;
           DFS-visit(nodeList, visited, start);
           schedule.prepend(v);
     }
  }
```

Is a topological ordering unique?

1. Yes
2. No
3. On Fridays.

Post-Order Depth-First Search

1. **on oven**
2. **shop**
3. beat
4. measure
5. mix
6. **clean**
7. in oven
8. **set timer**
9. take out



Topological Sort

Input:

- Directed Acyclic Graph (DAG)

Output:

- Total ordering of nodes, where all edges point forwards.

Algorithm:

- Post-order Depth-First Search
- O(V + E) time complexity

Topological Sort

Alternative algorithm:

Input: directed graph G

Repeat:

- S = all nodes in G that have *no* incoming edges.
- Add nodes in S to the topo-order
- Remove all edges adjacent to nodes in S
- Remove nodes in S from the graph

Time:

– O(V + E) time complexity

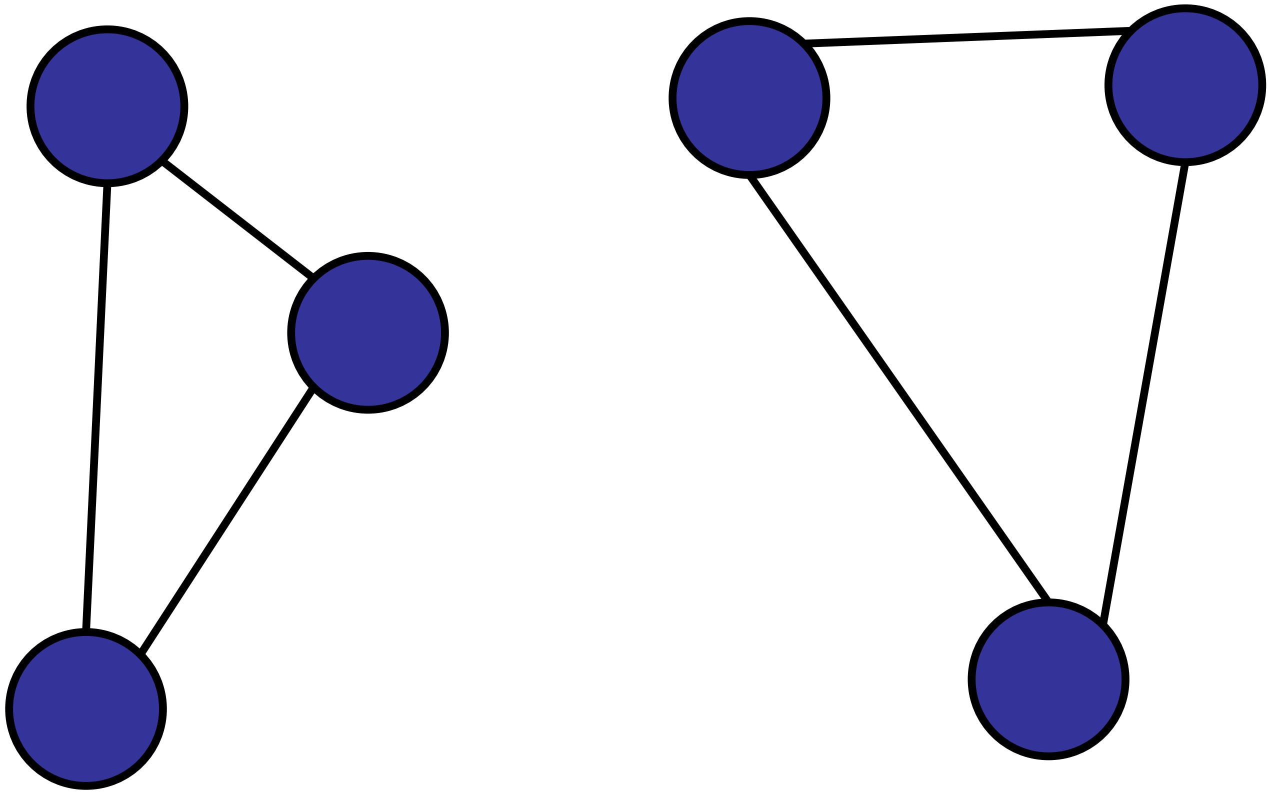
Roadmap

Directed Graphs

- What is a directed graph?
