CS2040S
Data Structures and Algorithms
(e-learning edition)

Welcome!

Plan of the Day

Trees

- Terminology
- Traversals
- Operations

Balanced Trees

- Height-balanced binary search trees
- AVL trees
- Rotations

Part 2

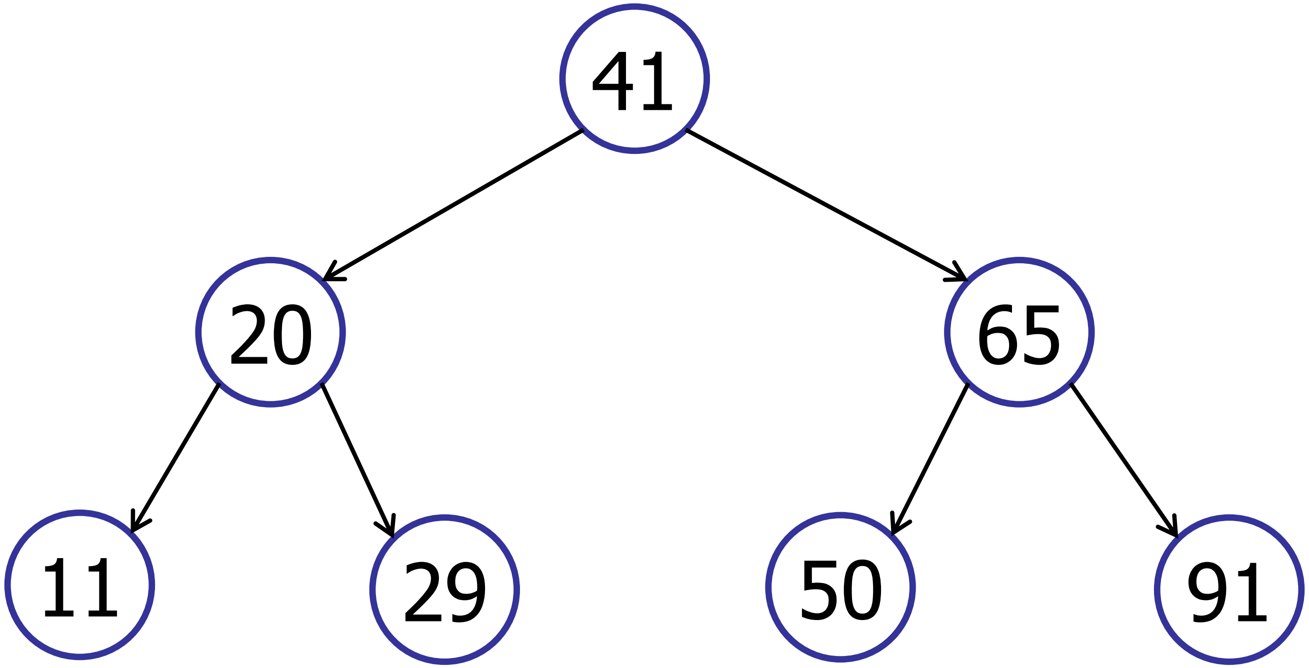
On the importance of being balanced

Part 2

On the importance of being balanced

- Height-balanced binary search trees
- AVL trees
- Rotations

Recap: Binary Search Trees



- Two children: v.left, v.right
- Key: v.key
- **BST Property**: all in left sub-tree < key < all in right sub-right

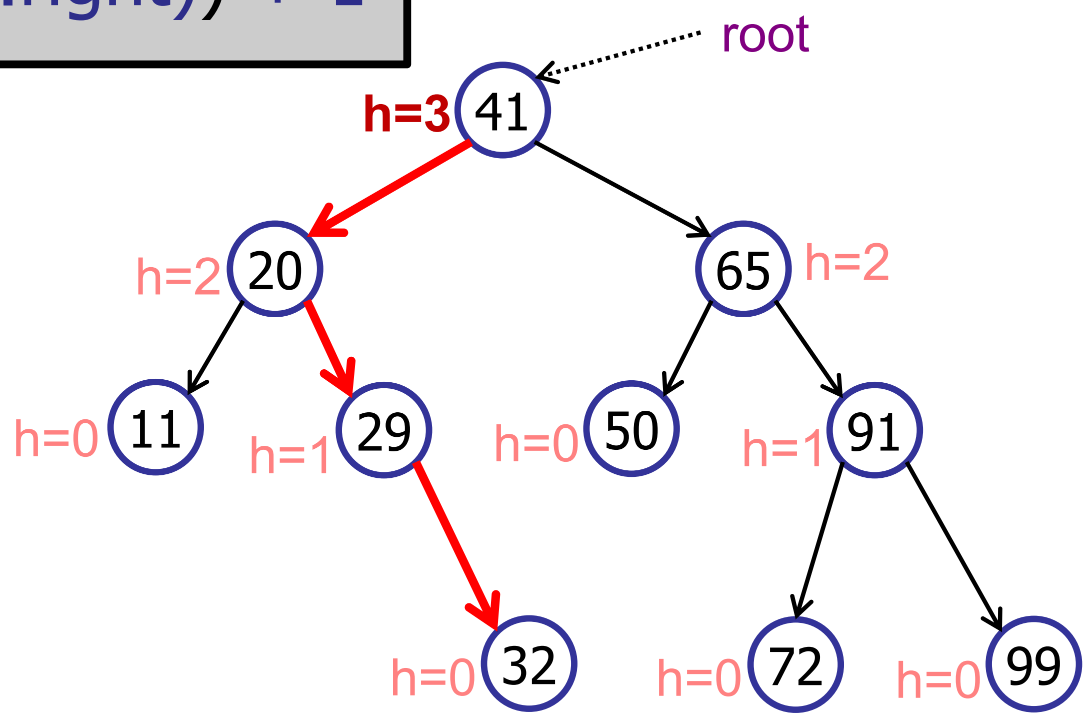
Binary Search Trees Heights

Height:

Number of edges on longest path from root to leaf.

$h(v) = 0$ (if v is a leaf)

$h(v) = \max(h(v.left), h(v.right)) + 1$



(For simplicity: $h(null) = -1$)

Binary Search Tree

Modifying Operations

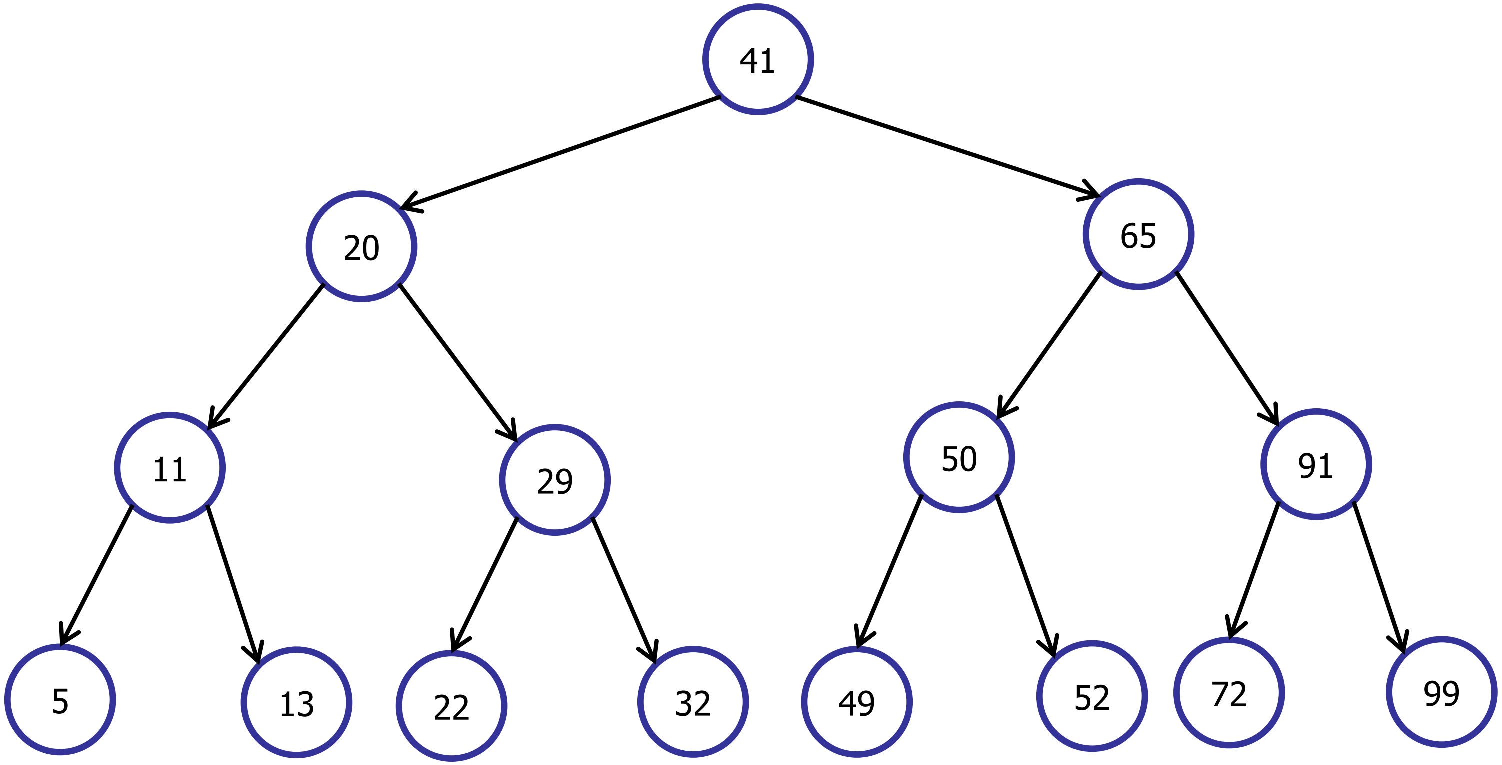
- insert: O(h)
- delete: O(h)

Query Operations:

- search: O(h)
- predecessor, successor: O(h)
- findMax, findMin: O(h)
- in-order-traversal: O(n)

The Importance of Being Balanced

Operations take O(h) time



What is the largest possible height h?

1. $\theta(1)$
2. $\theta(\log n)$
3. $\theta(\mathrm{sqrt}(n))$
4. $\theta(n)$
5. $\theta(n^2)$

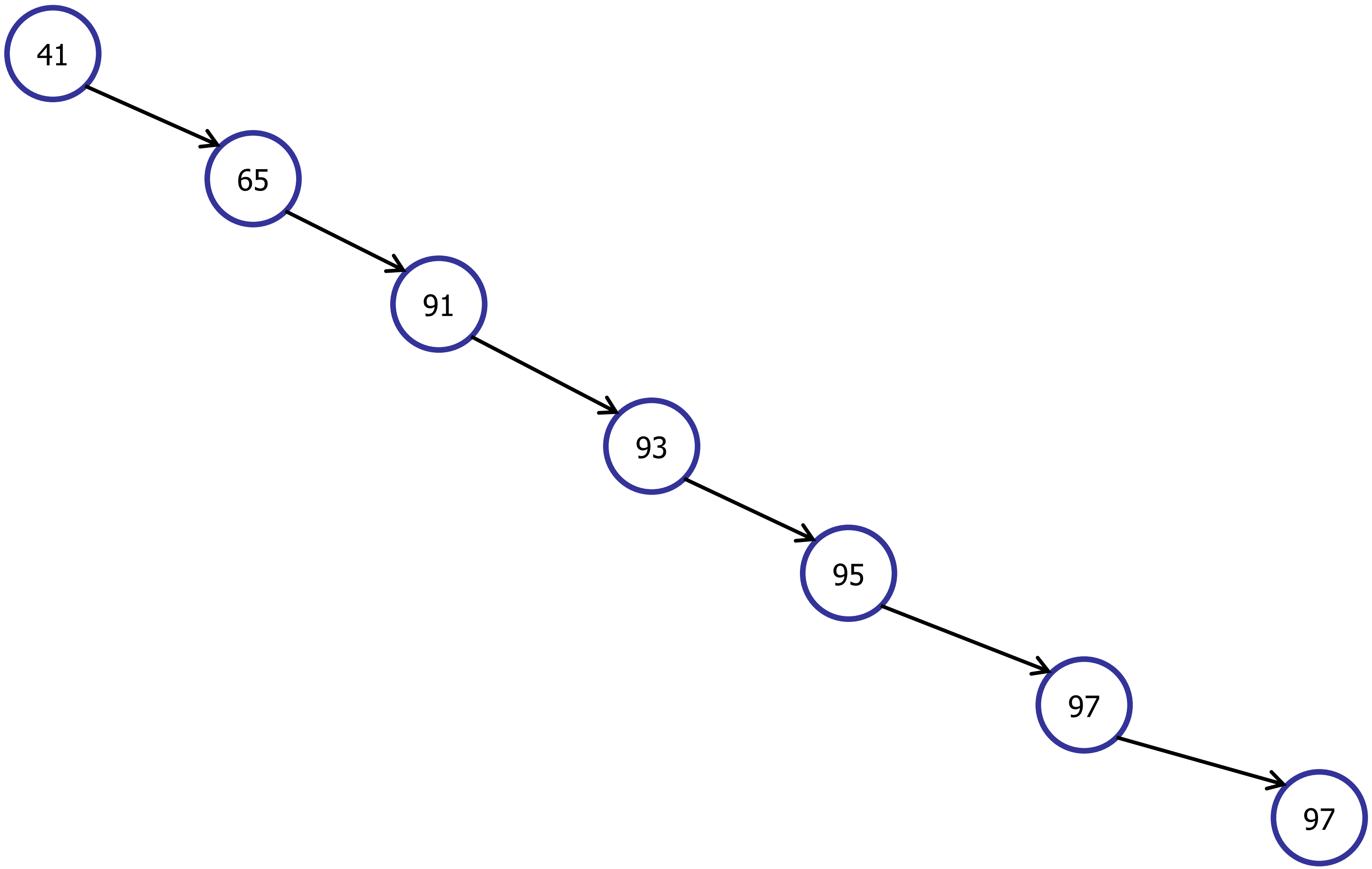
What is the largest possible height h?

1. $\theta(1)$
2. $\theta(\log n)$
3. $\theta(\mathrm{sqrt}(n))$
4. $\theta(n)$
5. $\theta(n^2)$

The Importance of Being Balanced

Operations take O(h) time

- $h \leq n$



What is the smallest possible height h?

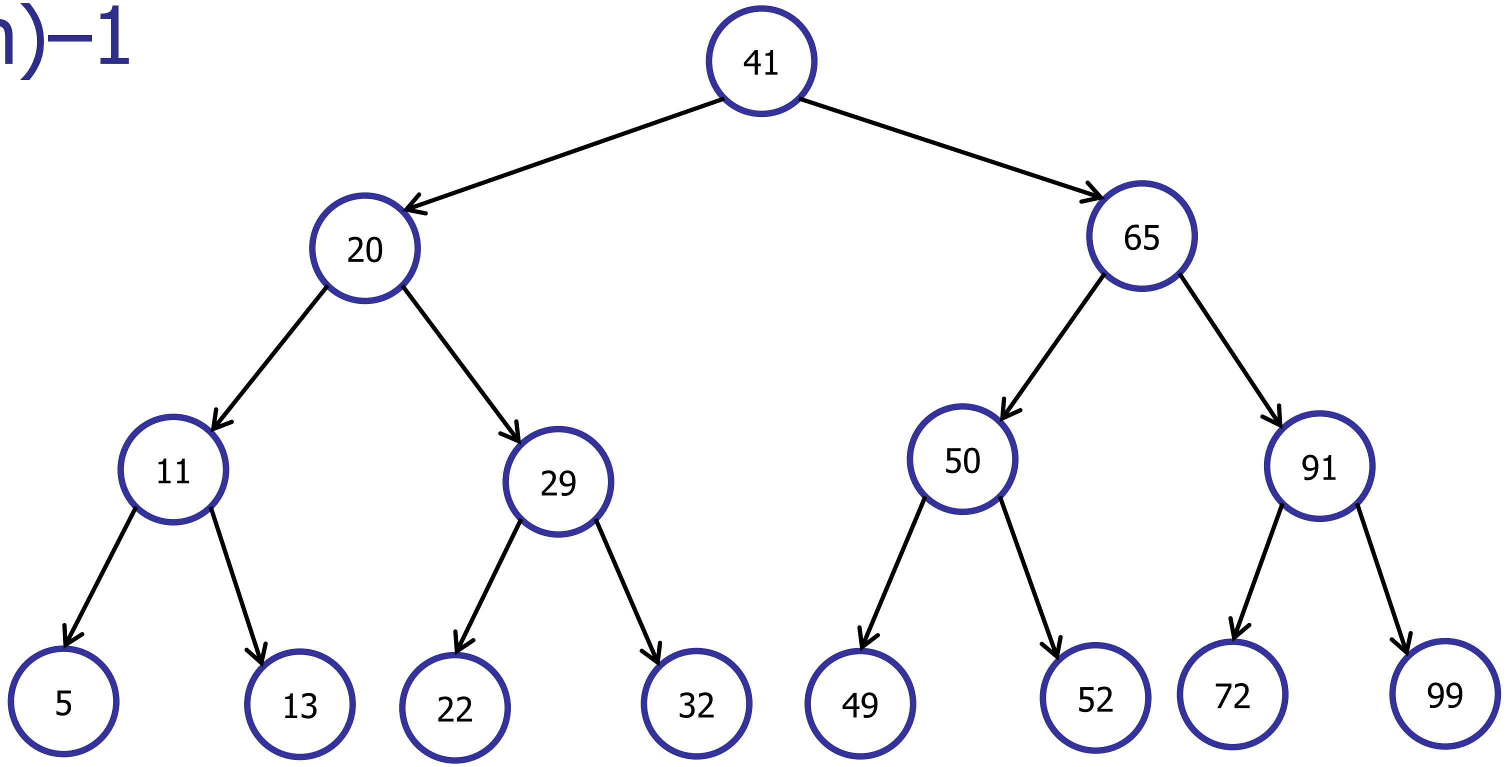
1. $\theta(1)$
2. $\theta(\log n)$
3. $\theta(\mathrm{sqrt}(n))$
4. $\theta(n)$
5. $\theta(n^2)$

What is the smallest possible height h?

1. $\theta(1)$
2. $\theta(\log n)$
3. $\theta(\mathrm{sqrt}(n))$
4. $\theta(n)$
5. $\theta(n^2)$

The Importance of Being Balanced

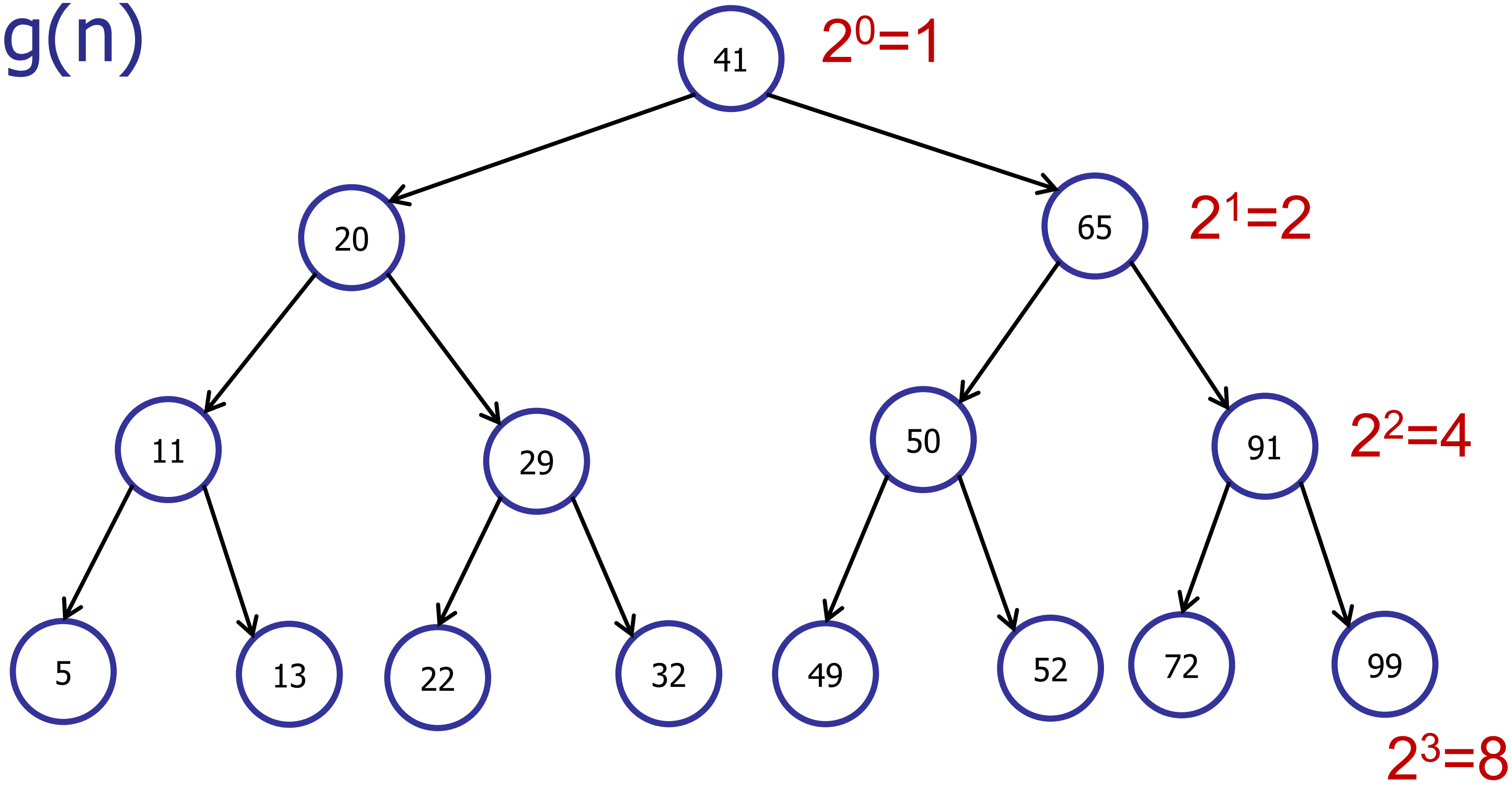
- Operations take O(h) time
  - $h \geq \log(n) - 1$



The Importance of Being Balanced

Operations take O(h) time

$h+1 \geq \log(n)$



$$n \leq 1 + 2 + 4 + \ldots + 2^h$$

$$\leq 2^0 + 2^1 + 2^2 + \ldots + 2^h < 2^{h+1}$$

The Importance of Being Balanced

Operations take O(h) time

$\log(n) - 1 \leq h \leq n$

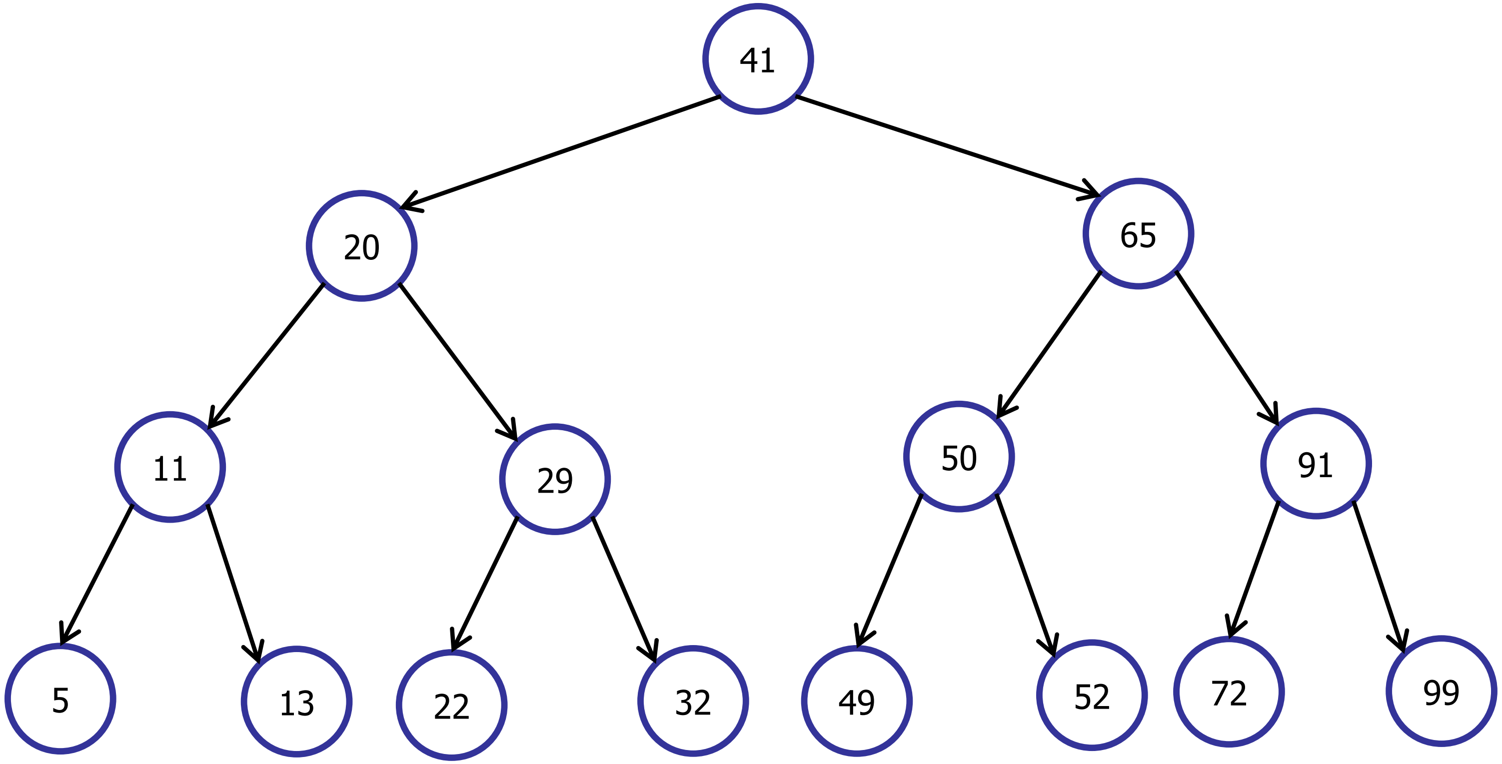
Key definition

A BST is balanced if $h = O(\log n)$

On a balanced BST: all operations run in $O(\log n)$ time.

