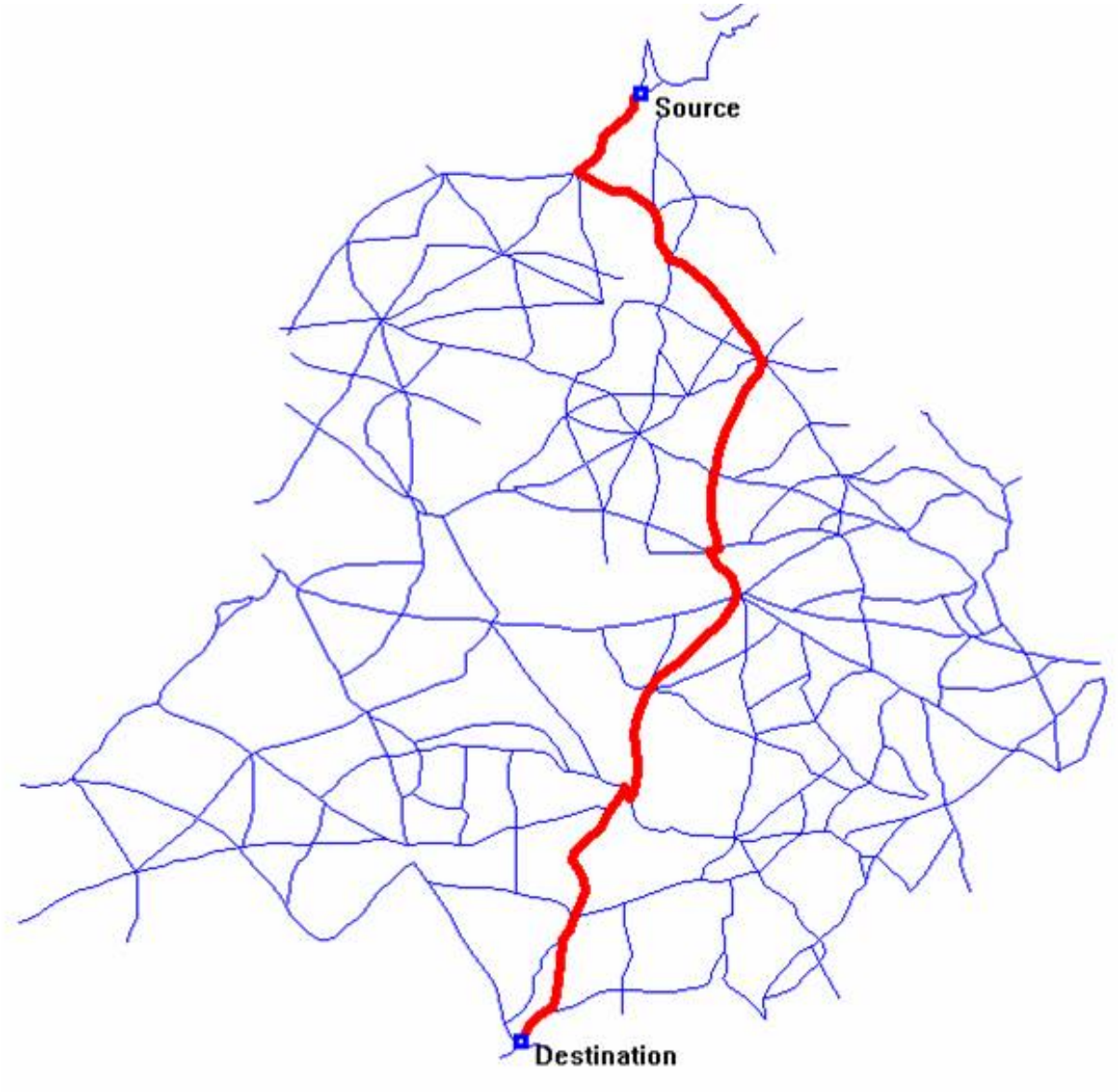
CS2040S
Data Structures and Algorithms
(e-learning edition)

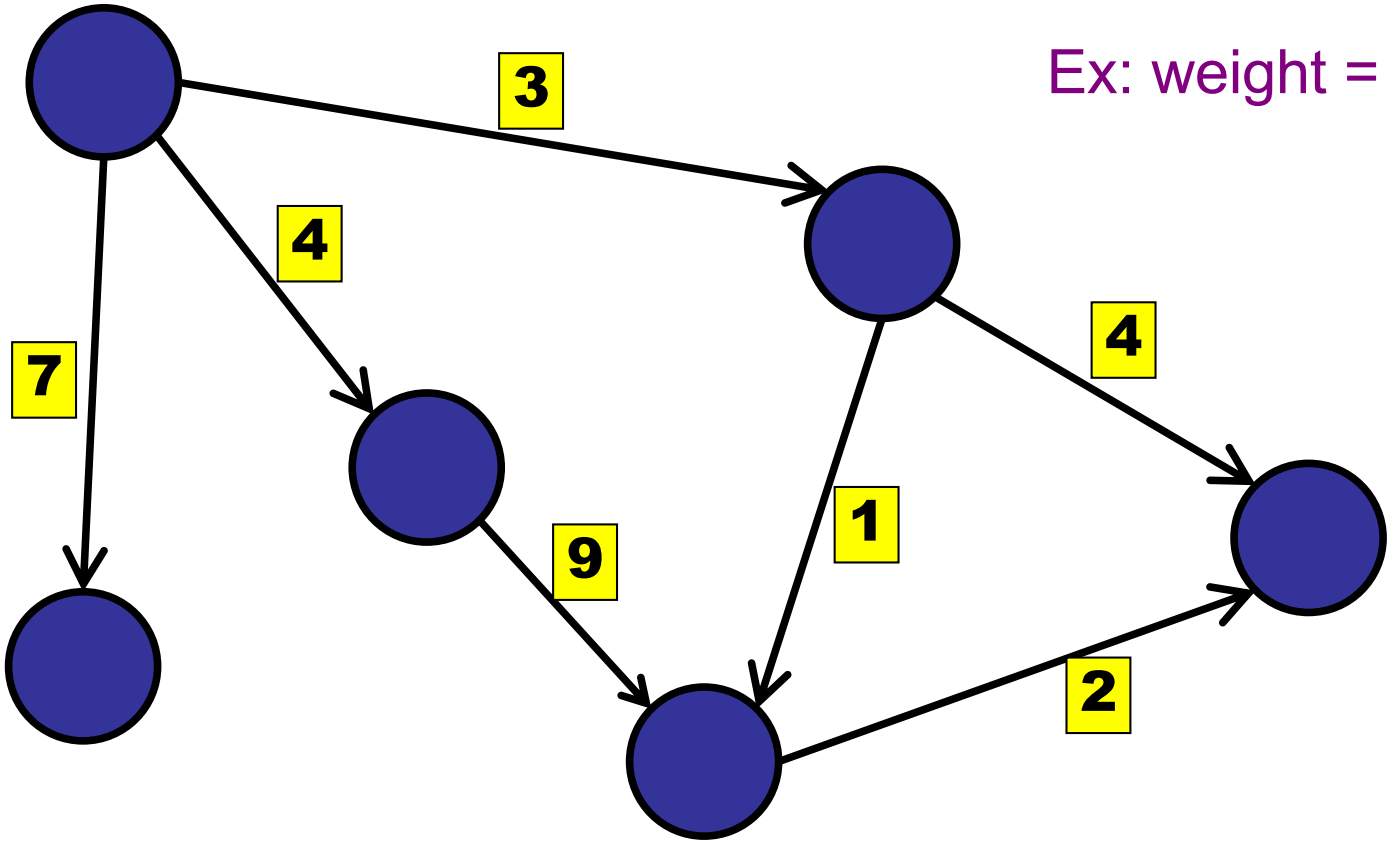
Shortest Paths!

SHORTEST PATHS



Weighted Graphs

Edge weights: w(e) : E ➔ R

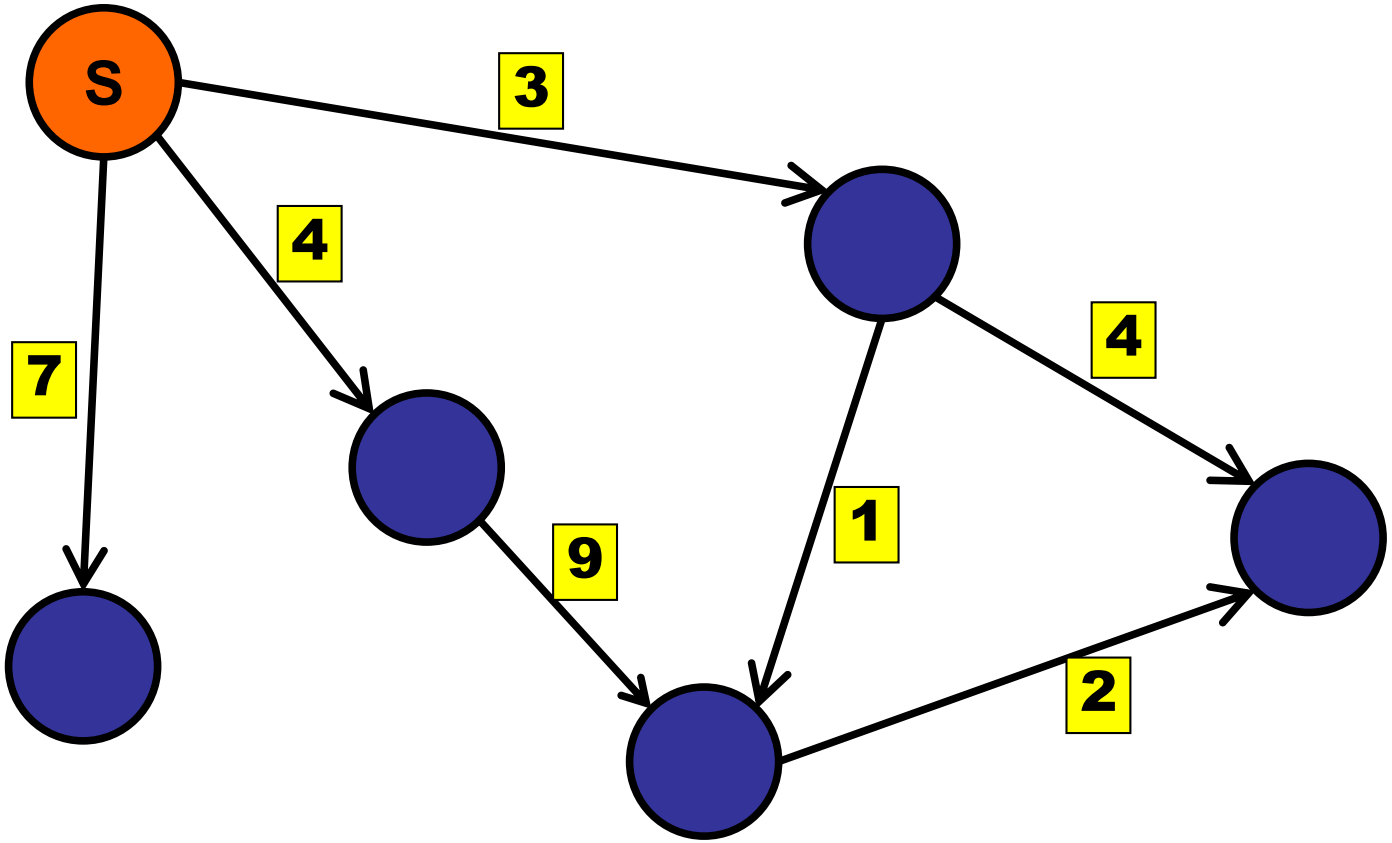


Ex: weight = distance

Adjacency list: stores weights with edge in NbrList

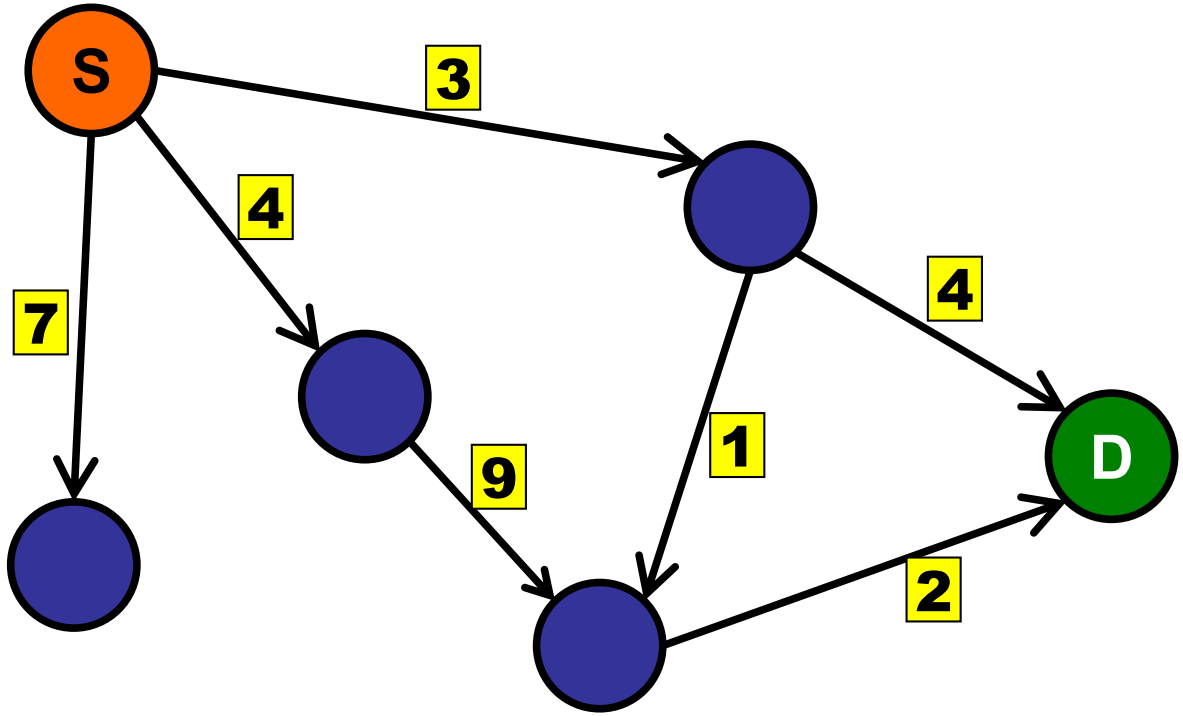
Shortest Paths

Distance from source?



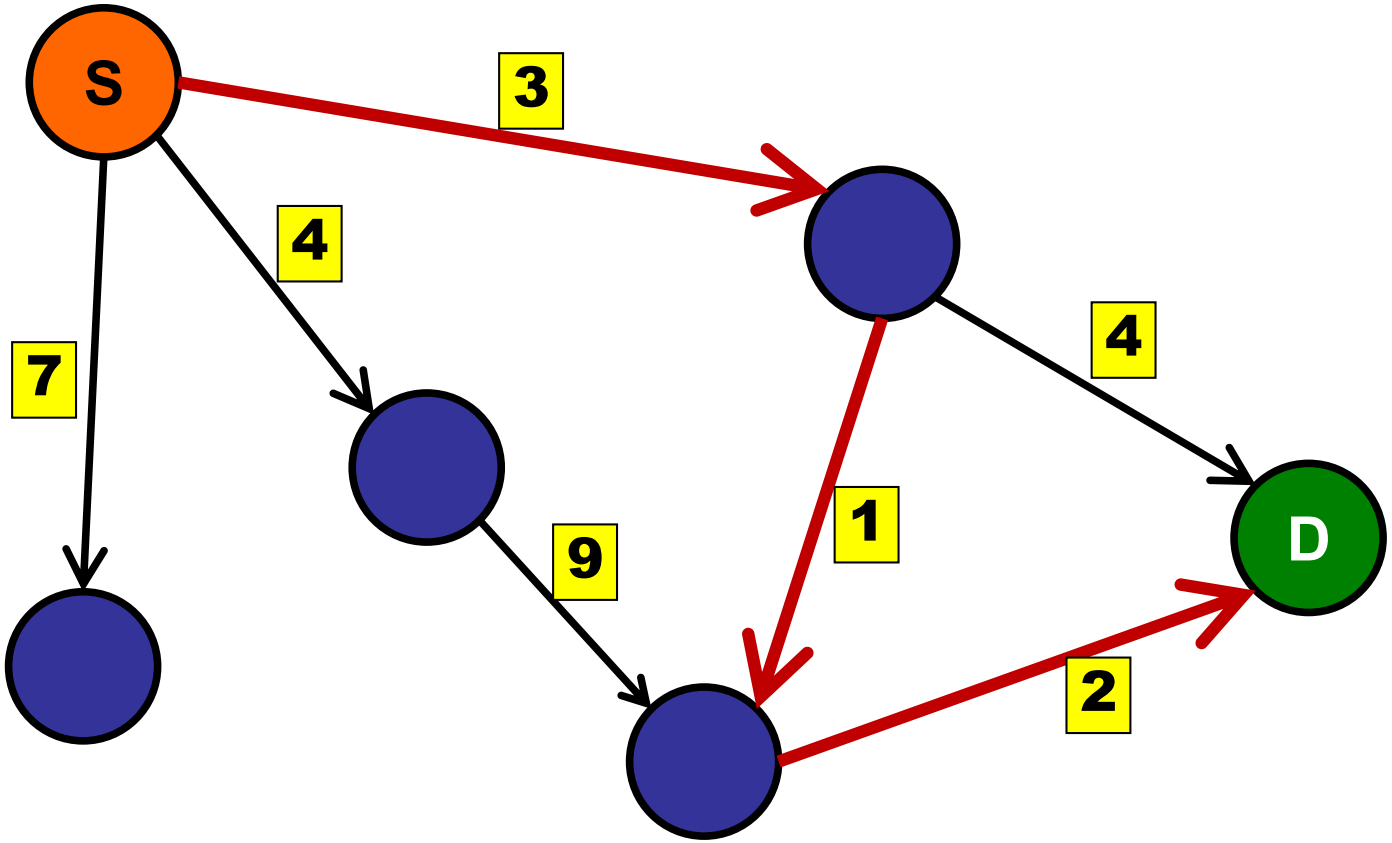
What is the distance from S to D?

