CS2040S – Data Structures and Algorithms

Lecture 17 – Connecting People

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Outline

Minimum Spanning Tree (MST)

- Motivating Example & Some Definitions

Two Algorithms to solve MST (you have a choice!)

- Prim's (greedy algorithm with PriorityQueue)
  - PriorityQueue is discussed in Lecture 09
- Kruskal's (greedy algorithm, uses sorting and UFDS)
  - UFDS is discussed in Lecture 10

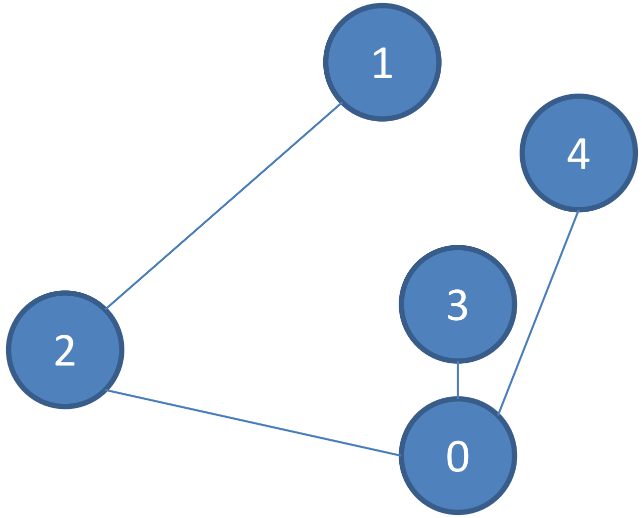
Review

Definitions that we have learned before

- **Tree T**
  - **T** is a **connected graph** that has **V** vertices and **V-1** edges
  - Important: One unique path between any two pair of vertices in **T**
- **Spanning** Tree **ST** of connected graph **G**
  - **ST** is a tree that spans (covers) every vertex in **G**
  - Recall the **BFS and DFS Spanning Tree**

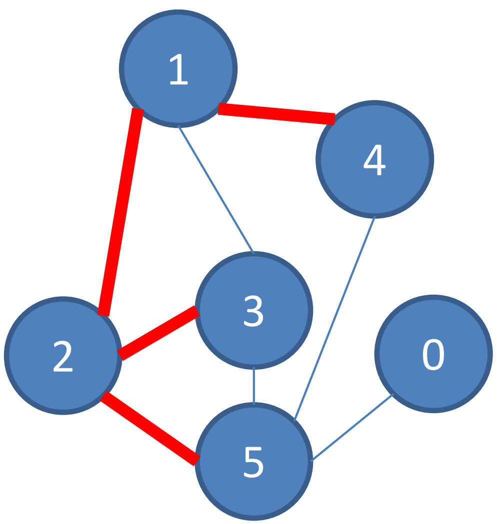
Is This A Tree?

1. Yes, why ___________
2. No, why ___________



Do the edges highlighted in red part form a spanning tree of the original graph?

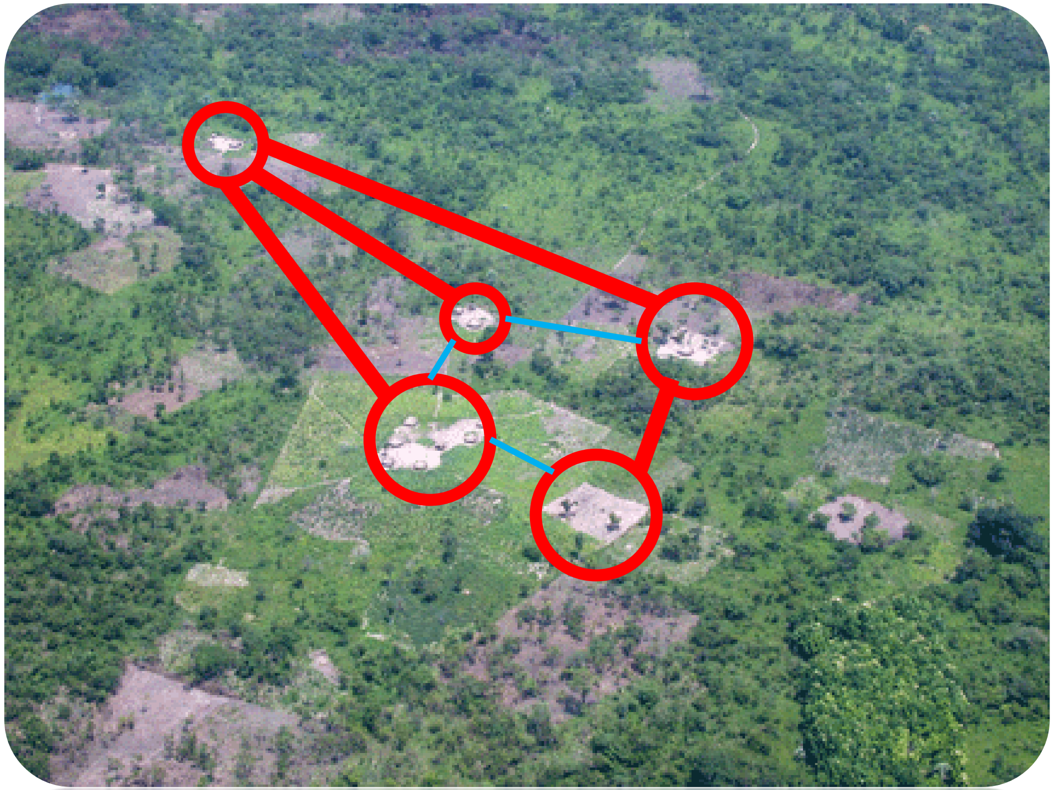
1. Yes, why ____________
2. No, why ____________



Motivating Example

Government Project

- Want to link rural villages with roads
- The cost to build a road depends on the terrain, etc
- You only have limited budget
- How are you going to build the roads?