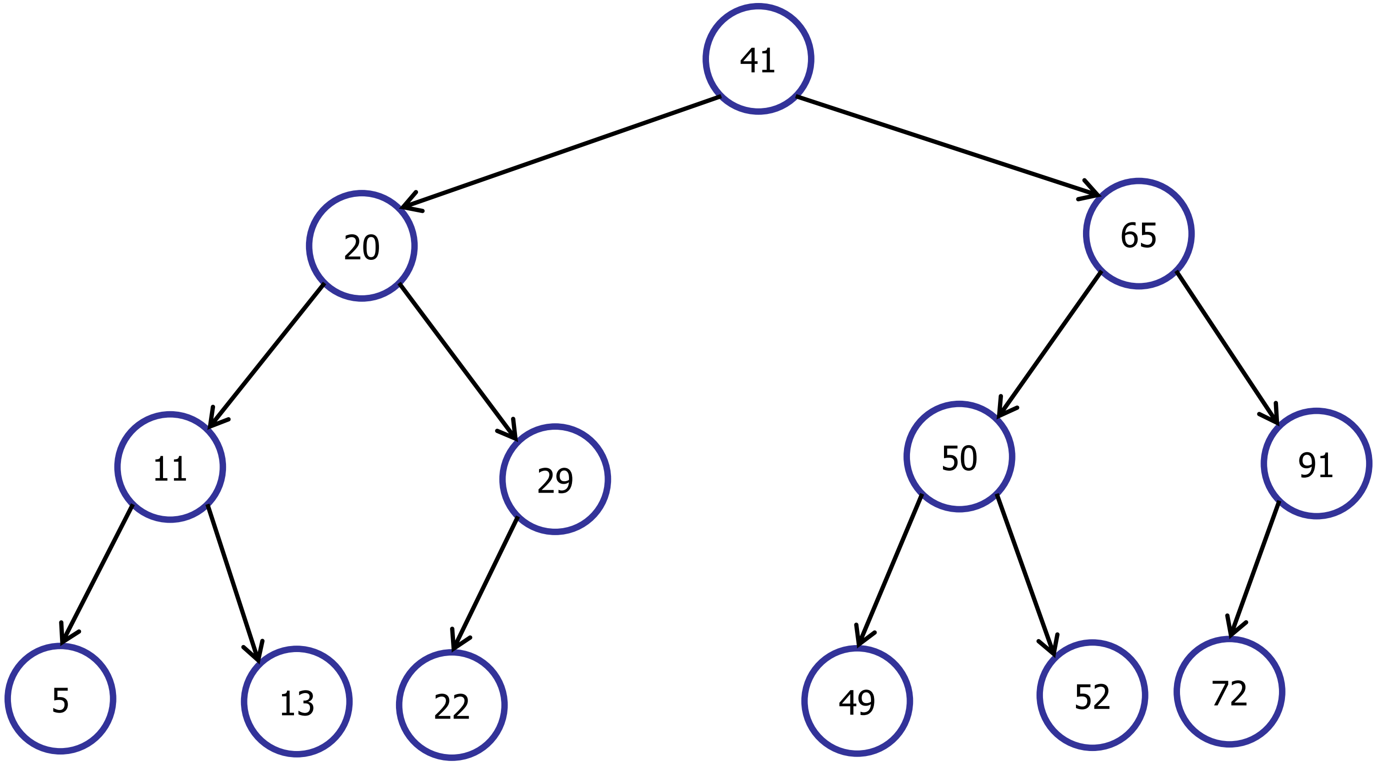
The Importance of Being Balanced

Perfectly balanced:



The Importance of Being Balanced

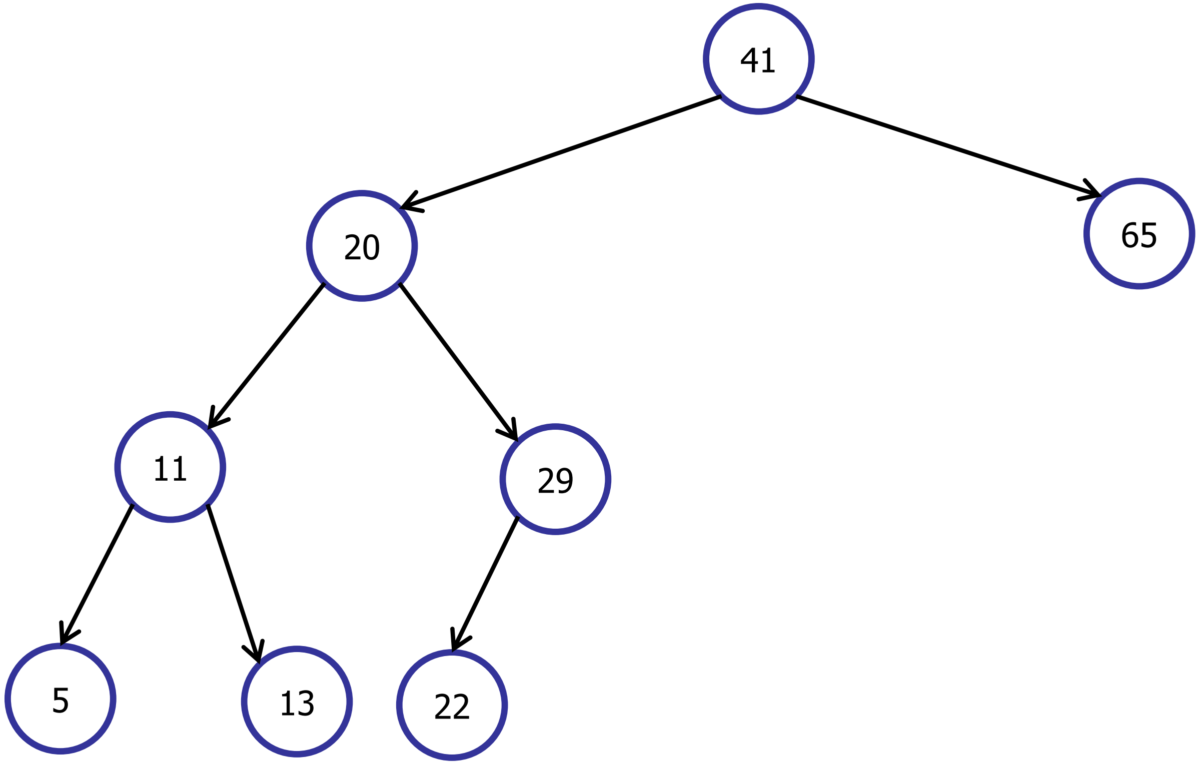
Almost perfectly balanced:



Every subtree has (almost) the same number of nodes.

The Importance of Being Balanced

Not perfectly balanced:



Left tree has 6, right tree has 1.

Balanced Search Trees

Many different flavors of balanced search trees

- AVL trees (Adelson-Velsii & Landis, 1962)
- B-trees / 2-3-4 trees (Bayer & McCreight, 1972)
- BB[α] trees (Nievergelt & Reingold 1973)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Treaps (Seidel and Aragon 1996)
- Skip Lists (Pugh 1989)
- Scapegoat Trees (Anderson 1989)

Balanced Search Trees

Many different flavors of balanced search trees

- AVL trees (Adelson-Velsii & Landis, 1962)
- B-trees / 2-3-4 trees (Bayer & McCreight, 1972)
- BB[α] trees (Nievergelt & Reingold 1973)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Treaps (Seidel and Aragon 1996)
- Skip Lists (Pugh 1989)
- Scapegoat Trees (Anderson 1989)

The Importance of Being Balanced

How to get a balanced tree:

- Define a good property of a tree.
- Show that if the good property holds, then the tree is balanced.
- After every insert/delete, make sure the good property still holds. If not, fix it.

Invariant

AVL Trees [Adelson-Velskii & Landis 1962]

AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

Step 1: Define Height Balance

Step 2: Maintain Balance

AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

- In every node v, store height:

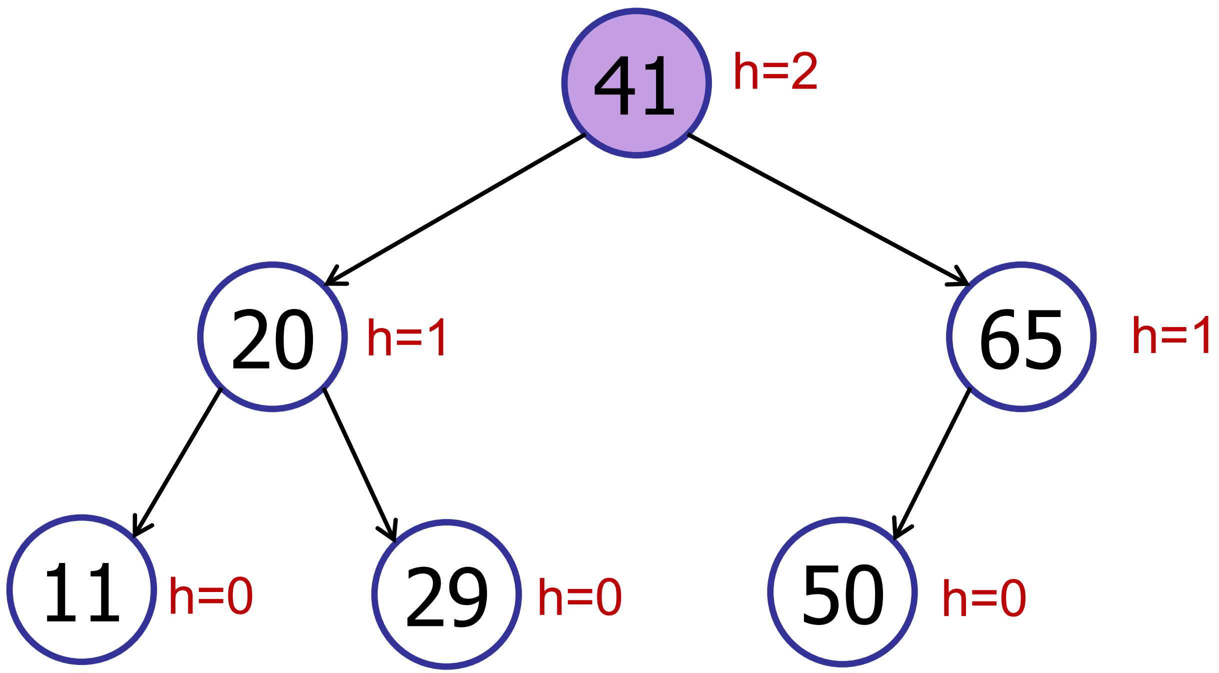
  v.height = h(v)

- On insert & delete update height:

```
insert(x)
    if (x < key)
        left.insert(x)
    else right.insert(x)
    height = max(left.height, right.height) + 1
```

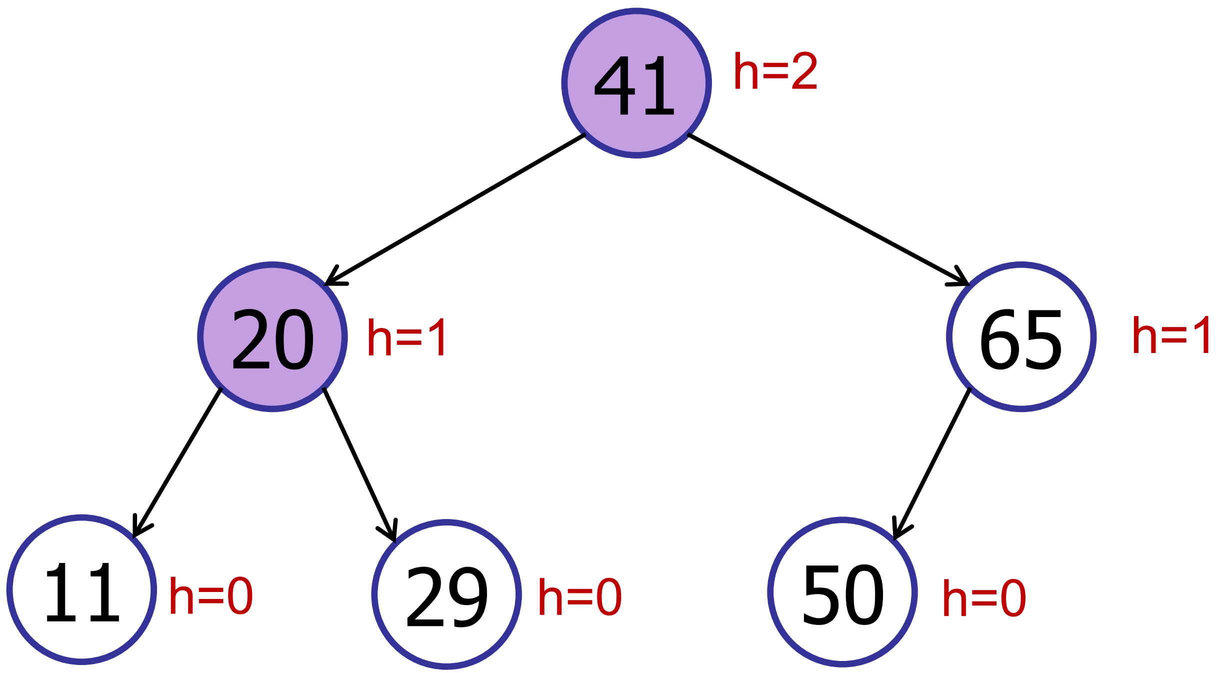
Binary Search Trees

insert(27)



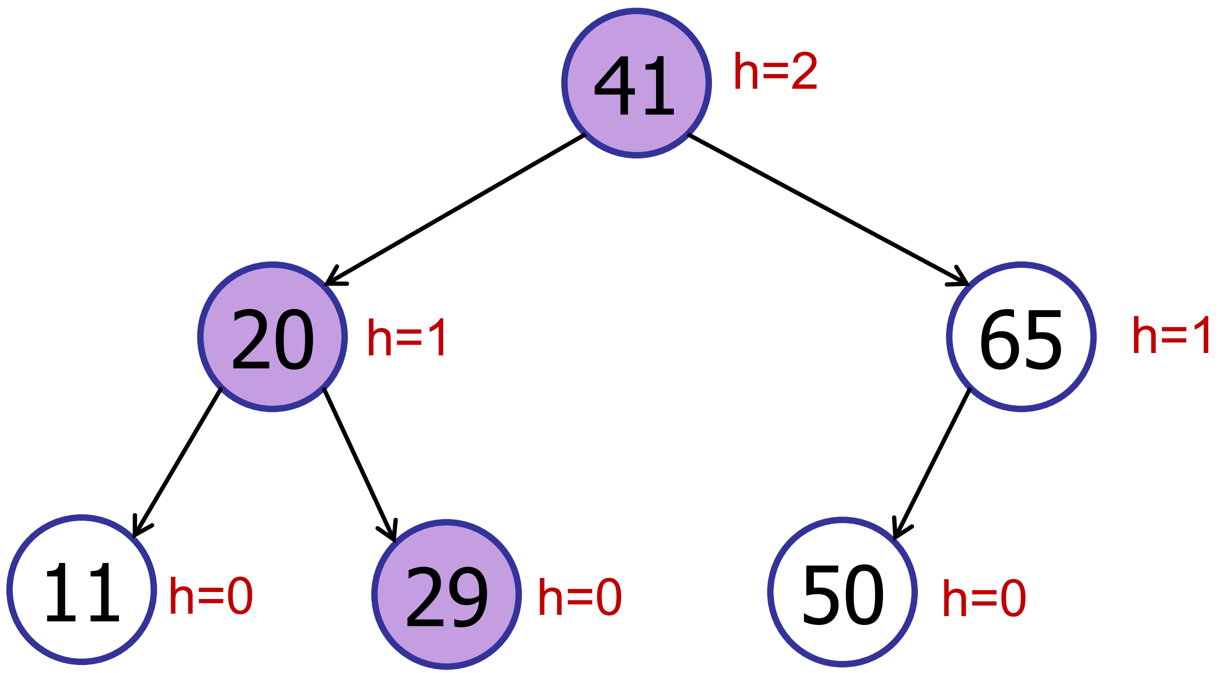
Binary Search Trees

insert(27)



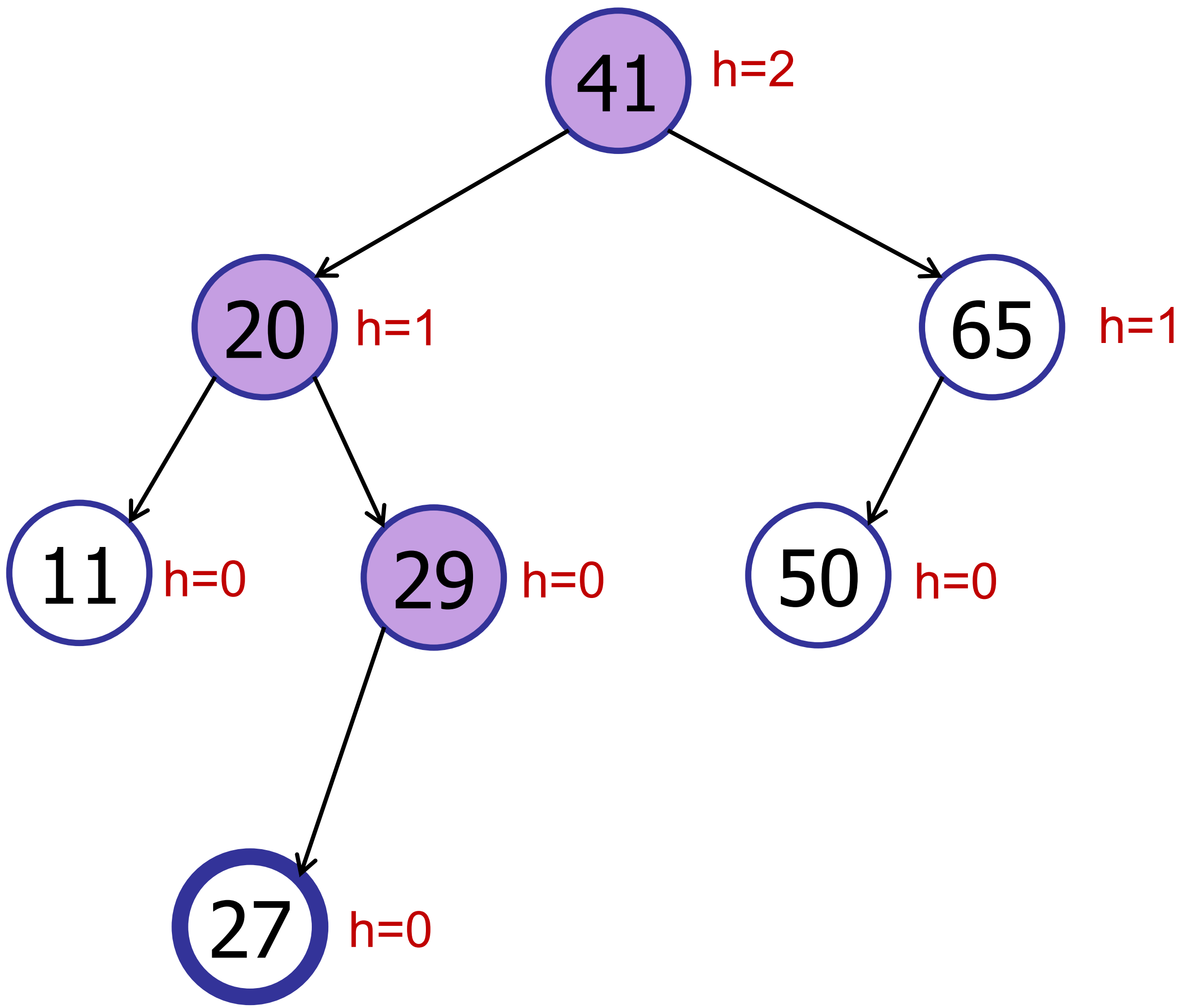
Binary Search Trees

insert(27)



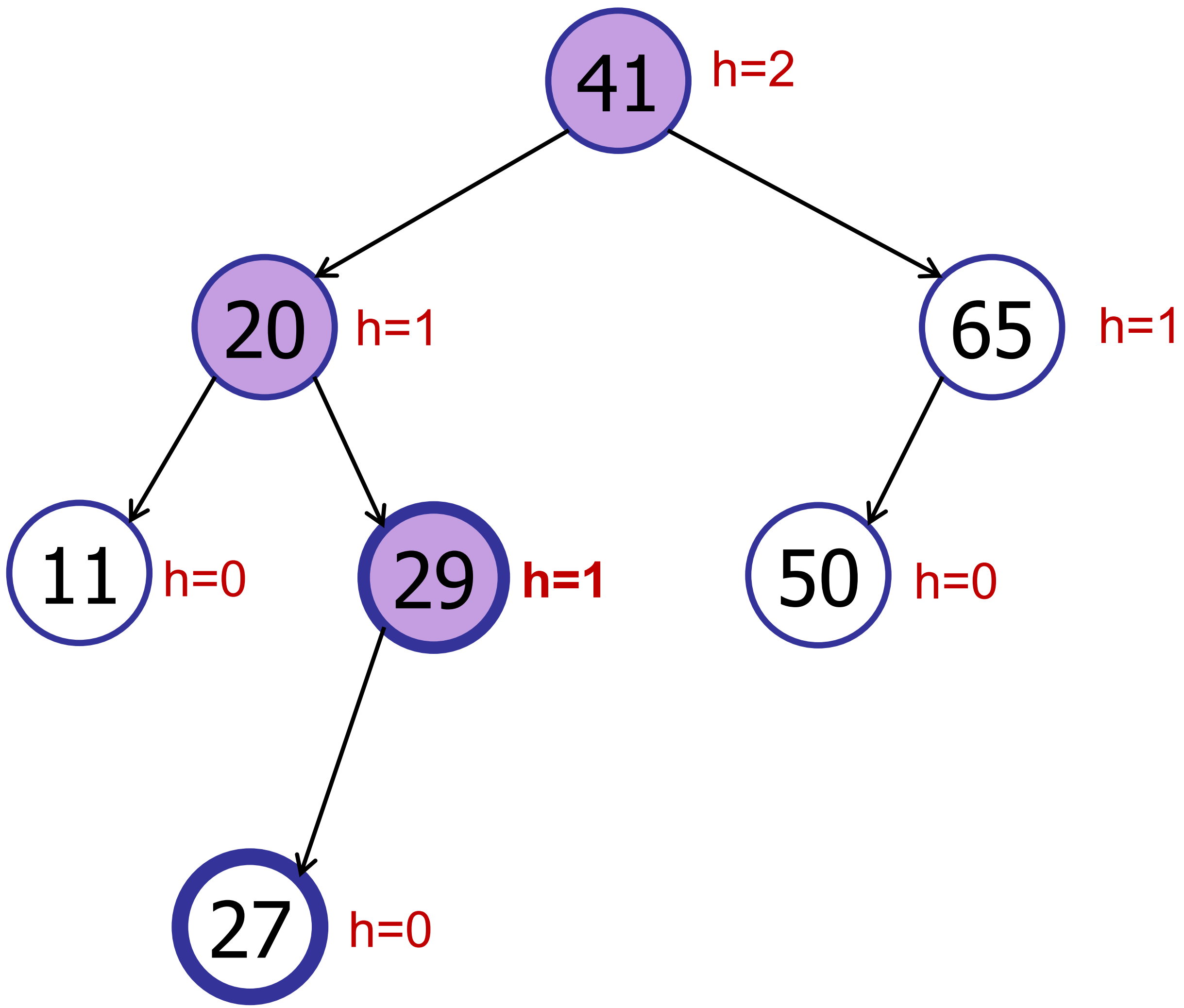
Binary Search Trees

insert(27)



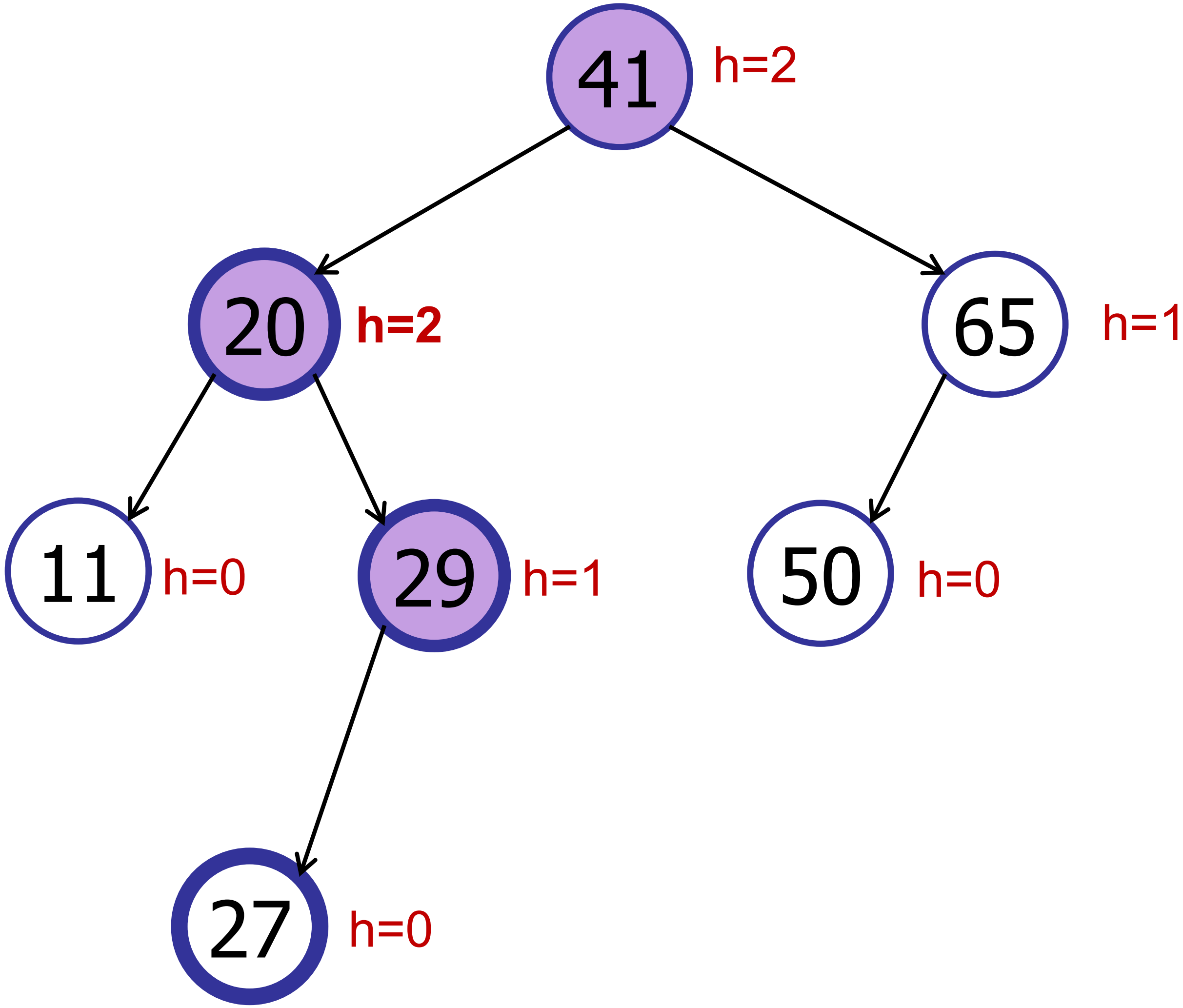
Binary Search Trees

insert(27)



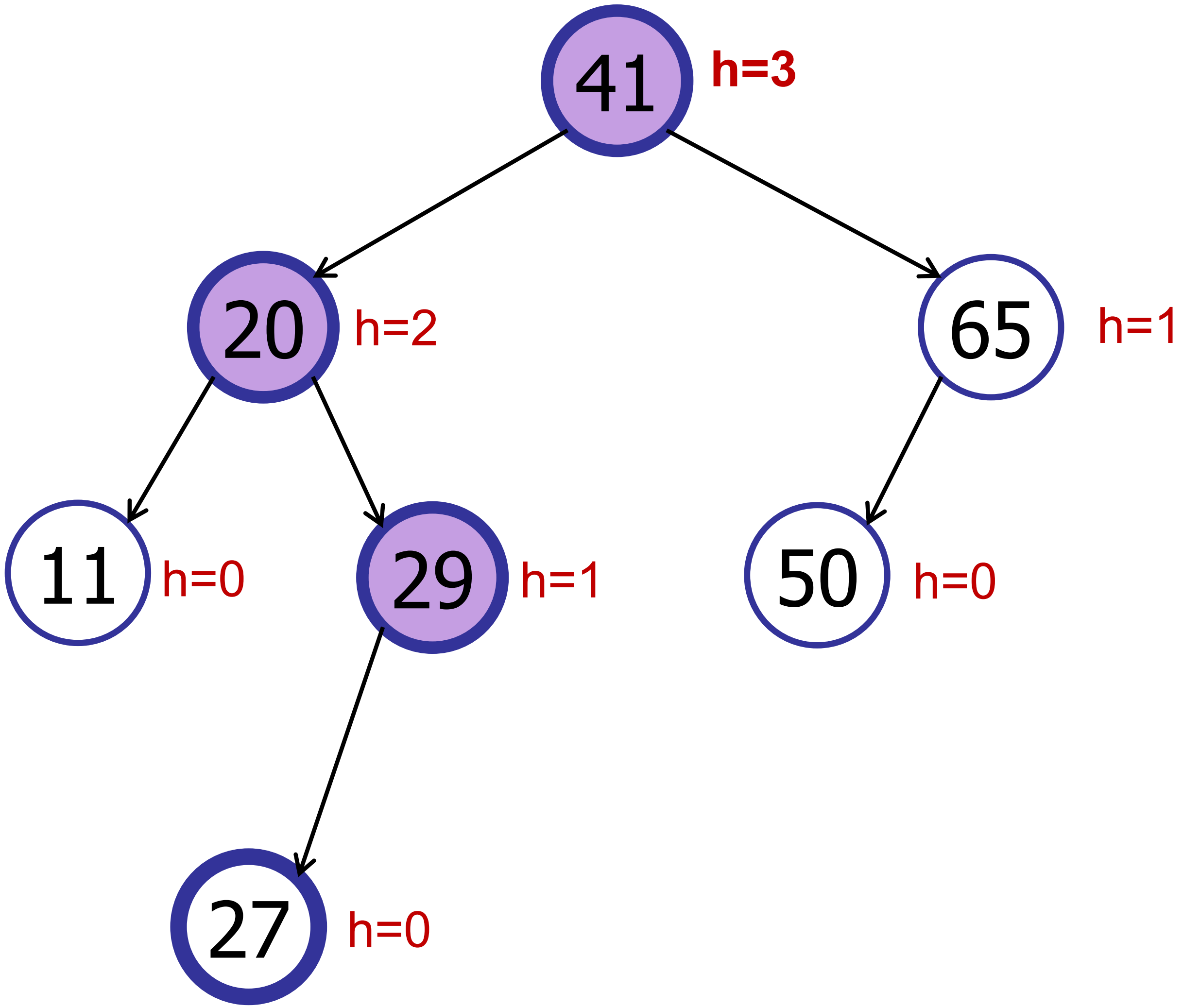
Binary Search Trees

insert(27)



Binary Search Trees

insert(27)



AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

- In every node v, store height:

$$v.height = h(v)$$

- On insert & delete update height:

```
insert(x)
        if (x < key)
                left.insert(x)
        else right.insert(x)
        height = max(left.height, right.height) + 1
```

AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

Step 1: Define Height Balance

Step 2: Maintain Balance

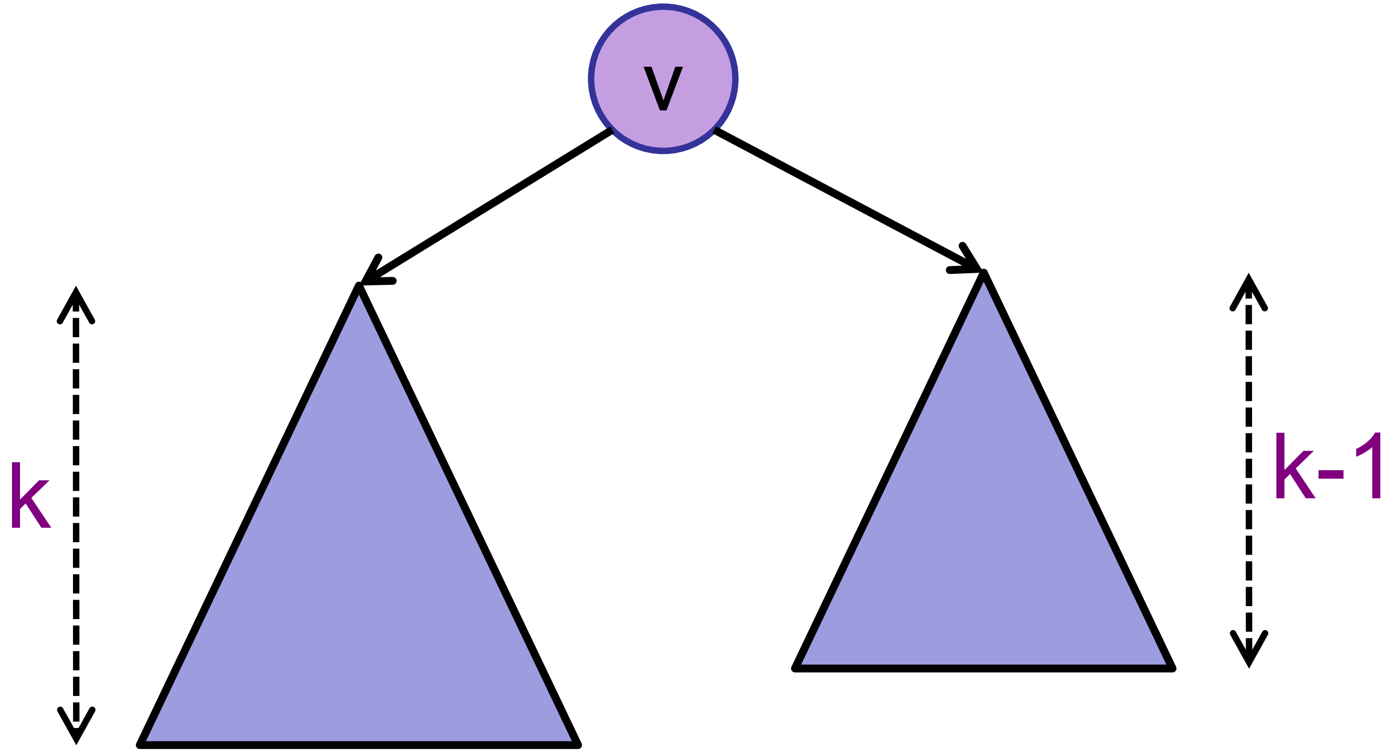
AVL Trees [Adelson-Velskii & Landis 1962]

Key definition

Step 1: Define Invariant

– A node v is **height-balanced** if:

$$|v.left.height - v.right.height| \leq 1$$



AVL Trees [Adelson-Velskii & Landis 1962]

Step 1: Define Invariant

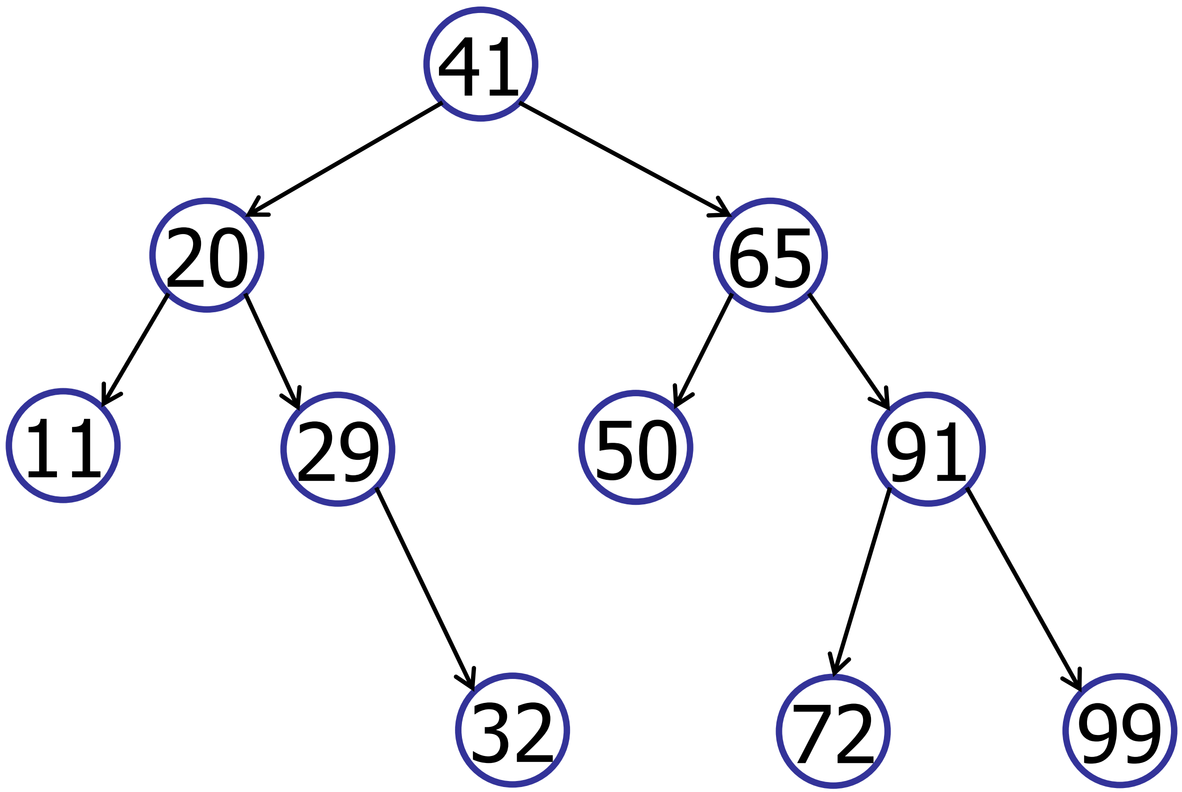
- A node v is height-balanced if:

$$|v.left.height - v.right.height| \leq 1$$

- A binary search tree is height balanced if **every** node in the tree is height-balanced.

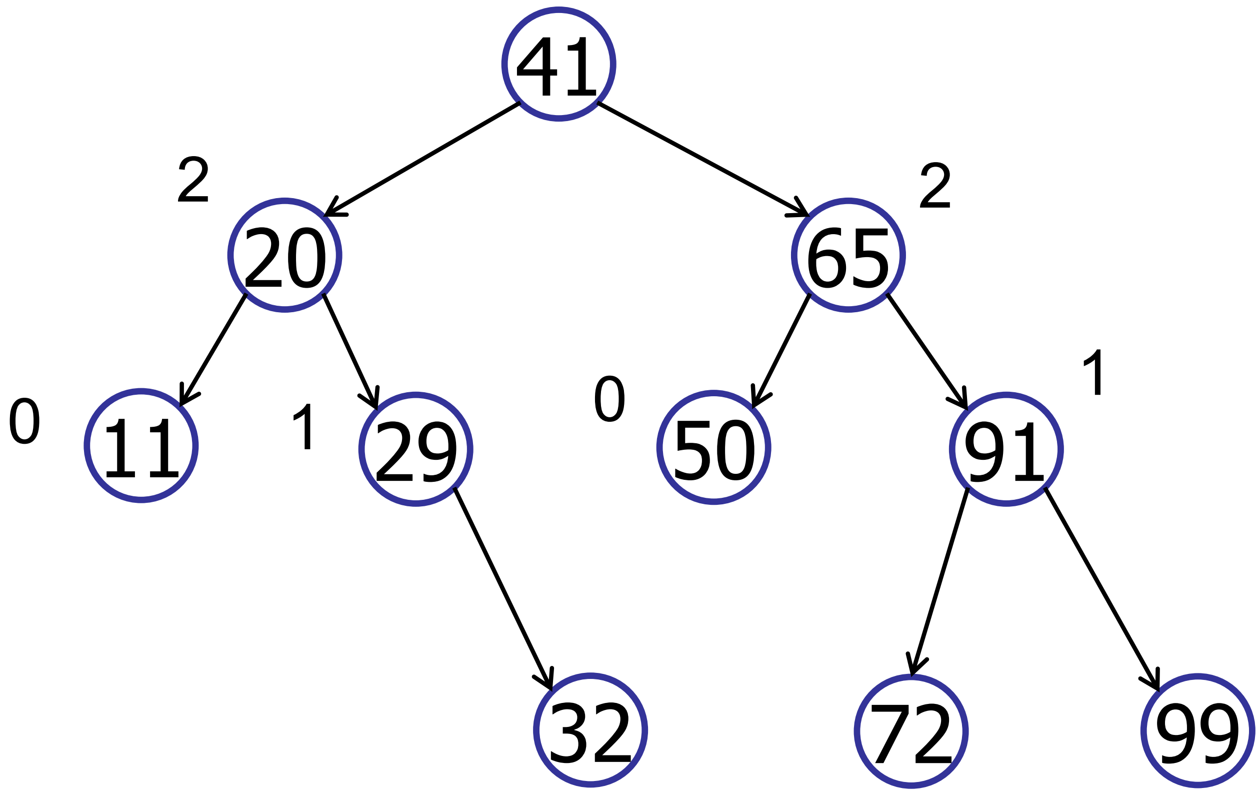
Is this tree height-balanced?

1. Yes
2. No
3. I'm confused.



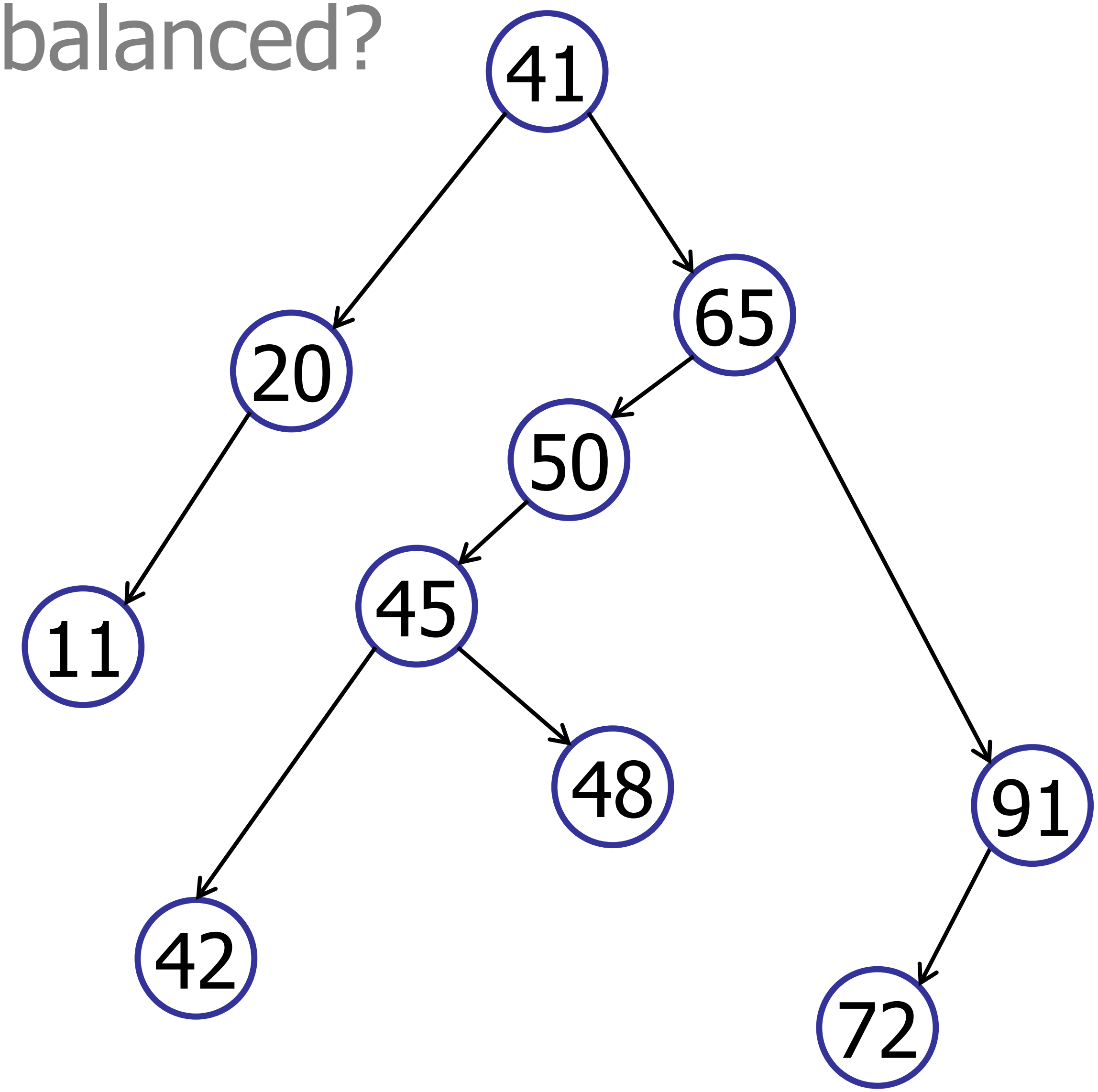
Is this tree height-balanced?

1. Yes
2. No
3. I'm confused.



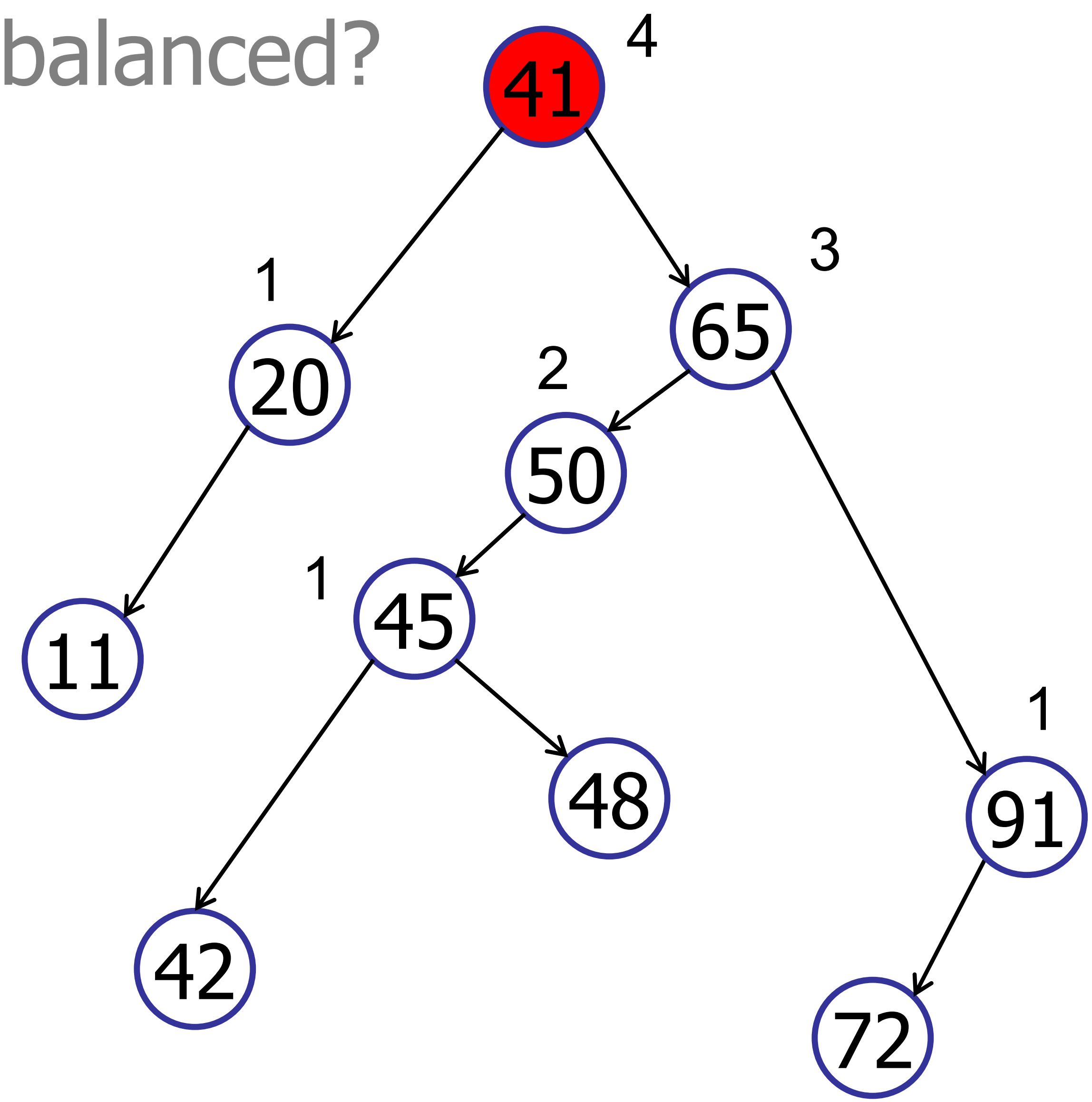
Is this tree height-balanced?

1. Yes
2. No
3. I’m confused.



Is this tree height-balanced?

1. Yes
2. No
3. I'm confused.



Height-Balanced Trees

Claim:

A height-balanced tree with n nodes has **at most** height $h < 2\log(n)$.

$\Leftrightarrow h/2 < \log(n)$

$\Leftrightarrow 2^{h/2} < 2^{\log(n)}$

$\Leftrightarrow 2^{h/2} < n$

A height-balanced tree with height h has **at least** $n > 2^{h/2}$ nodes

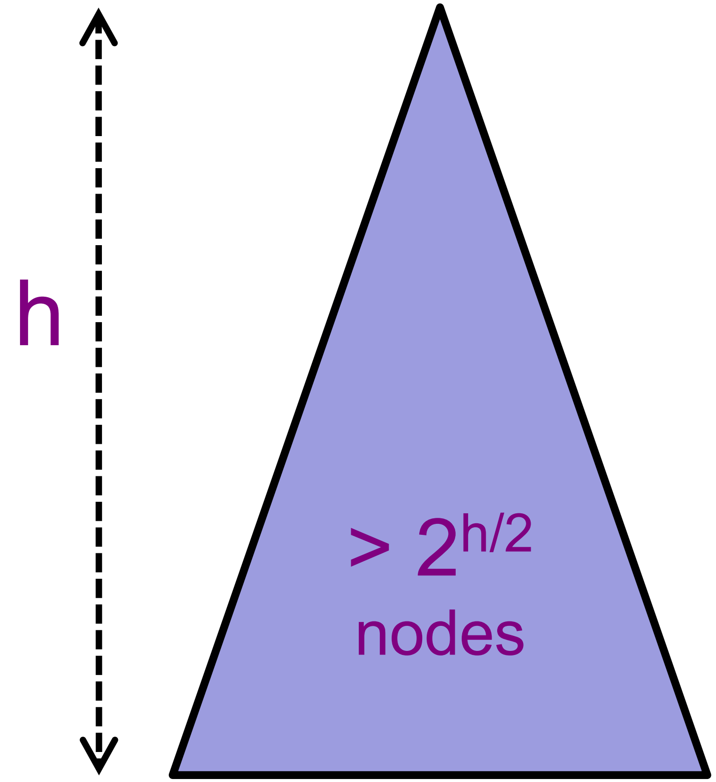
Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

Show:

$n_h > 2^{h/2}$

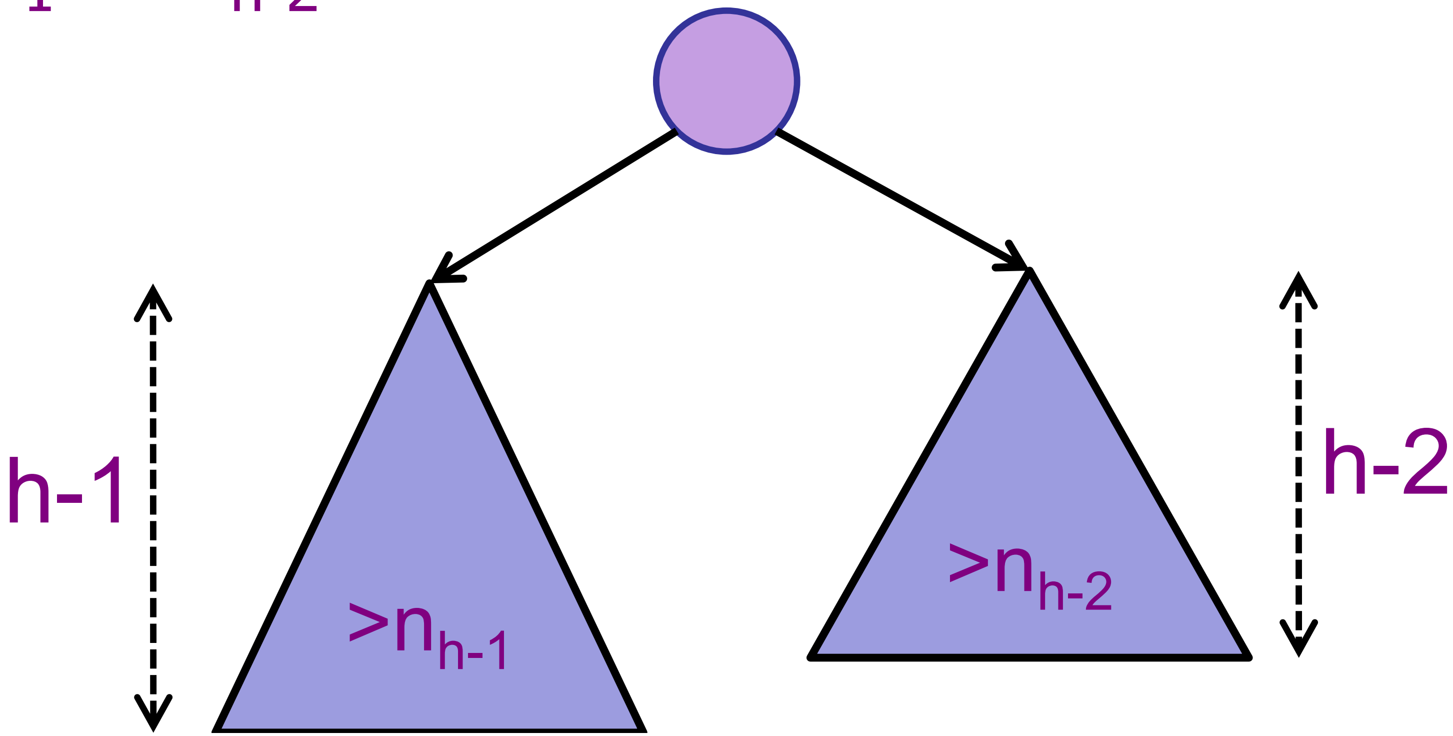


Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$n_h \geq 1 + n_{h-1} + n_{h-2}$



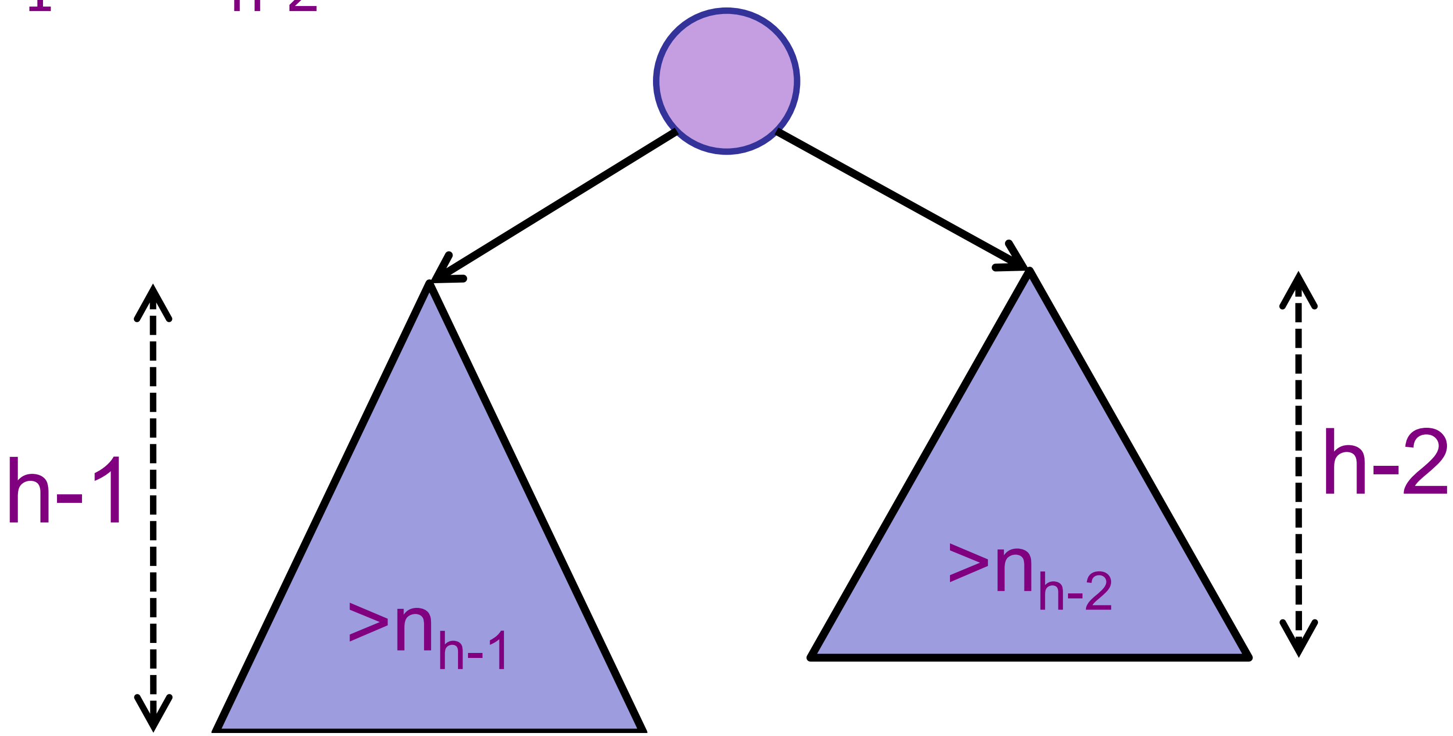
Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$n_h \geq 1 + n_{h-1} + n_{h-2}$

$\geq 2n_{h-2}$



Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2n_{h-2}$$

$$\geq 4n_{h-4}$$

$$\geq 8n_{h-6}$$

$$\geq \ldots$$

How many times?

Base case:
$n_0 = 1$

Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$$n_h \geq 1 + n_{h-1} + n_{h-2}$$

$$\geq 2^1 n_{h-2}$$

$$\geq 2^2 n_{h-4}$$

$$\geq 2^3 n_{h-6}$$

$$\geq \ldots \geq 2^k n_0$$

What is k?

Base case:

$n_0 = 1$

Height-Balanced Trees

Proof:

Let n_h be the minimum number of nodes in a height-balanced tree of height h.

$$
\begin{aligned}
n_h &\geq 1 + n_{h-1} + n_{h-2} \\
&\geq 2n_{h-2} \\
&\geq 2^{h/2}\, n_0 \\
&\geq 2^{h/2}
\end{aligned}
$$

Base case:
$n_0 = 1$

Height-Balanced Trees

Claim:

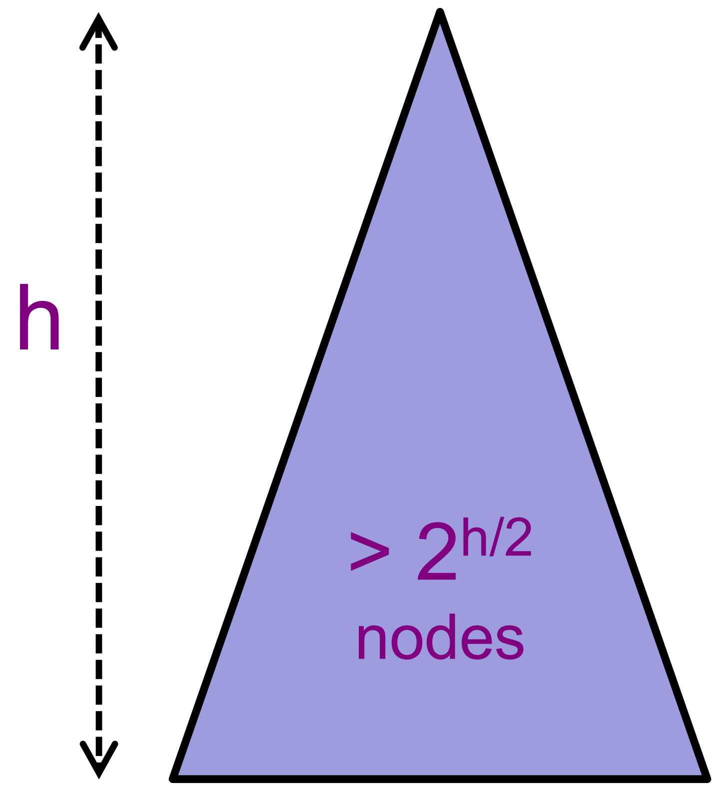
A height-balanced tree with n nodes has height $h < 2\log(n)$.