- Searching directed graphs (DFS / BFS)
- Topological Sort
- Connected Components

Connected Components

Undirected graphs

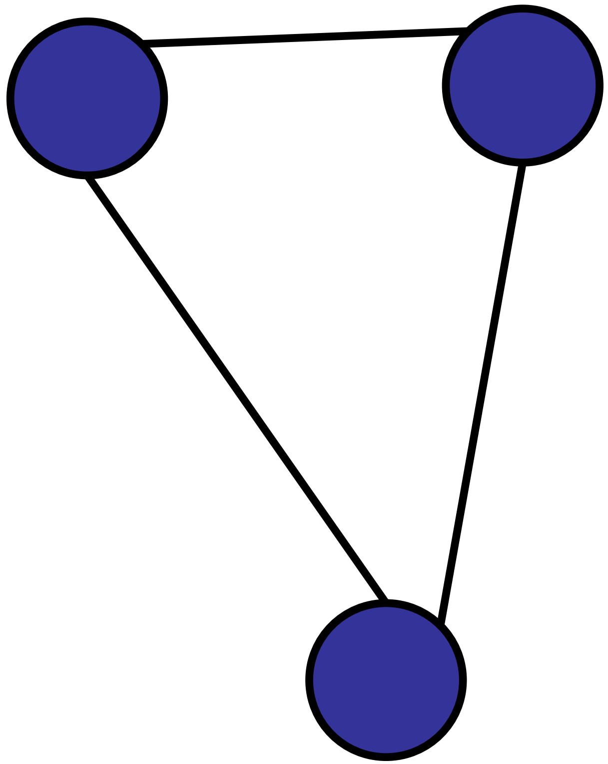


Two connected components

Connected Components

Undirected graphs

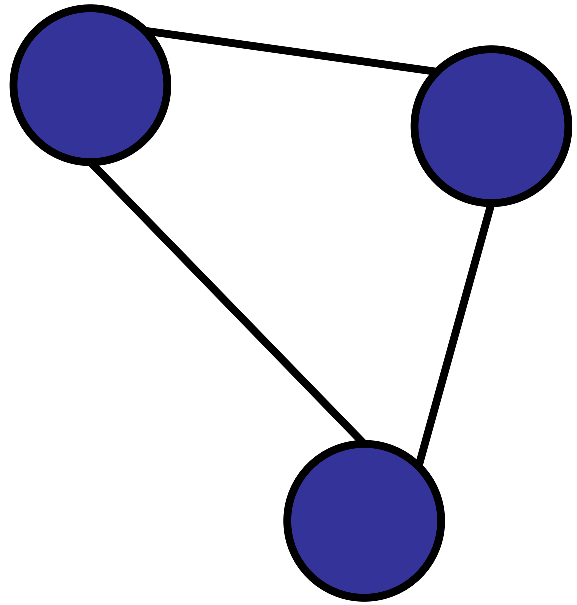
Vertex v and w are in the same connected component if and only if there is a path from v to w.



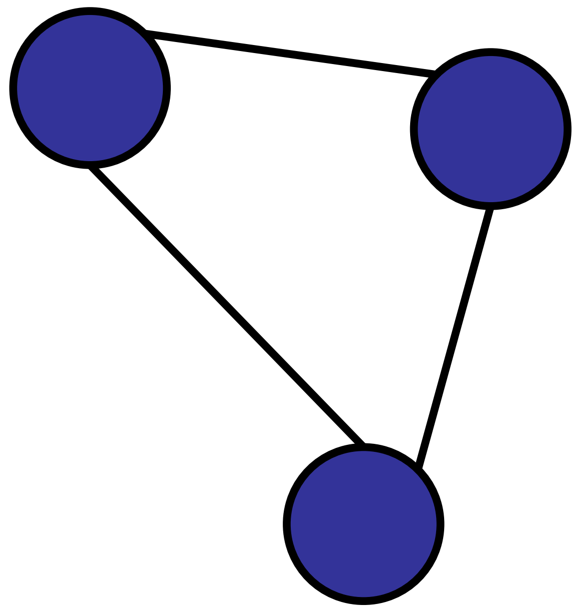
Connected Components

Undirected graphs

Vertex v and w are in the same connected component if and only if there is a path from v to w.

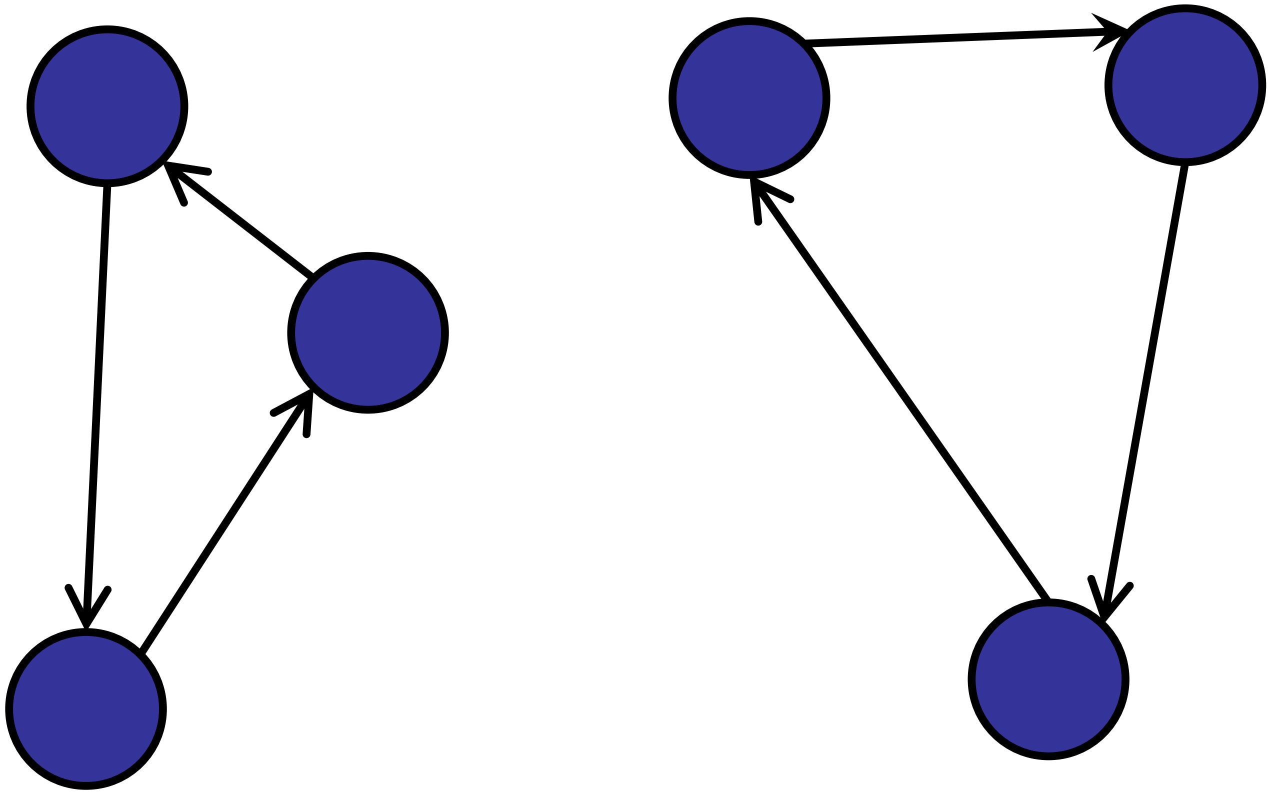


There is a set $\{v_1, v_2, \ldots, v_k\}$ where there is no path from any v_i to v_j if and only if there are k connected components.