Definitions (1)

- Vertex set **V** (e.g. street intersections, houses, etc)
- Edge set **E** (e.g. streets, roads, avenues, etc)
  - Generally undirected (e.g. bidirectional road, etc)
  - Weighted (e.g. distance, time, toll, etc)
- Weight function **w(a, b): E→R**
  - Sets the weight of edge from **a** to **b**
- **Weighted** Graph **G**: **G(V, E), w(a, b): E→R**
- **Connected** undirected graph **G**
  - There is a path from any vertex **a** to any other vertex **b** in **G**
- The graph **G** we're concerned with is **connected undirected** and **weighted** when dealing with **MST**

More Definitions (2)

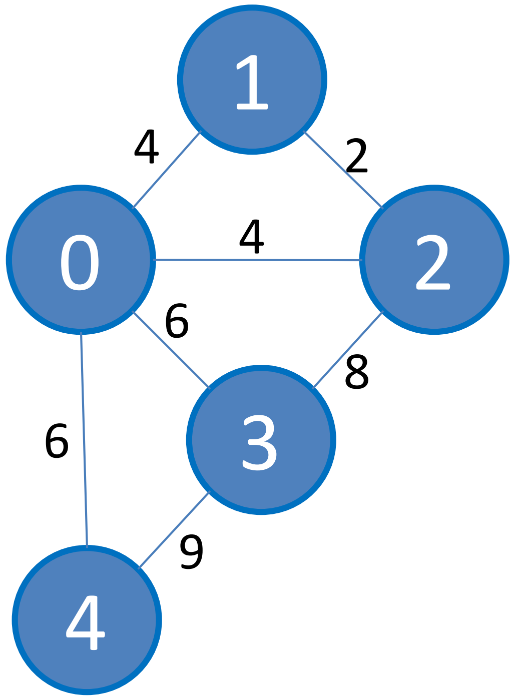
- Spanning Tree **ST** of connected undirected weighted graph **G**
  - Let **w(ST)**, weight of **ST**, denotes the total weight of edges in **ST**

$$w(ST) = \sum_{(a,b) \in ST} w(a,b)$$

- **Minimum Spanning Tree (MST)** of connected undirected weighted graph **G**
  - **MST** of **G** is an **ST** of **G** with the minimum possible **w(ST)**

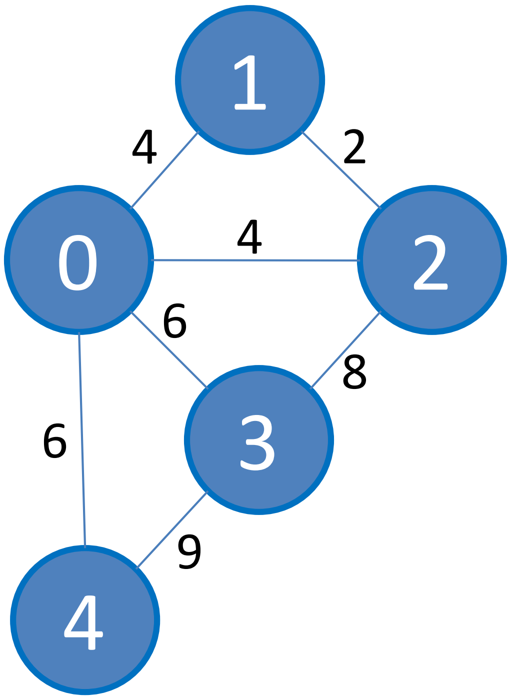
More Definitions (3)

- **The (standard) MST Problem**
  - Input: A connected undirected weighted graph **G(V, E)**
  - Select some edges of **G** such that the graph forms a spanning tree, but with minimum total weight
  - Output: Minimum Spanning Tree **(MST)** of **G**