Show:

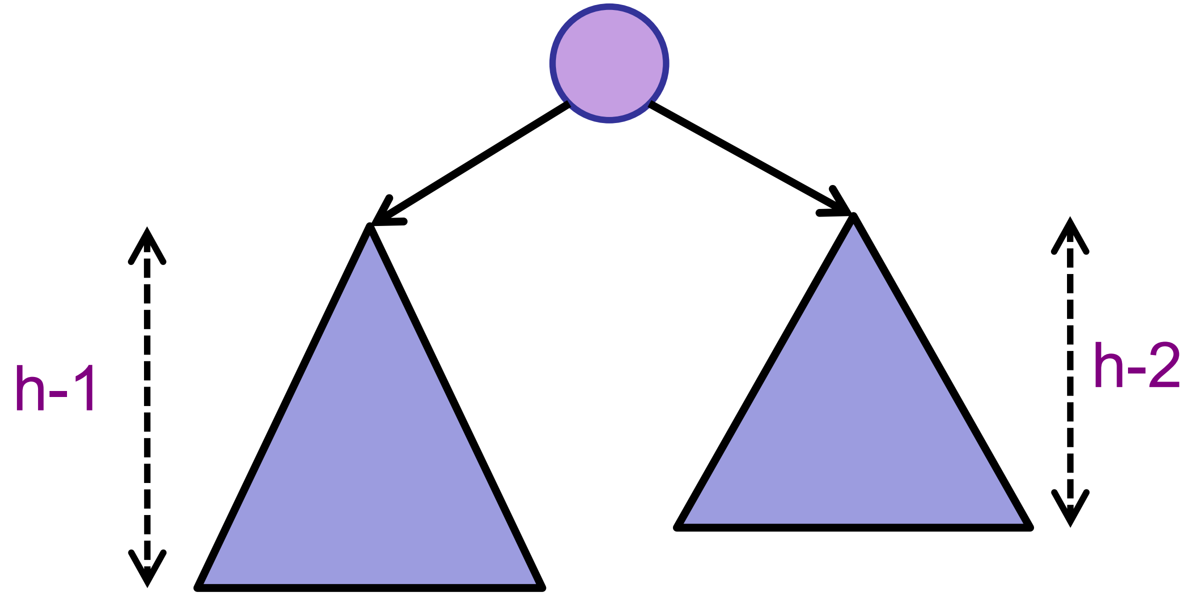
$n_h > 2^{h/2}$

$\Leftrightarrow$

$h < 2\log(n_h)$



Height-Balanced Trees



Show (induction):

F_n = n^{th} Fibonacci number

$n_h = F_{h+2} - 1 \cong \phi^{h+1}/\sqrt{5} - 1$ (rounded to nearest int)

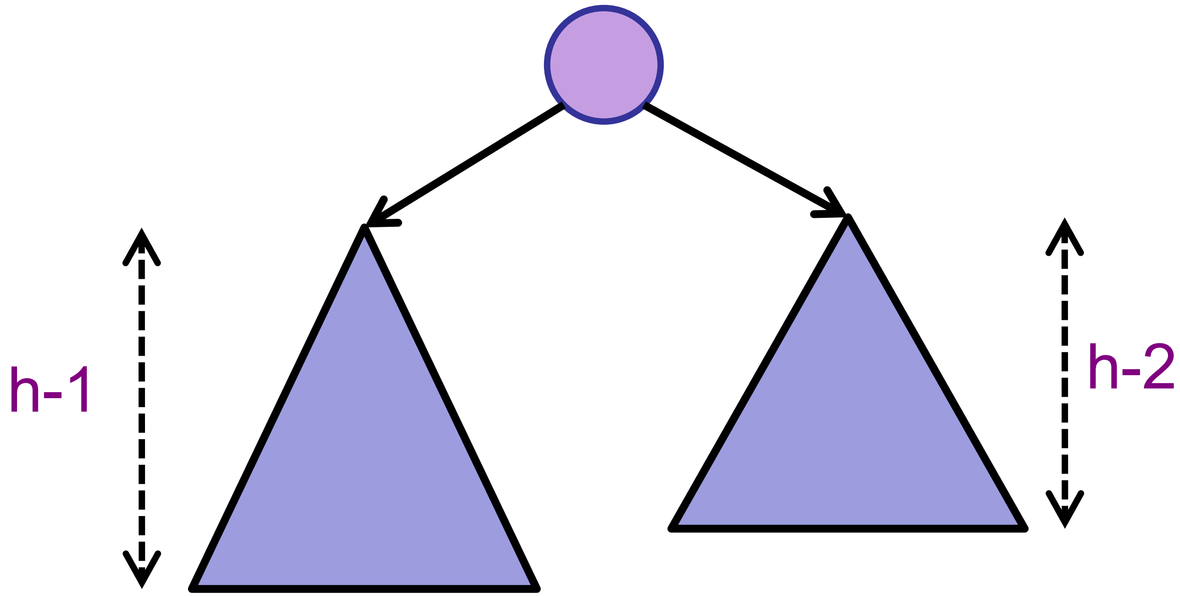
$h \cong \log(n) / \log(\phi)$ $\phi \cong 1.618$

$h \cong 1.44 \log(n)$

Height-Balanced Trees

Claim:

A height-balanced tree is balanced, i.e., has height $h = O(\log n)$.



AVL Trees [Adelson-Velskii & Landis 1962]

Step 0: Augment

Step 1: Define Height Balance

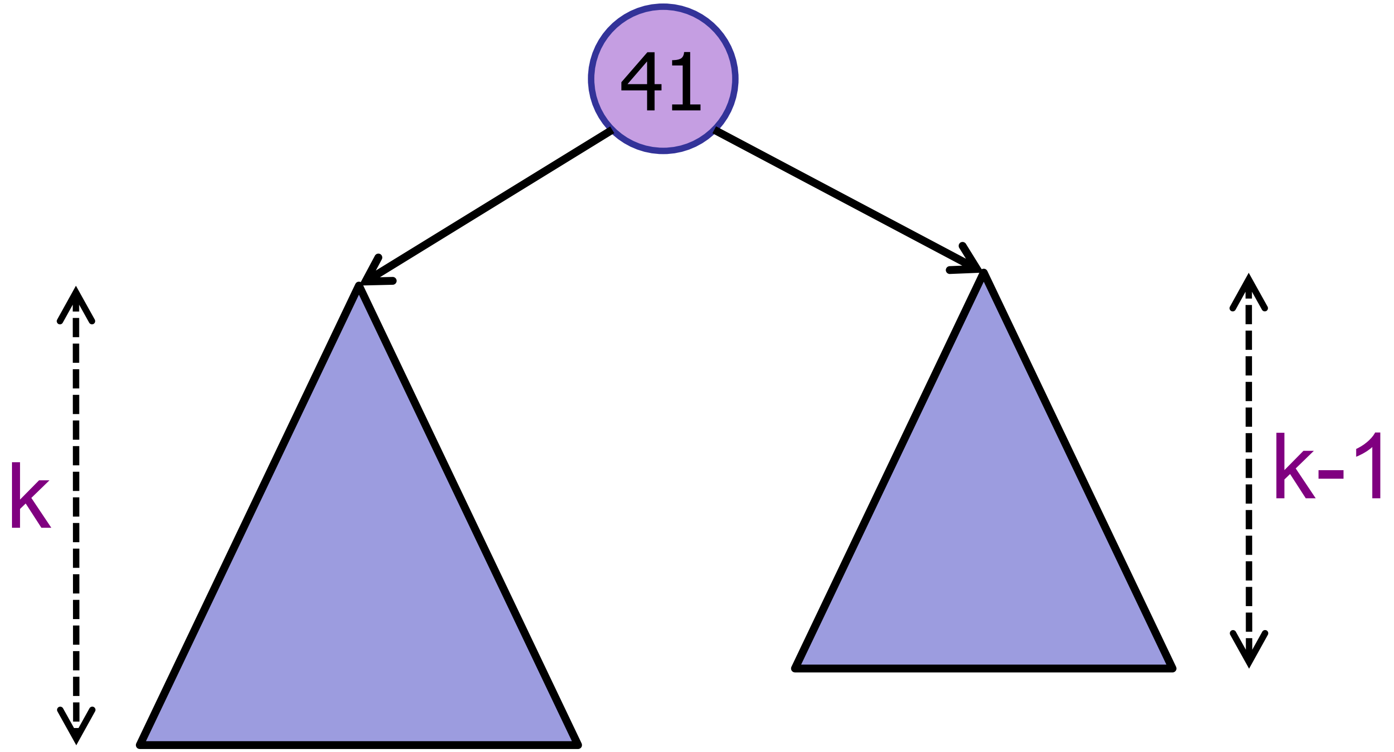
Step 2: Maintain Balance

It’s good that we don’t have to

Balance perfectly

AVL Trees [Adelson-Velskii & Landis 1962]

Step 2: Show how to maintain height-balance



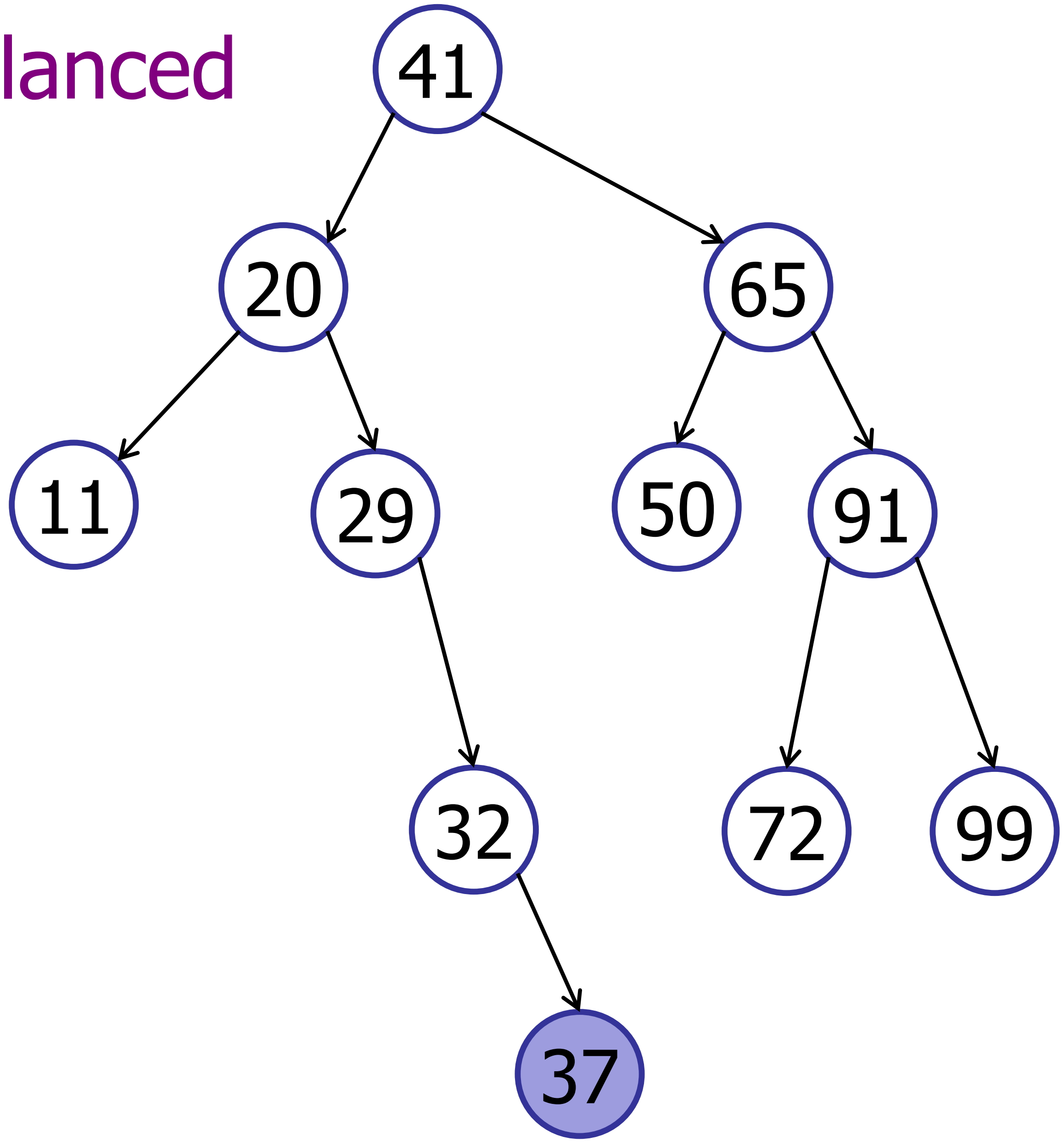
Inserting in an AVL Tree

Before insertion, balanced

insert(37)

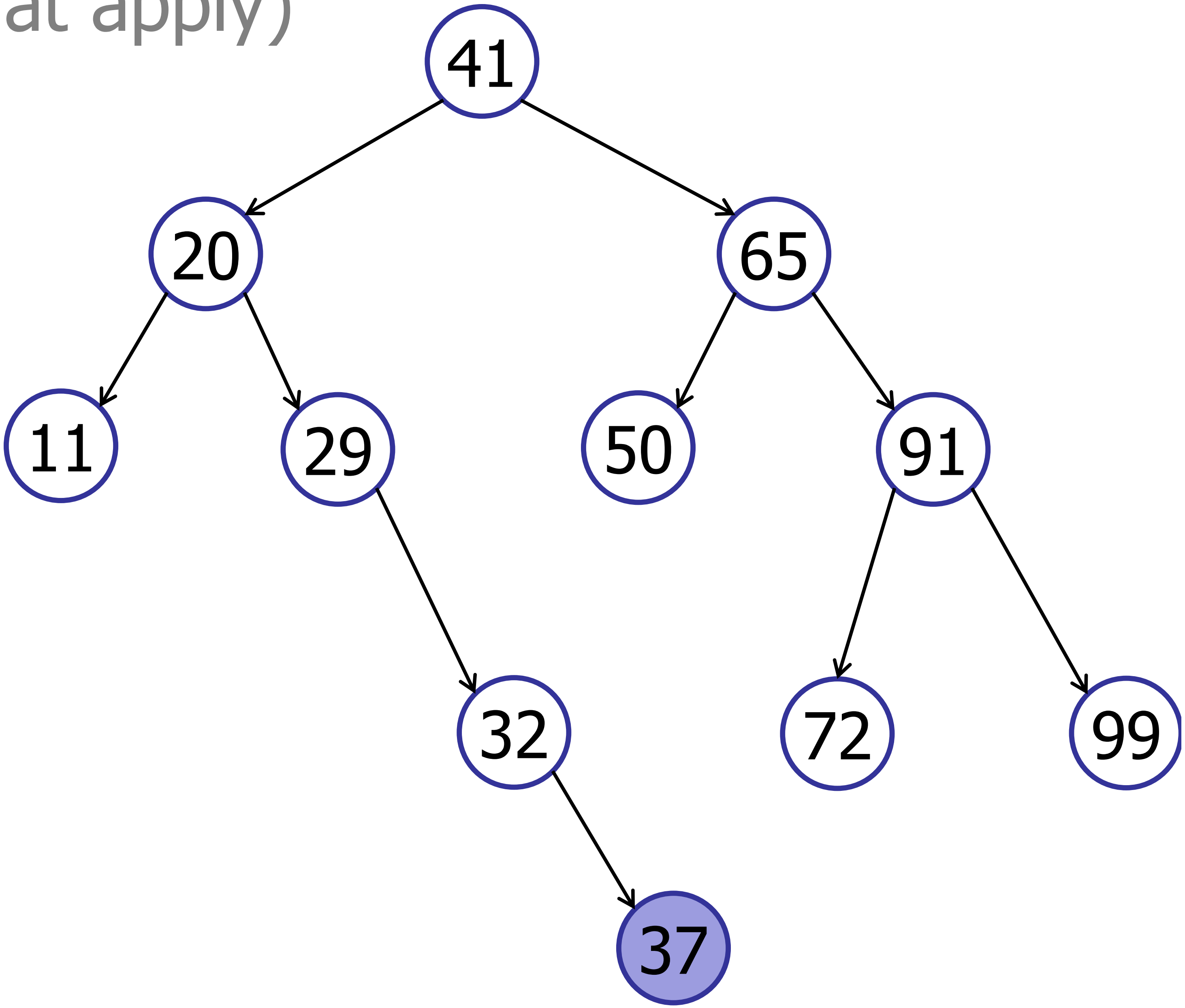
No longer balanced
after insertion!

Need to rebalance!



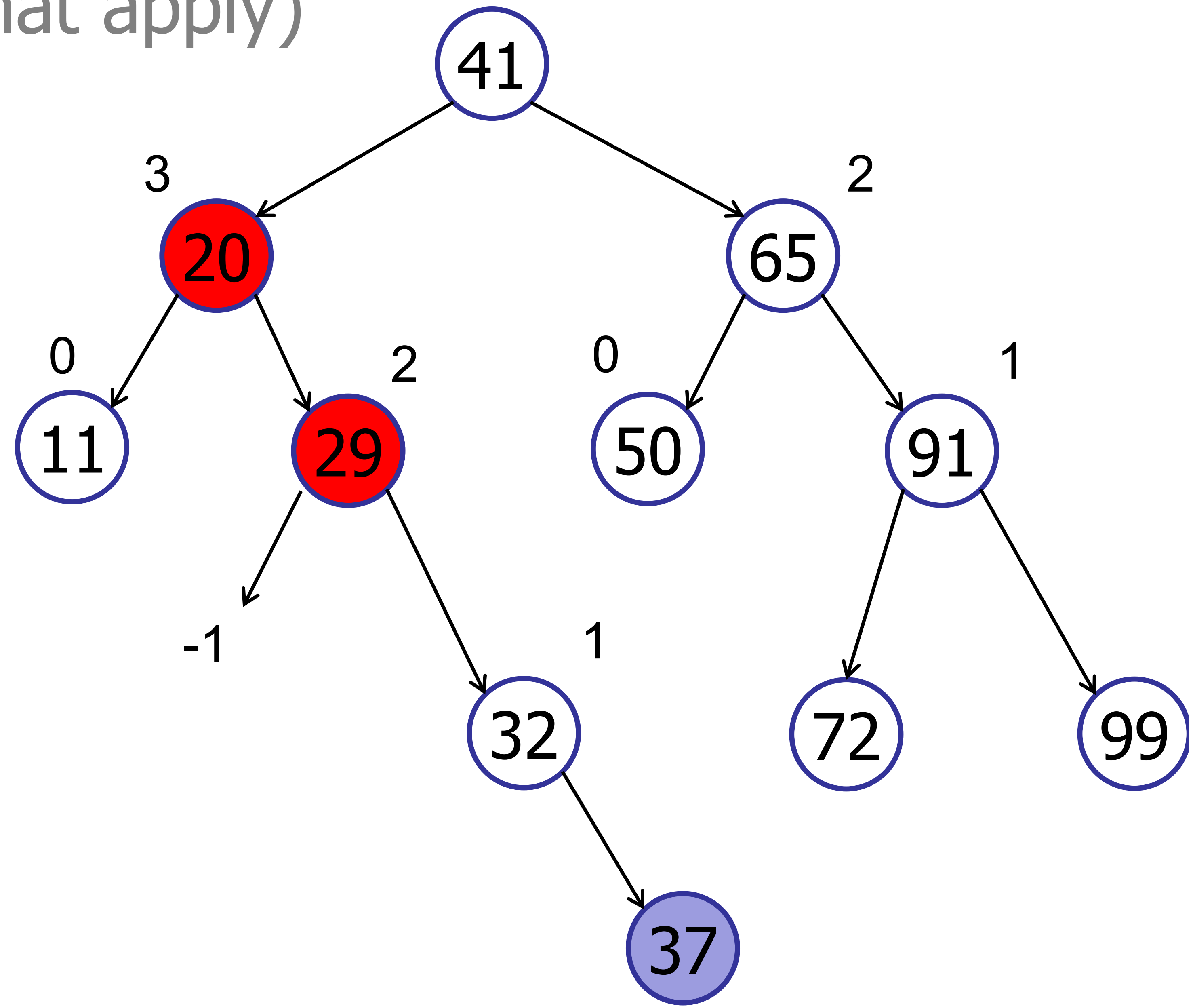
Which nodes need rebalancing?
(click all that apply)

1. 41
2. 20
3. 11
4. 29
5. 32
6. 37
7. 65



Which nodes need rebalancing?
(click all that apply)

1. 41
2. 20
3. 11
4. 29
5. 32
6. 37
7. 65

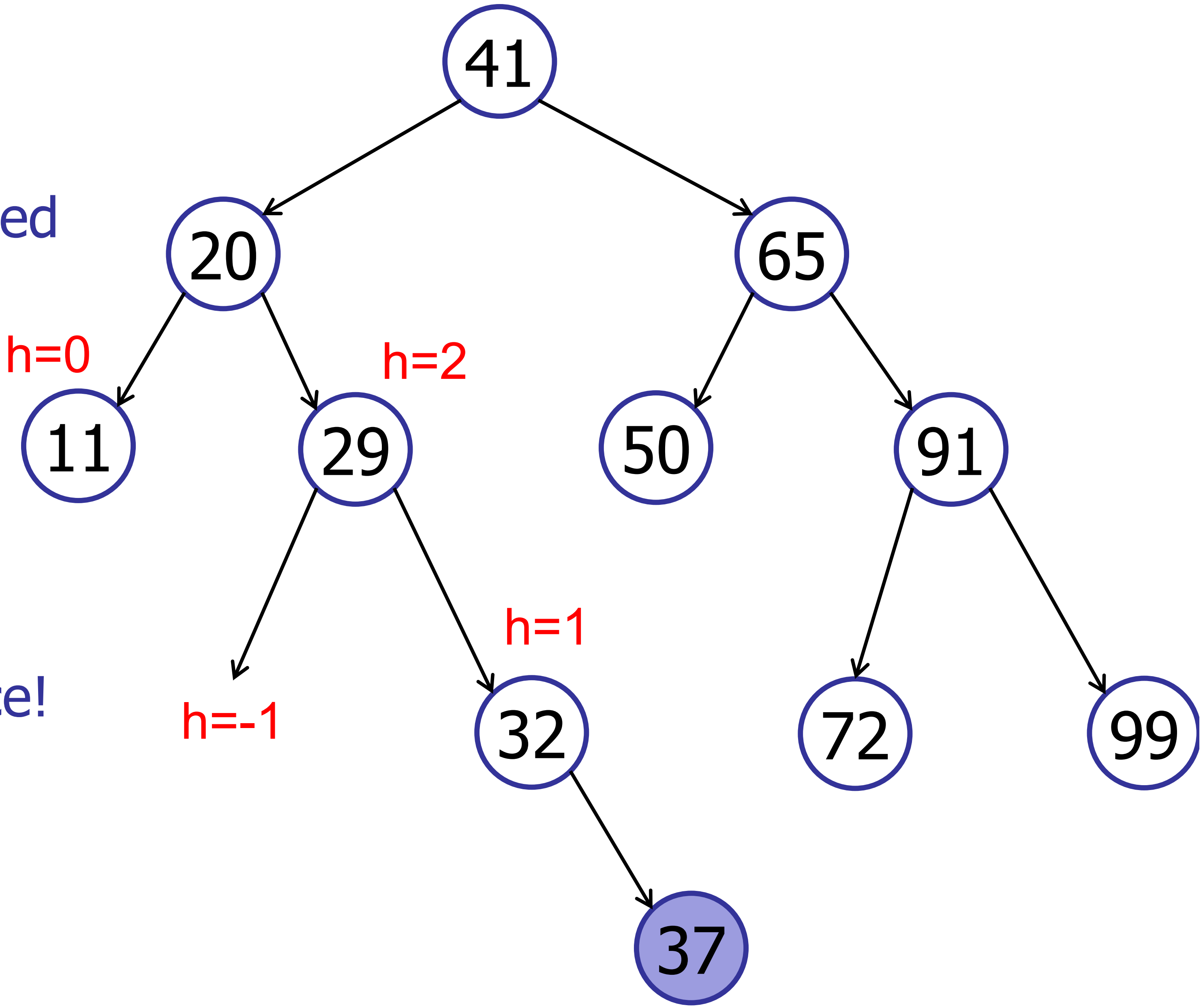


Inserting in an AVL Tree

insert(37)

No longer balanced after insertion!

Need to rebalance!

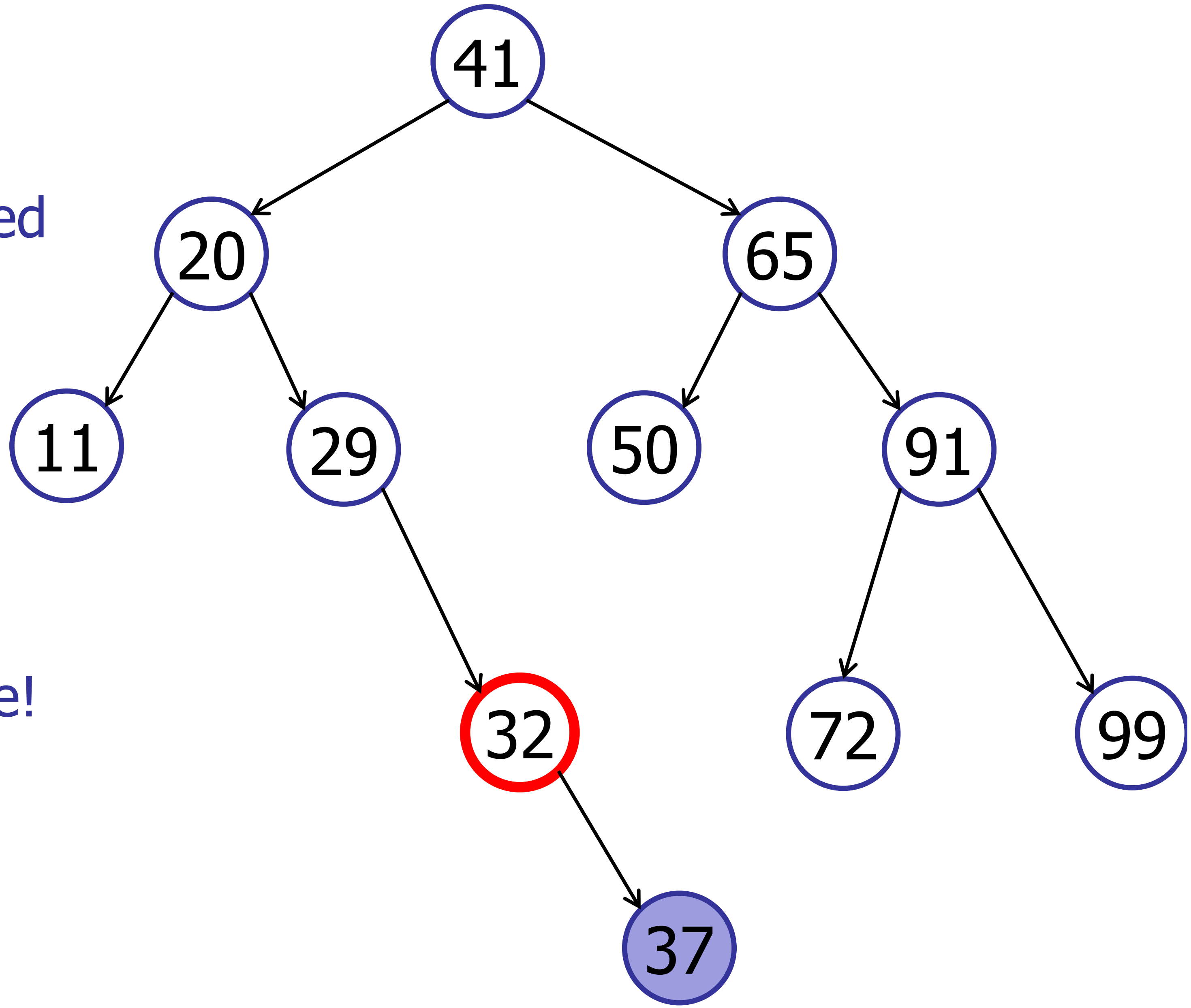


Inserting in an AVL Tree

insert(37)

No longer balanced after insertion!

Need to rebalance!