Connected Components

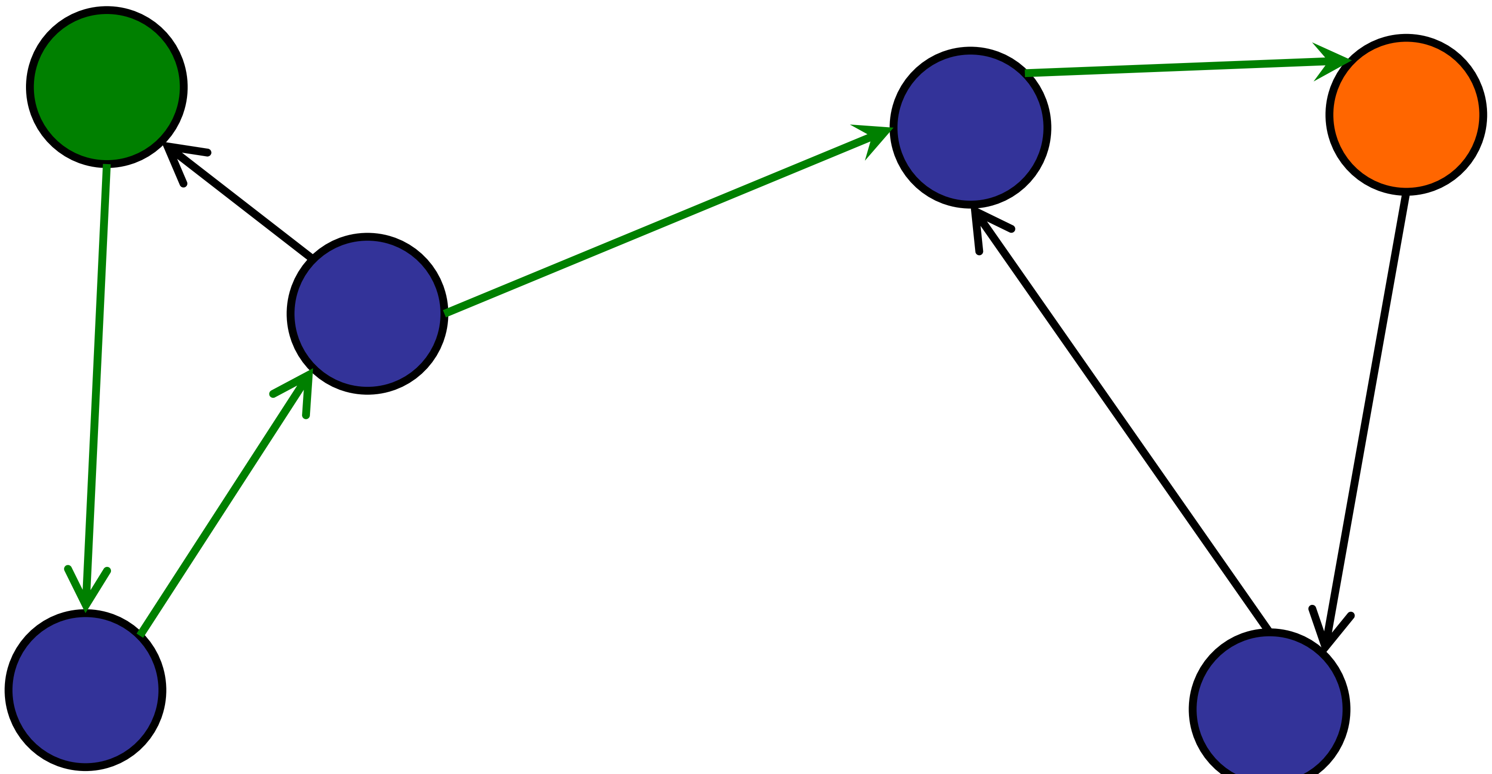
Directed graphs



Two connected components

Connected Components