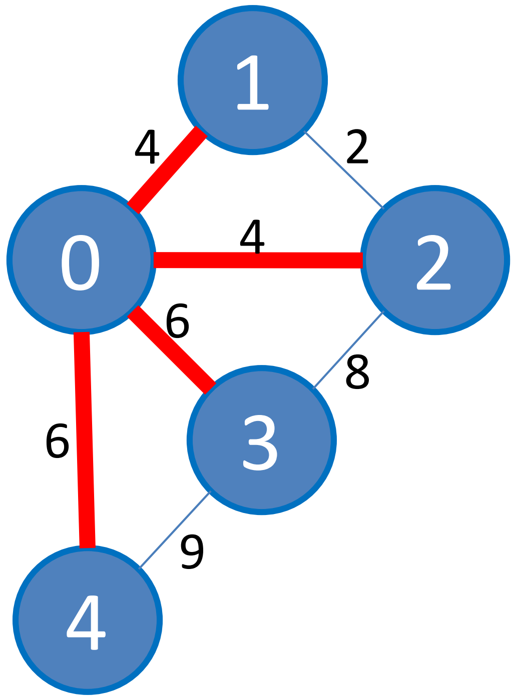


Example

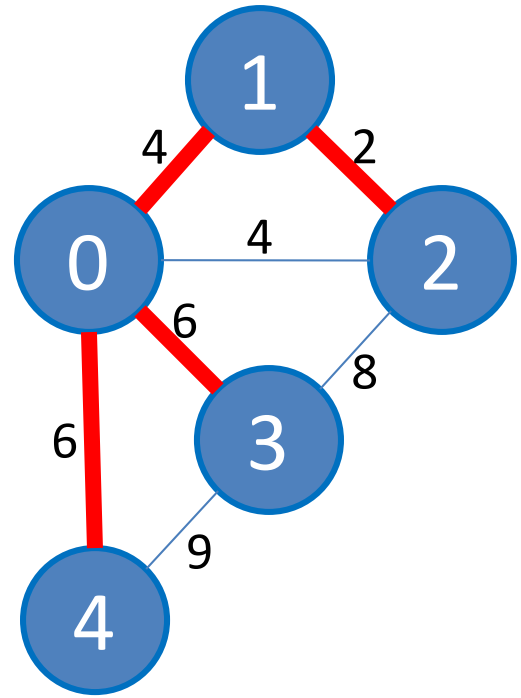
The Original Graph



A Spanning Tree
Cost: 4+4+6+6 = 20

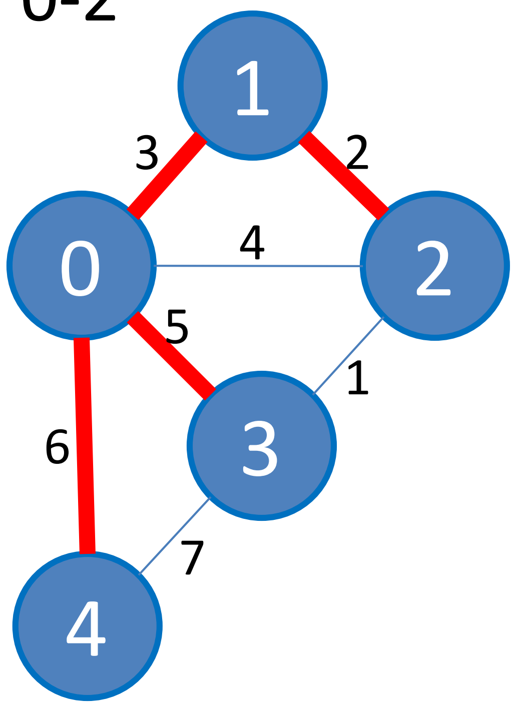


An MST
Cost: 4+6+6+2 = 18



Do the edges highlighted in red part form an MST of the original graph?

1. No, we must replace edge 0-3 with edge 2-3
2. No, we must replace edge 1-2 with 0-2
3. Yes



Brute force/Complete Search Solution?

- Consider all cycles in the graph and break them!
  - For each cycle remove the largest edge
  - If 1 or more edges in a cycle has already been removed previously move on to the next cycle

- Cycle property: For any cycle C in graph G(V,E), if weight of an edge e is larger than every other edge in C, e cannot be included in the MST of G(V,E)

- How to get all cycles in the graph?
  - Not so easy except for some special graphs … (Can you think of a way to do this?)
  - Can have up to $O(2^N)$ different cycles!
  - Listing down one by one is slow !

MST Algorithms

MST is a <u>well-known</u> Computer Science problem

Several efficient (polynomial) algorithms:

- Jarnik’s/Prim’s greedy algorithm
  - Uses PriorityQueue Data Structure taught in Lecture 09
- Kruskal’s greedy algorithm
  - Uses Union-Find Data Structure taught in Lecture 10
- Boruvka’s greedy algorithm (not discussed here)
- And a few more advanced variants/special cases…

Do you still remember Prim's/Kruskal's algorithms from CS1231/S?

1. Yes and I also know how to ***implement*** them
2. Yes, but I have not try implementing them yet
3. I forgot that particular CS1231/S material…
   but I know it exists
4. Eh?? These two algorithms were covered before in CS1231/S??
5. I didn't take CS1231/S ☹