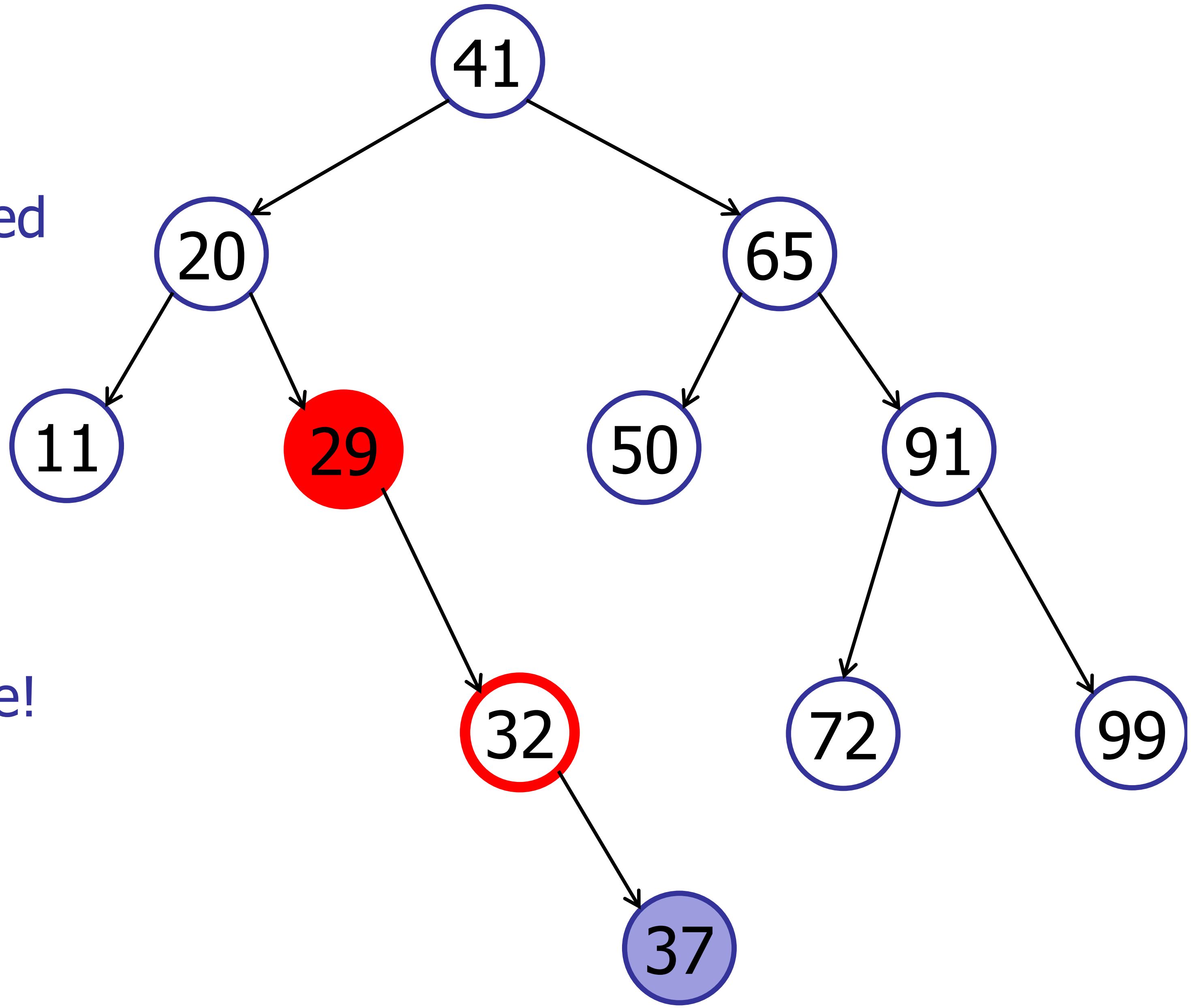


Inserting in an AVL Tree

insert(37)

No longer balanced after insertion!

Need to rebalance!

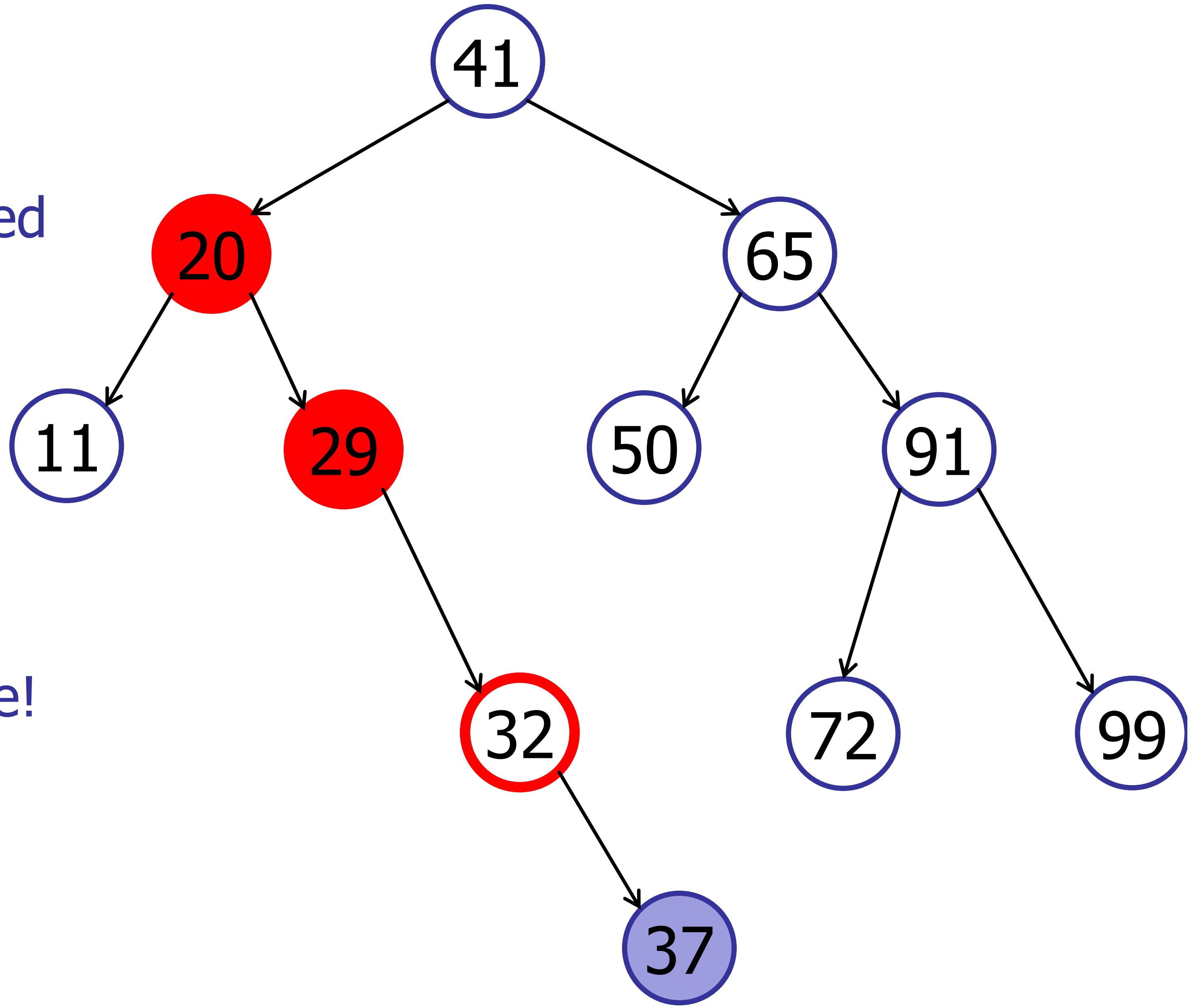


Inserting in an AVL Tree

insert(37)

No longer balanced after insertion!

Need to rebalance!

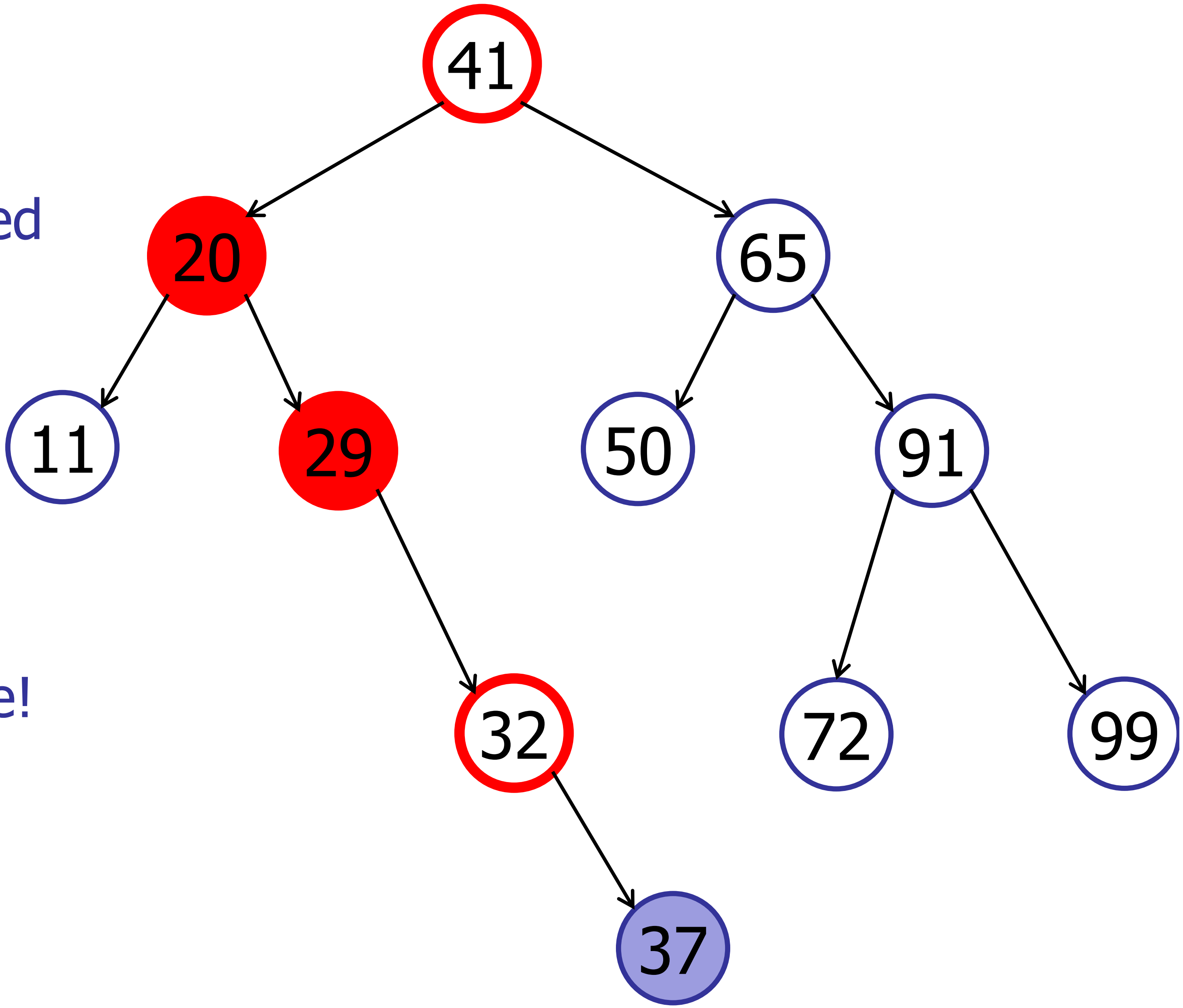


Inserting in an AVL Tree

insert(37)

No longer balanced after insertion!

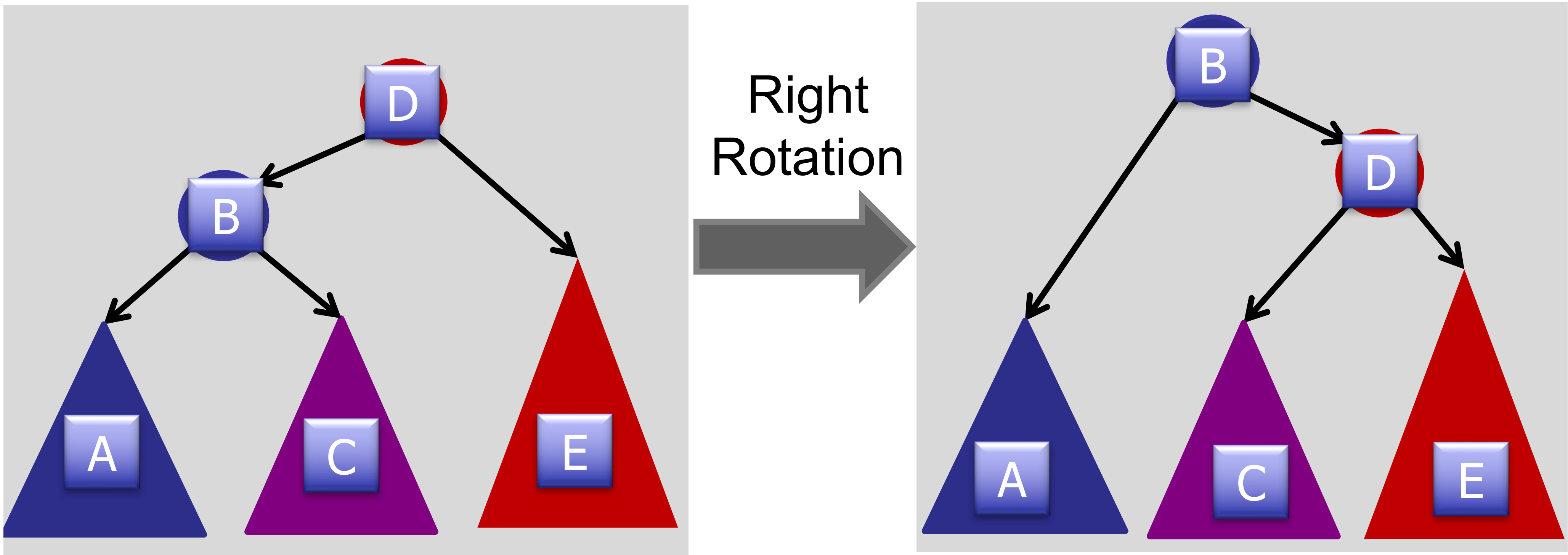
Need to rebalance!



Trick to rebalance the tree

Tree rotation!

Tree Rotations

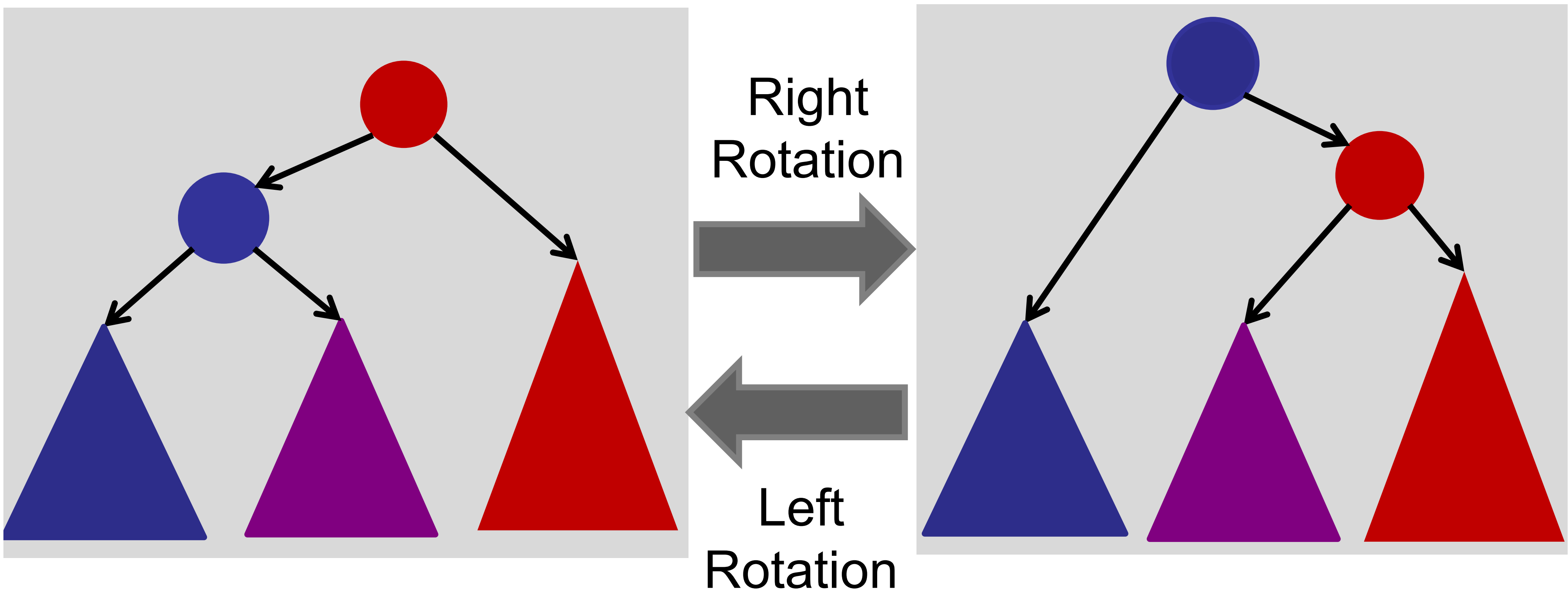


A < B < C < D < E

Rotations maintain ordering of keys.

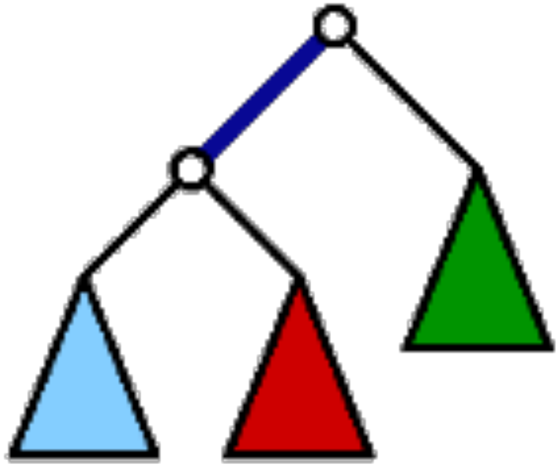
$\Rightarrow$ Maintains BST property.

Tree Rotations

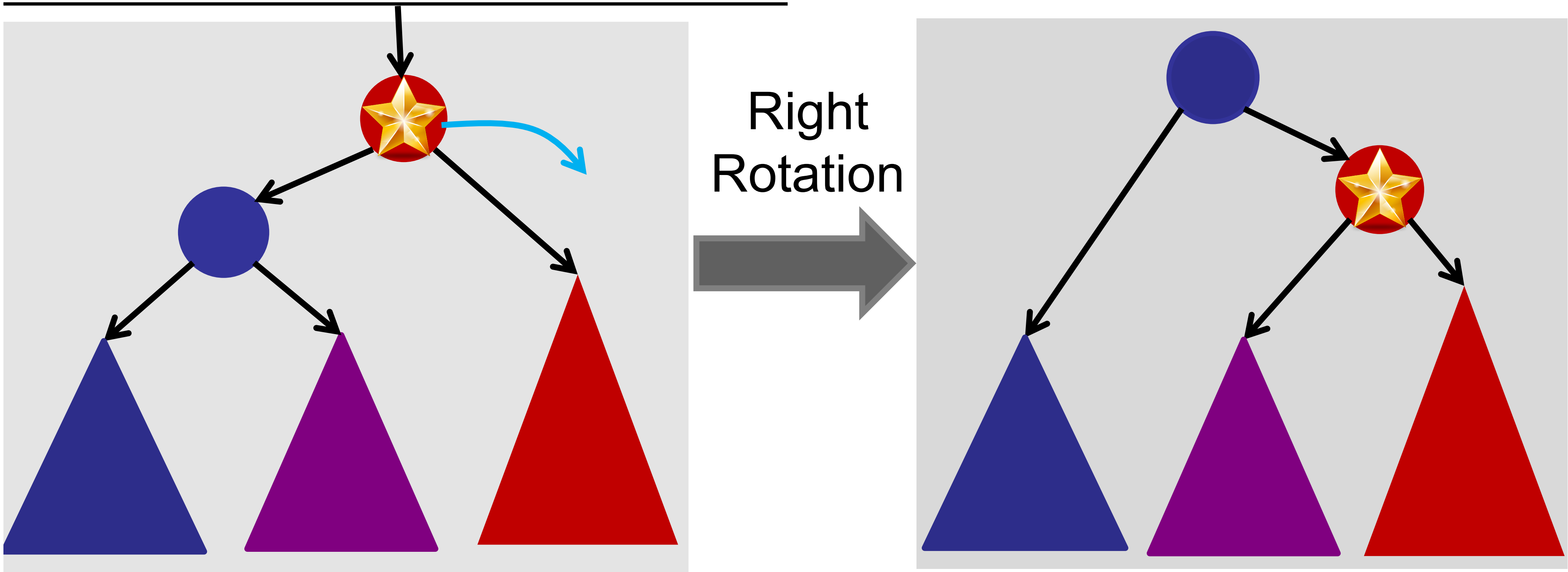


Wait….

What is a left rotation and what is a right rotation!?

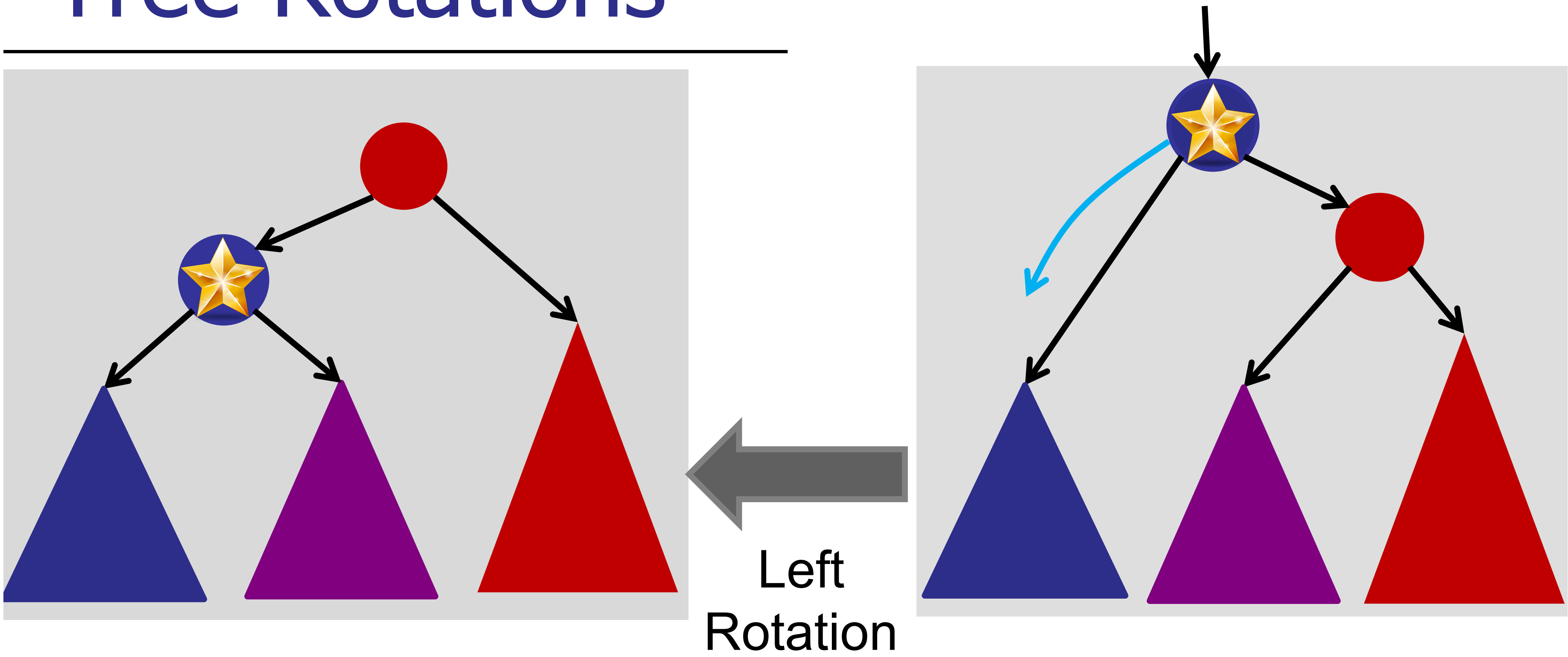


The way to remember it



The root of the subtree moves right

Tree Rotations

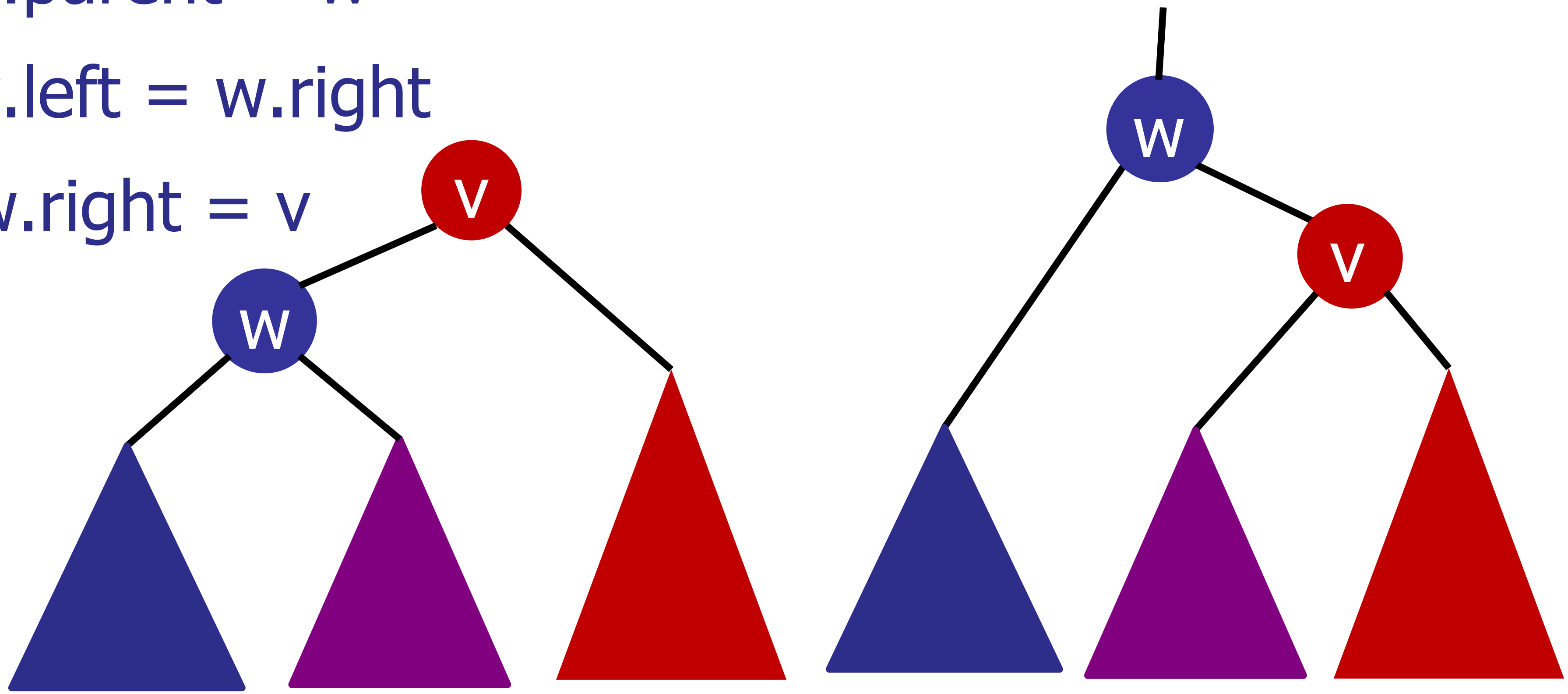


The root of the subtree moves left

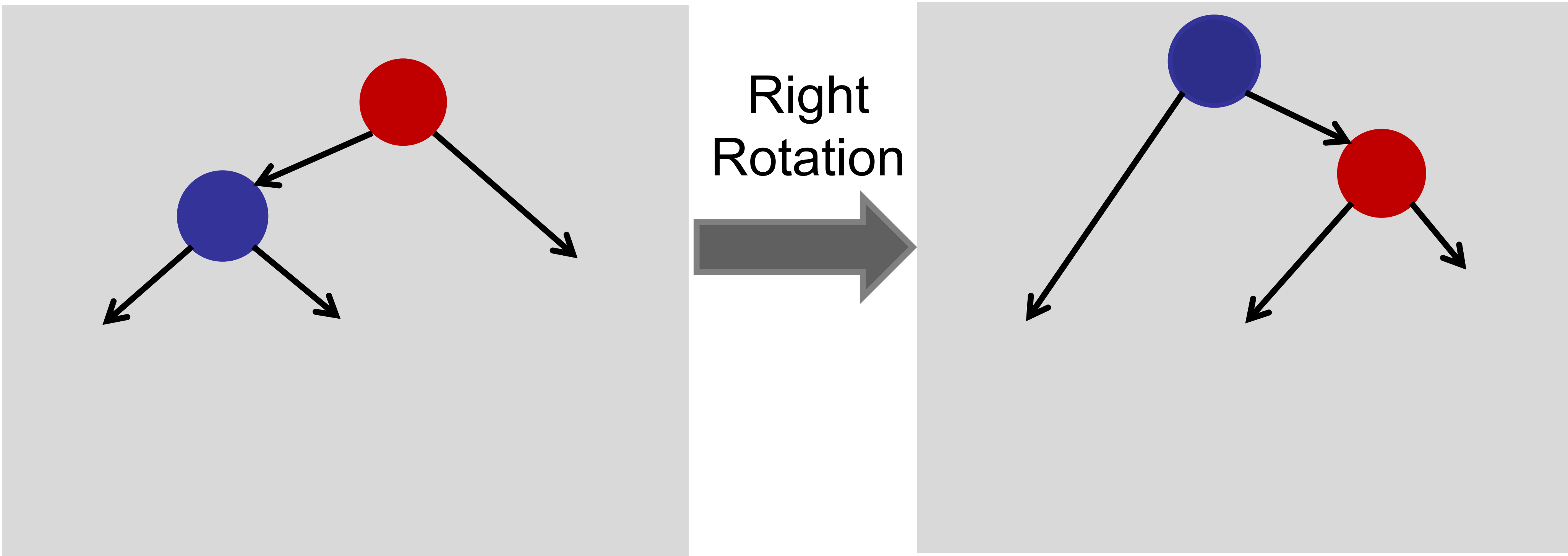
Rotations

right-rotate(v) // assume v has left != null

```
w = v.left
w.parent = v.parent
v.parent = w
v.left = w.right
w.right = v
```



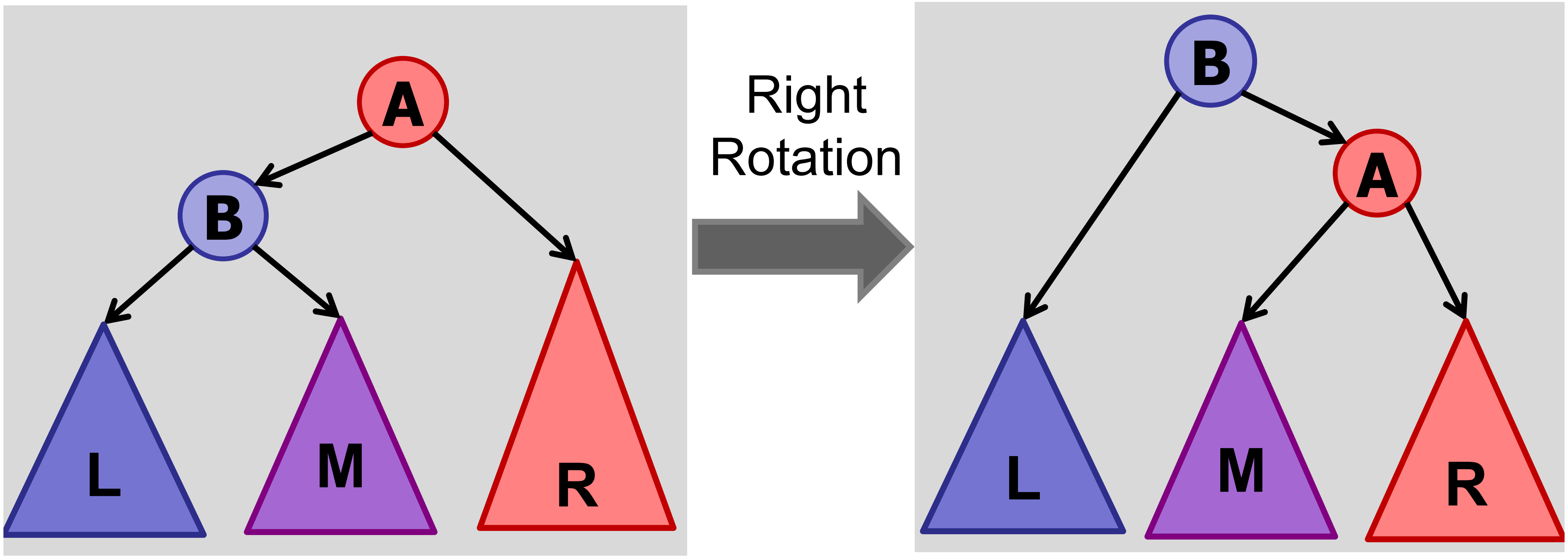
Tree Rotations



rotate-right requires a left child

rotate-left requires a right child

Tree Rotations



After insert:

Use tree rotations to restore balance.