1. 2
2. 4
3. ✓ 6
4. 7
5. 9
6. Infinite



Shortest Paths

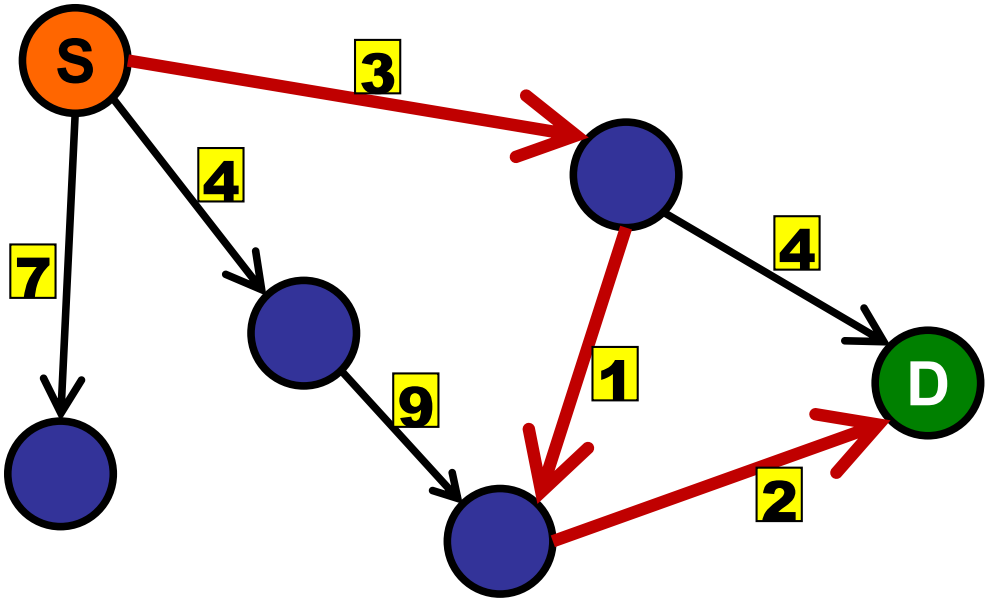
Distance from source?



Shortest Paths

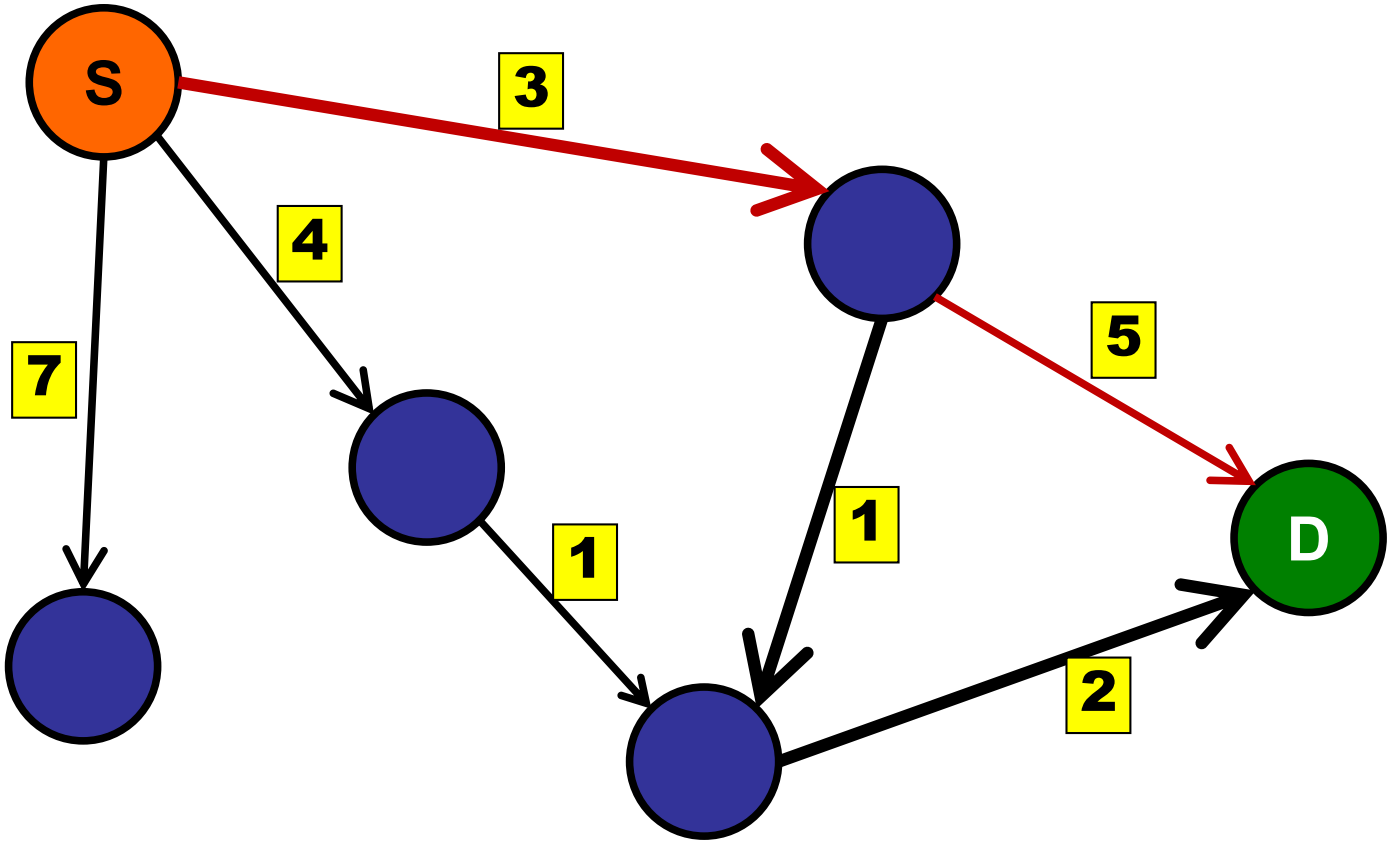
Questions:

- How far is it from S to D?
- What is the shortest path from S to D?
- Find the shortest path from S to every node.
- Find the shortest path between every pair of nodes.



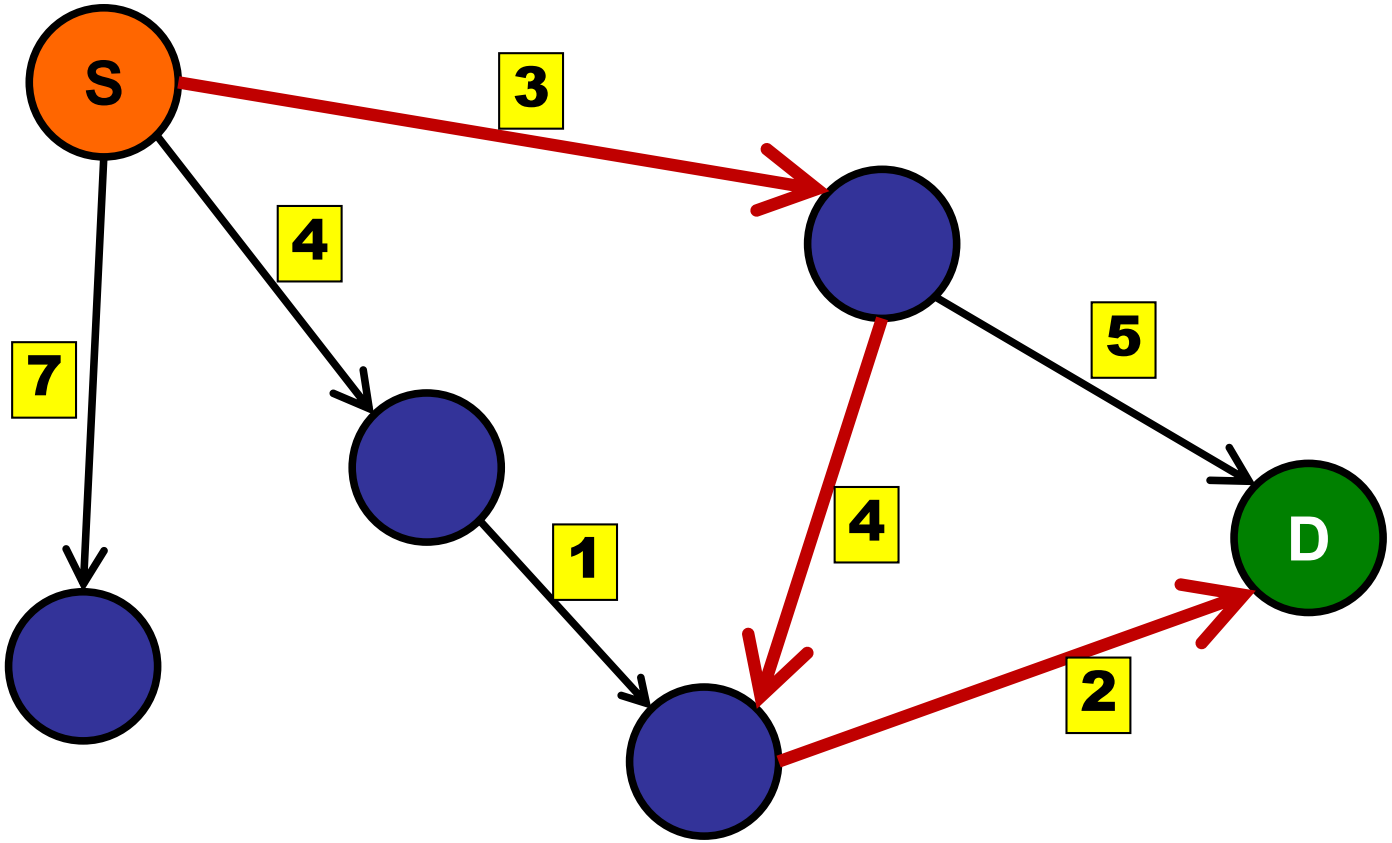
Shortest Paths

Common mistake: “Why can’t I use BFS?”



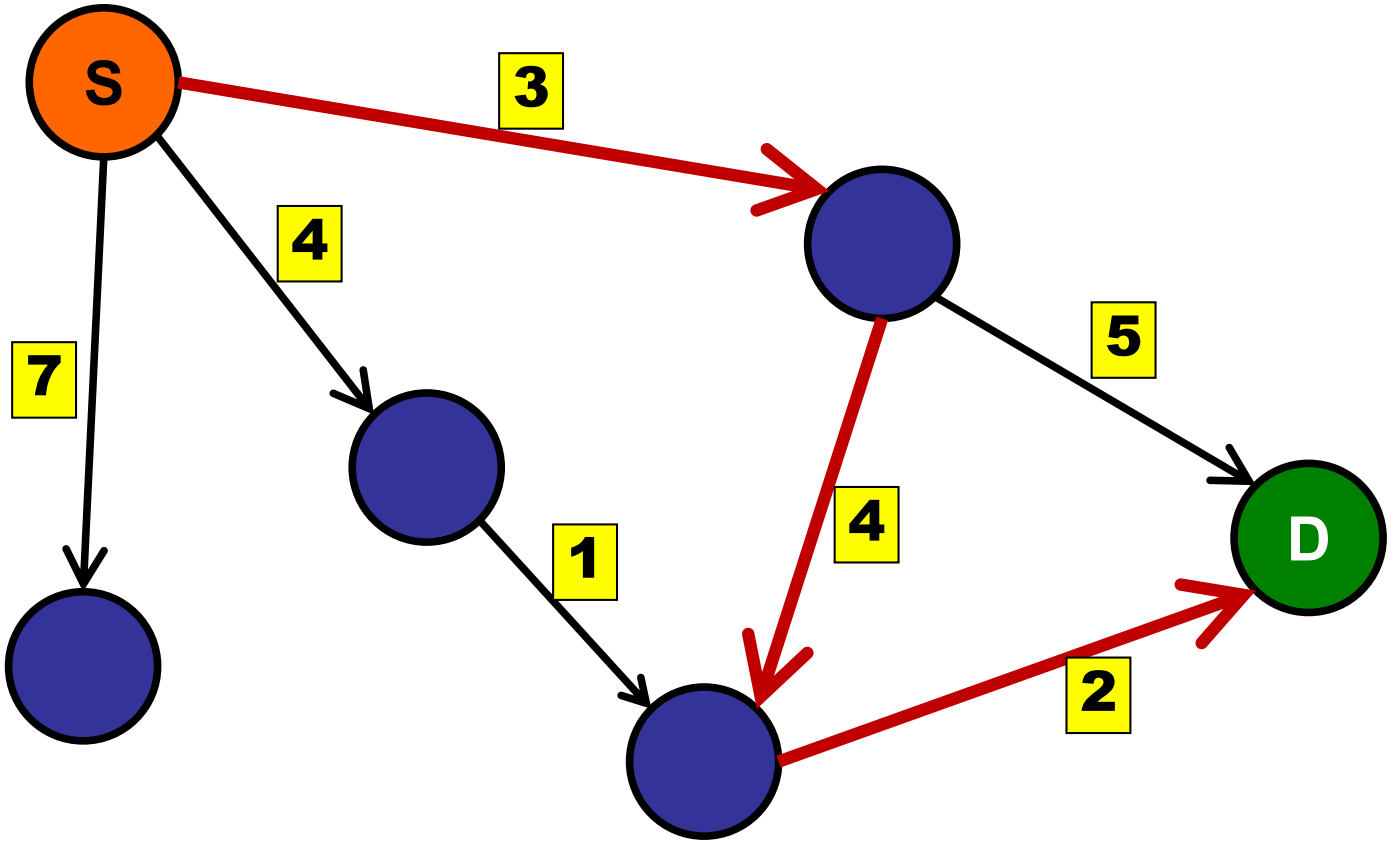
Shortest Paths

Common mistake: “Why can’t I use BFS?”



Shortest Paths

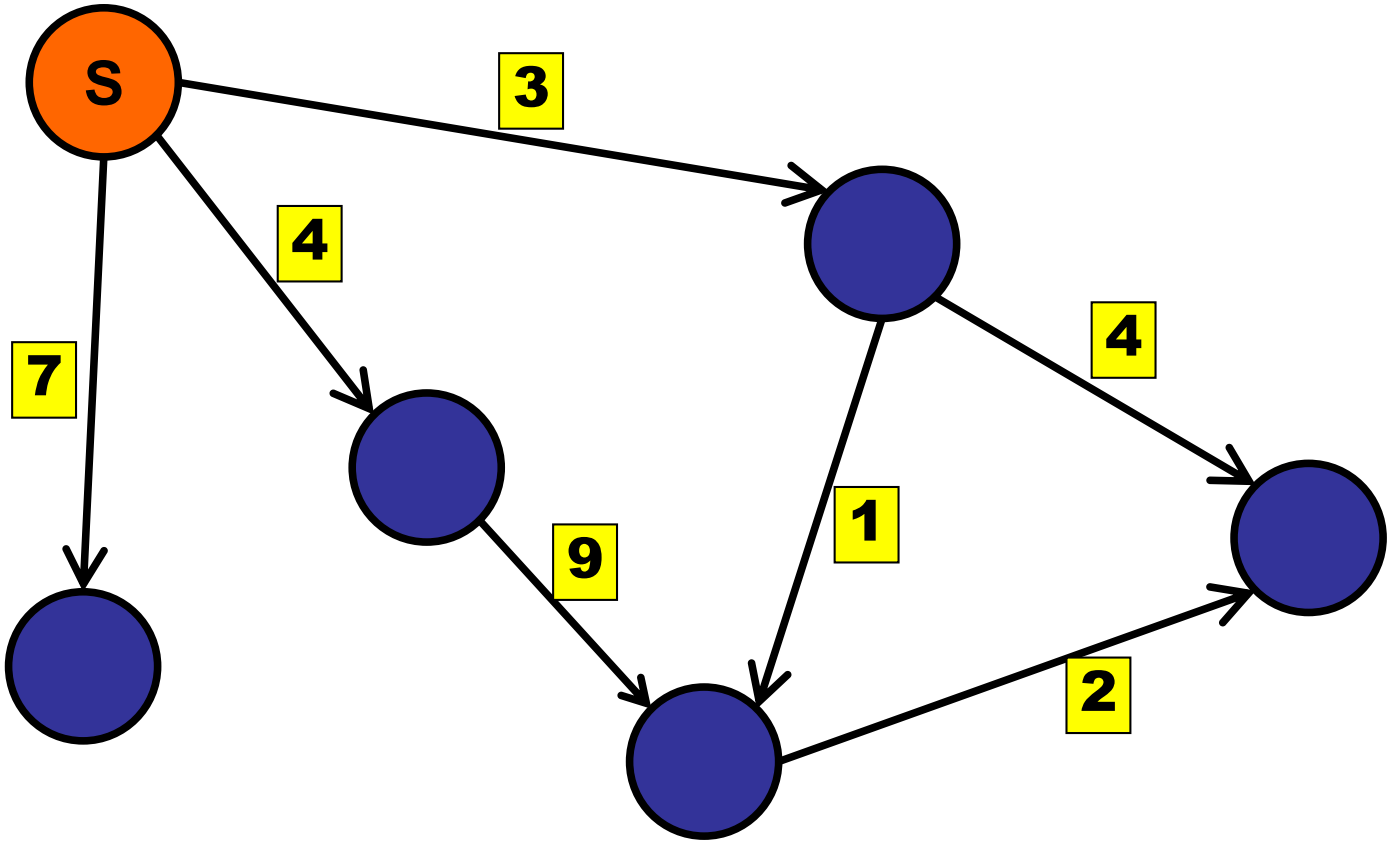
Common mistake: “Why can’t I use BFS?”