Prim's Algorithm

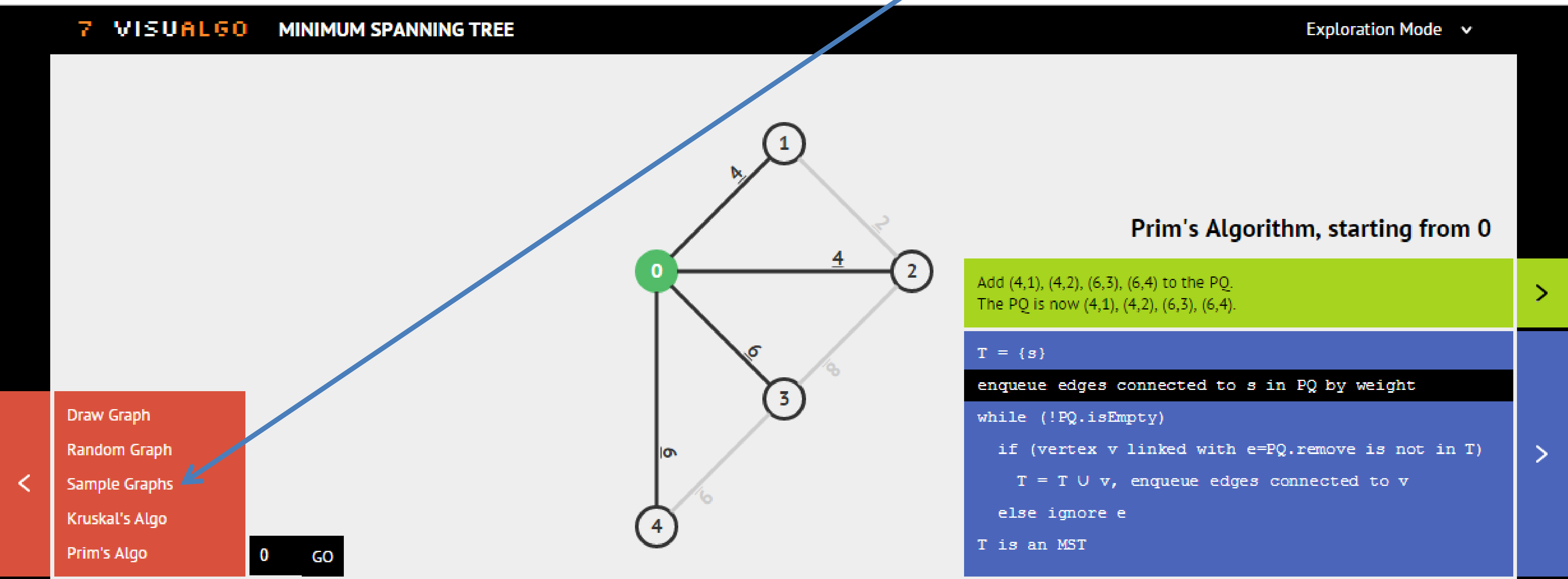
Very simple pseudo code

```
1. T ← {s}, a starting vertex s (usually vertex 0)

2. enqueue edges connected to s (only the other ending
   vertex and edge weight) into a priority queue PQ
   that orders elements based on increasing weight

3. while there are unprocessed edges left in PQ
     take out the front most edge e
     if vertex v linked with this edge e is not taken yet
       T ← T ∪ v (including this edge e)
       enqueue each edge adjacent to v into the PQ if it
       is not already in T

4.T is an MST
```

MST Algorithm: Prim's

Ask VisuAlgo to perform Prim's *from various sources* on the sample Graph (CP3 4.10), *then try other graphs*

In the screen shot below, we show the start of **Prim(0)**



Easy Java Implementation

You need to use two known Data Structures to be able to implement Prim's algorithm:

1. A priority queue **PQ** (we can use Java PriorityQueue), and
2. A boolean array **taken** (to decide if a vertex has been taken or not)

With these DSes, we can run Prim's in O(**E** log **V**) using Adjacency list

- We only process each edge once (enqueue and dequeue it), O(**E**)
  - Each time, we enqueue/dequeue from a PQ in O(log **E**)
  - As $\mathbf{E} = O(\mathbf{V}^2)$, we have $O(\log \mathbf{E}) = O(\log \mathbf{V}^2) = O(2 \log \mathbf{V}) = O(\log \mathbf{V})$
  - Total time $O(\mathbf{E}) * O(\mathbf{logV}) = O(\mathbf{E}\log\mathbf{V})$

Let's have a quick look at PrimDemo.java

Why Does Prim's Work? (1)

First, we have to realize that **Prim's algorithm** is a **greedy algorithm**

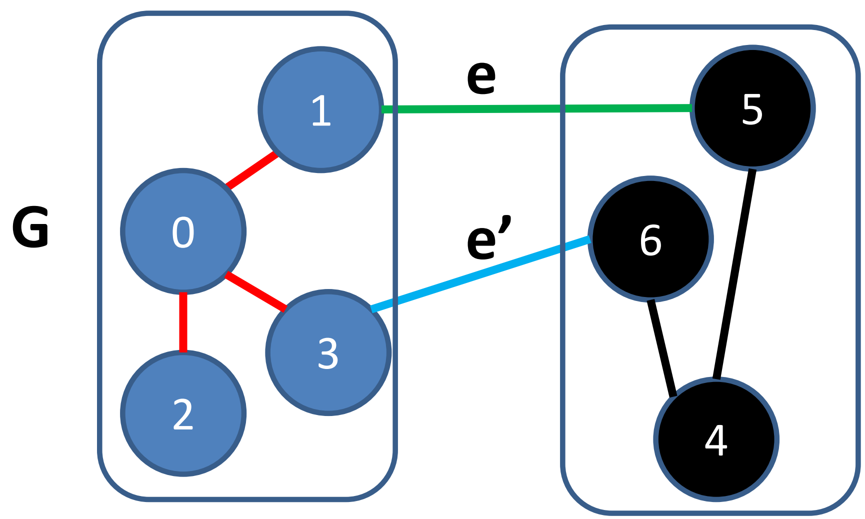
This is because **at each step**, it always try to select the next valid edge e with **minimal weight** (greedy!)

Greedy algorithm is usually simple to implement

- However, it usually requires "proof of correctness"
- Here, we will just see a quick proof

Cut Property of a Connected Graph G

- **Cut of a connected graph**: any partition of vertices of G into 2 disjoint subset (vertices in one set is not in the other). An example is shown below.
- **Cut Set**: The set of edges that cross a cut (e and e’ in the example)
- **Cut Property of a connected graph**: For any cut of the graph, if the weight of an edge **e** in the cut-set is strictly smaller than the weights of other edges of the Cut Set, then **e** belongs to all MSTs of the graph.