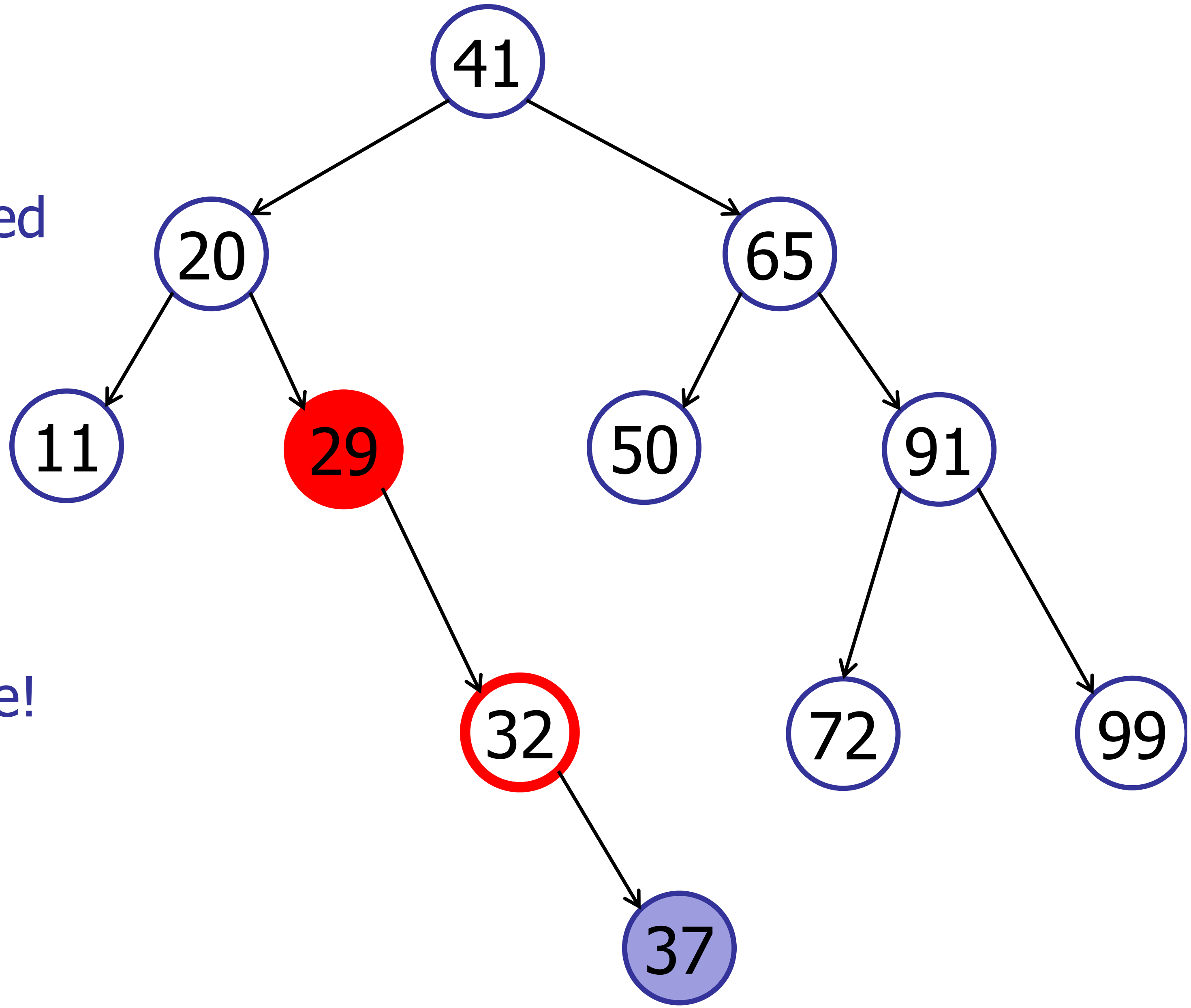
Height is out-of-balance by 1

Inserting in an AVL Tree

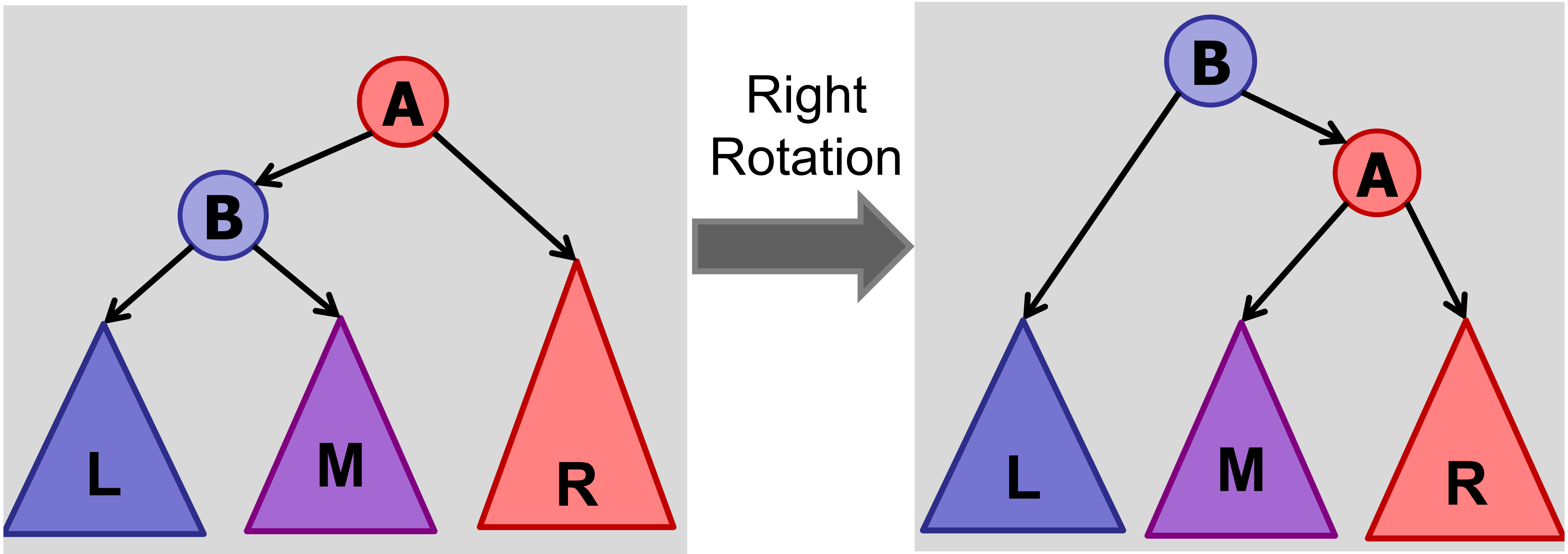
insert(37)

No longer balanced after insertion!

Need to rebalance!



Tree Rotations

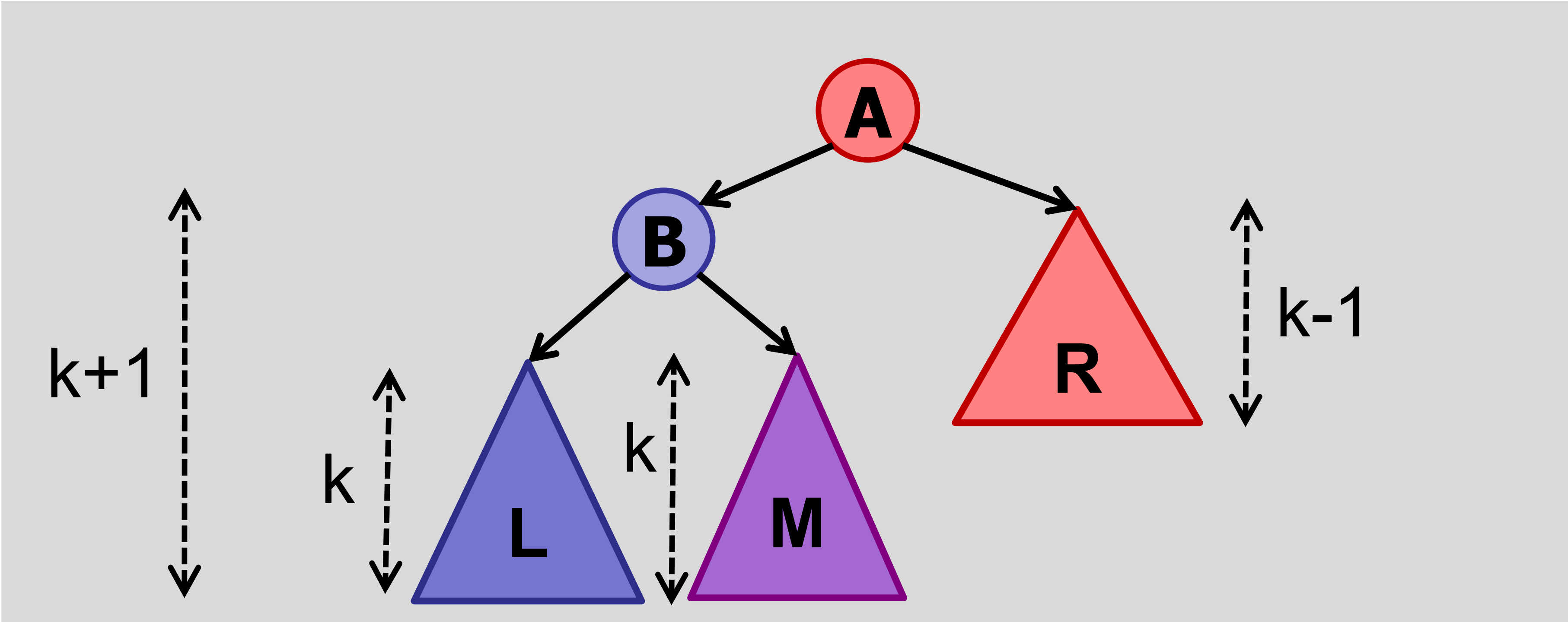


Use tree rotations to restore balance.

After insert, start at bottom, work your way up.

Assume tree is **LEFT-heavy**.

Tree Rotations (Left Heavy)

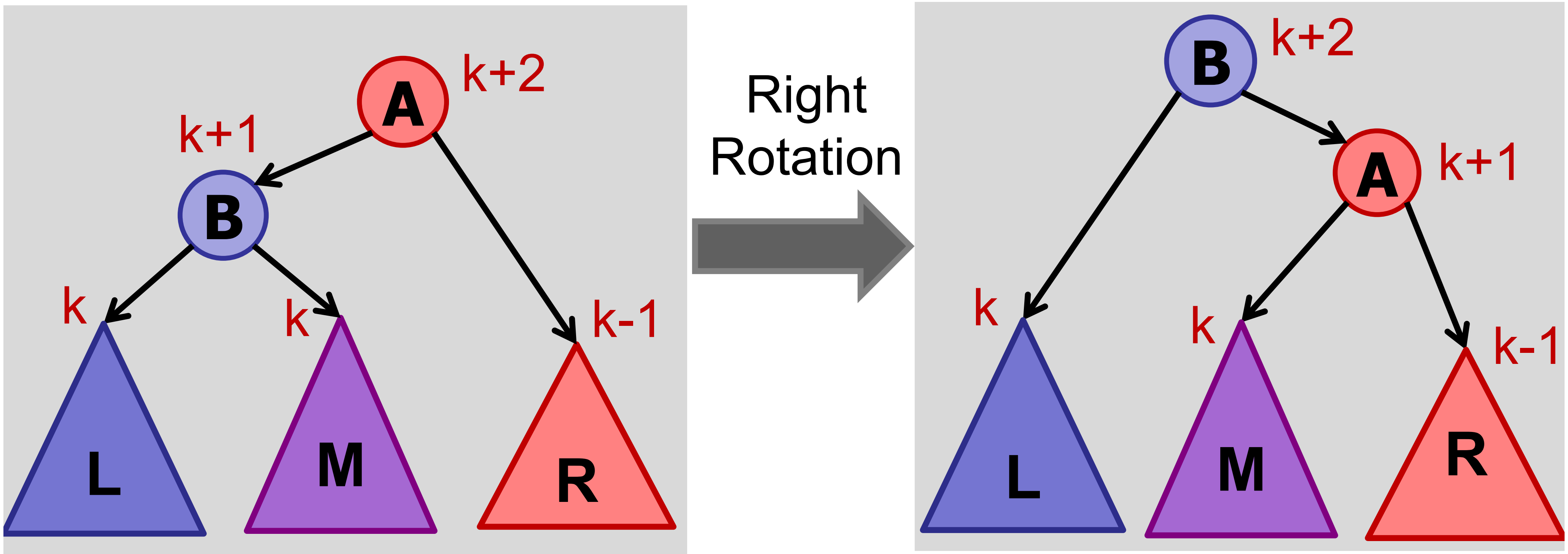


Assume **A** is the lowest node in the tree violating balance property.

Case 1: **B** is balanced : h(**L**) = h(**M**)

h(**R**) = h(**M**) – 1

Tree Rotations

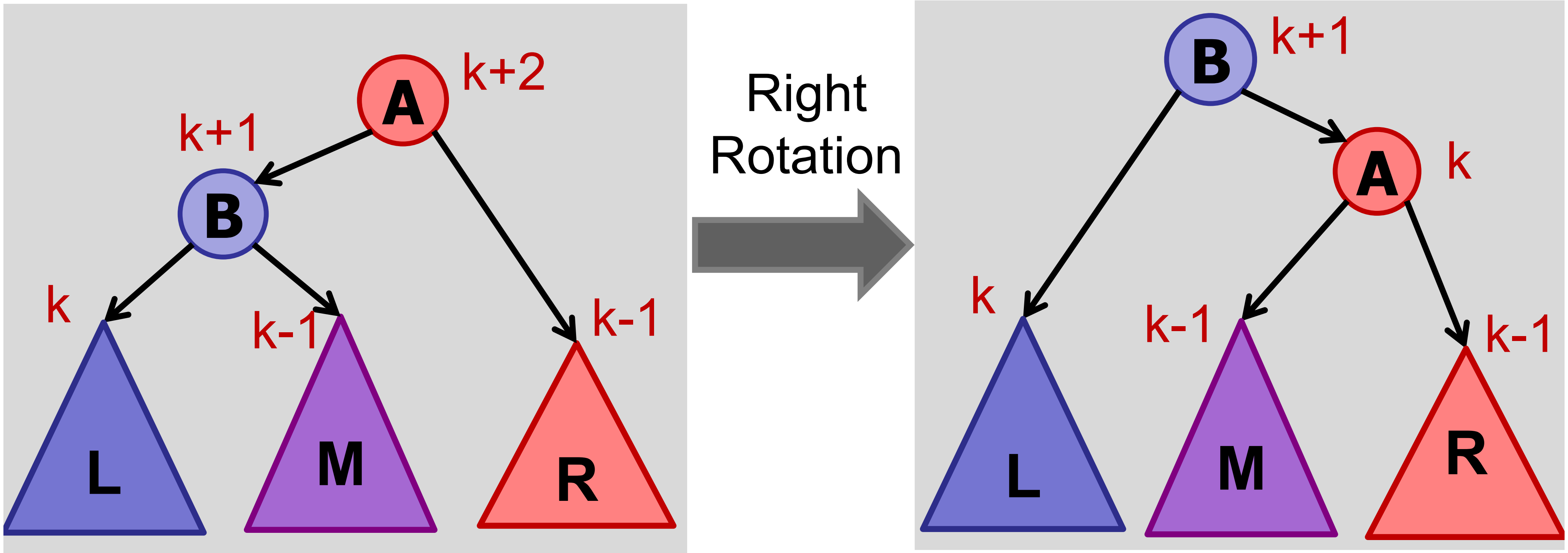


right-rotate:

Case 1: **B** is balanced : h(**L**) = h(**M**)

h(**R**) = h(**M**) – 1

Tree Rotations

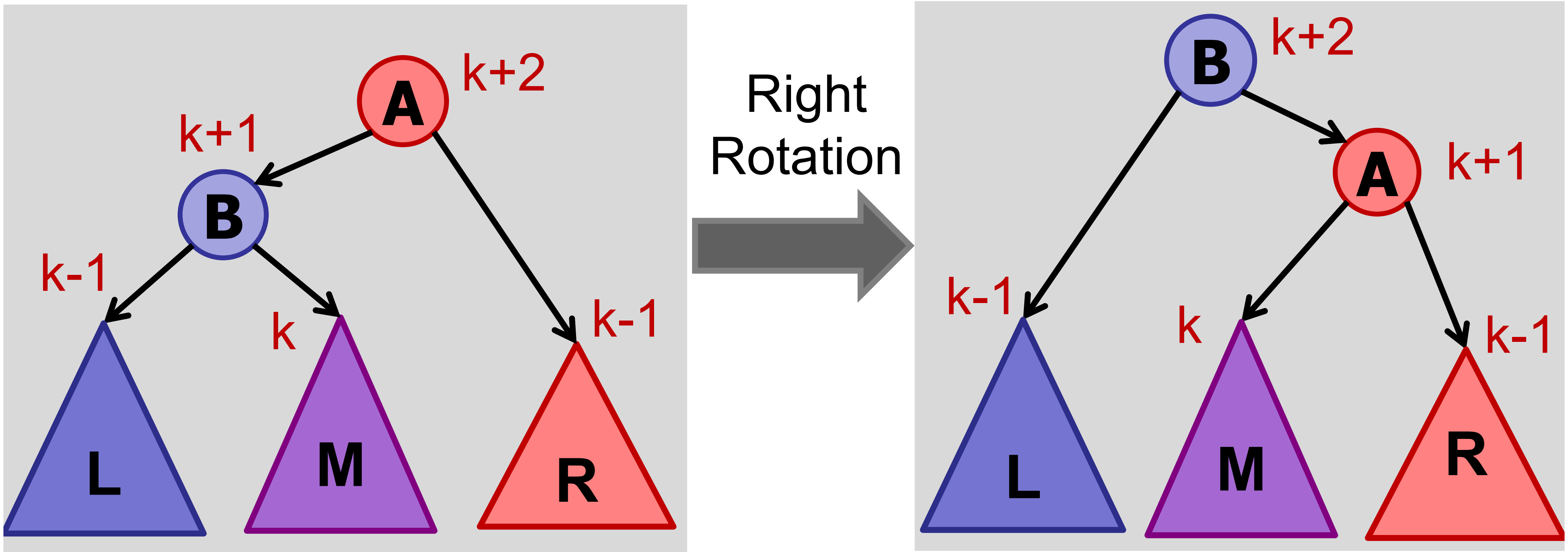


right-rotate:

Case 2: **B** is left-heavy: h(**L**) = h(**M**) +1

h(**R**) = h(**M**)

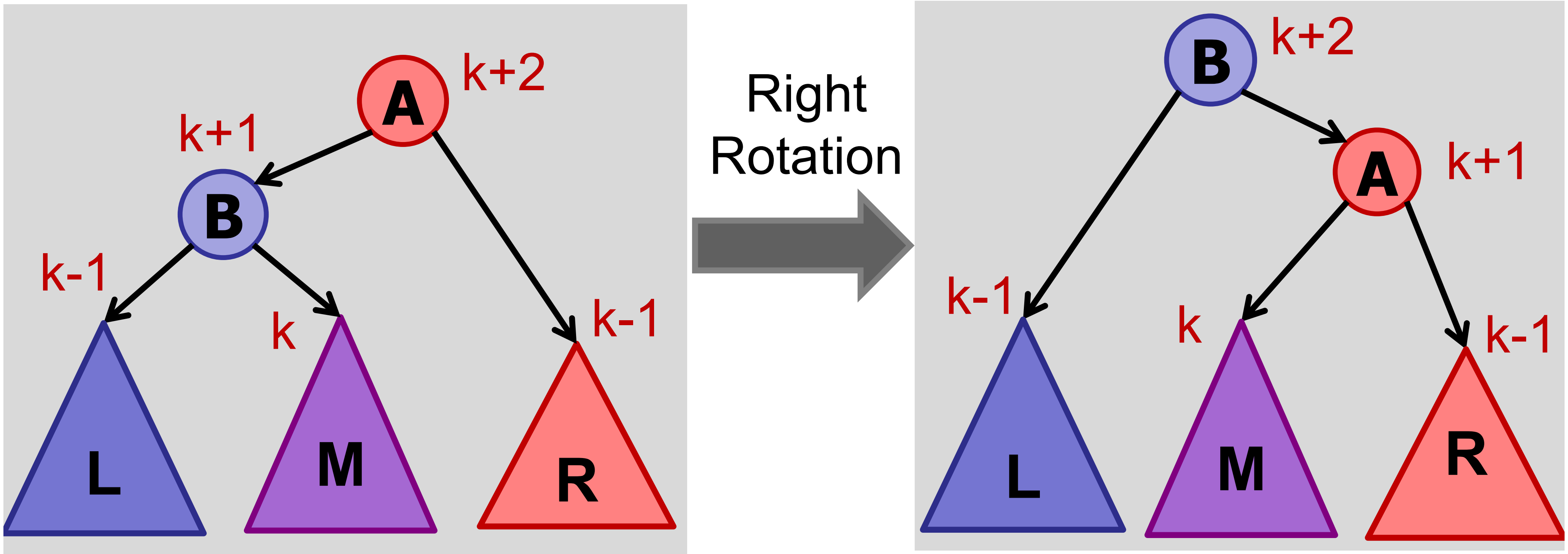
Tree Rotations



right-rotate:

Case 3: **B** is right-heavy: h(**L**) = h(**M**) – 1

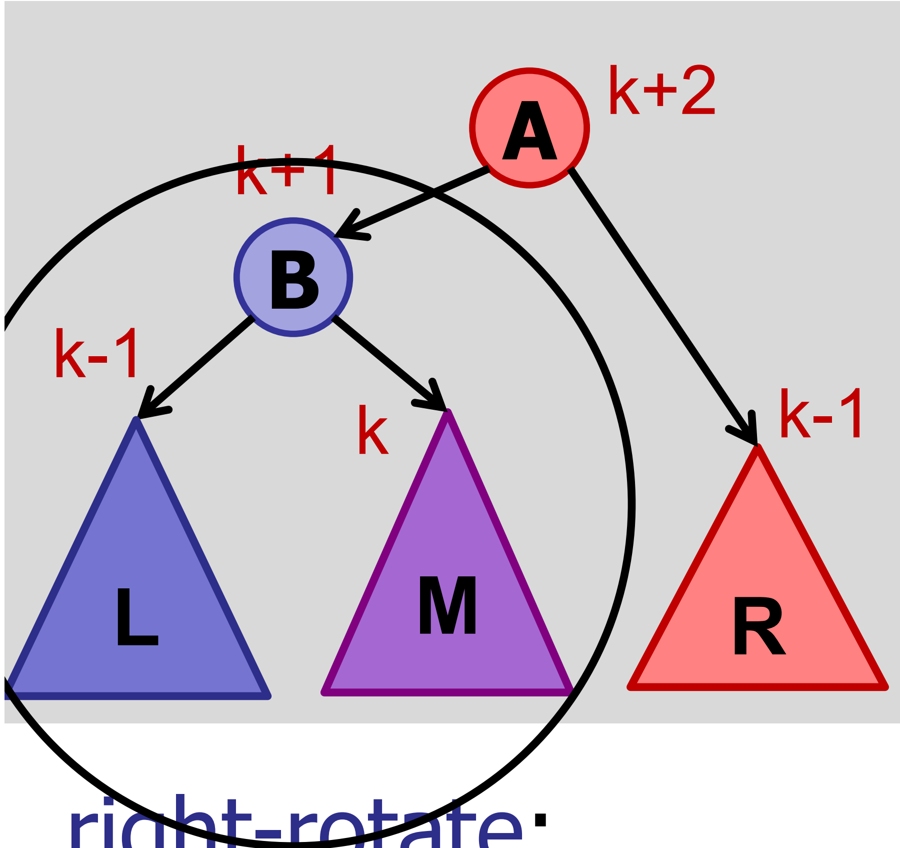
h(**R**) = h(**L**)



Are we done?