BFS finds minimum number of HOPS not minimum DISTANCE.

Shortest Paths

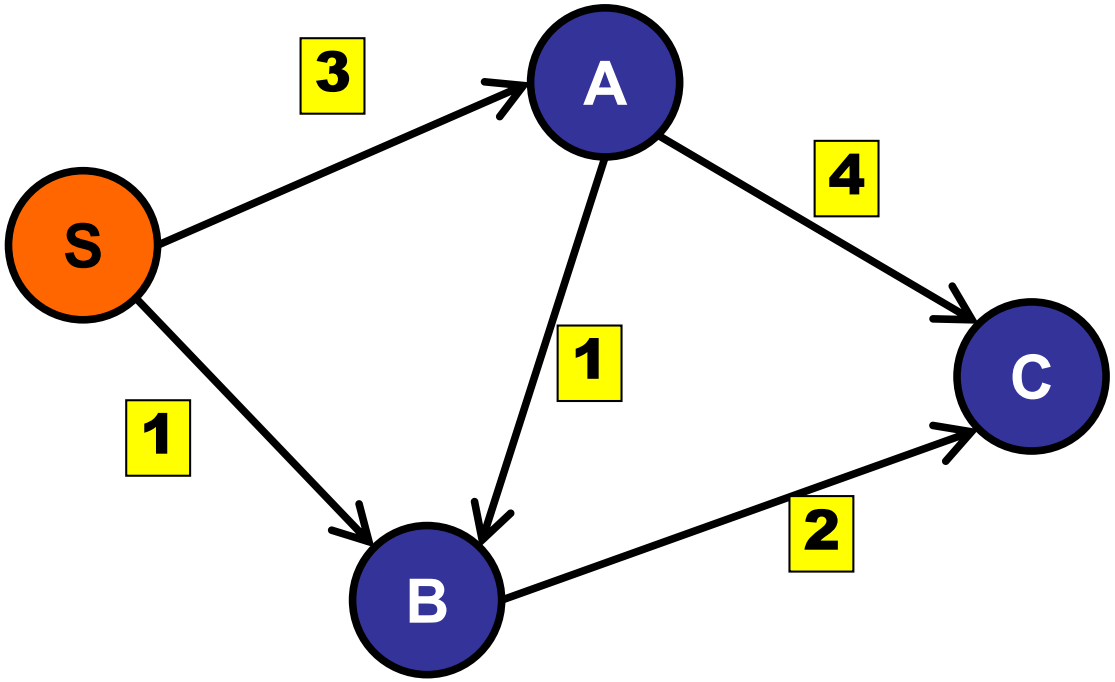
Notation: $\delta(u,v)$ = distance from u to v



Shortest Paths

Key idea: triangle inequality

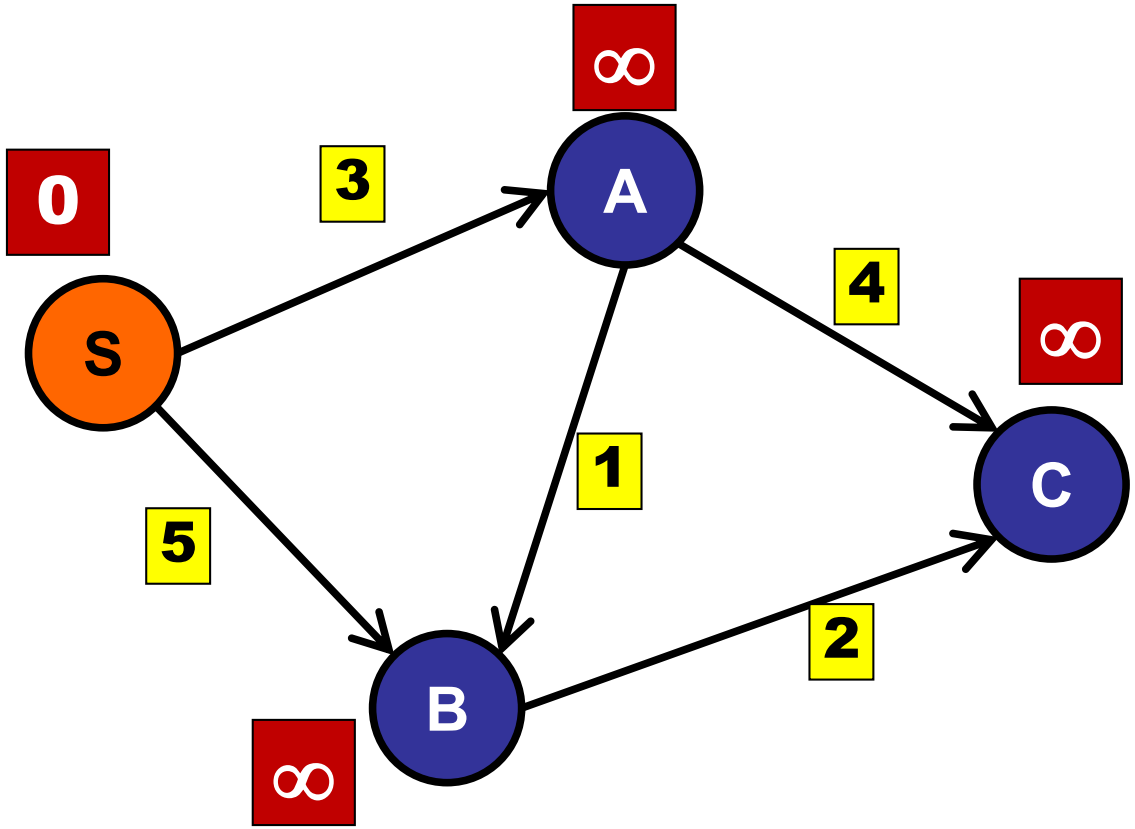
$\delta(S, C) \leq \delta(S, A) + \delta(A, C)$



Shortest Paths

Maintain estimate for each distance:

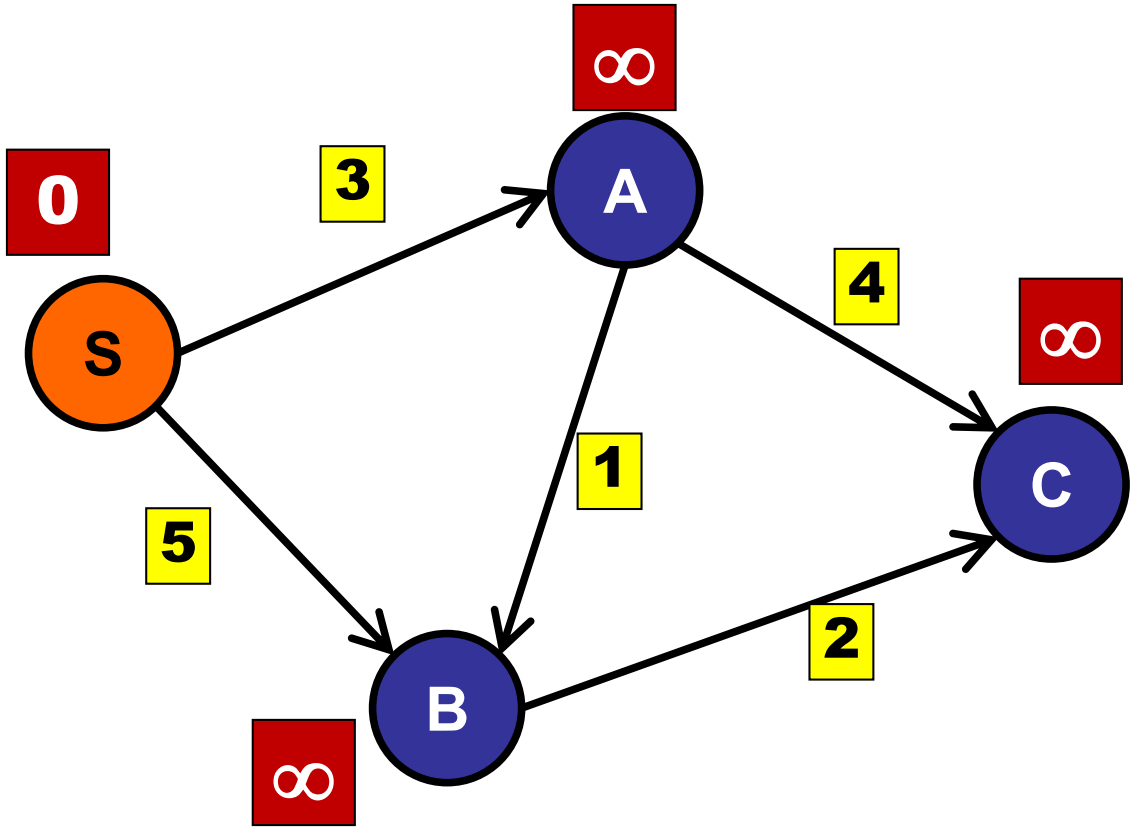
```
int[] dist = new int[V.length];

Arrays.fill(dist, INFTY);

dist[start] = 0;
```



Shortest Paths

Maintain estimate for each distance:

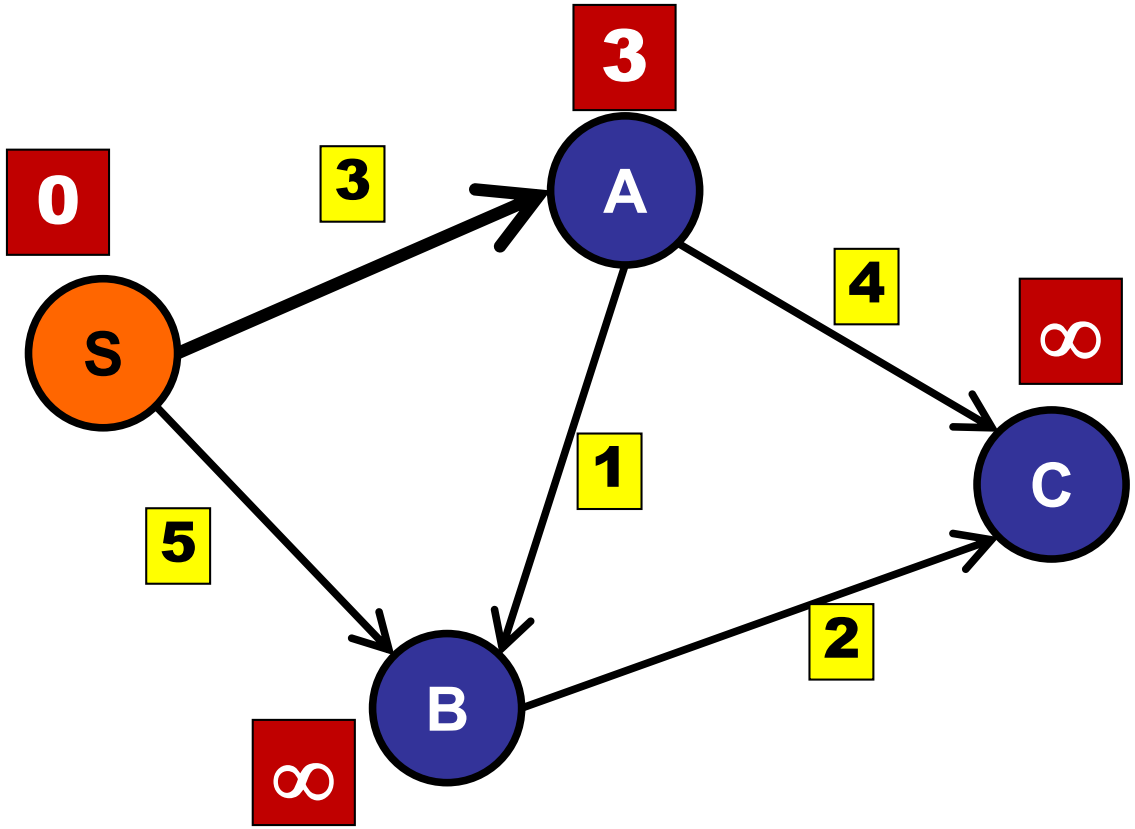
- Reduce estimate
- Invariant: estimate $\geq$ distance



Shortest Paths

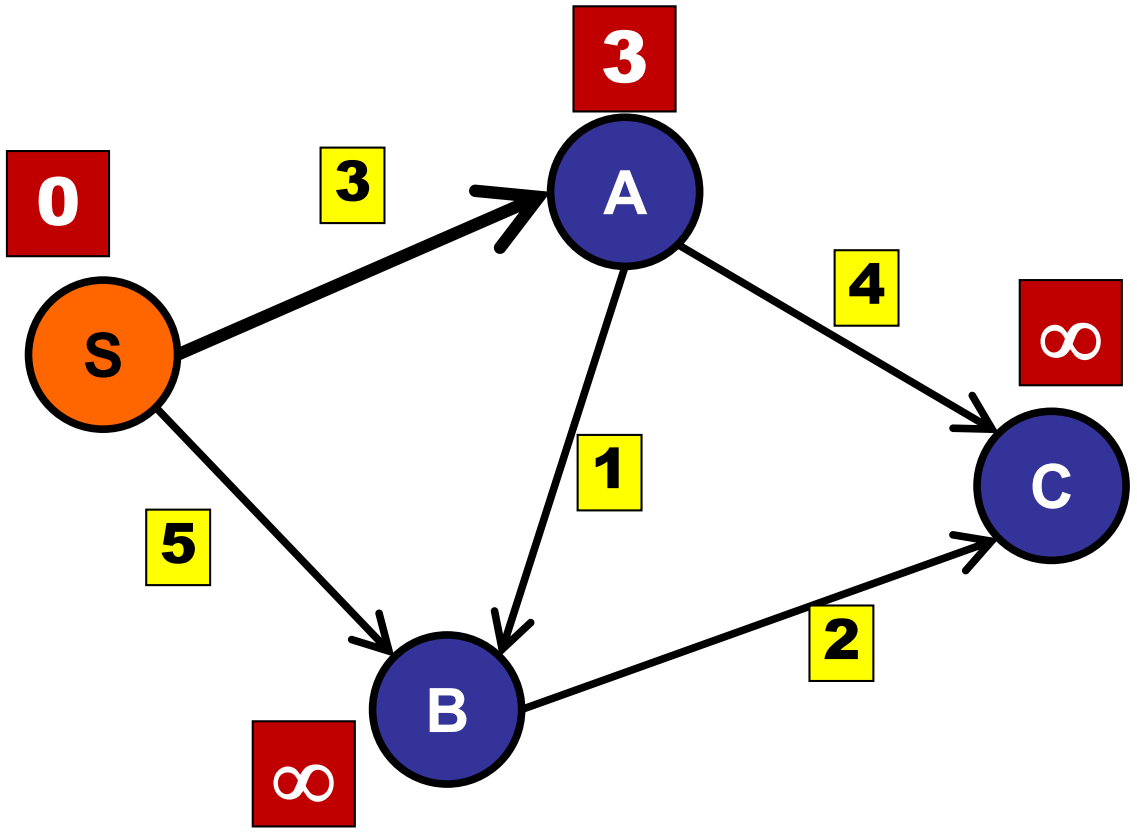
Maintain estimate for each distance:

relax(S, A)



Shortest Paths

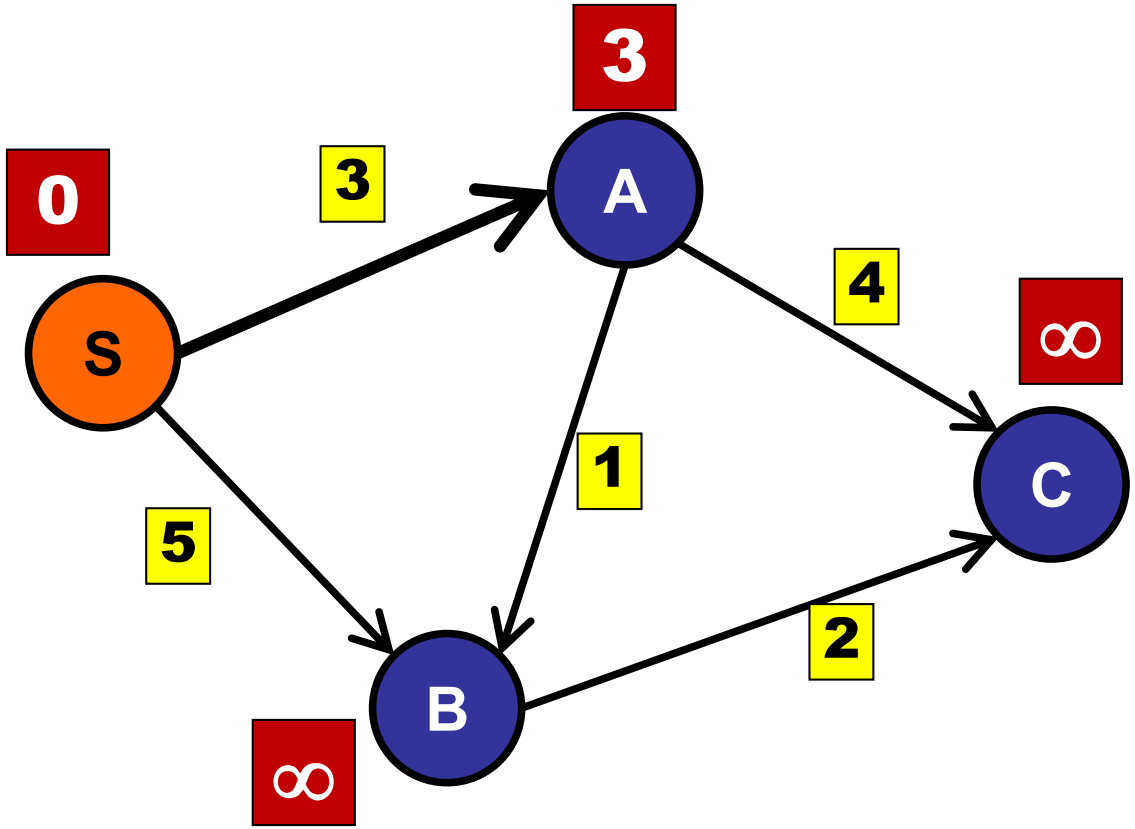
```
relax(int u, int v){
     if (dist[v] > dist[u] + weight(u,v))
          dist[v] = dist[u] + weight(u,v);
}
```