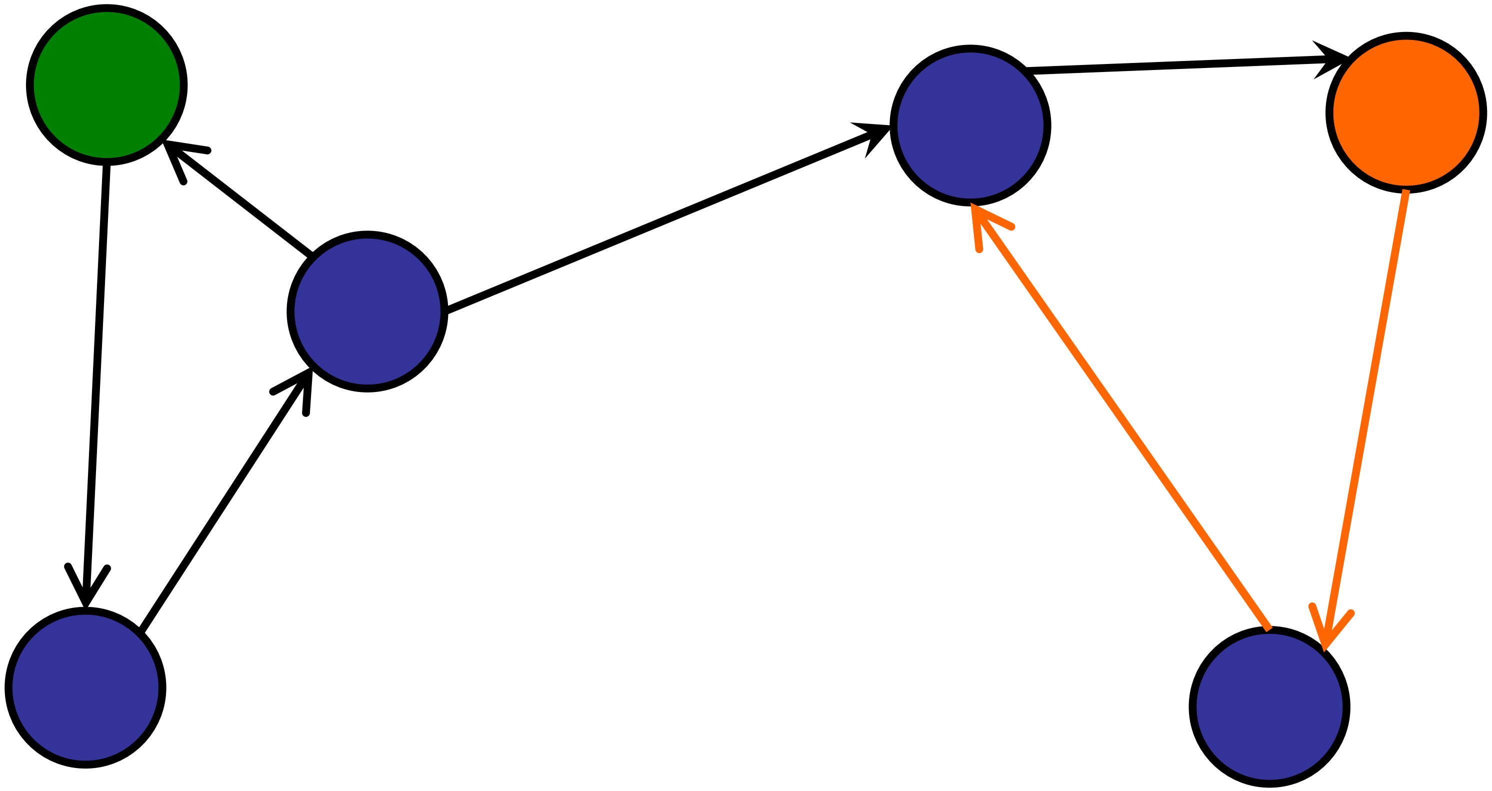
Directed graphs



Two connected components??

Connected Components

Directed graphs



Two connected components??