1. Yes.
2. No.
3. Maybe.

Tree Rotations



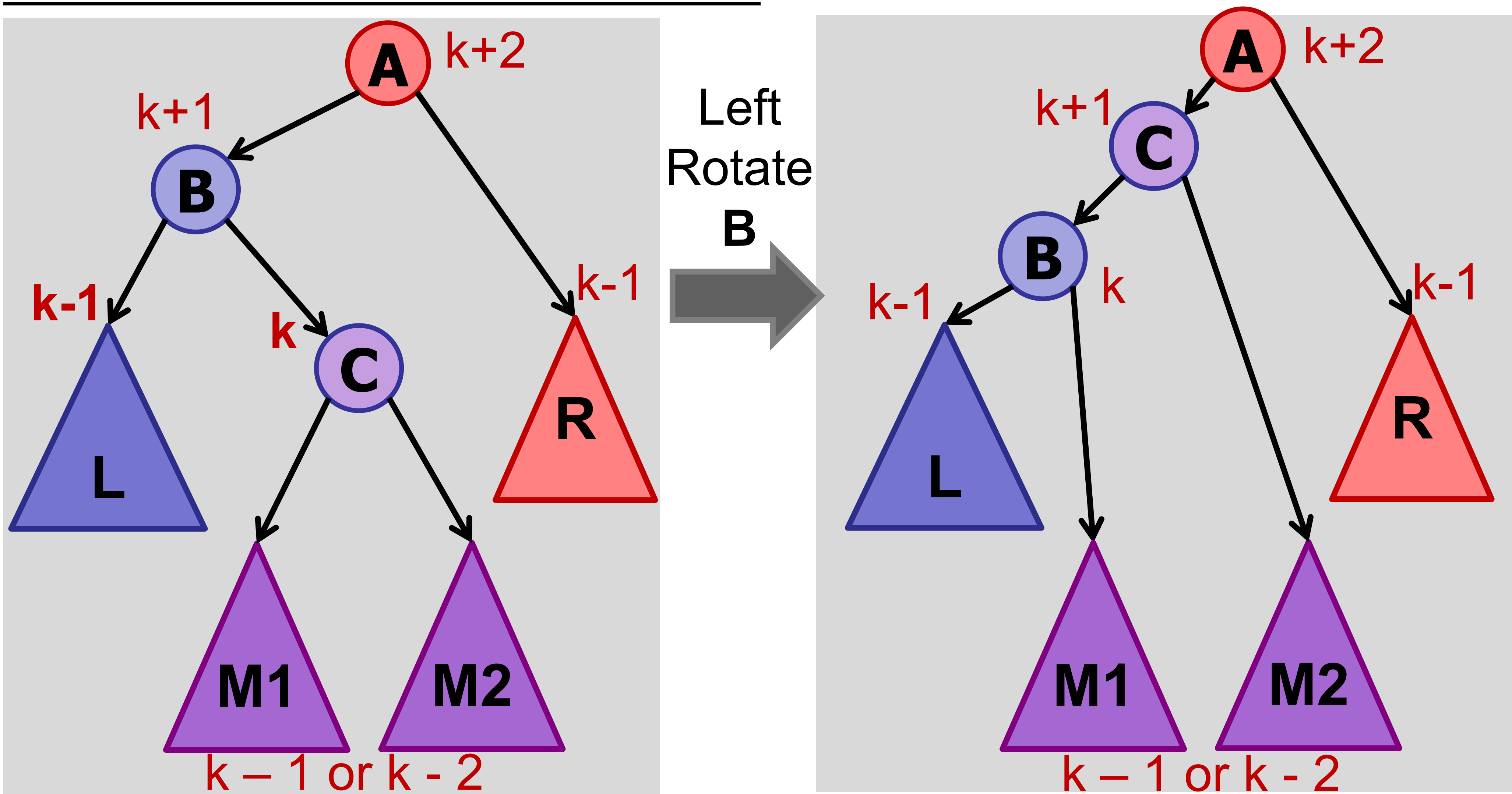
Let's do something first before we right-rotate(A)

right-rotate:

Case 3: **B** is right-heavy: h(**L**) = h(**M**) – 1

h(**R**) = h(**L**)

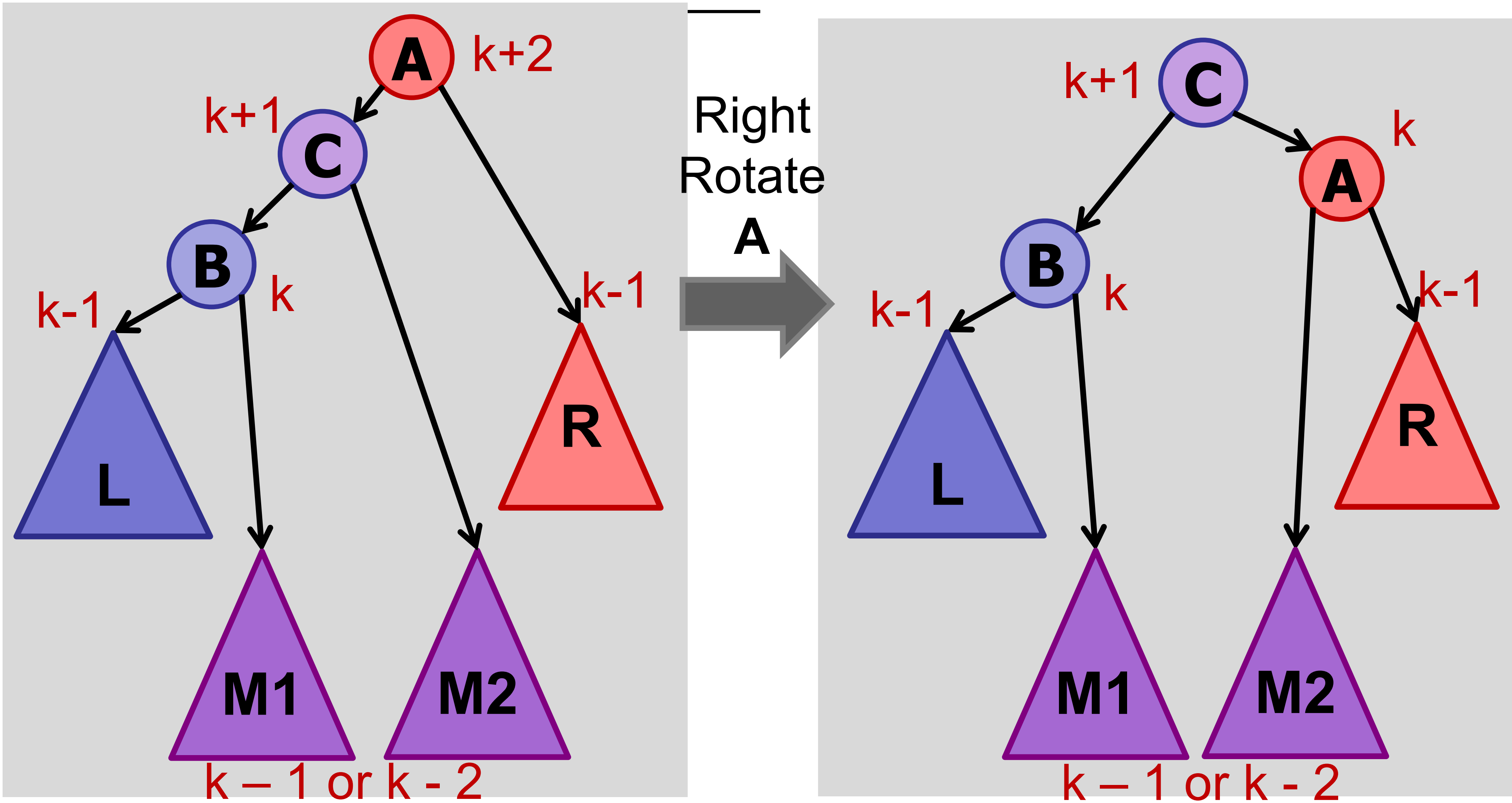
Tree Rotations



Left-rotate B

After left-rotate B: **A** and **C** still out of balance.

Tree Rotations



After right-rotate A: all in balance.

Rotations

Summary:

If v is out of balance and left heavy:

1. v.left is balanced: right-rotate(v)
2. v.left is left-heavy: right-rotate(v)
3. v.left is right-heavy: left-rotate(v.left)
   right-rotate(v)

If v is out of balance and right heavy:

Symmetric three cases….

How many rotations do you need after an insertion (in the worst case)?

1. 1
2. 2
3. 4
4. log(n)
5. 2log(n)
6. n

How many rotations do you need after an insertion (in the worst case)?

1. 1
2. 2
3. 4
4. log(n)
5. 2log(n)
6. n

<u>Question:</u>

Why isn't it 2log(n)?

Insert in AVL Tree

Summary:

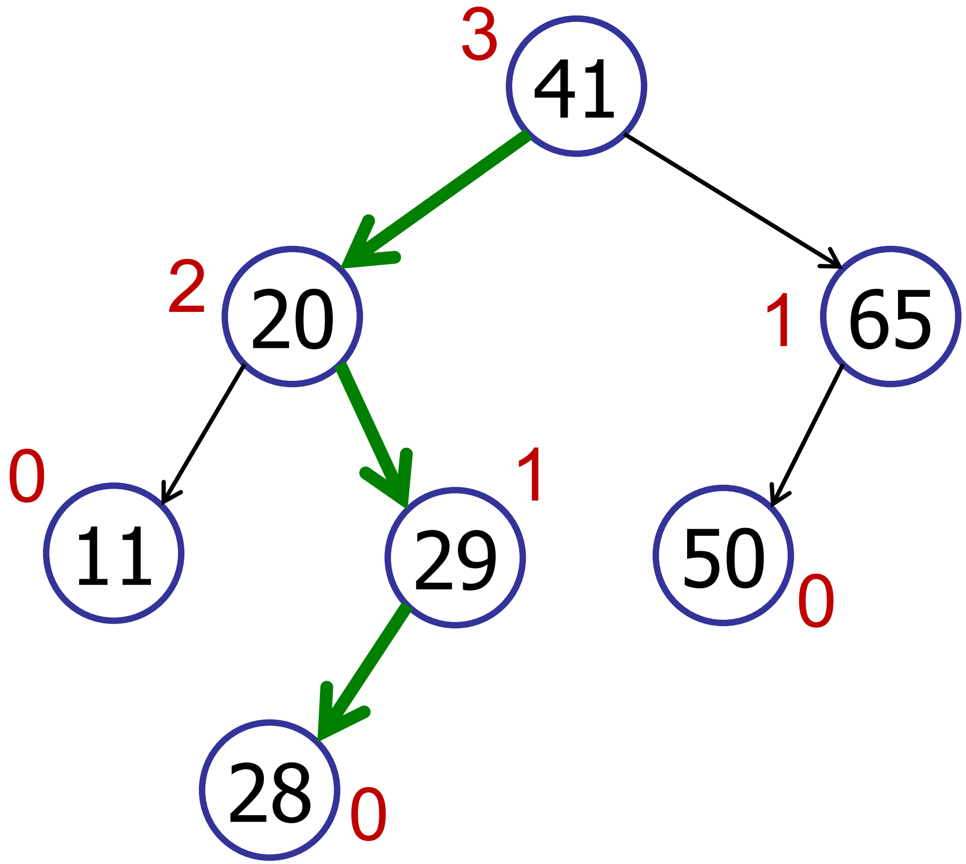
- Insert key in BST.
- Walk up tree:
  - At every step, check for balance.
  - If out-of-balance, use rotations to rebalance.

Note: only need to perform two rotations

- Why?
- In each case, reduce height of sub-tree by 1
- What about Case 1, above?

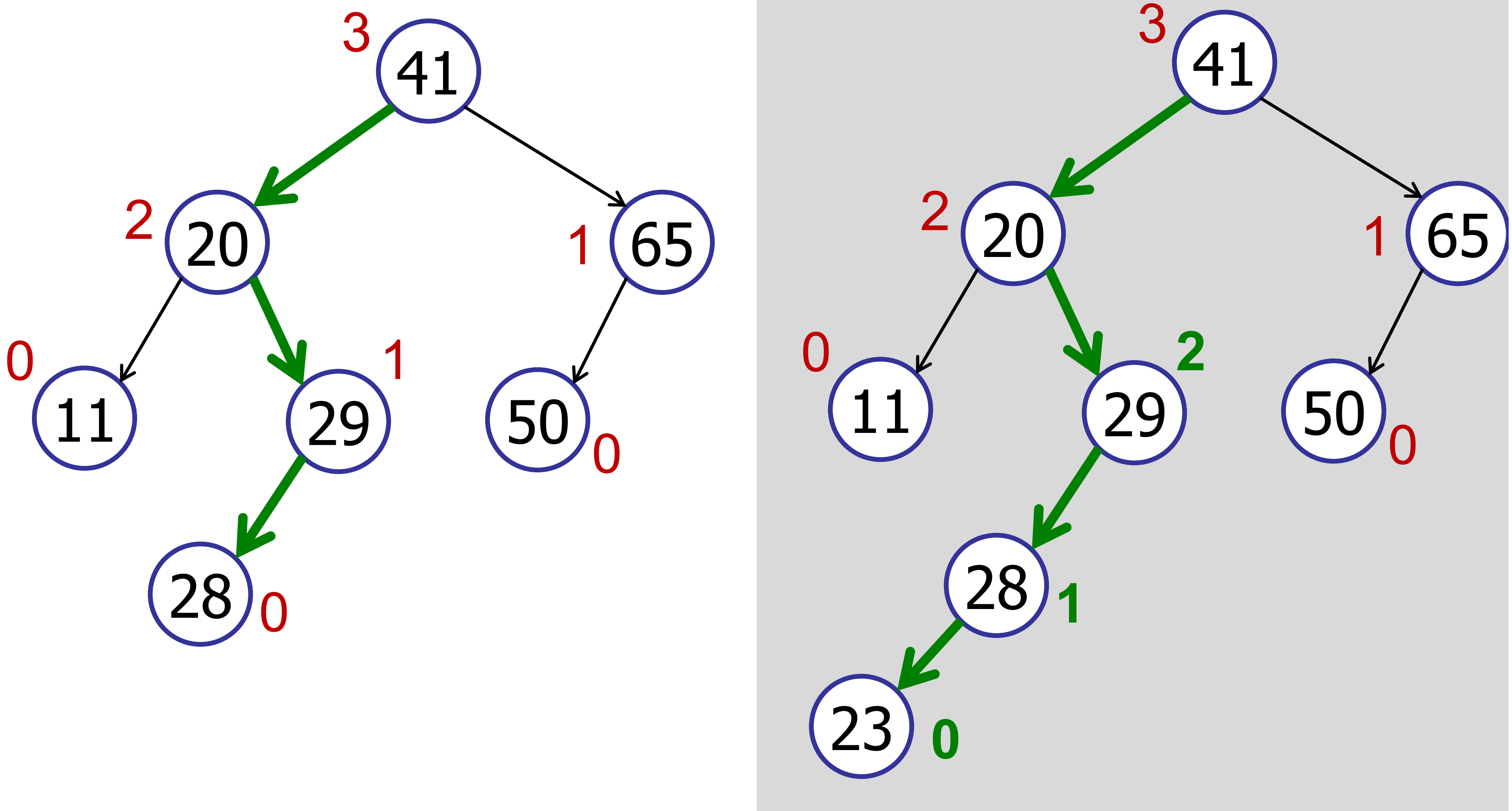
Example

insert(23)



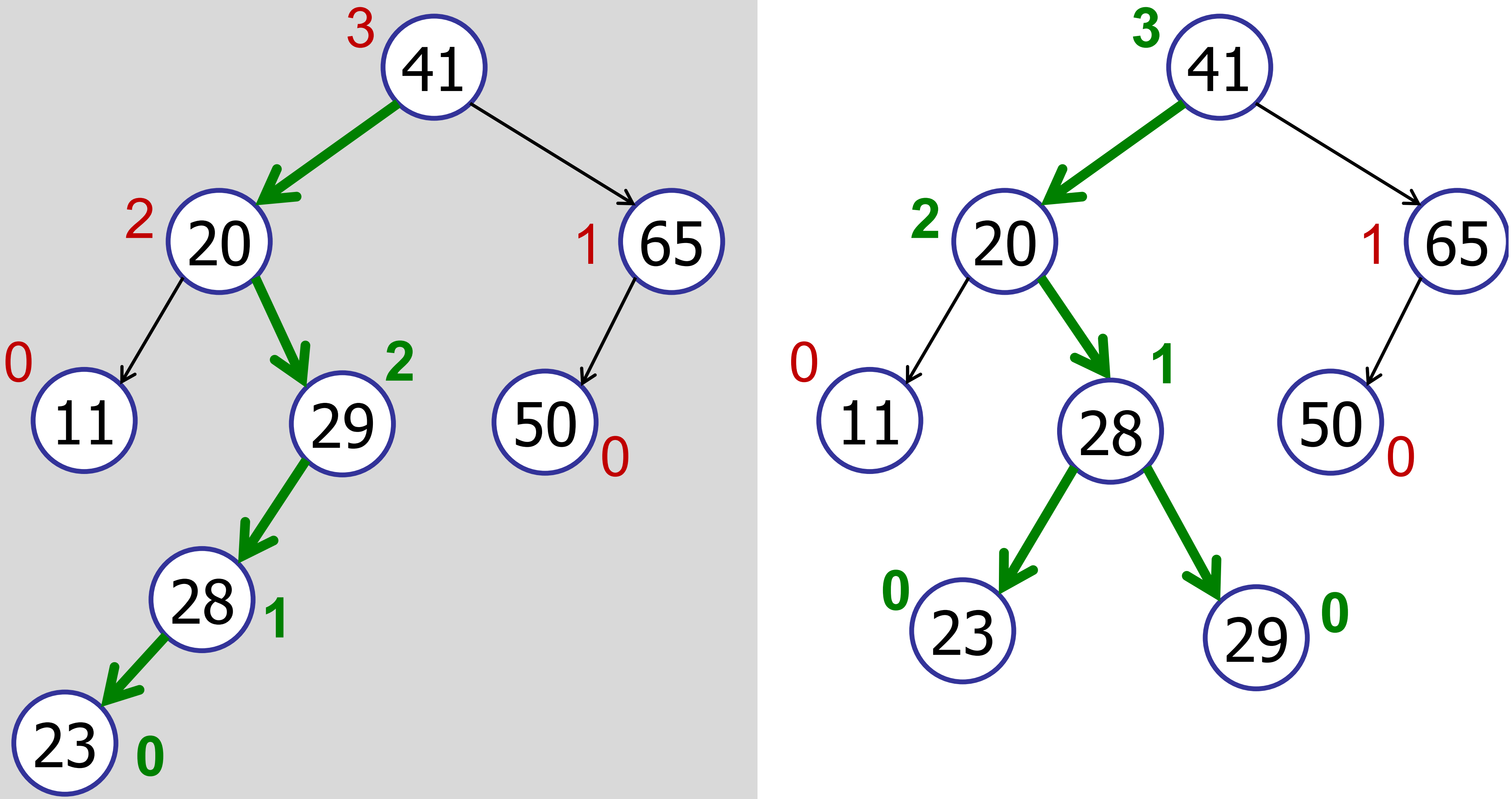
Example

insert(23)



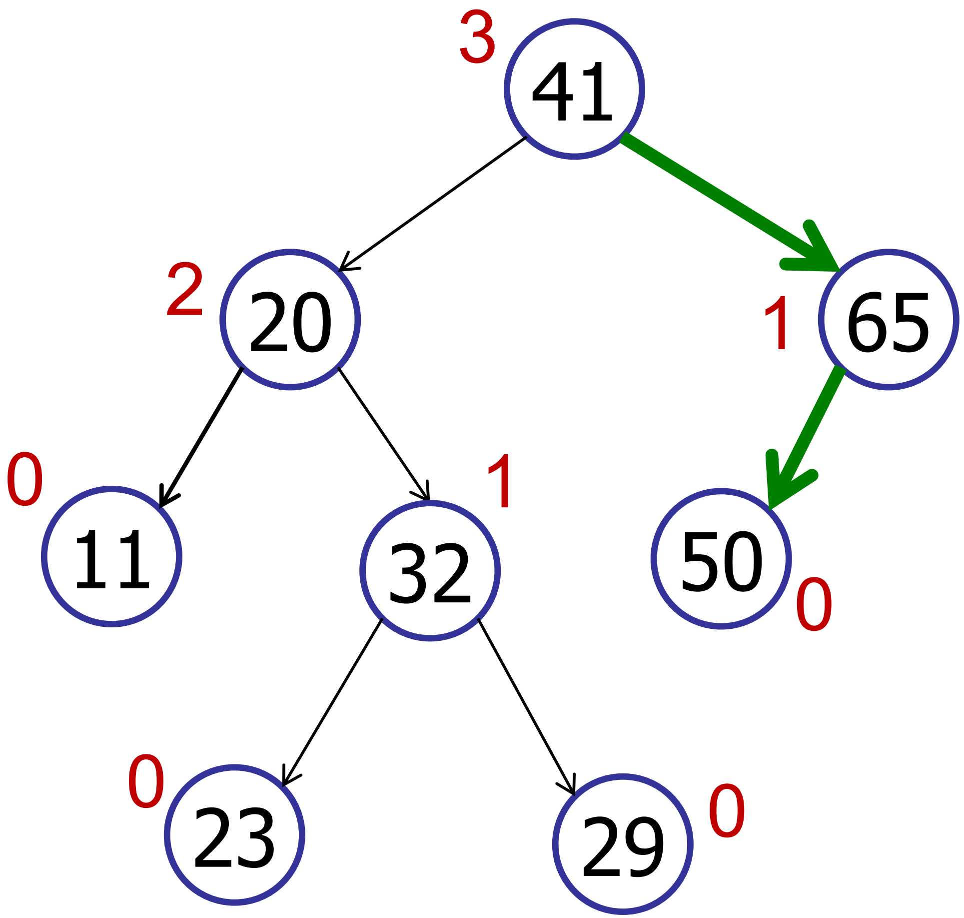
Example

right-rotate(29)



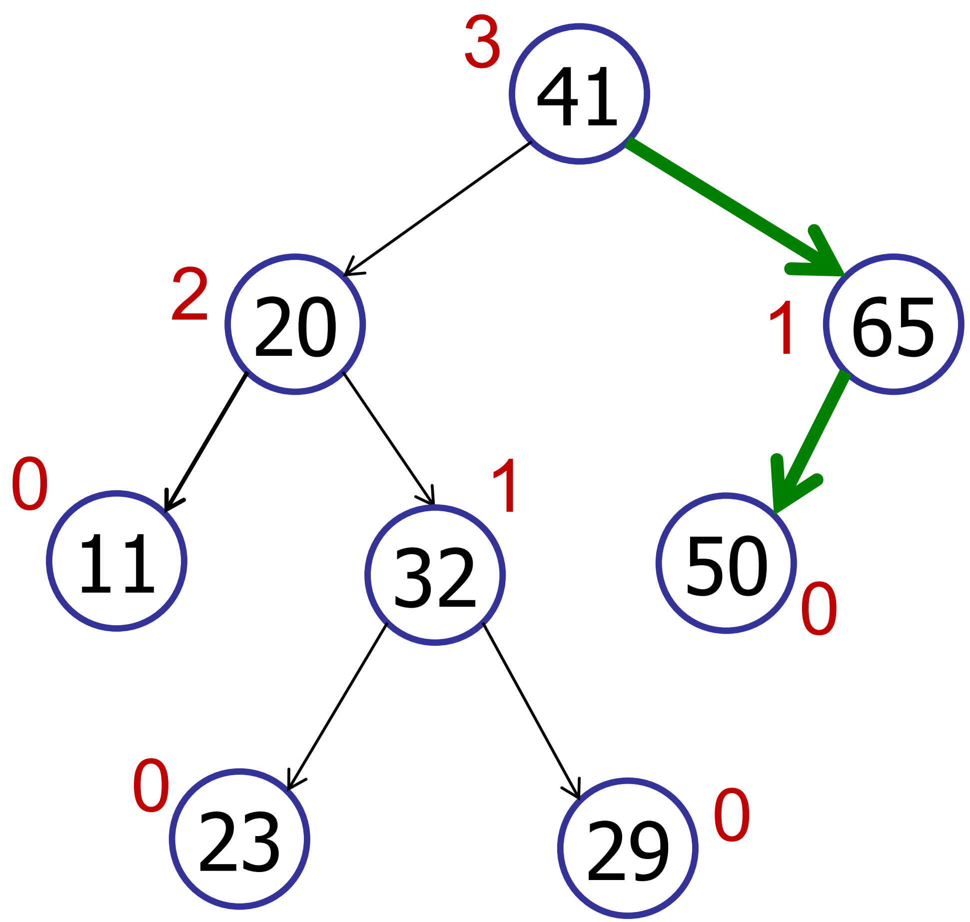
Example

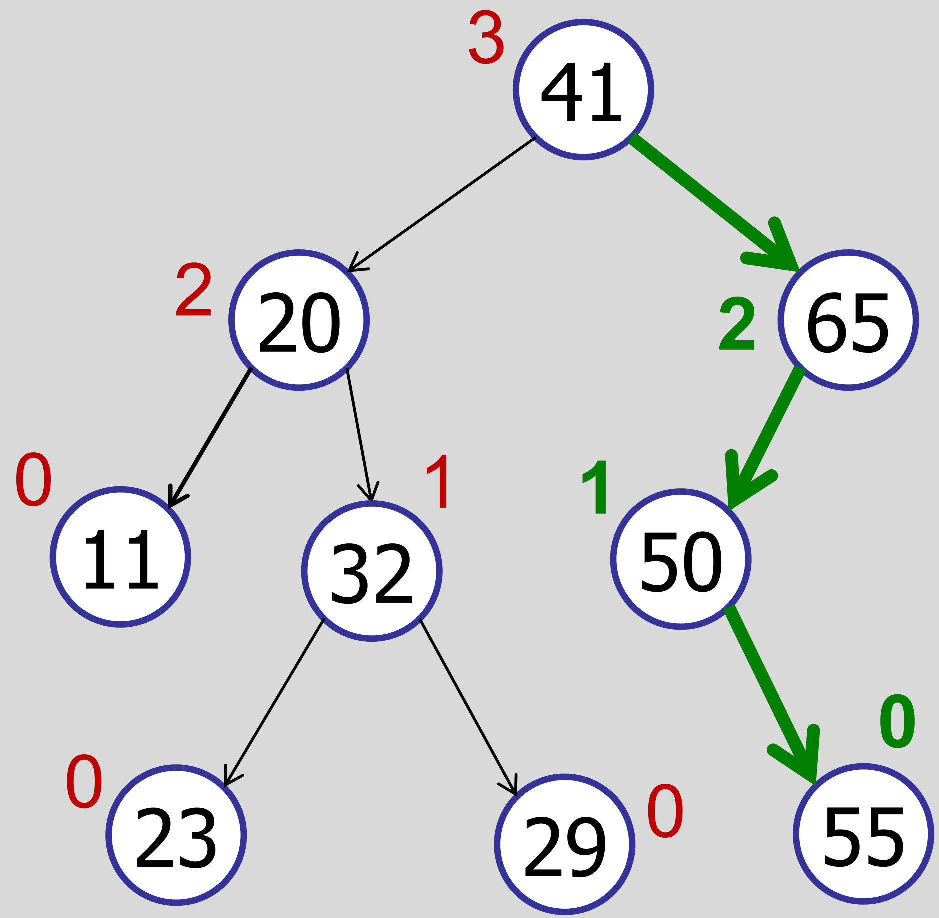
insert(55)



Example

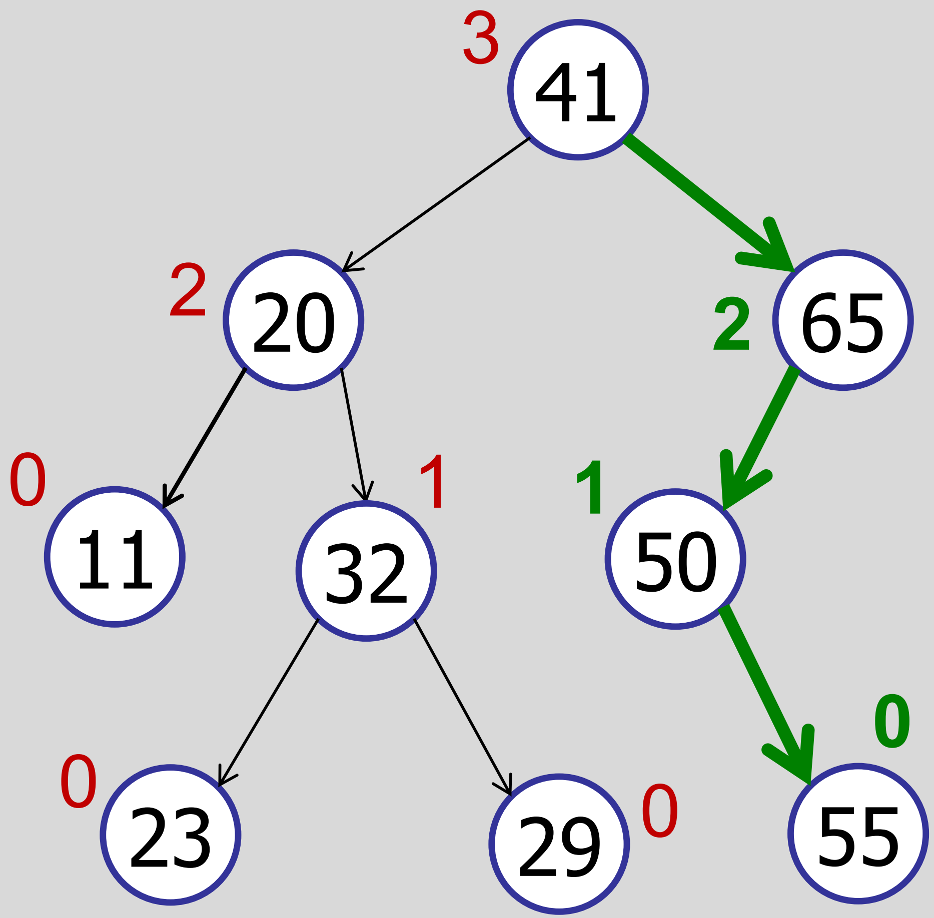
insert(55)