Shortest Paths

Maintain estimate for each distance:

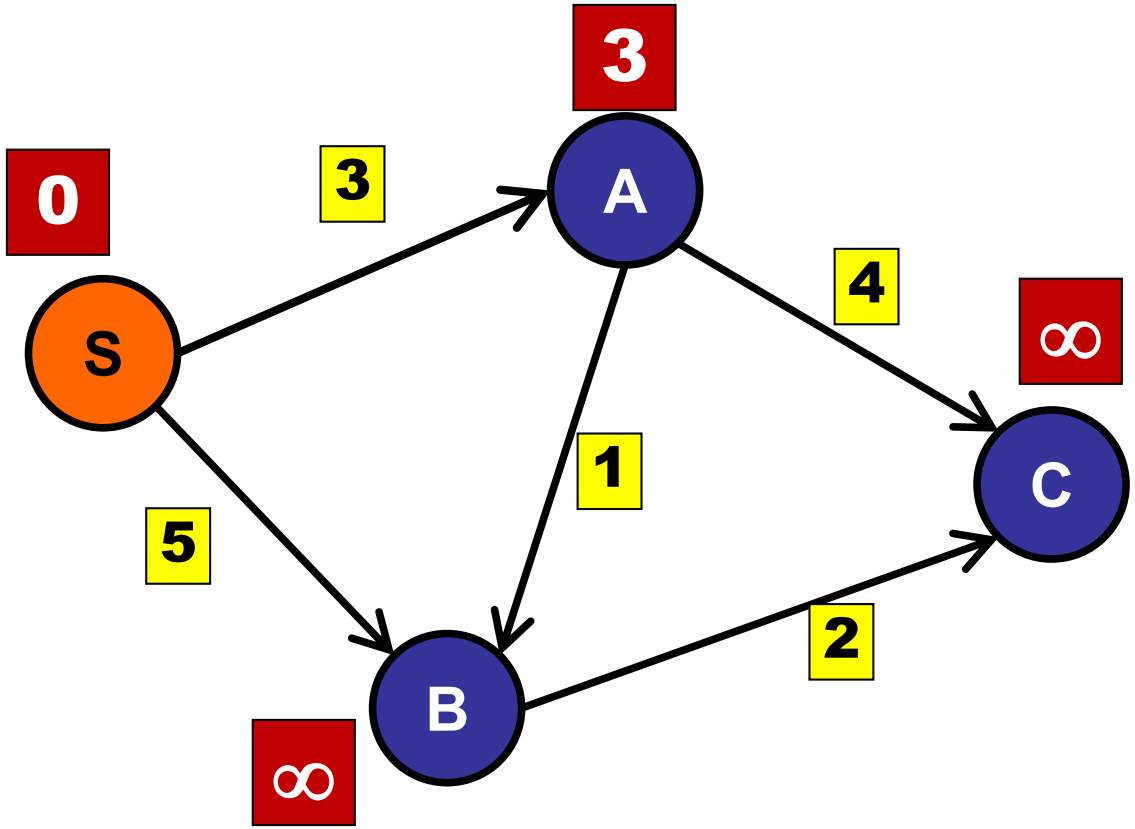
relax(S, A)



Shortest Paths

Maintain estimate for each distance:

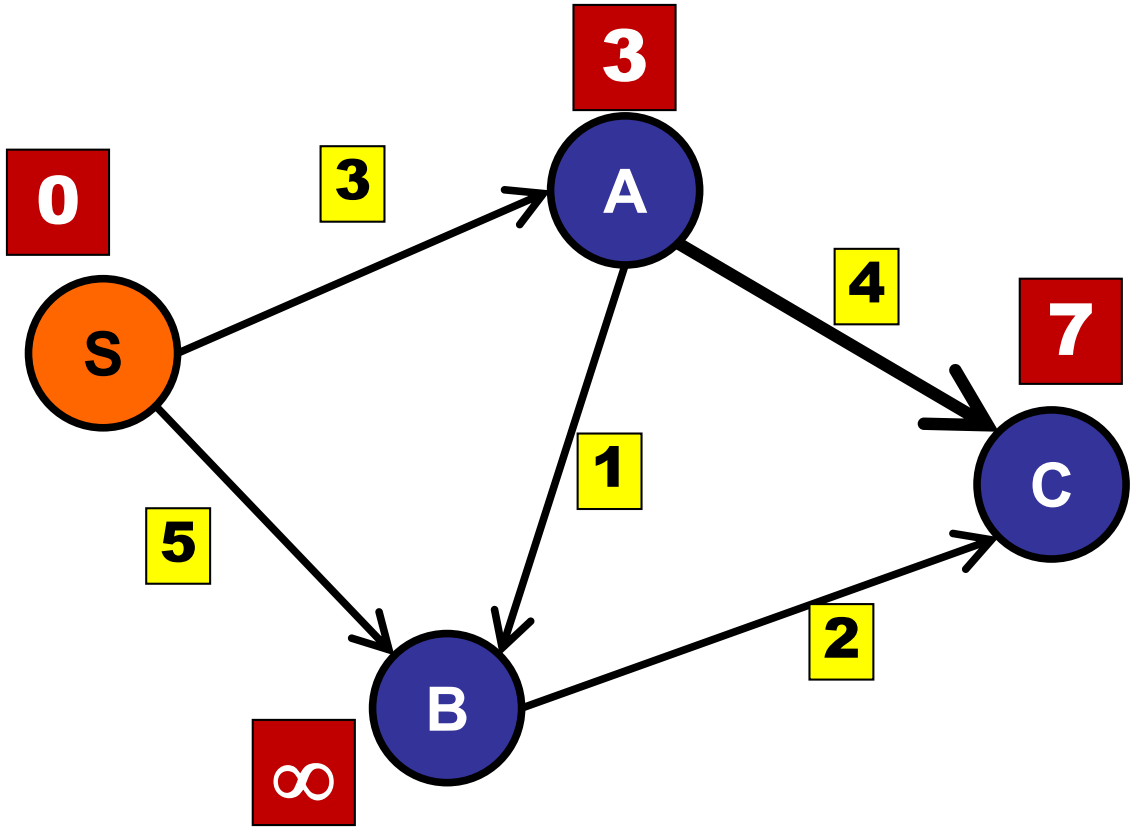
relax(A, C)



Shortest Paths

Maintain estimate for each distance:

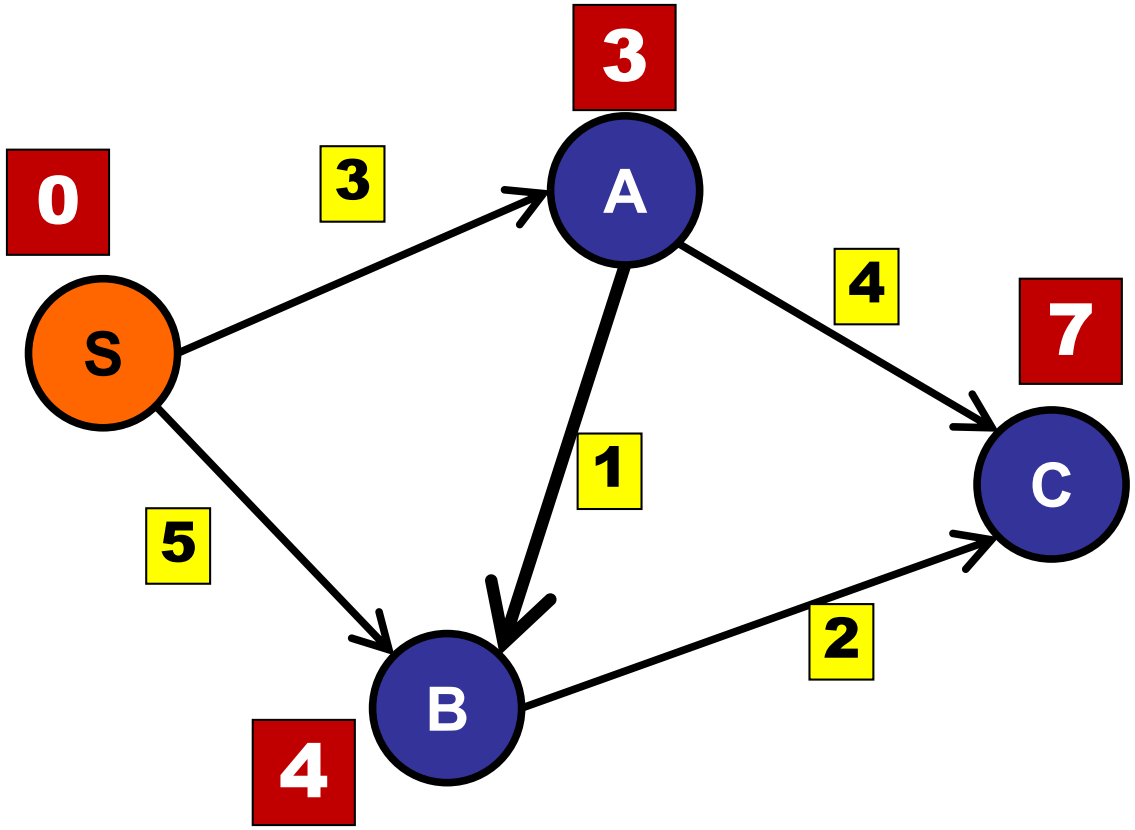
relax(A, C)



Shortest Paths

Maintain estimate for each distance:

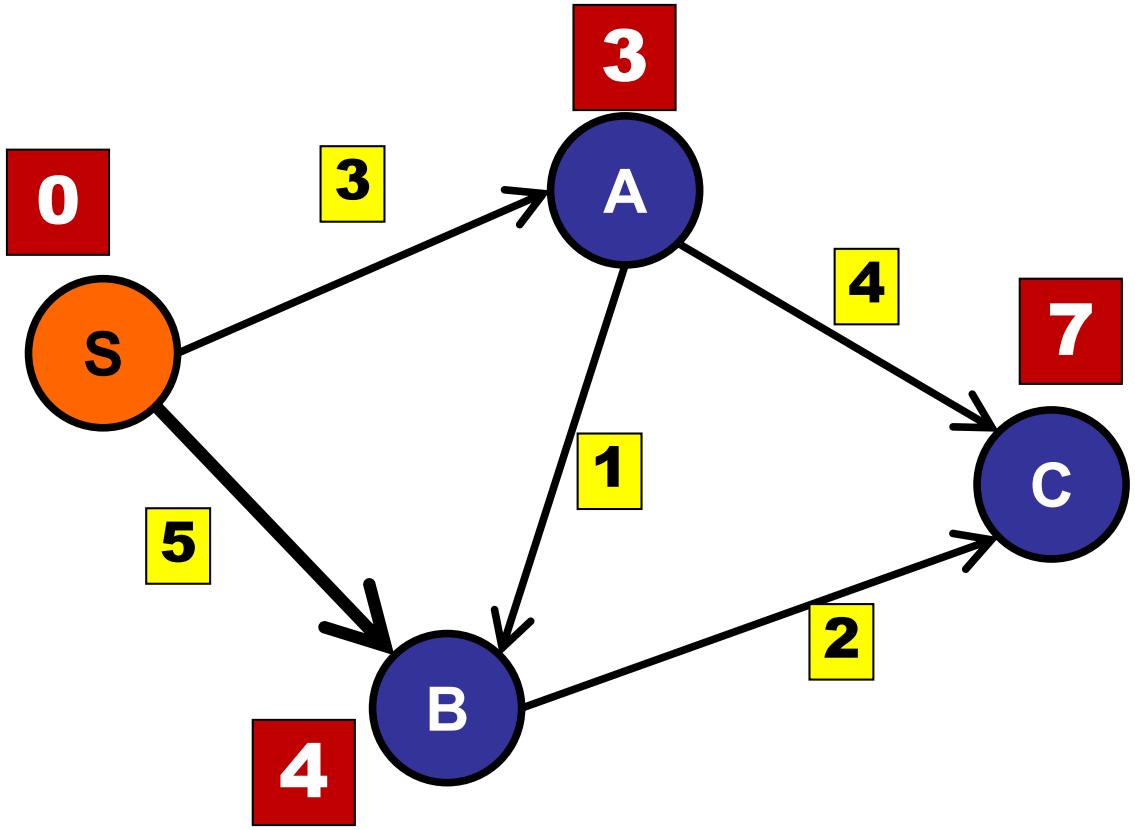
relax(A, B)



Shortest Paths

Maintain estimate for each distance:

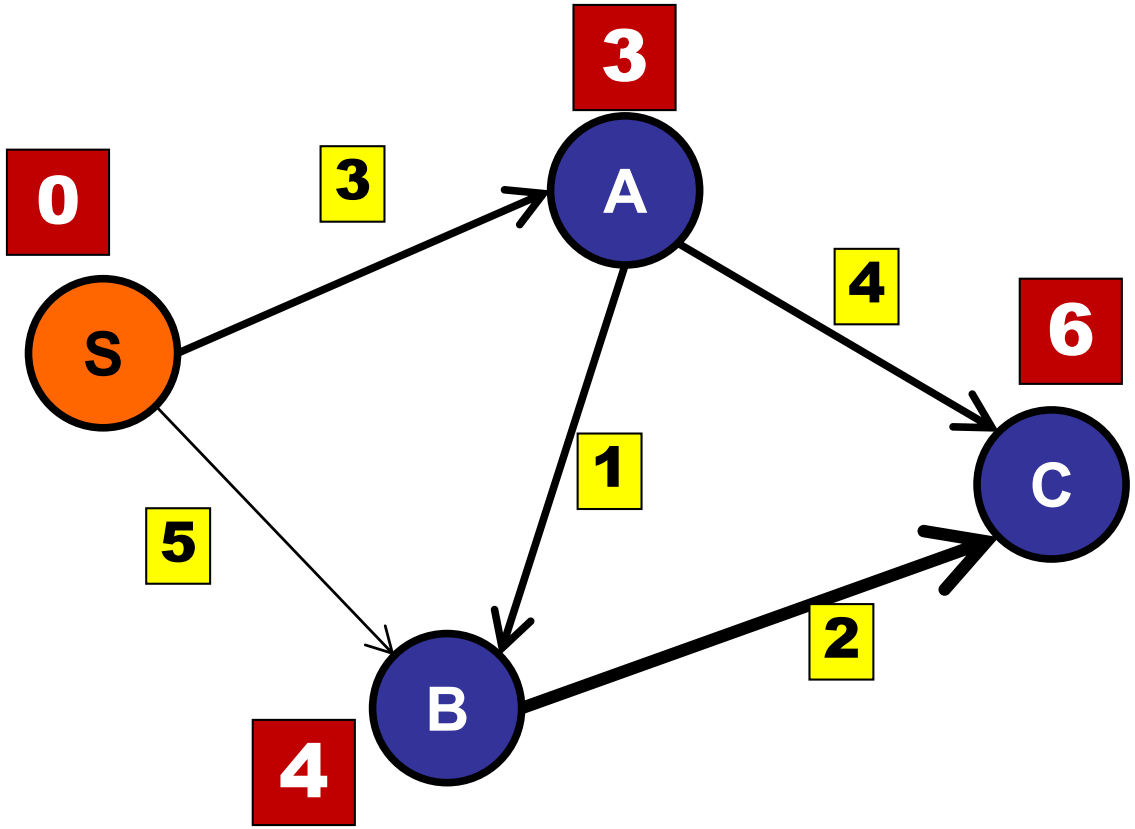
relax(S, B)