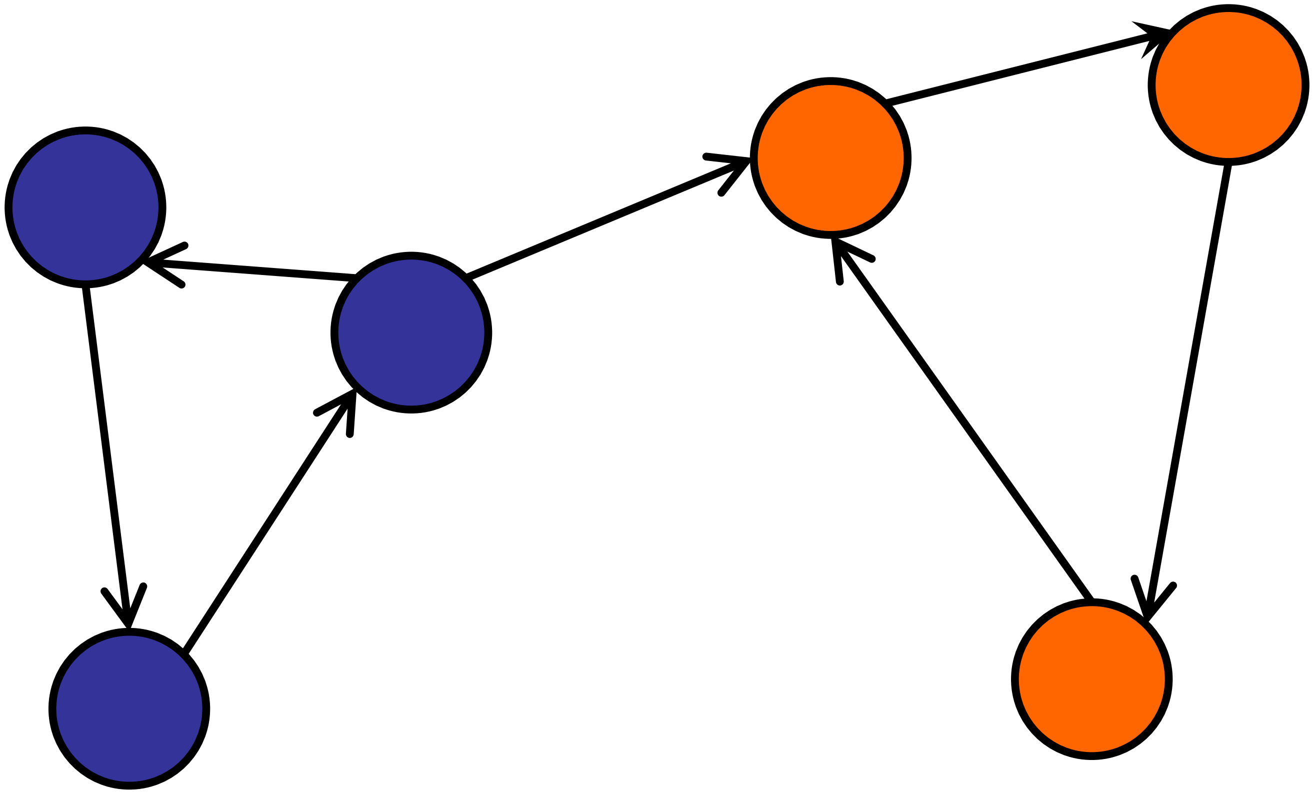
Connected Components

Strongly connected component

For every vertex v and w:

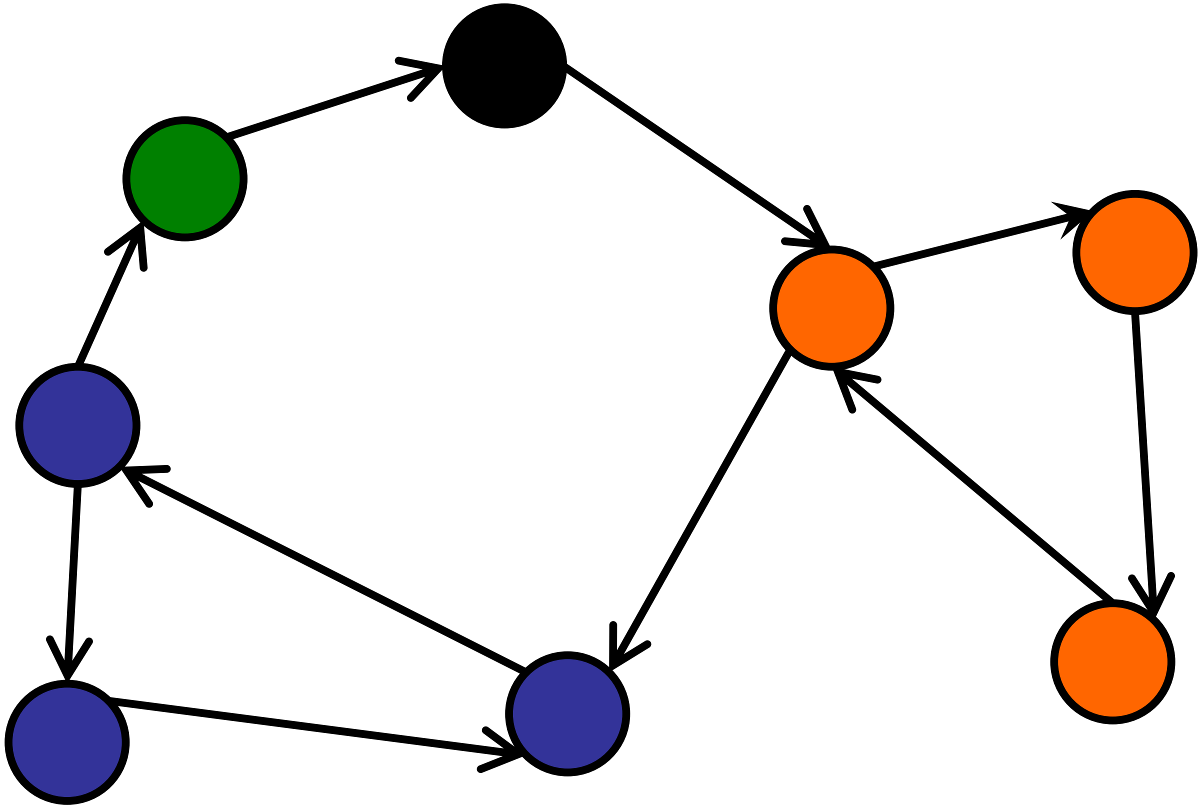
–There is a path from v to w.