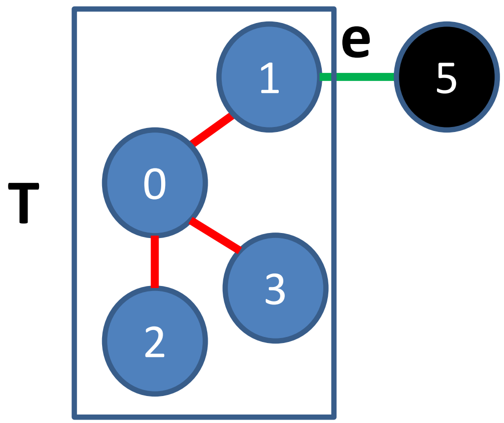


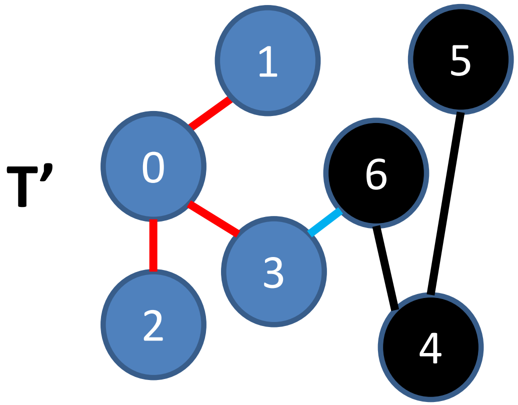
Why Does Prim's Work? (2)

with visual explanation

Proof by contradiction:

- Assume that edge **e** is the first edge at iteration k chosen by Prim's which is not in any valid MST.
- Let **T** be the tree generated by Prim's before adding **e**.
- Now **T** must be a subtree of some valid MST **T'**

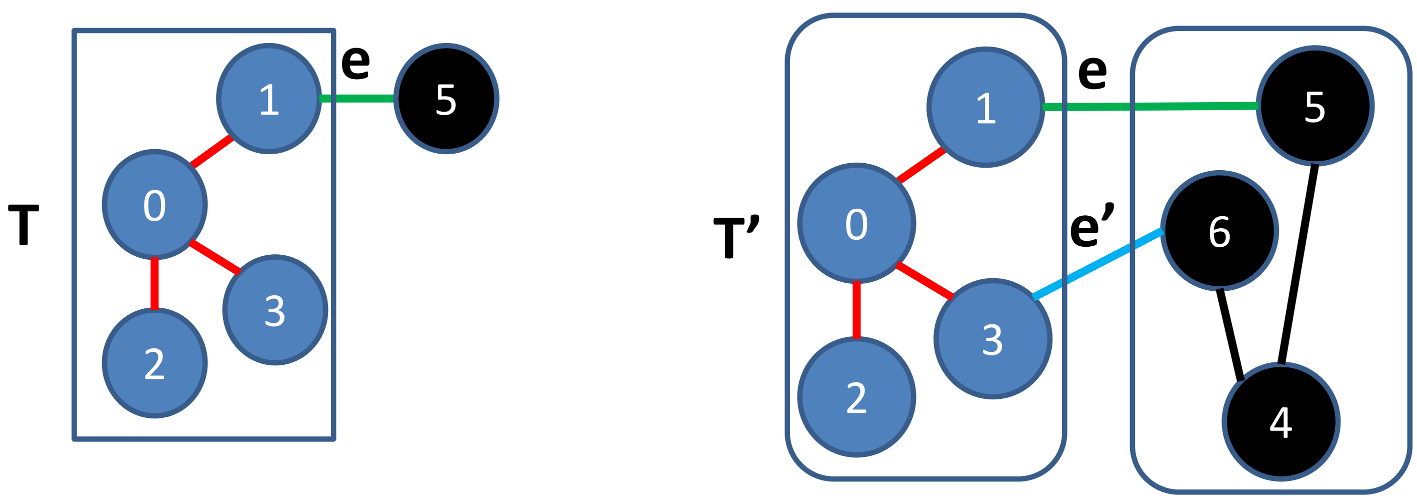




Why Does Prim’s Work? (3)

with visual explanation

- Adding edge **e** to **T’** will now create a cycle.
- Since e has 1 endpoint in **T** (the valid endpoint) and one endpoint outside **T**, trace around this cycle in **T’** until we get to some edge **e’** that goes back to **T**
- Vertices of **T** (blue) and vertices outside **T** (black) forms 2 disjoints subsets of **T’**. This is a cut of **T’**, where {**e**,**e’**} is the cut set



Why Does Prim's Work? (4)

with visual explanation

- By Prim's algorithm **e** and **e'** must be candidate edges at iteration k, but **e** was chosen meaning w(**e**) ≤ w(**e'**) by the **cut property**
- Now replacing **e'** with **e** in **T'** must give us tree **T''** covering all vertices of the graph s.t w(**T''**) ≤ w(**T'**)
- Contradiction that **e** is first edge chosen wrongly

*Prim's variant for Dense Graphs (1)

- For dense graphs where <u>$E = O(V^2)$</u>, time complexity of Prim's is $O(E\log V)$ = **$O(V^2\log V)$**