Shortest Paths

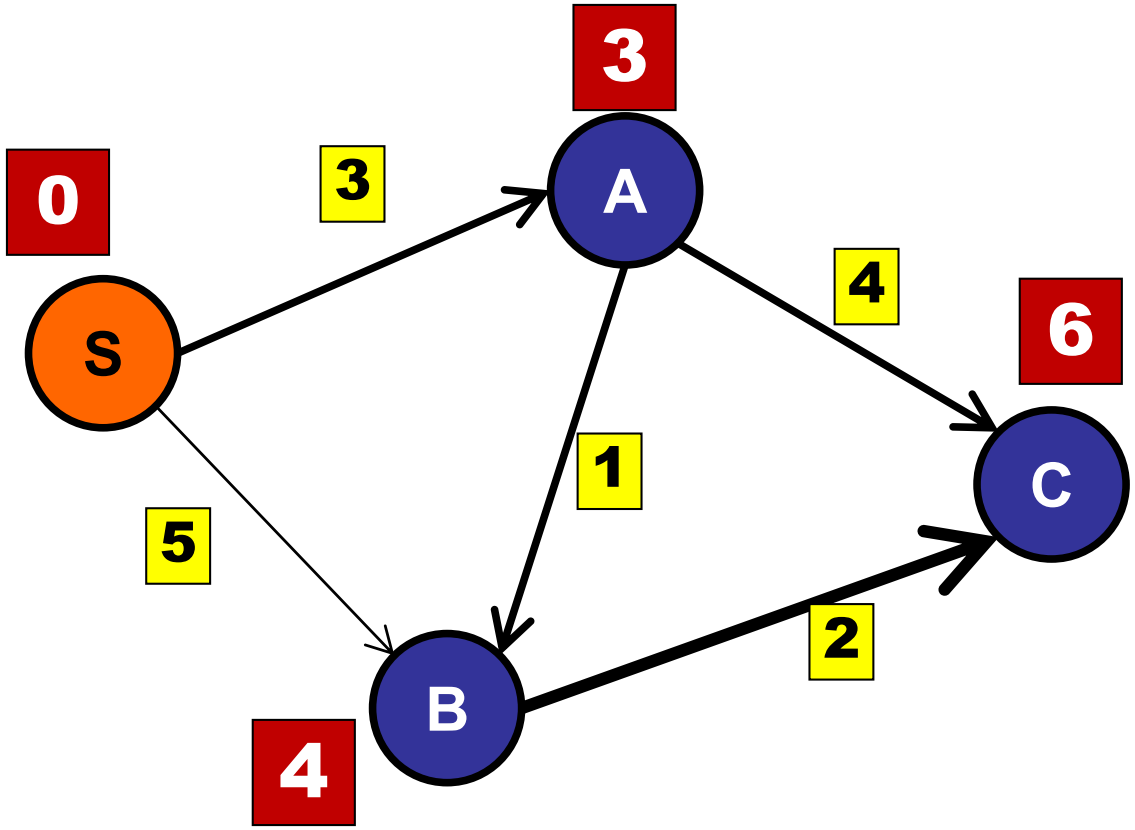
Maintain estimate for each distance:

relax(B, C)



Shortest Paths

```
for (Edge e : graph)
    relax(e)
```

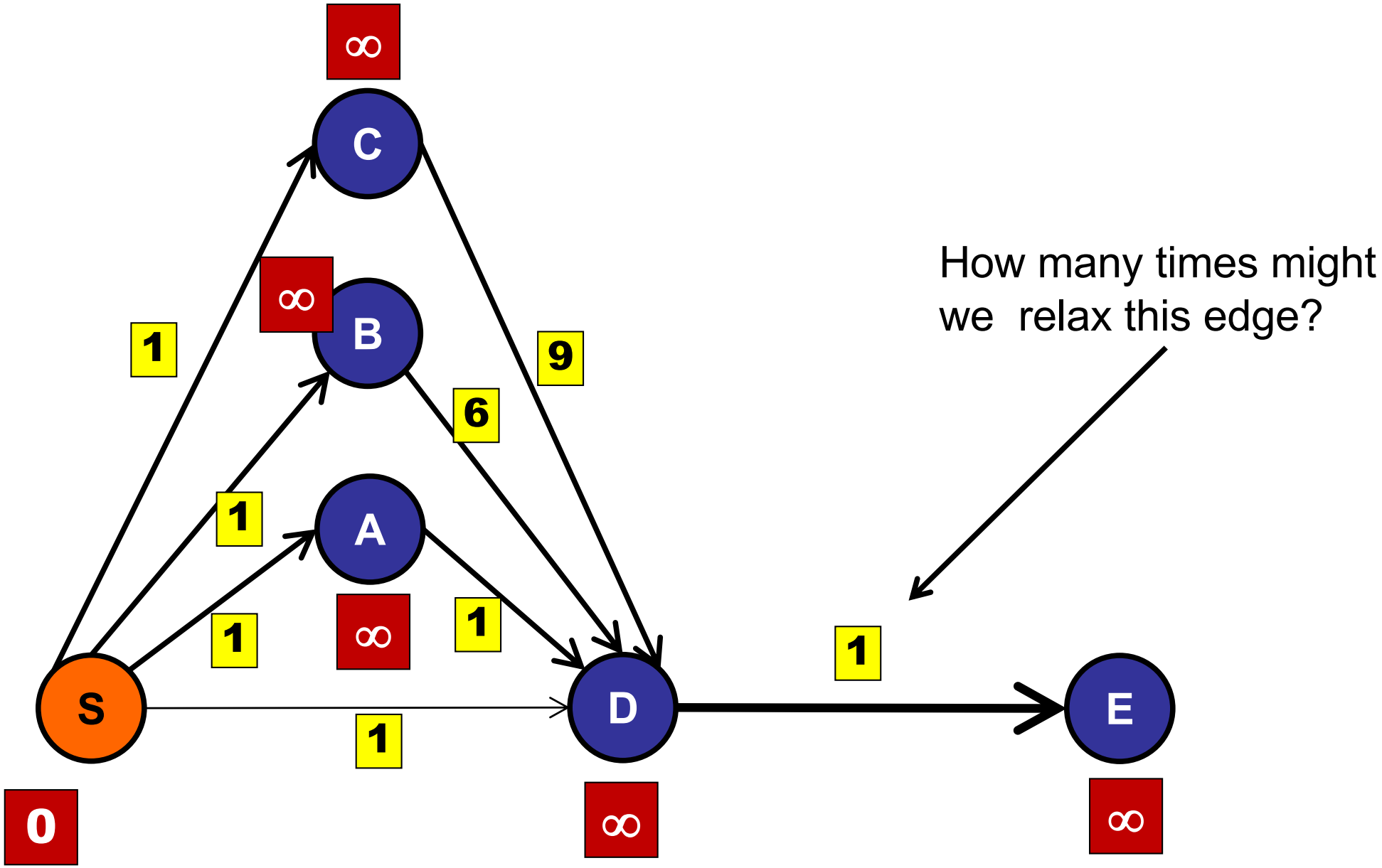


Does this algorithm work:

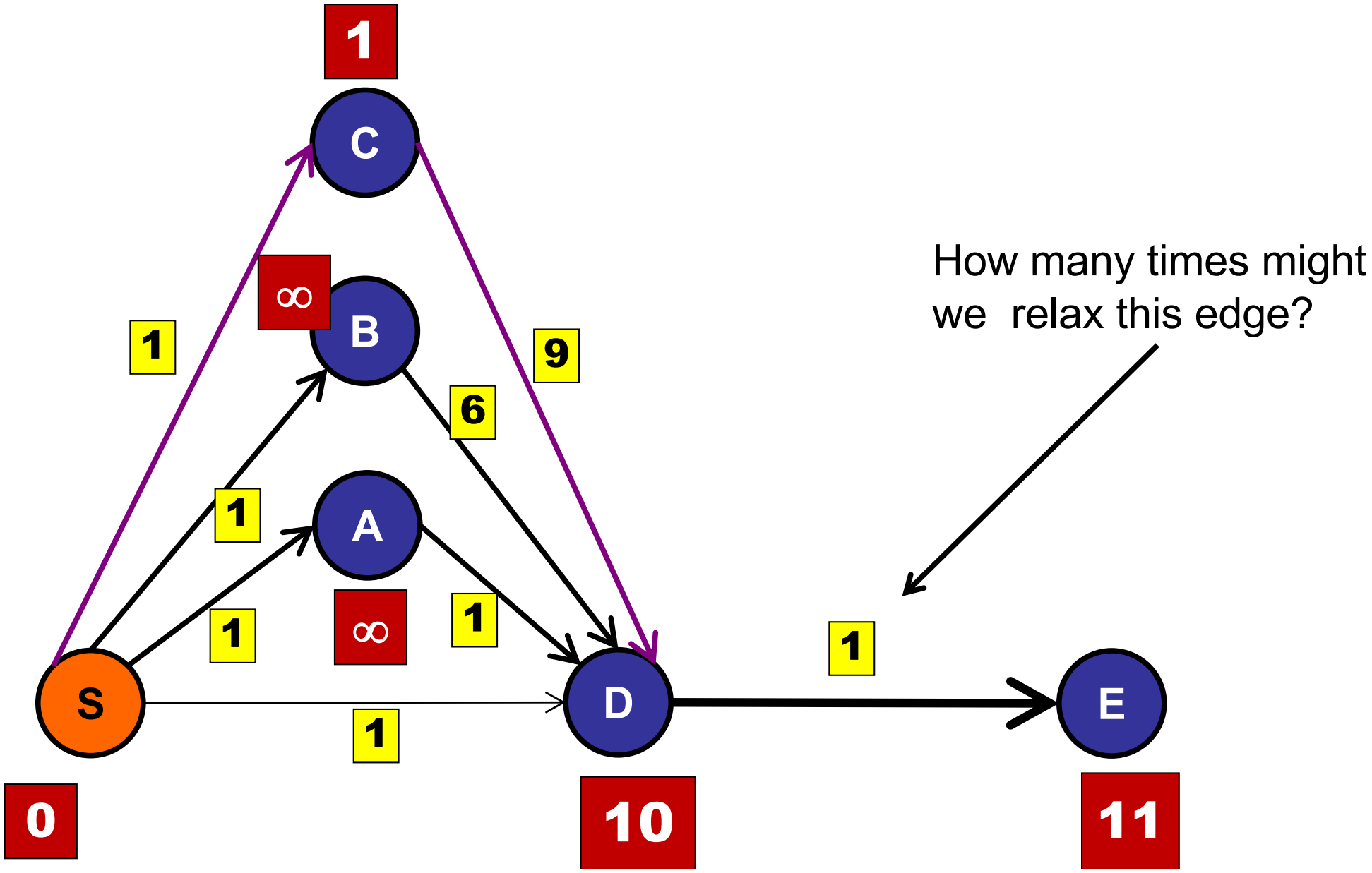
for every edge e: relax(e)