– There is a path from w to v.



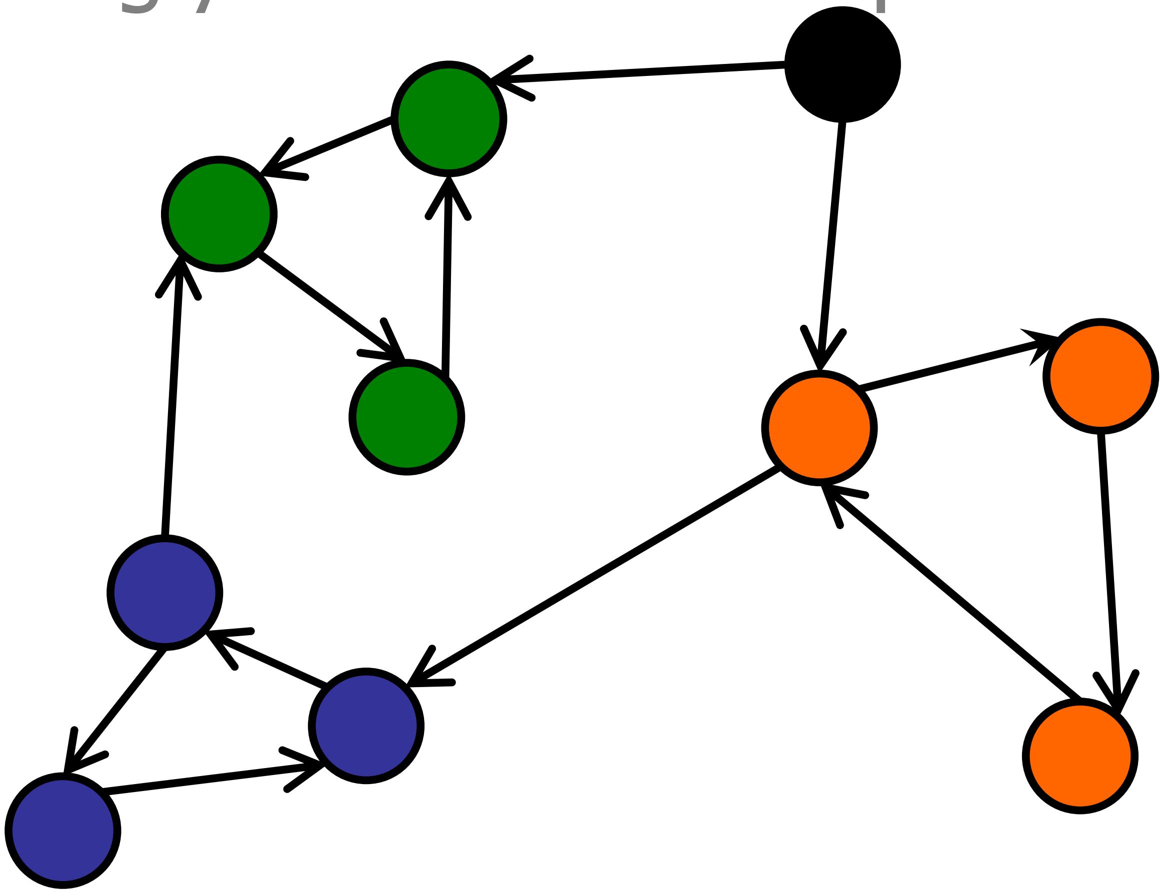
How many strongly connected components?

1. 1
2. 2
3. 3
4. 4
5. 5
6. Other

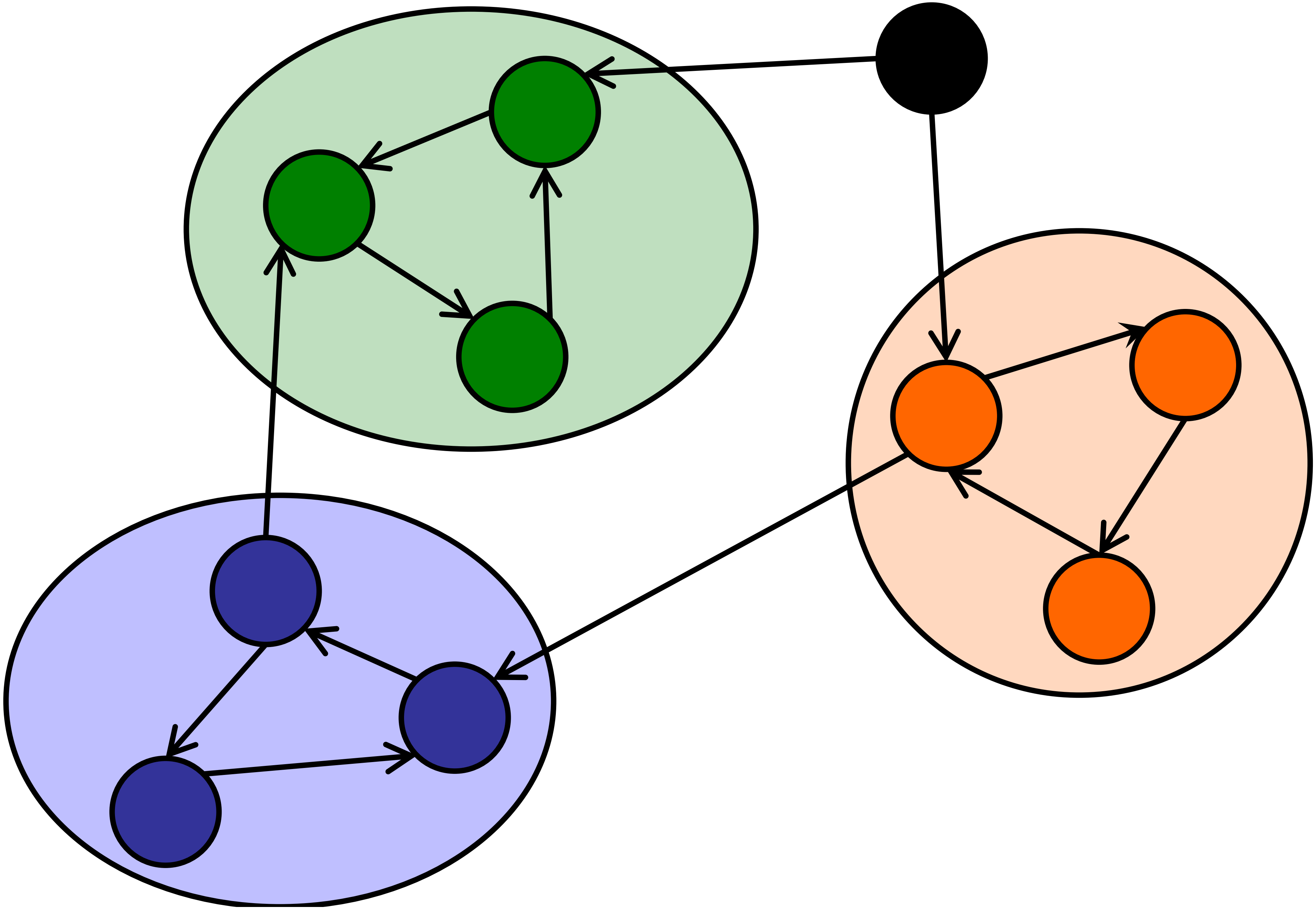


How many strongly connected components?