- We can improve this time complexity …. ironically by replacing the PQ with a simple array **A** of size V
  - For each **A**[v], store a pair info <w(u,v),u> of the smallest weighted edge (u,v) to v among all edges to v that has been explored as Prim's is executed
  - *Why only track the smallest weighted explored edge to a vertex v?*

**this variant is not in Visualgo*

Prim’s variant for Dense Graphs (2)

Pseudo code is as follows

```
1.  Initialize A to <+inf,s>
2.  A[s] = <0,s>
3.  While not all vertices are in T
      Scan A to get v where A[v].first is minimum in A
      T 🡐 T ∪ v and the adjoining edge
      A[v] = <+inf, A[v].second> // will not choose it again
      for all u adjacent to v // all neighbors of v
        if (u is not in T && A[u].first > w(v,u))
          A[u] = <w(v,u),v>

4. T is an MST
```

Prim’s variant for Dense Graphs (3)

Time complexity of Prim’s variant for dense graphs

```
1.  Initialize A to <+inf,s>
2.  A[s] = <0,s>
3.  While not all vertices are in T // O(V) iterations
      scan A to get v where A[v].first is min. in A // O(V)
      T 🡐 T ∪ v and the adjoining edge
       A[v] = <+inf, A[v].second>
      for all u adjacent to v // O(V) due to O(V) neighbors
        if (u is not in T && A[u].first > w(v,u))
          A[u] = <w(v,u),v>

4. T is an MST
```

Total Time Complexity = **O(V^2)**, better than O(V^2logV) for standard Prim’s algorithm

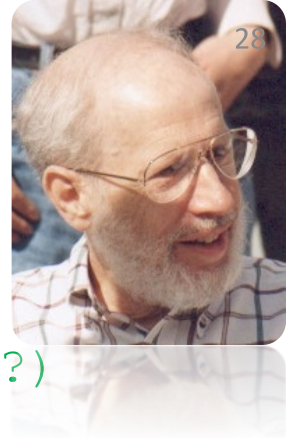
Coming up next: Kruskal’s algorithm

Kruskal's Algorithm

Very simple pseudo code

```
sort the set of E edges by increasing weight
T ← {}
while there are unprocessed edges left
  pick an unprocessed edge e with min cost
  if adding e to T does not form a cycle
    add e to T
T is an MST
```

Kruskal's Implementation (1)



```
sort the set of E edges by increasing weight // O(?)
T ← {}
while there are unprocessed edges left // O(E)
  pick an unprocessed edge e with min cost // O(?)
  if adding e to T does not form a cycle // O(?)
    add e to the T // O(1)
T is an MST
```

To sort the edges:

- We use **EdgeList** to store graph information
- Then use "any" sorting algorithm that we have seen before

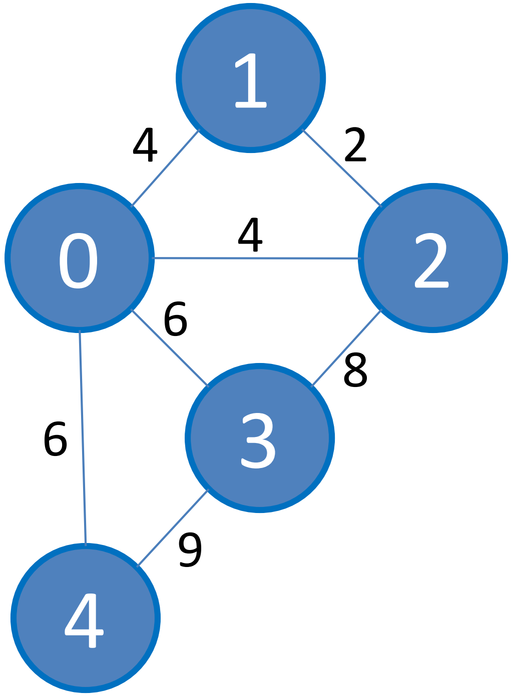
To test for cycles:

- We use **Union-Find Disjoint Sets**

Sorting Edges in Edge List

Adjacency Matrix/List that we have learned previously are *not suitable* for edge-sorting task!