1. Yes
2. Sometimes
3. No

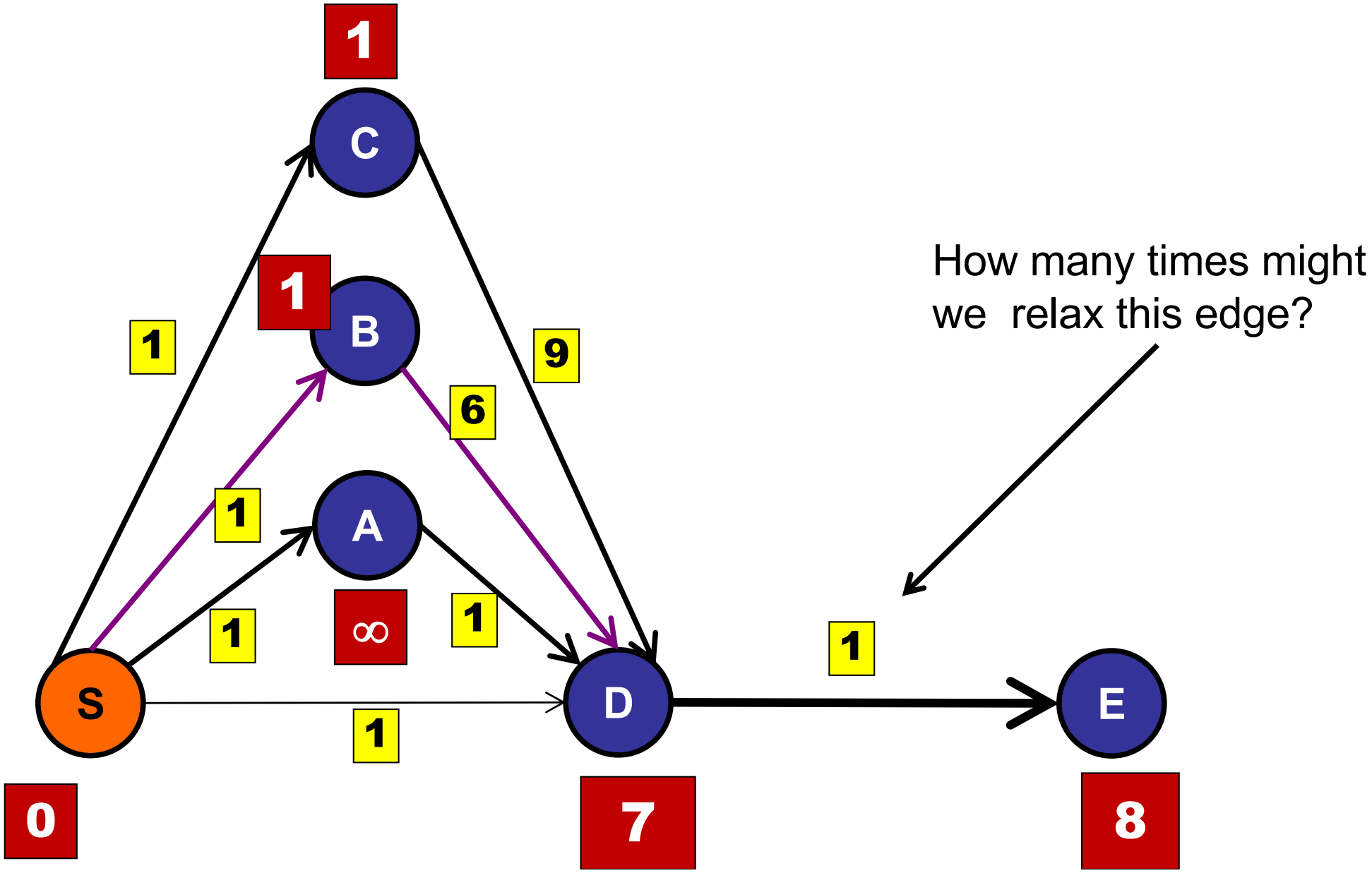
Shortest Paths



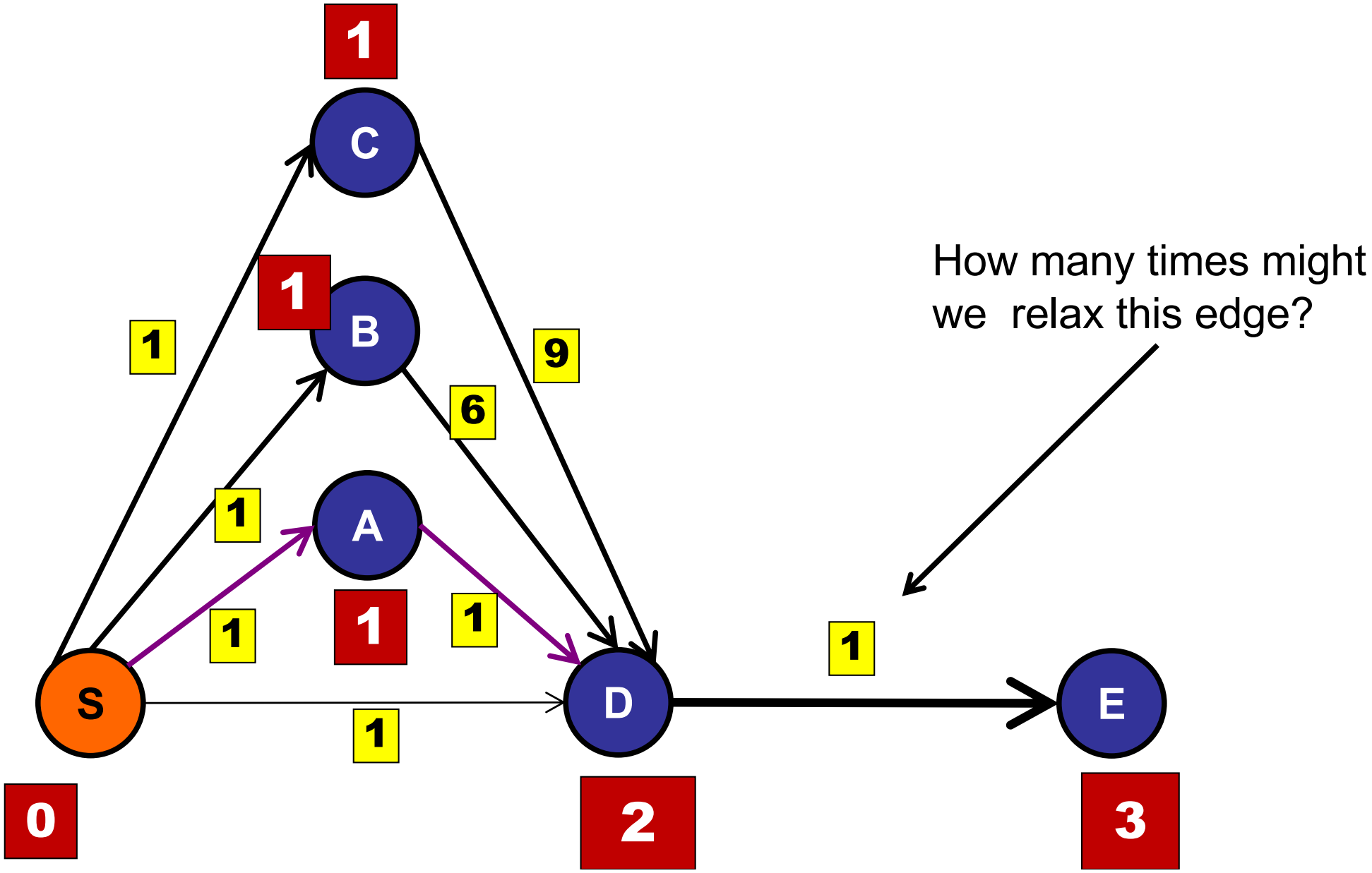
Shortest Paths



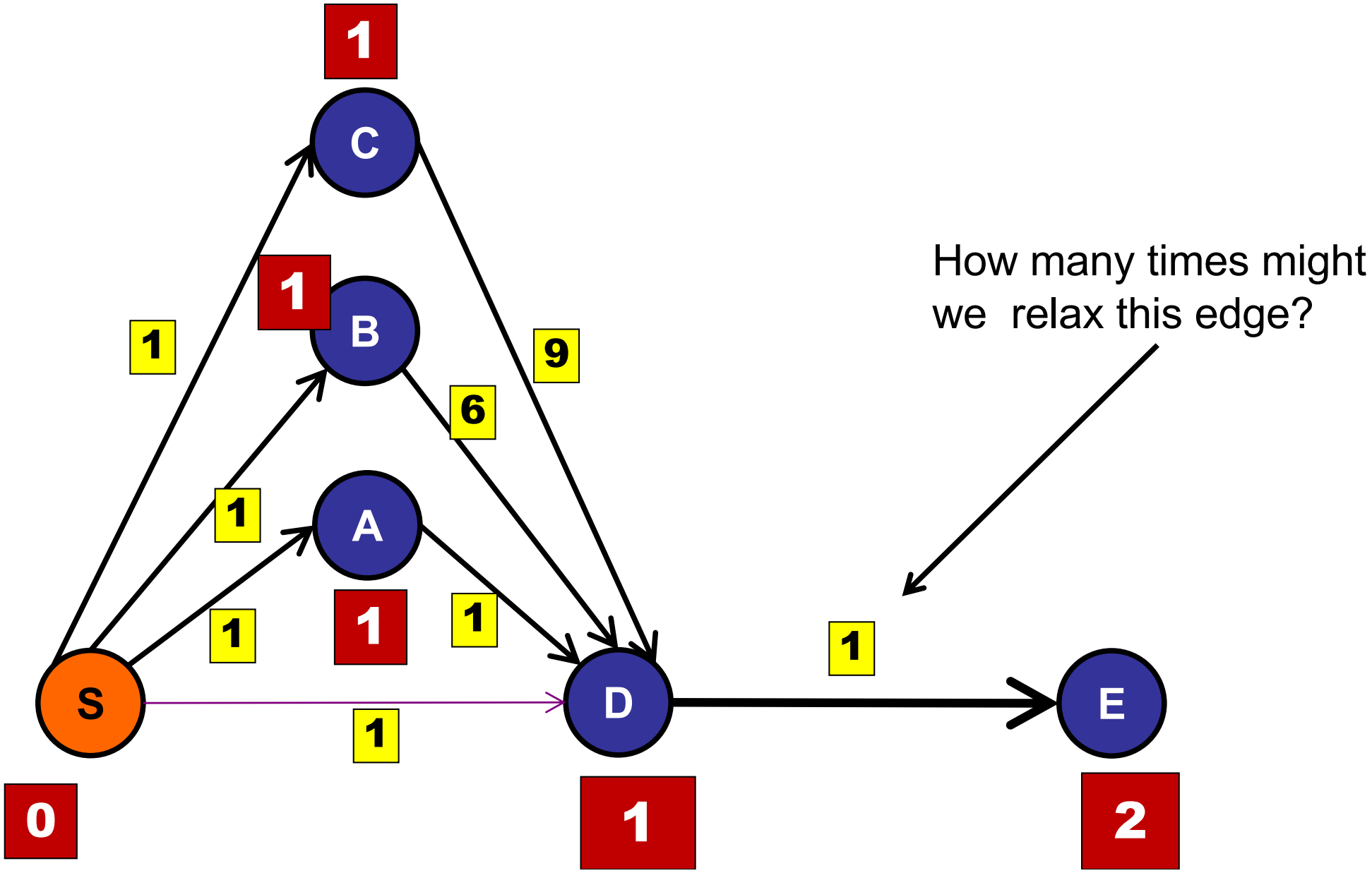
Shortest Paths



Shortest Paths

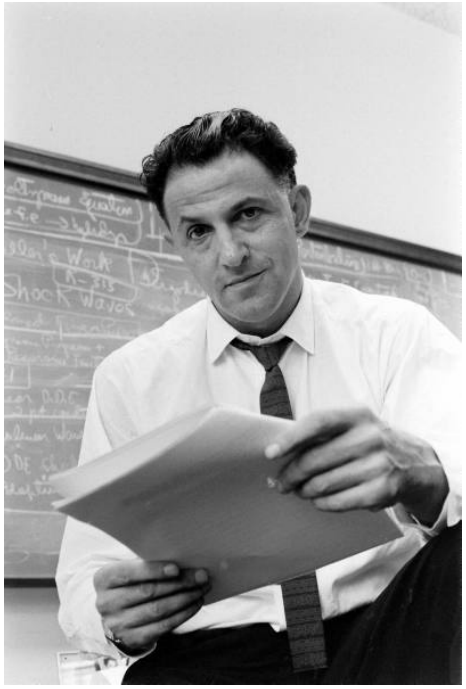


Shortest Paths

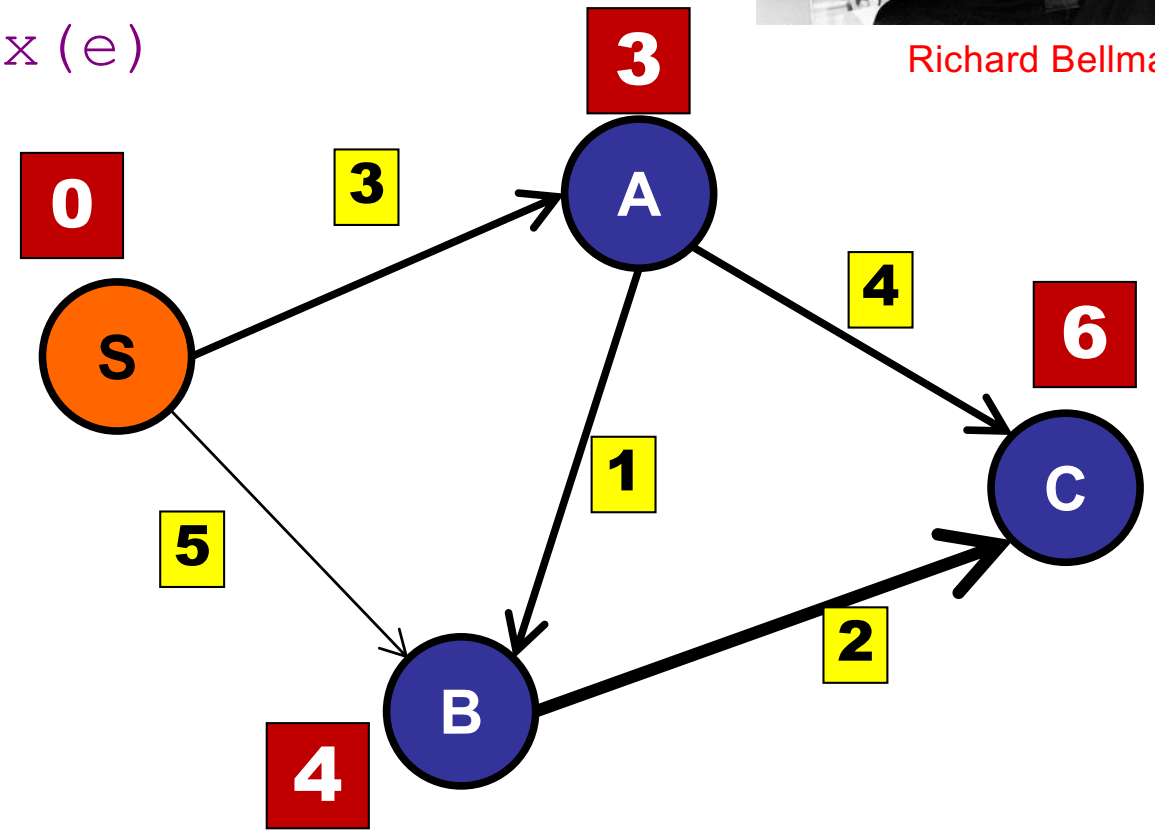


Bellman-Ford

```
n = V.length;

for (i=0; i<n; i++)

    for (Edge e : graph)

                relax(e)
```



Richard Bellman



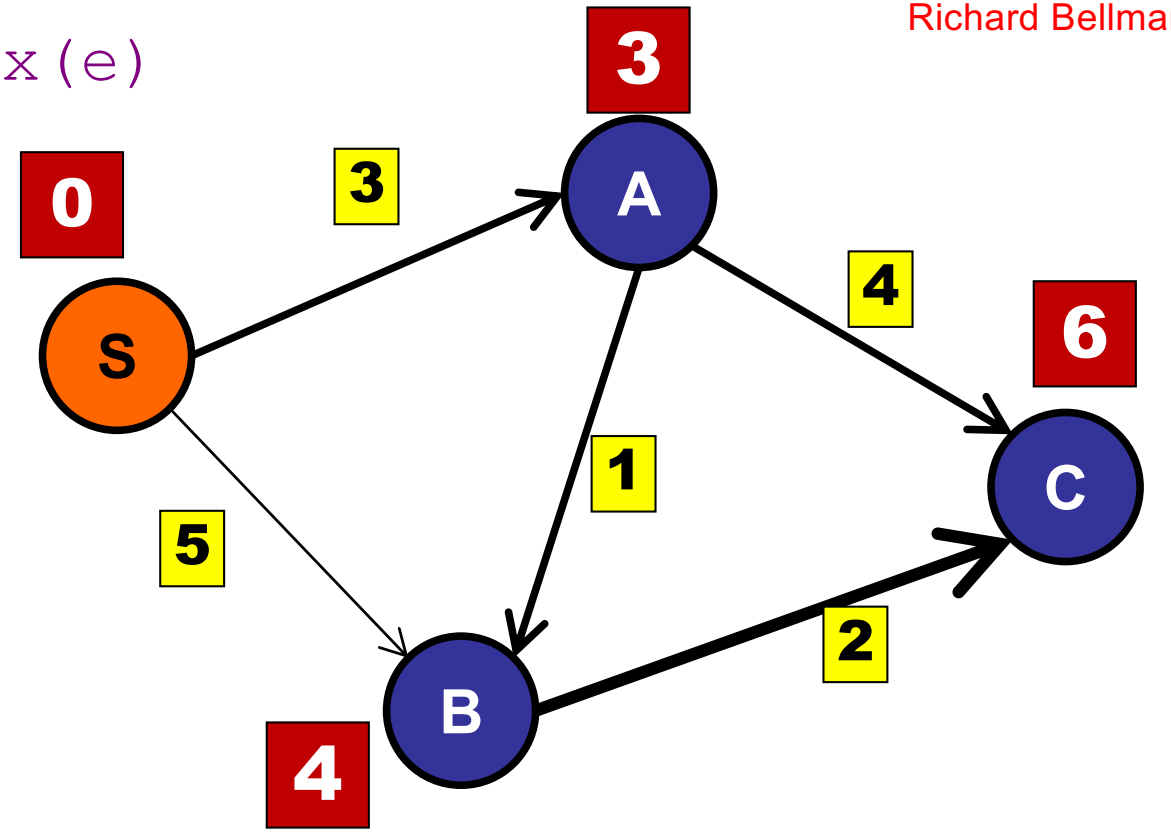
When can you terminate early?

1. When a relax operation has no effect.
2. When two consecutive relax operations have no effect.
3. When an entire sequence of |E| relax operations have no effect.
4. Never. Only after |V| complete iterations.

Bellman-Ford

```
n = V.length;

for (i=0; i<n; i++)

    for (Edge e : graph)

                relax(e)
```

Richard Bellman



What is the running time of Bellman-Ford?