1. 1
2. 2
3. 3
4. 4
5. 5
6. Other

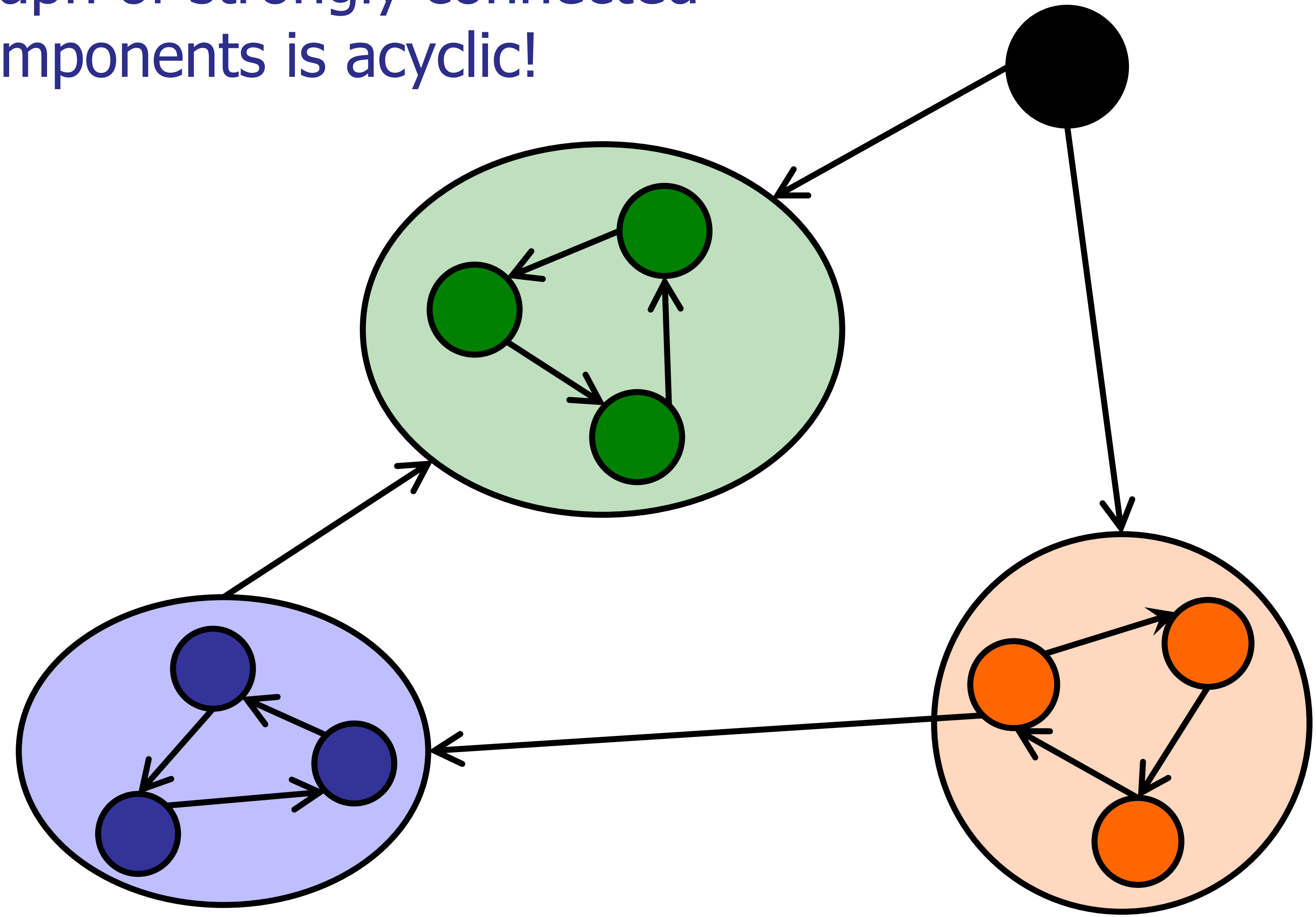


Connected Components



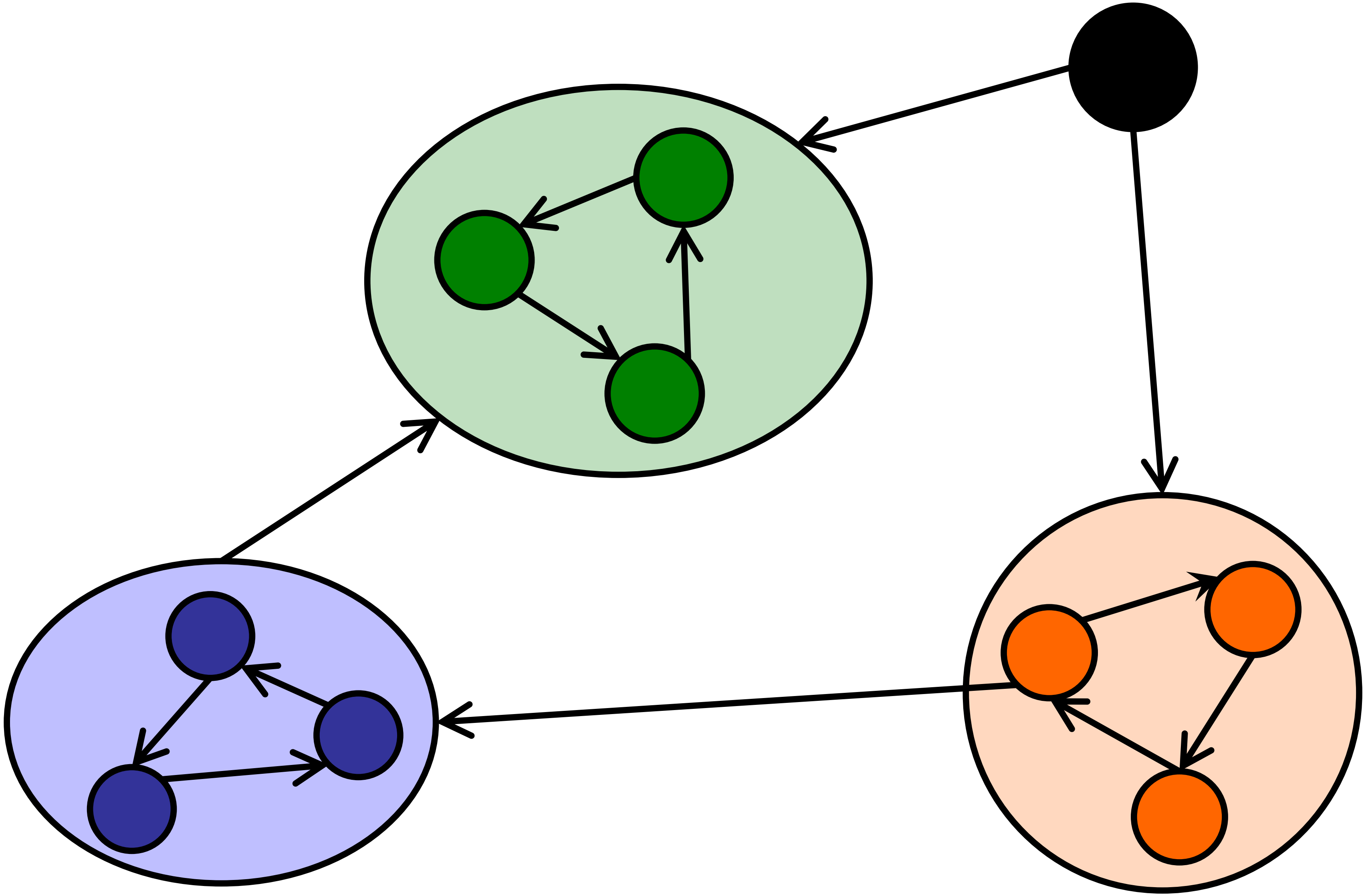
Connected Components

Graph of strongly connected components is acyclic!



Connected Components

Challenge: find all strongly connected components.



Roadmap

Directed Graphs

- What is a directed graph?
- Searching directed graphs (DFS / BFS)
- Topological Sort
- Connected Components