To sort **EdgeList**, we use ***one liner* Java Collections.sort …**



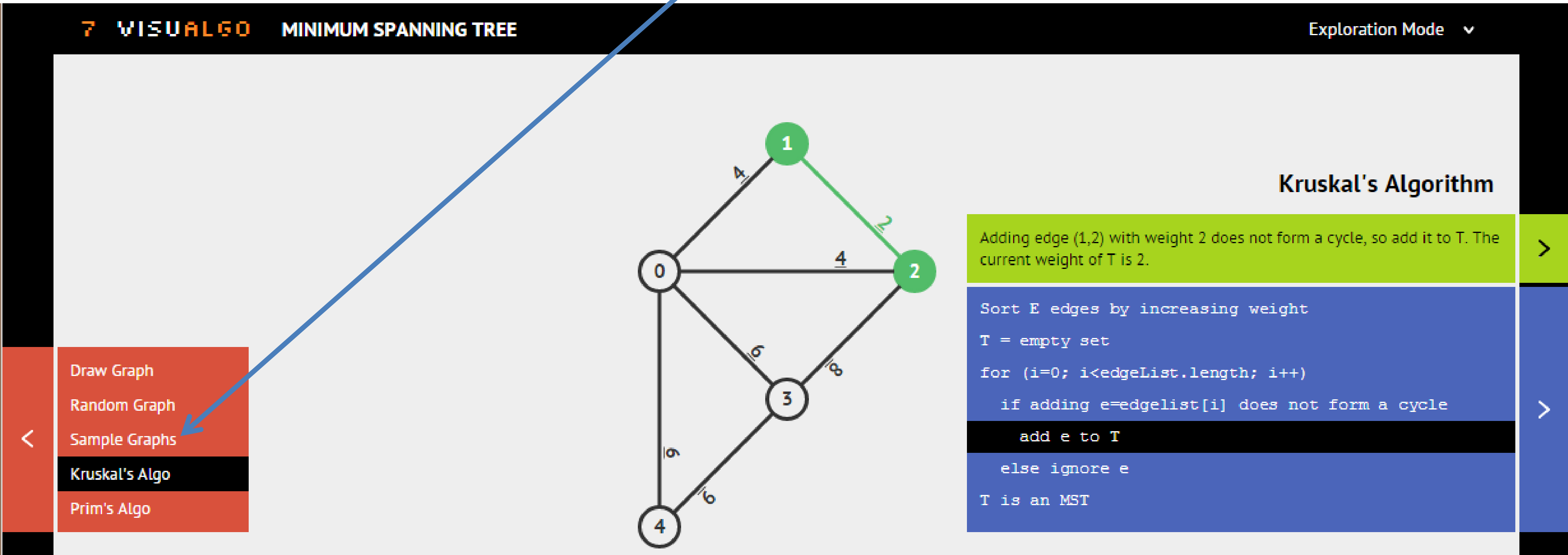
i	w	u	v
0	2	1	2
1	4	0	1
2	4	0	2
3	6	0	3
4	6	0	4
5	8	2	3
6	9	3	4

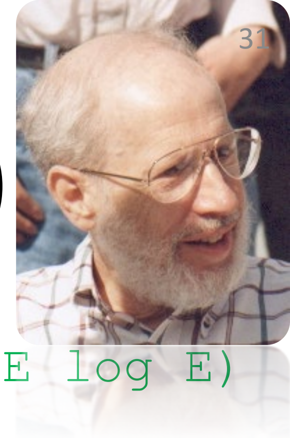
MST Algorithm: Kruskal's

Ask VisuAlgo to perform Kruskal's on the sample Graph (CP3 4.10), *<u>then try other graphs</u>*

In the screen shot below, we show the start of **Kruskal**

(there is no parameter for this algorithm)





Kruskal's Implementation (2)

```
sort the set of E edges by increasing weight // O(E log E)
T ← {}
while there are unprocessed edges left // O(E)
  pick an unprocessed edge e with min cost // O(1)
  if adding e to T does not form a cycle // O(α(V)) = O(1)
    add e to the T // O(1)
T is an MST
```

To sort the edges, we need O(**E** log **E**)

To test for cycles, we need O(α(**V**)) – small, assume constant O(**1**)

In overall

- Kruskal's runs in O(**E** log **E** + ~~**E** α(**V**)~~) // E log E dominates!
- As **E** = O(**V²**), thus Kruskal's runs in O(**E** log **V²**) = O(**E log V**)

Let's have a quick look at KruskalDemo.java

Why Does Kruskal’s Work? (1)

Kruskal’s algorithm is also a **greedy algorithm**

Because **at each step**, it always try to select the next unprocessed edge **e** with **minimal weight** (greedy!)

Simple proof on how this greedy strategy works

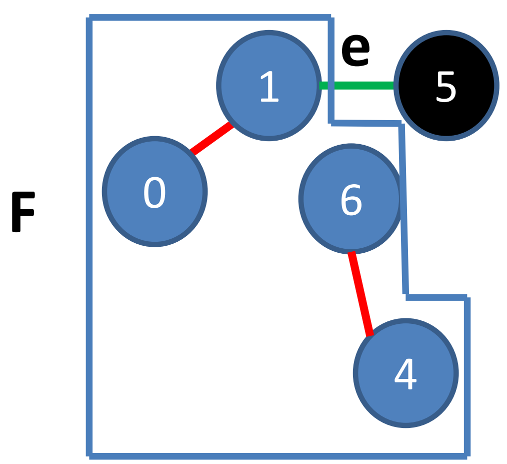
- Almost the same as that for Prim’s

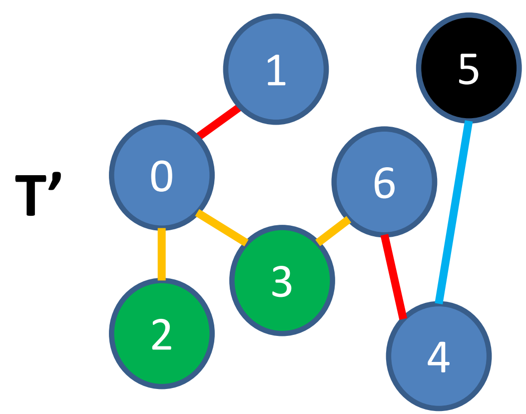
Why Does Kruskal’s Work? (2)

with visual explanation

Proof by contradiction:

- Assume that edge **e** is the first edge at iteration k chosen by Kruskal’s which is not in any valid MST.
- Let **F** be the forest generated by Kruskal’s before adding **e**.
- Now **F** must be a part of some valid MST **T’**