1. O(V)
2. O(E)
3. O(V+E)
4. O(E log V)
5. ✓ O(EV)

Bellman-Ford

```
n = V.length;

for (i=0; i<n; i++)

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```

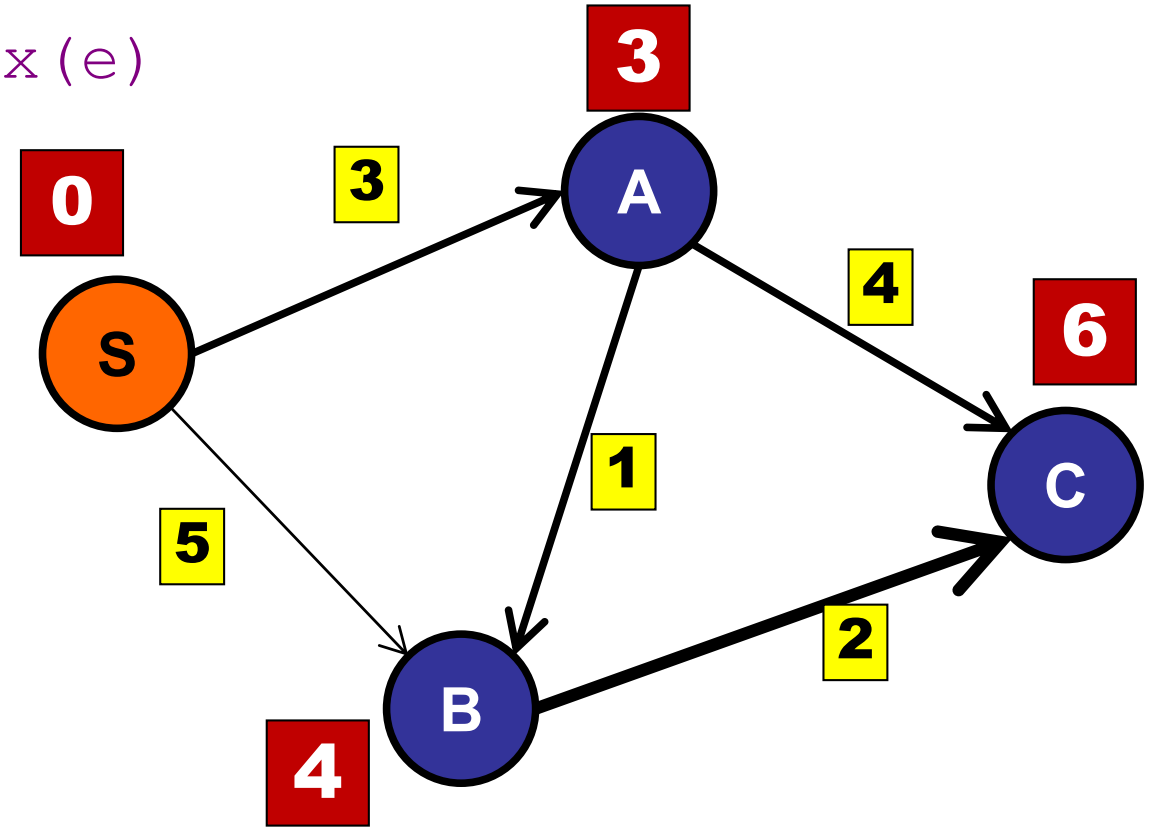

Bellman-Ford

Why does this work?

Bellman-Ford

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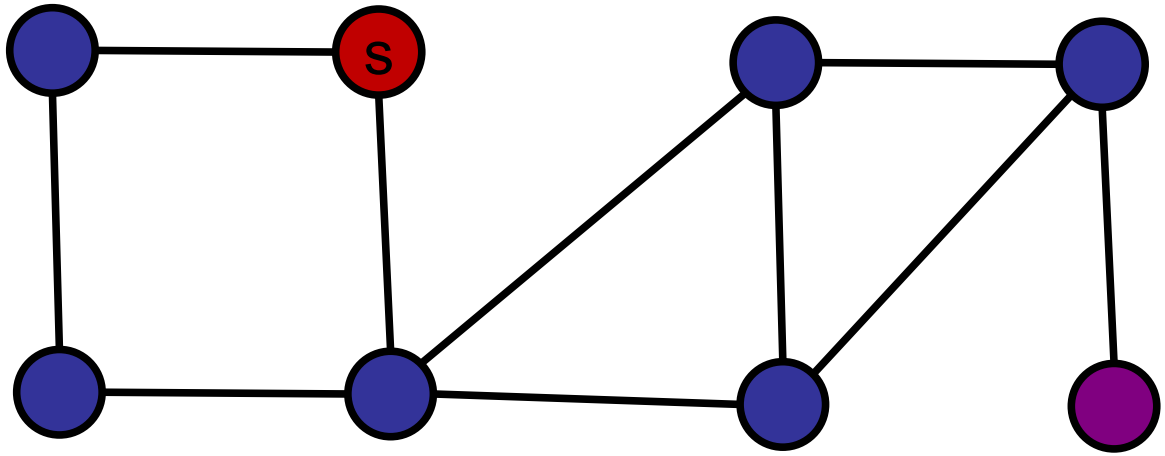

Bellman-Ford

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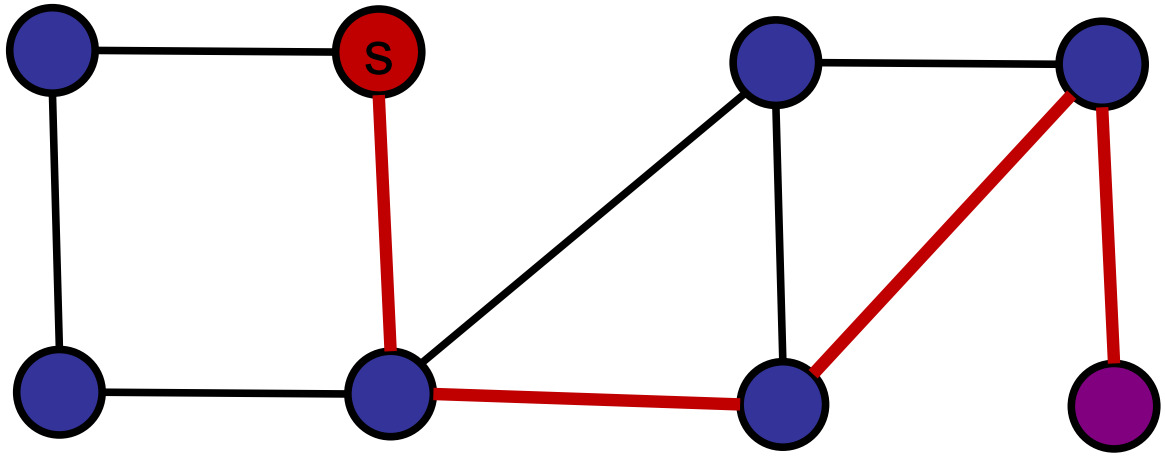
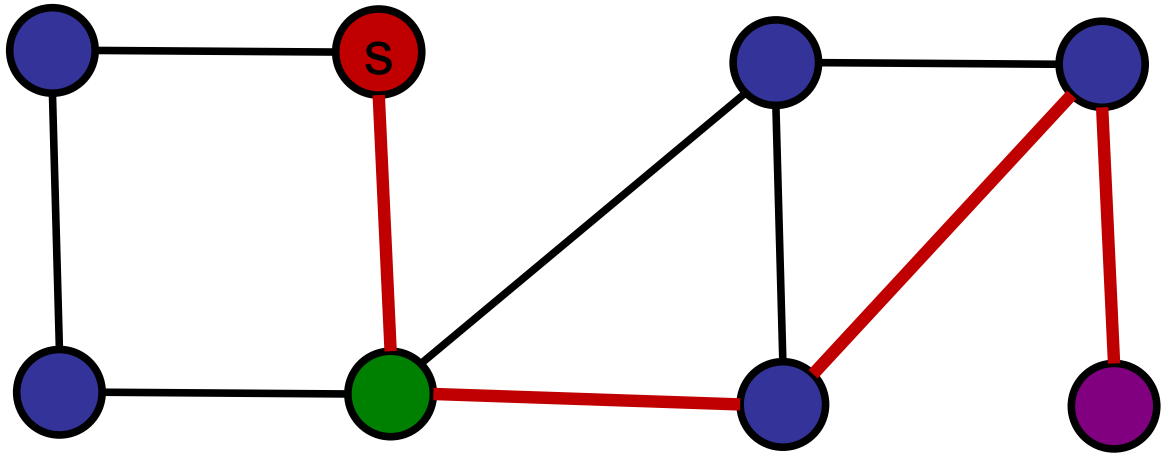


Look at minimum weight path from S to D.

(Path is simple: no loops.)

Bellman-Ford

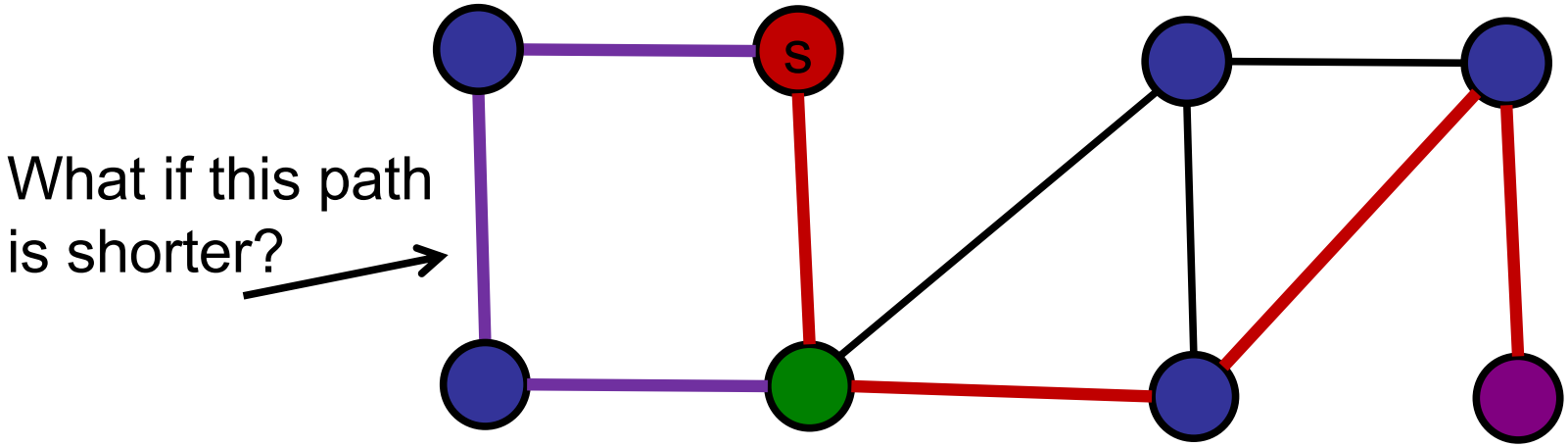
Why does this work?



After 1 iteration, 1 hop estimate is correct.

Bellman-Ford

Why does this work?



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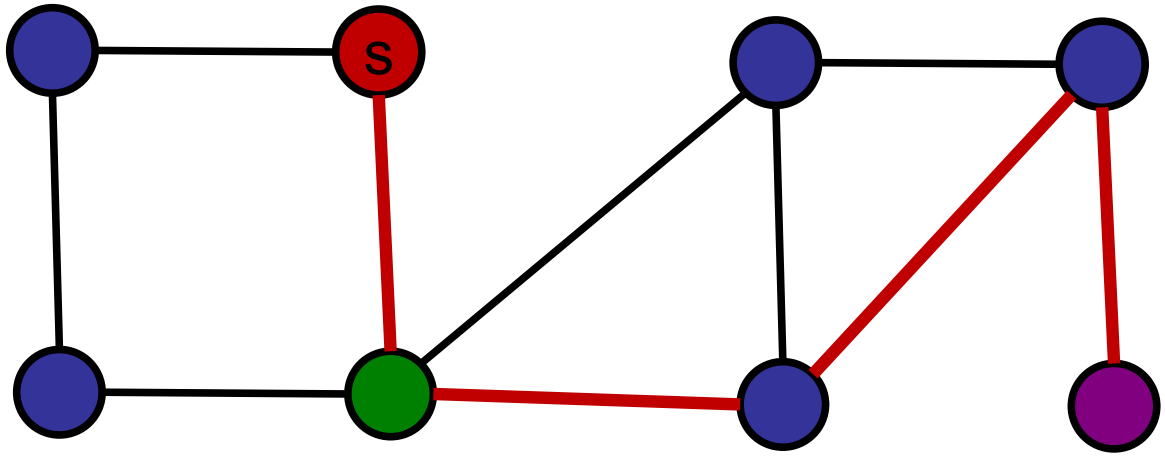

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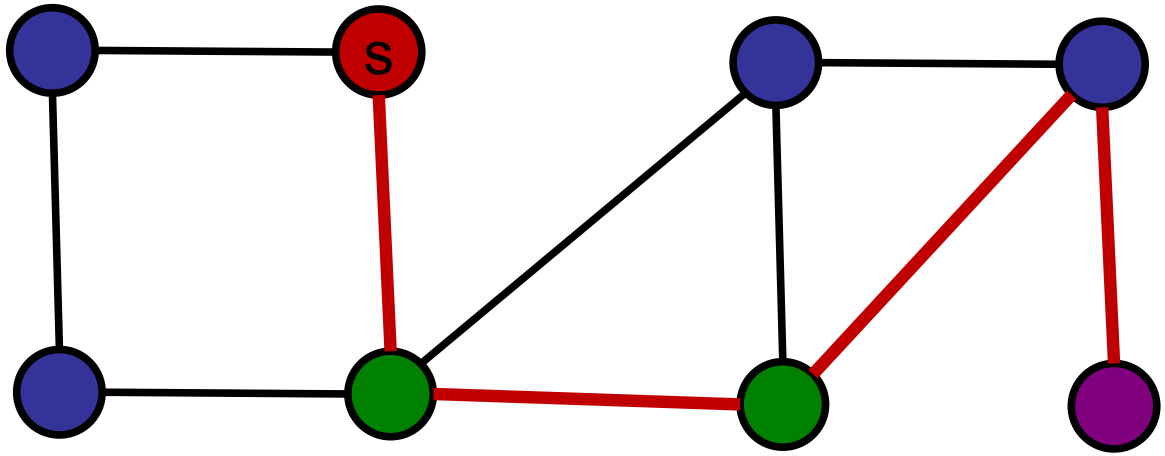
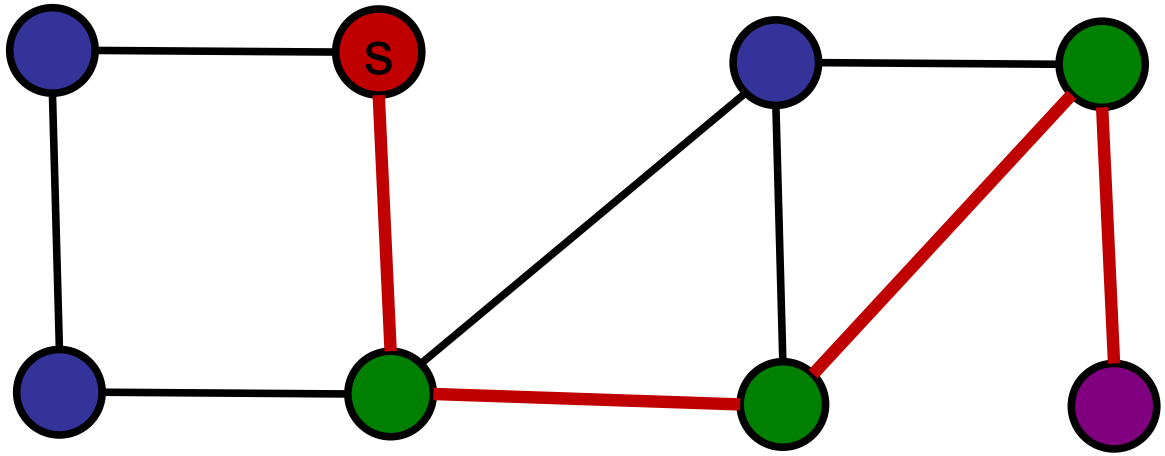


After 2 iterations, 2 hop estimate is correct.

Bellman-Ford

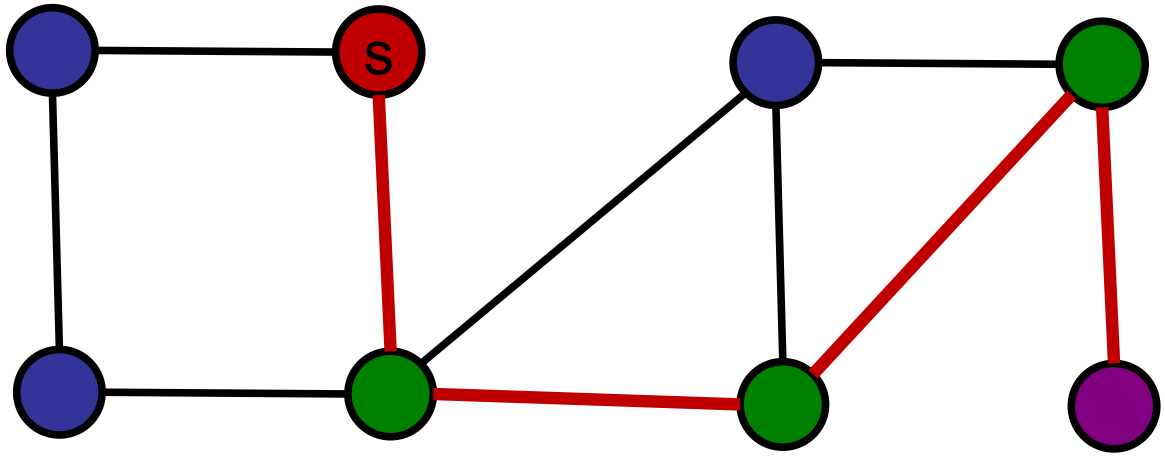
Why does this work?



After 3 iterations, 3 hop estimate is correct.

Bellman-Ford

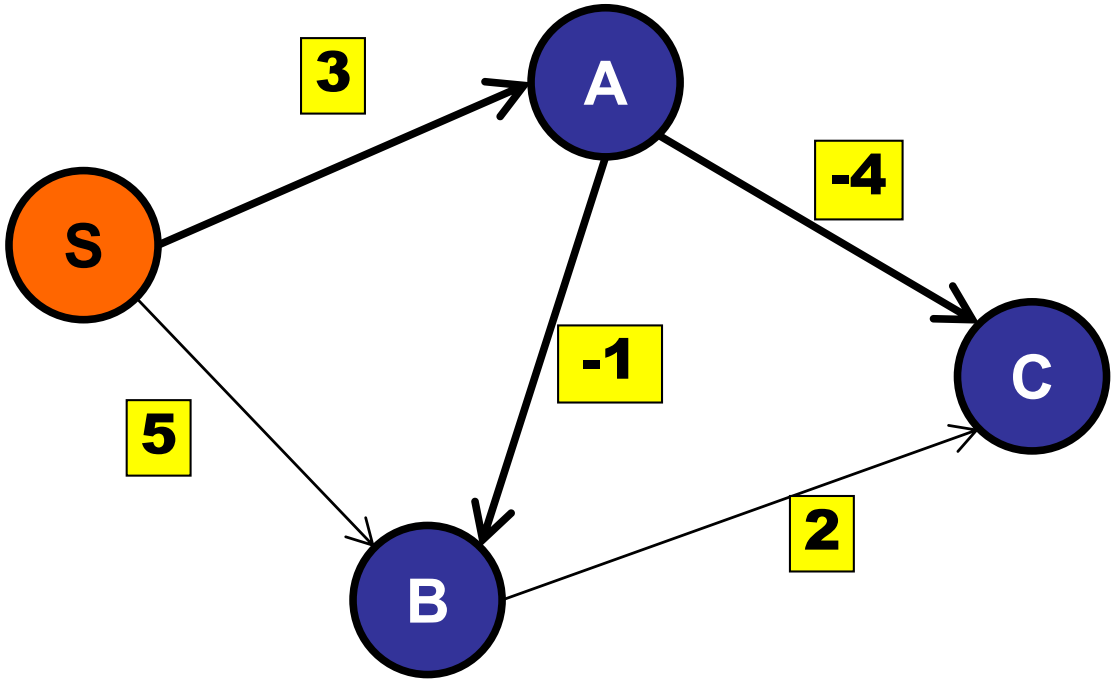
Why does this work?



After 4 iterations, D estimate is correct.