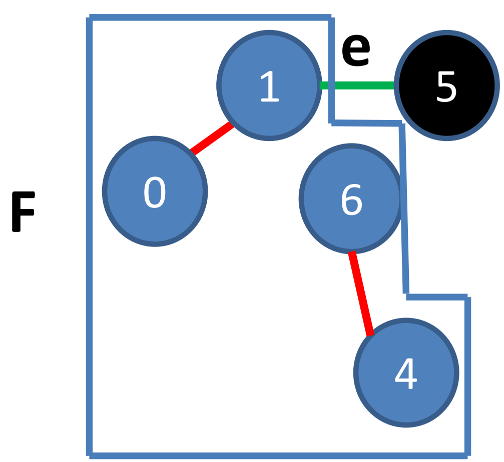


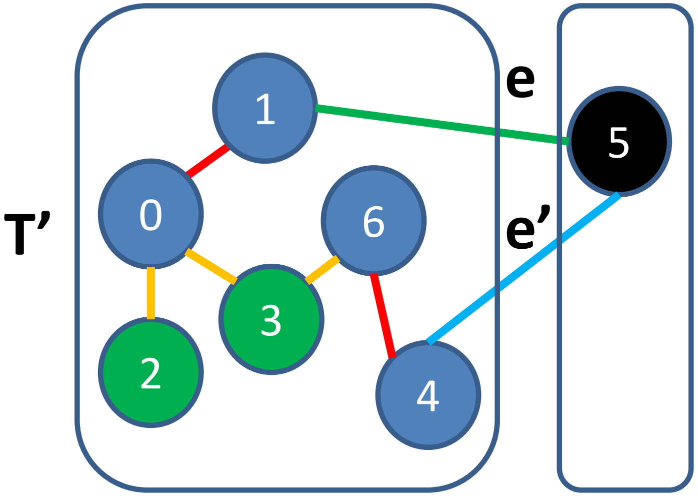


Why Does Kruskal's Work? (3)

with visual explanation

- Putting **e** into **T'** will create a cycle.
- Tracing the cycle, let **V'** be the set of vertices encountered in the cycle that is outside of **F** {only 5 in the example}, where **e'** is the edge leading back into **F**
- We can create a cut of **T'** where the 2 partitions are {all vertices except **V'**} ({0,1,2,3,4,6} in the example) and **V'** ({5} in the example), and the cut set is {**e,e'**} in the example

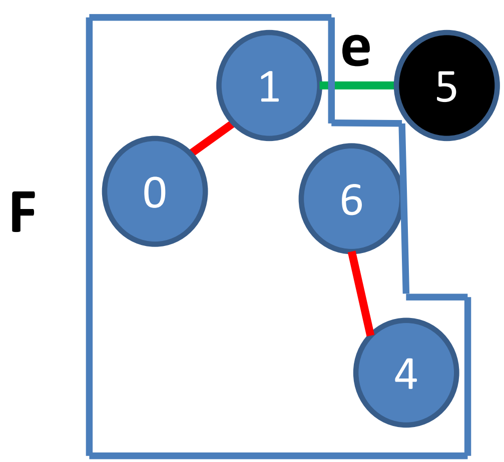


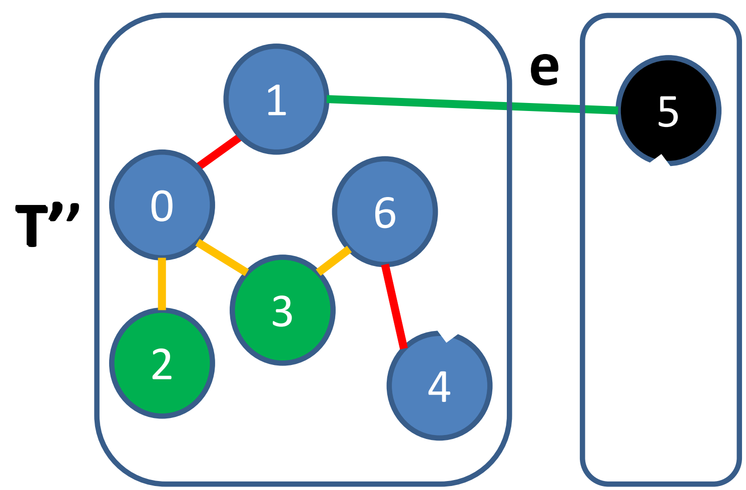


Why Does Kruskal's Work? (4)

with visual explanation

- At iteration k, both **e** and **e'** are candidate (they are not chosen and do not form a cycle if chosen).
- Since **e** was chosen, w(**e**) ≤ w(**e'**) by the **cut property**
- Now replacing **e'** with **e** in **T'** must give us tree **T''** covering all vertices of the graph s.t w(**T''**) ≤ w(**T'**)
- Contradiction that **e** is first edge chosen wrongly





If given an MST problem, I will…

1. Use/code Kruskal’s algorithm
2. Use/code Prim’s algorithm
3. No preference…

Summary

Introduce the MST problem (covered briefly in CS1231/S)

Discuss 2 algorithms

- Prim's algorithm (uses PriorityQueue ADT) & a variant for dense graphs where $E=O(V^2)$ (uses an array instead of PQ)
- Kruskal's algorithm (uses Edge List and UFDS)
- Can view the above 2 algorithms as making use of the Cut Property as opposed to the Cycle Property to construct the MST of any given connected weighted graph

You *may* learn MST/Prim's/Kruskal's again in CS3230