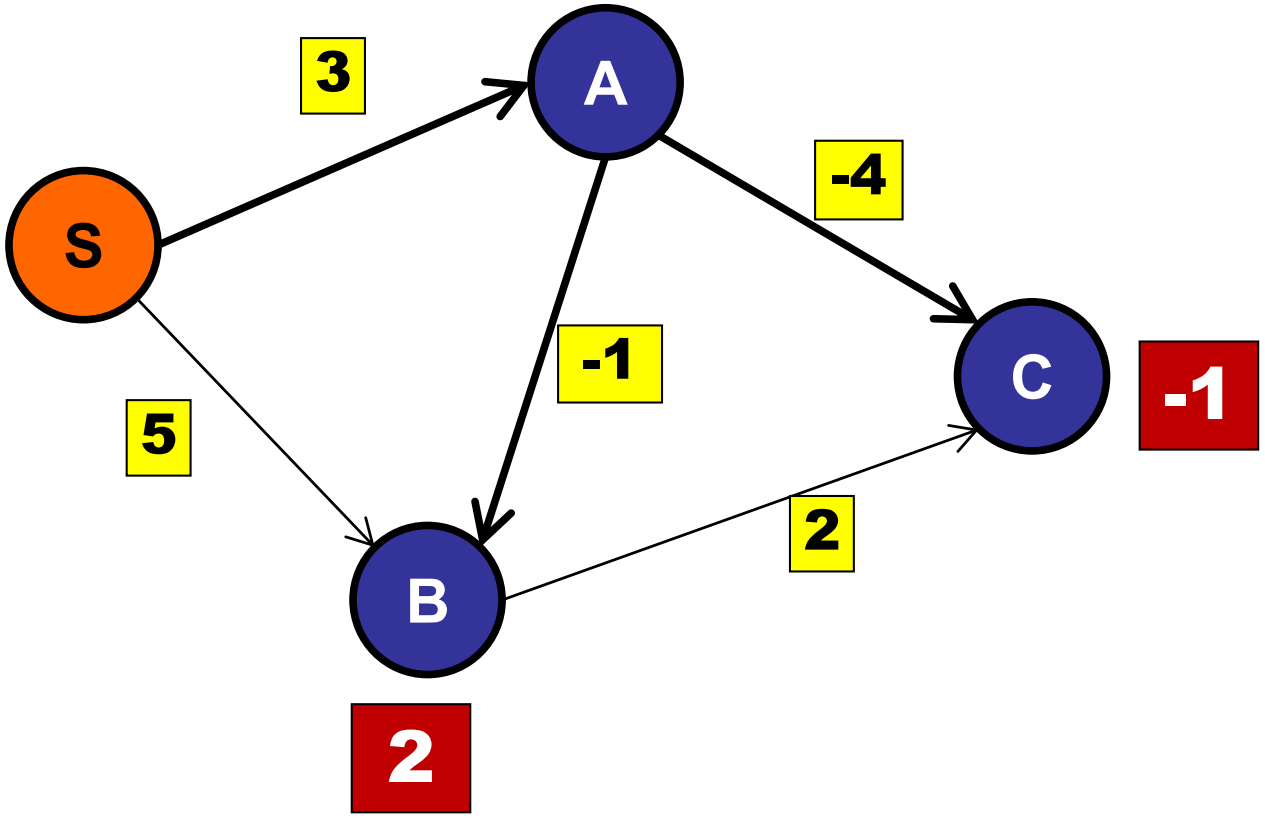
Bellman-Ford

What if edges have negative weight?



Bellman-Ford

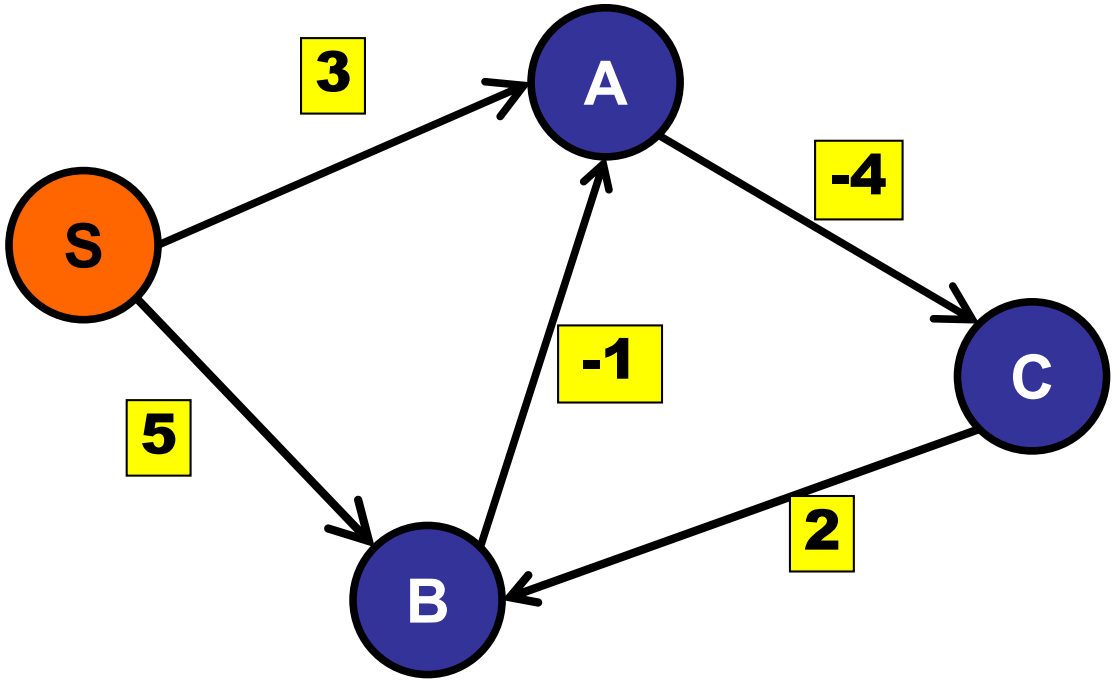
What if edges have negative weight?



No problem!

Bellman-Ford

What if edges have negative weight?



Bellman-Ford

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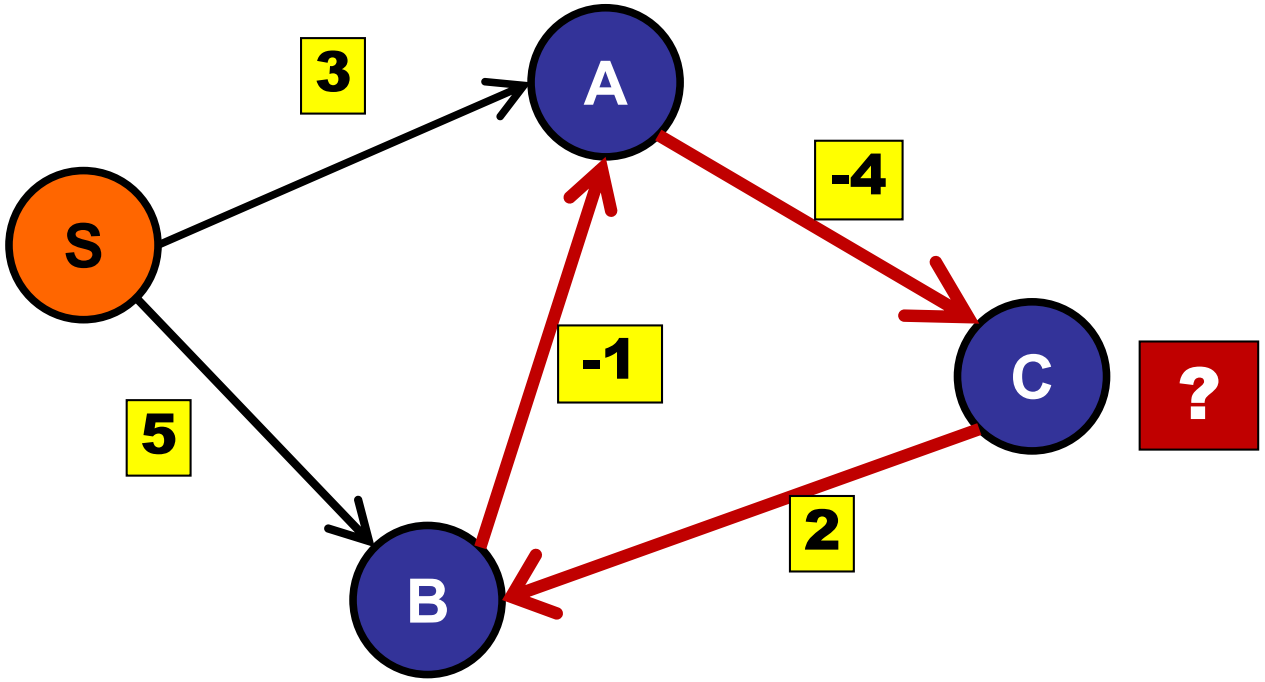
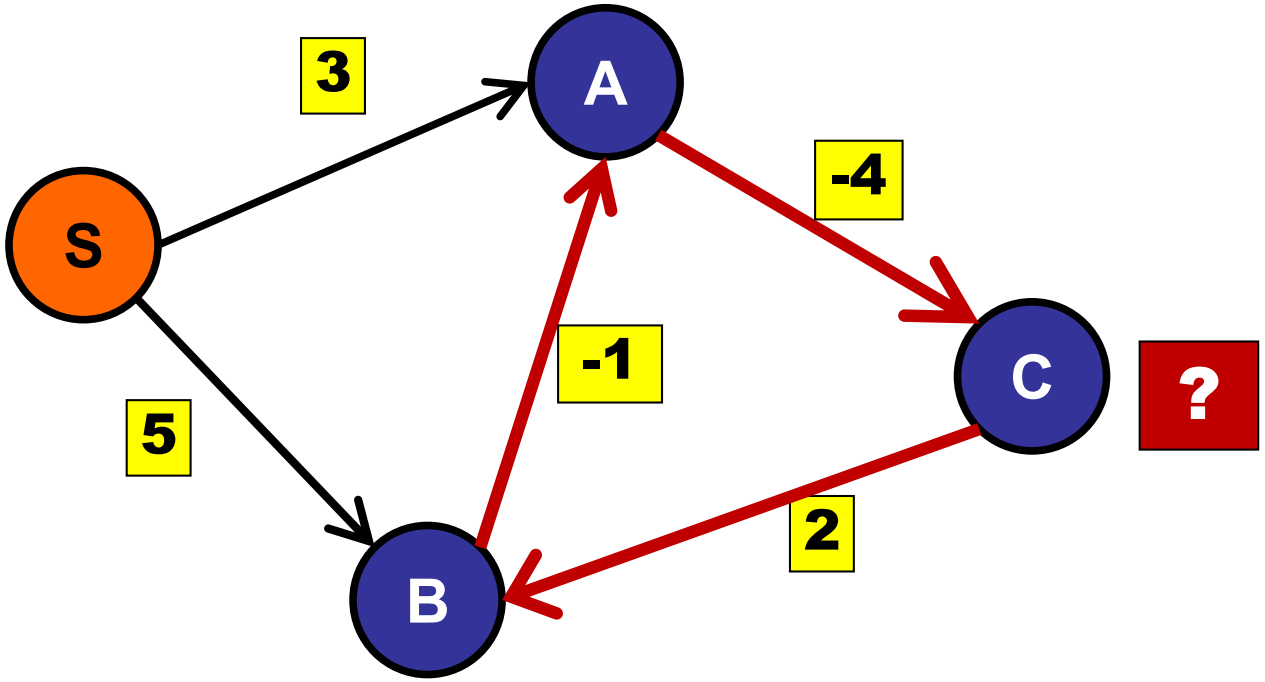


d(S,C) is infinitely negative!

Negative weight cycles

How to detect negative weight cycles?



Negative weight cycles

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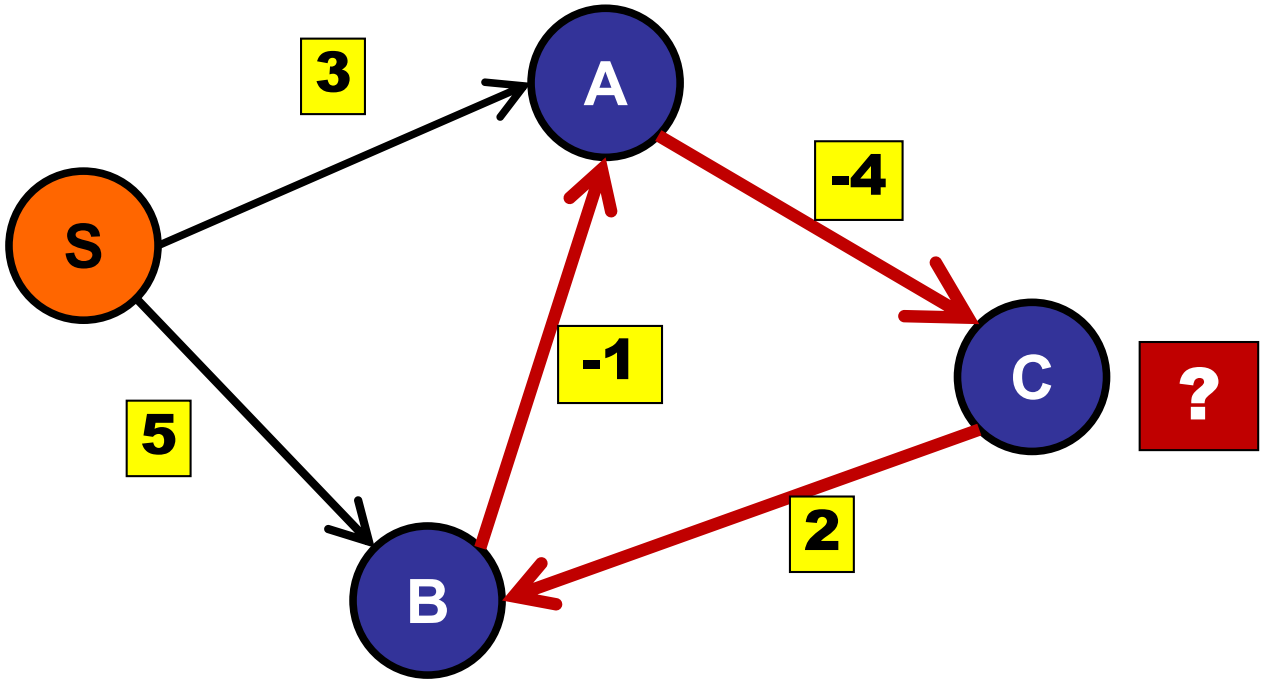
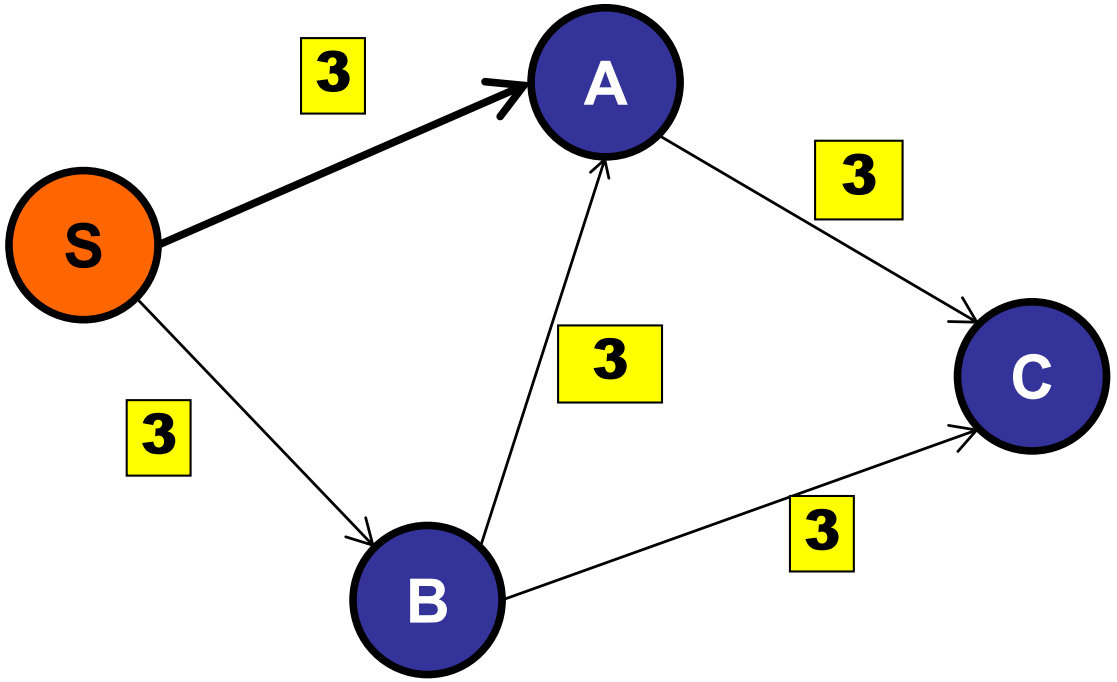


Run Bellman-Ford for |V|+1 iterations.

If an estimate changes in the last iteration…
then negative weight cycle.

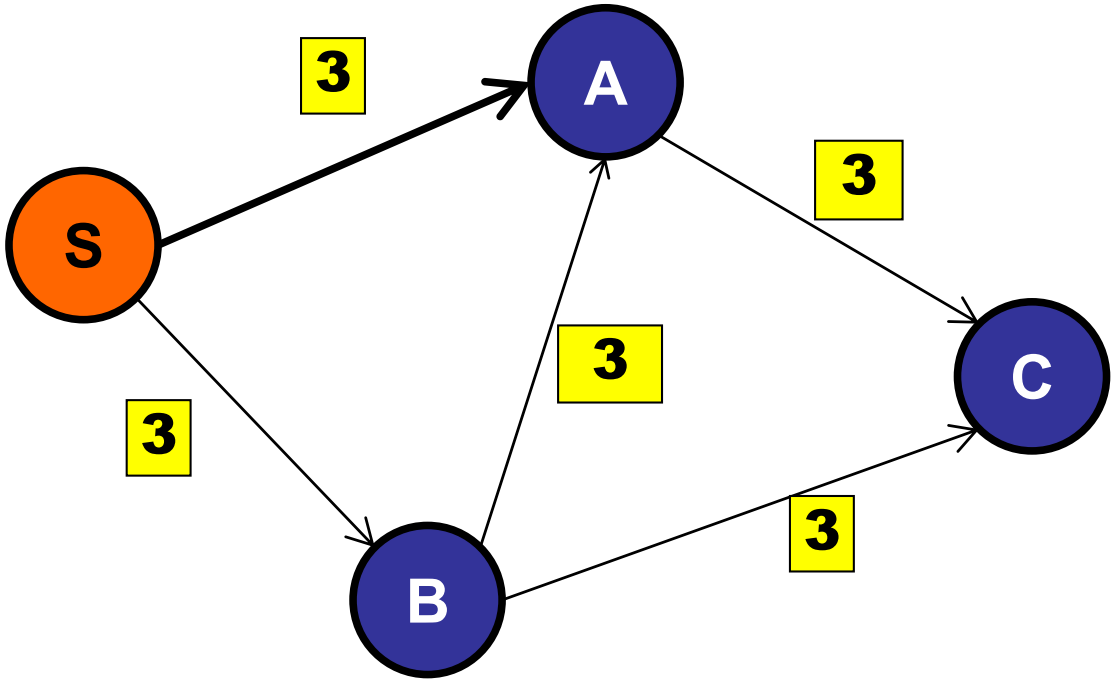
Bellman-Ford

Special case: all edges have the same weight



Bellman-Ford

Special case: all edges have the same weight.



Use regular Breadth-First Search.

Bellman-Ford Summary

Basic idea:

- Repeat |V| times: relax every edge
- Stop when "converges".
- O(VE) time.

Special issues:

- If negative weight-cycle: impossible.
- Use Bellman-Ford to detect negative weight cycle.
- If all weights are the same, use BFS.