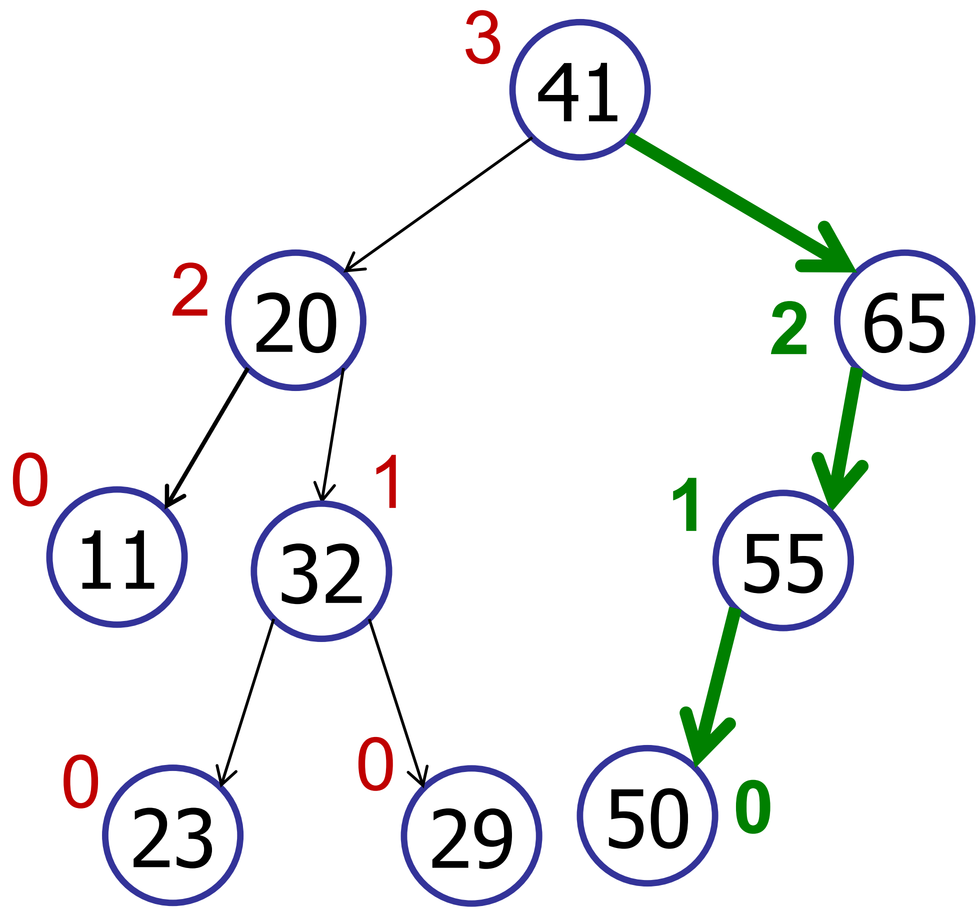




Example

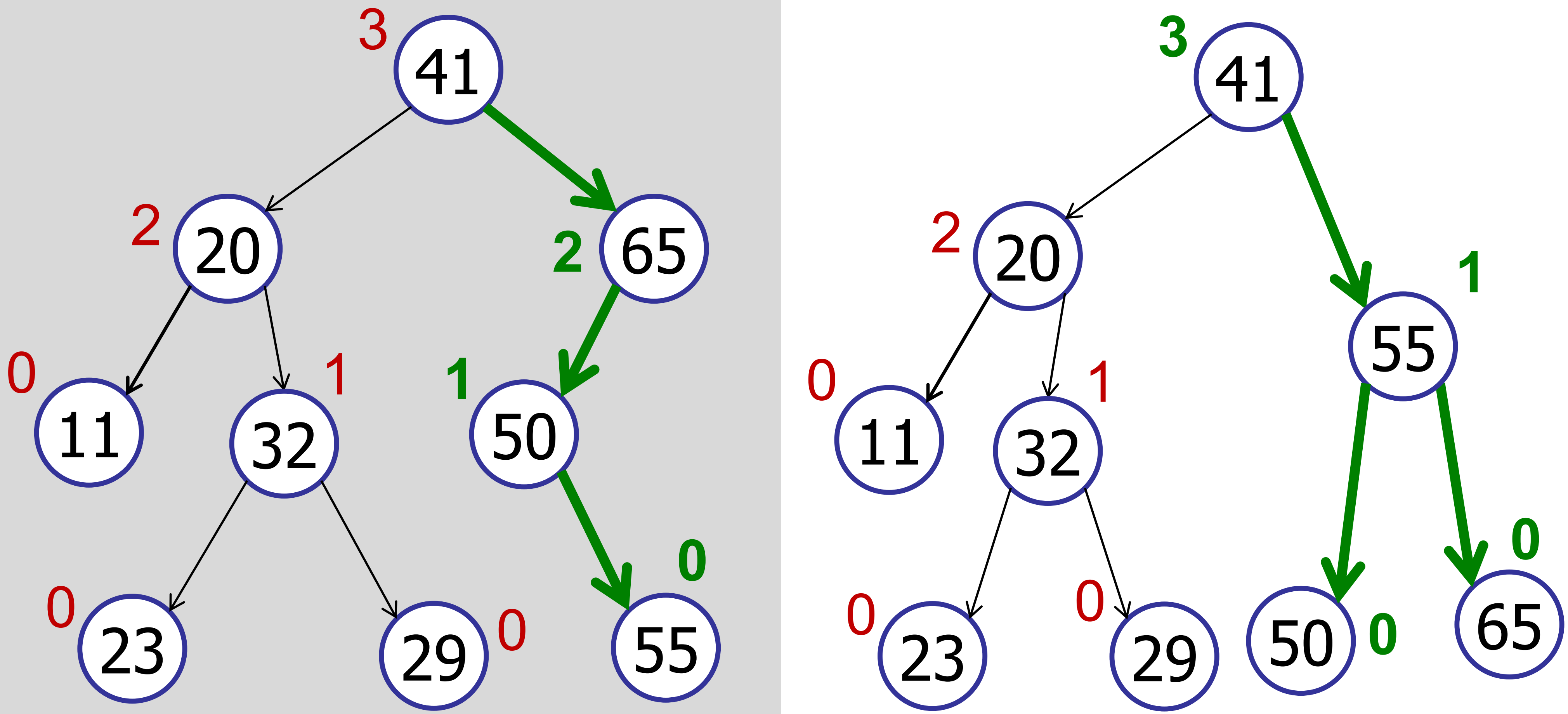
left-rotate(50)





Example

right-rotate(65)

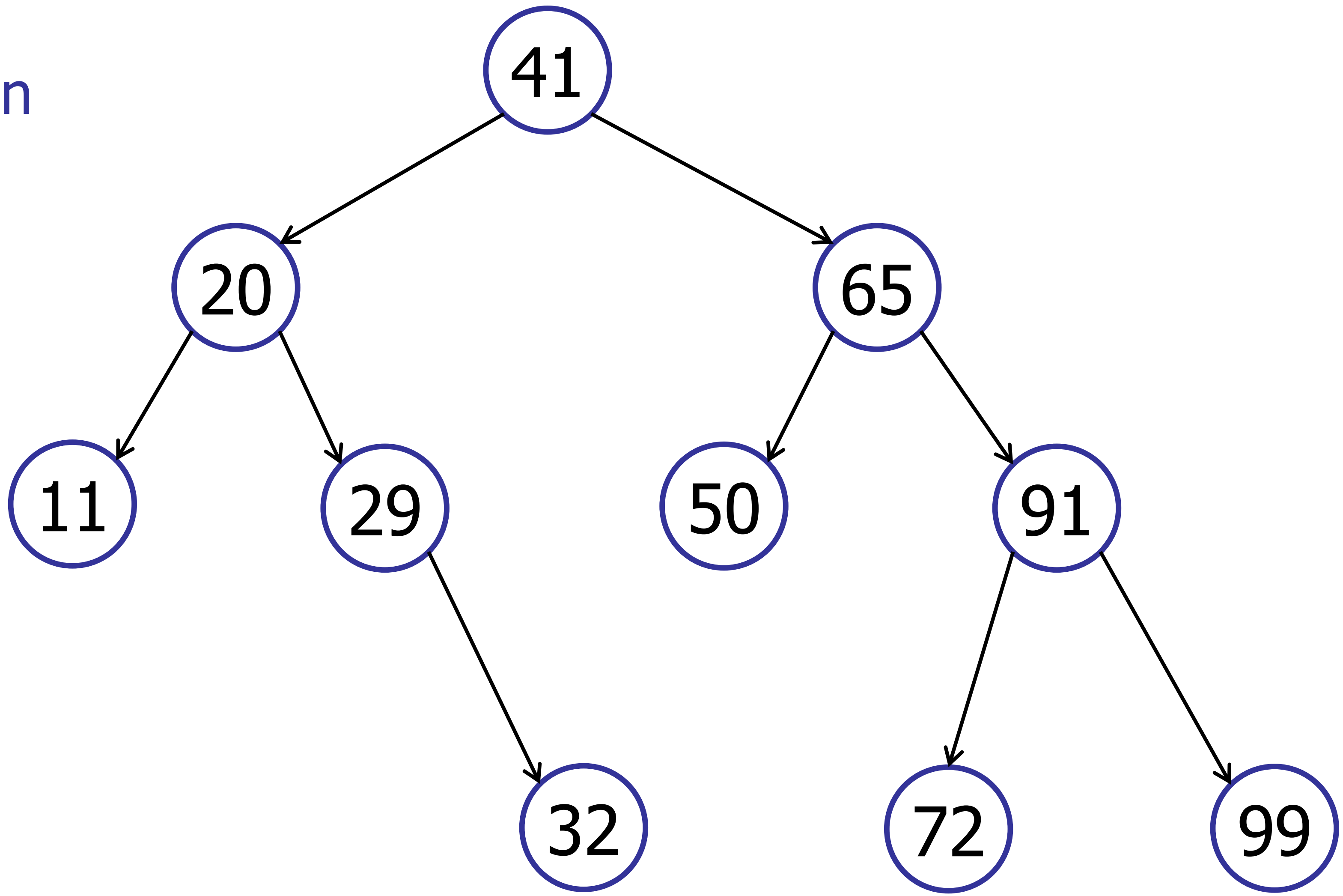


Binary Search Tree

delete(v)

Three cases:

1. No children
2. 1 child
3. 2 children



Binary Search Tree

delete(v)

Three cases:

1. No children:
   - remove v

2. 1 child:
   - remove v
   - connect child(v) to parent(v)

3. 2 children
   - Swap v with x = successor(v)
   - delete(v)
     - (which is in the original position of the successor)

Will this cause more calls for the function delete()?

Binary Search Tree

delete(v)

1. If v has two children, swap it with its successor.

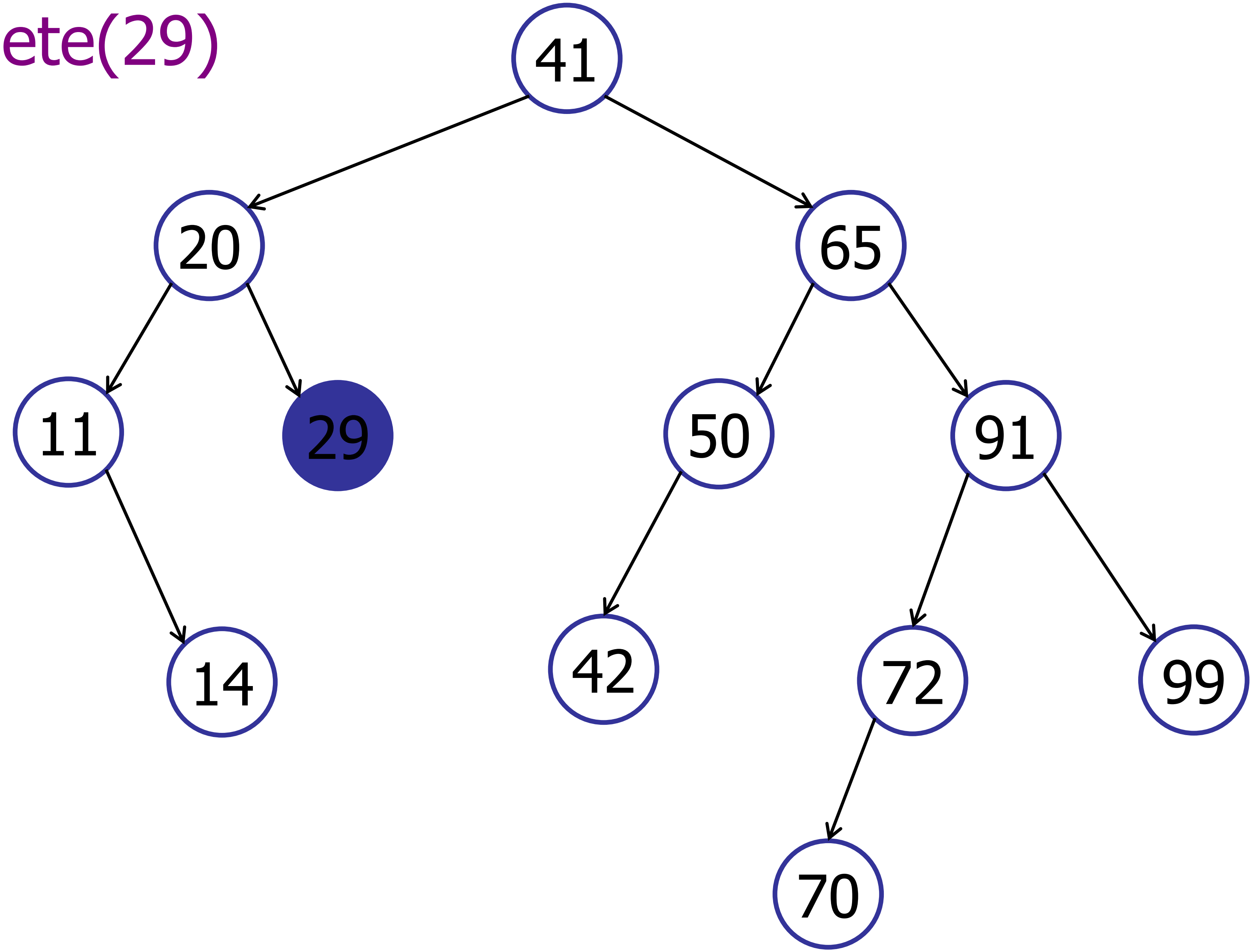
2. Delete node v from binary tree (and reconnect children).

3. For every ancestor of the deleted node:
   - Check if it is height-balanced.
   - If not, perform a rotation.
   - Continue to the root.

Deletion may take up to log(n) rotations.

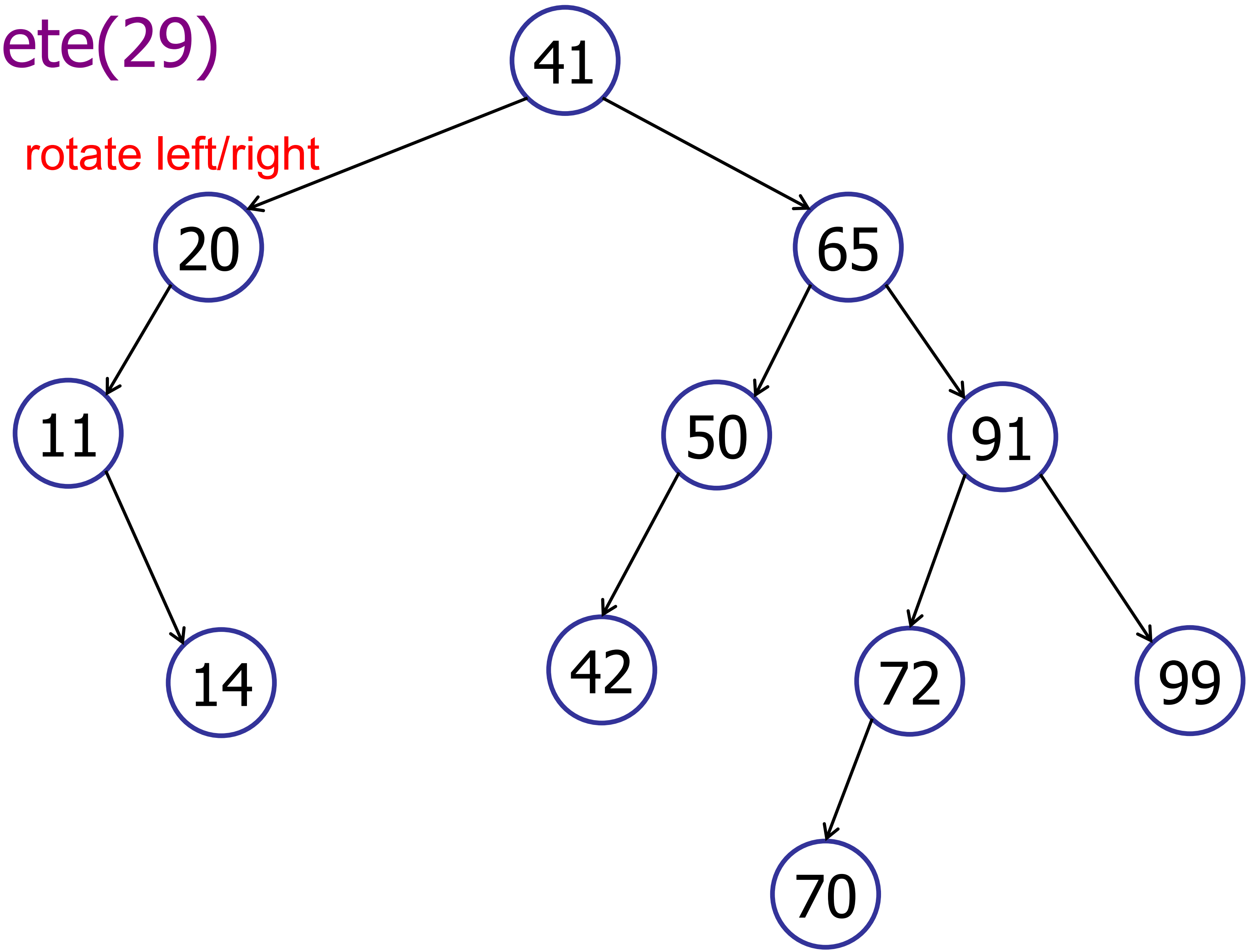
Binary Search Tree

delete(29)



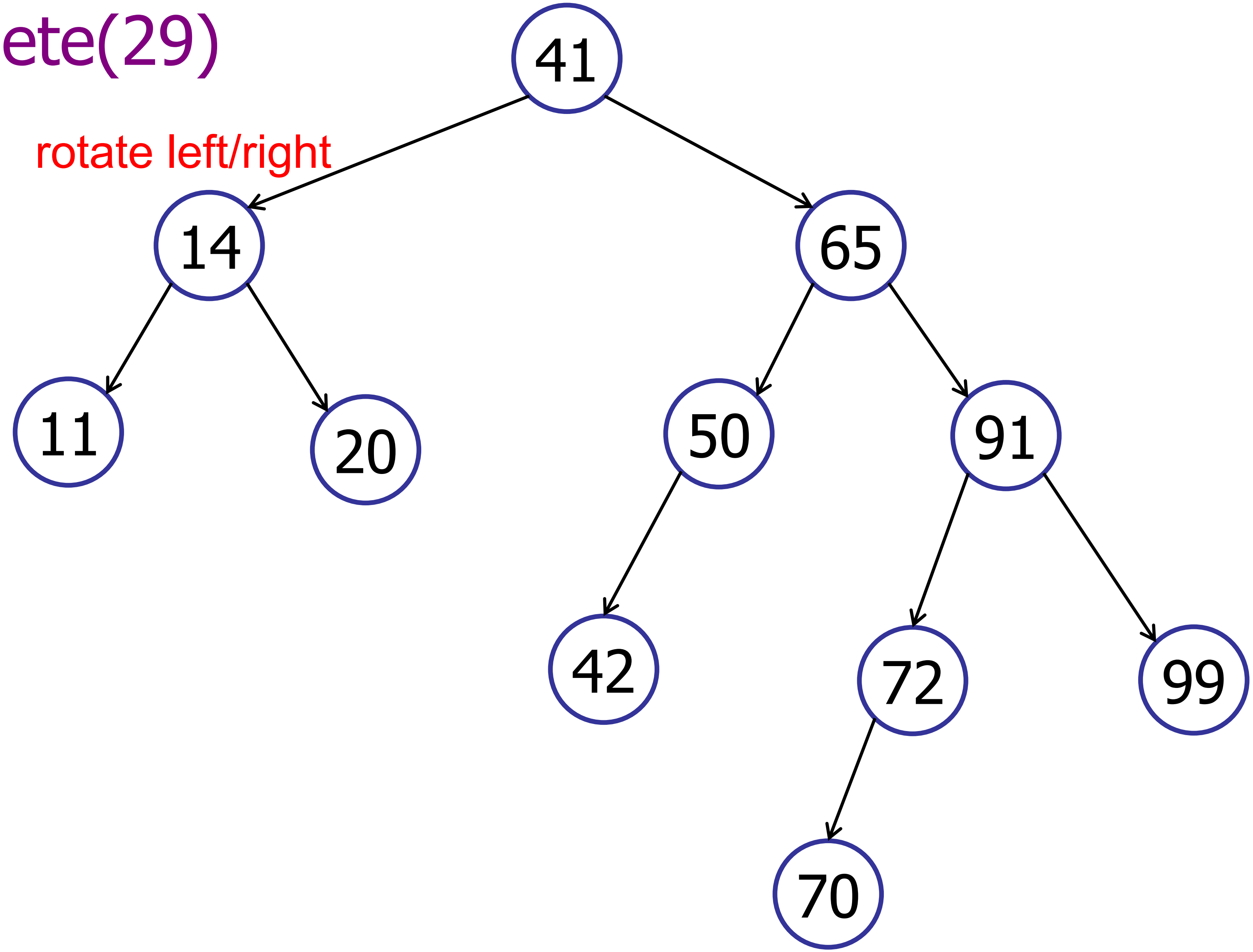
Binary Search Tree

delete(29)



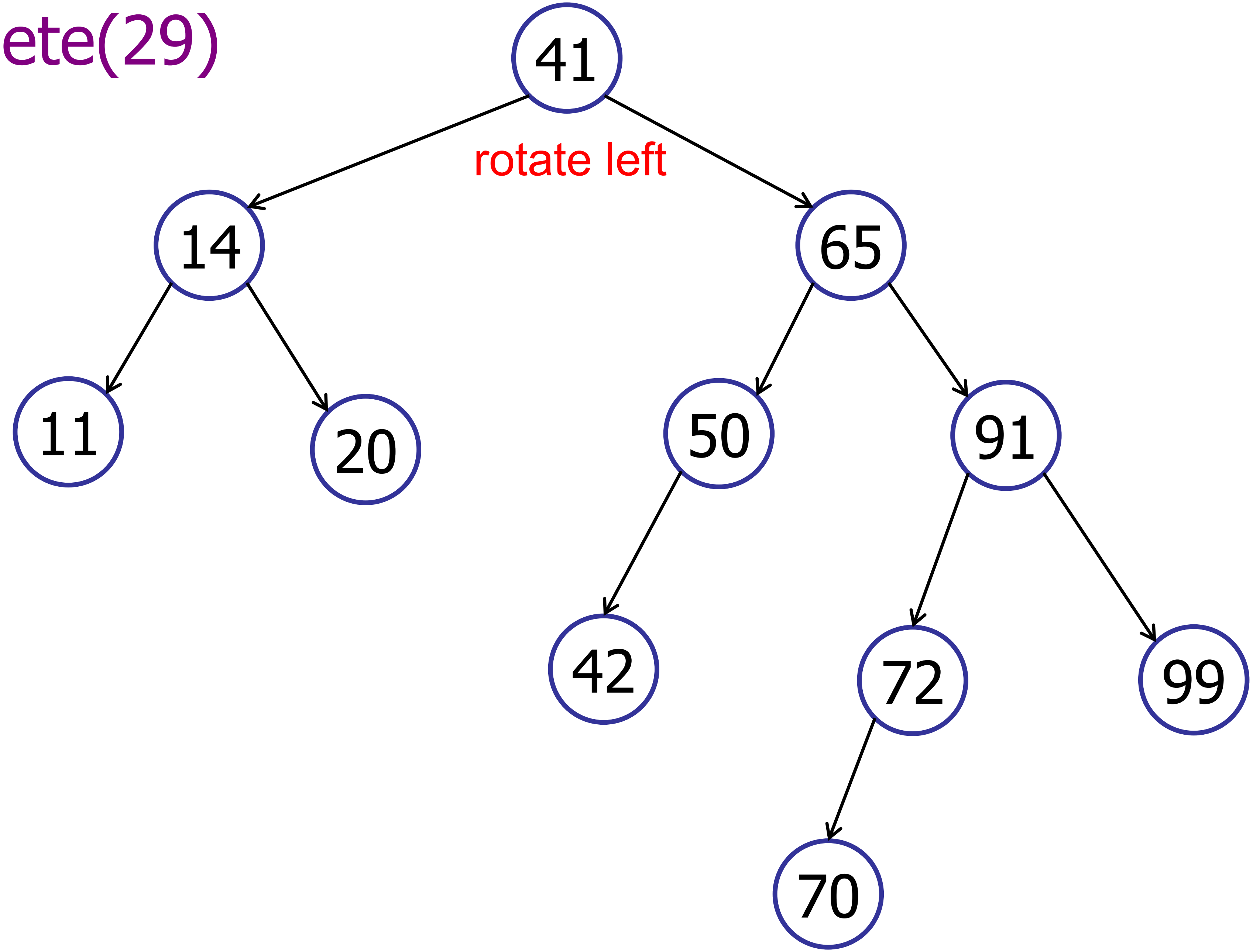
Binary Search Tree

delete(29)



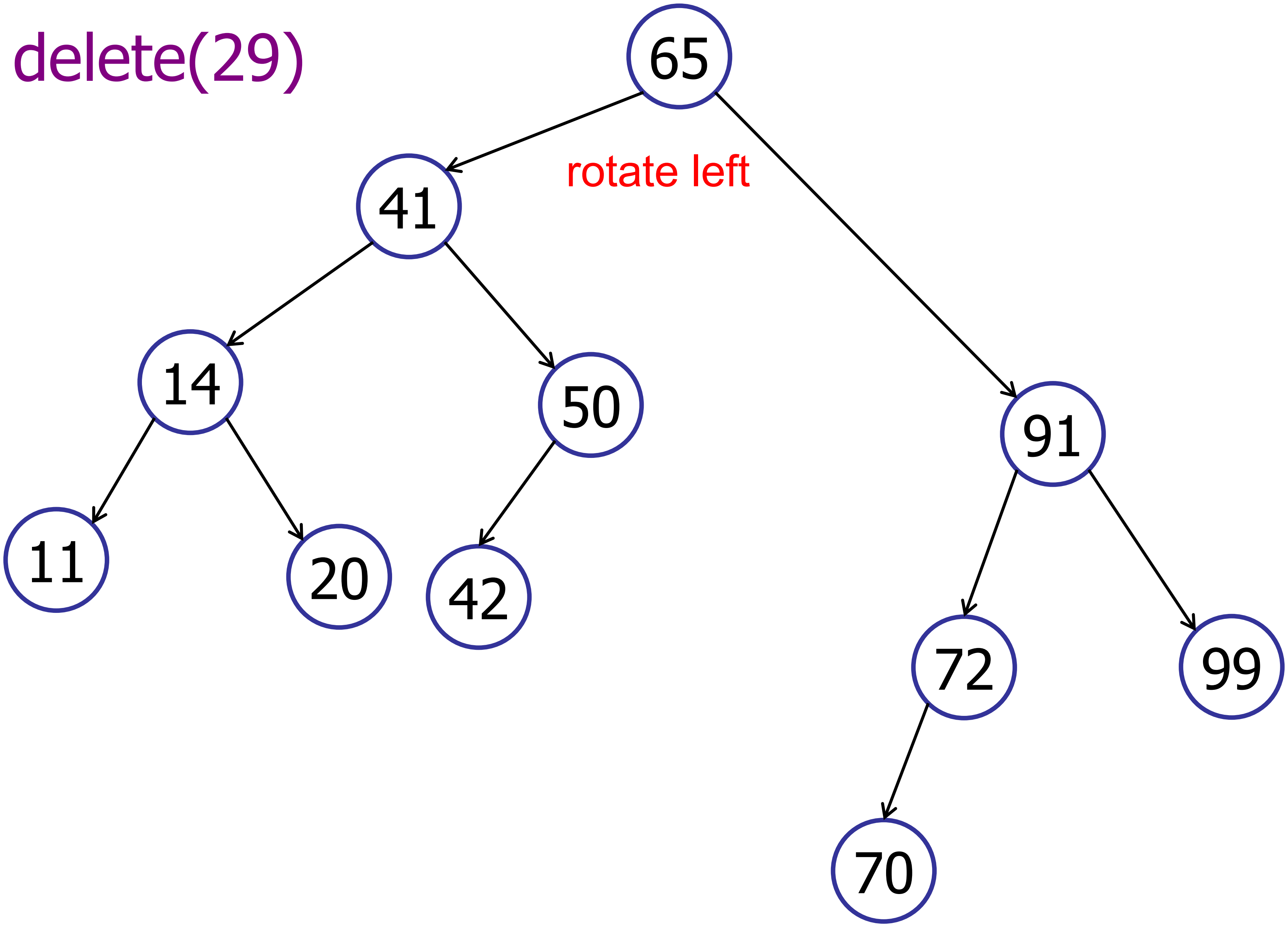
Binary Search Tree

delete(29)



Binary Search Tree

delete(29)



Quick review: a rotation costs:

1. $O(1)$
2. $O(\log n)$
3. $O(n)$
4. $O(n^2)$
5. $O(2^n)$

Every insertion requires 1 or 2 rotations?

1. Yes
2. No
3. I don’t know

A tree is balanced if every node’s children differ in height be at most 1?

✓ 1. Yes
2. No
3. I don’t know

AVL Trees

What if you do not remove deleted nodes?

– Mark a node "deleted" and leave it in the tree.

Logical deletes:

– Performance degrades over time.

– Clean up later? (Amortized performance…)

AVL Trees

What if you do not want to store the height in every node?

- Only store difference in height from parent.

Balanced Search Trees

Many different flavors of balanced search trees

- AVL trees (Adelson-Velsii & Landis, 1962)
- B-trees / 2-3-4 trees (Bayer & McCreight, 1972)
- BB[α] trees (Nievergelt & Reingold 1973)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Treaps (Seidel and Aragon 1996)
- Skip Lists (Pugh 1989)

Balanced Search Trees

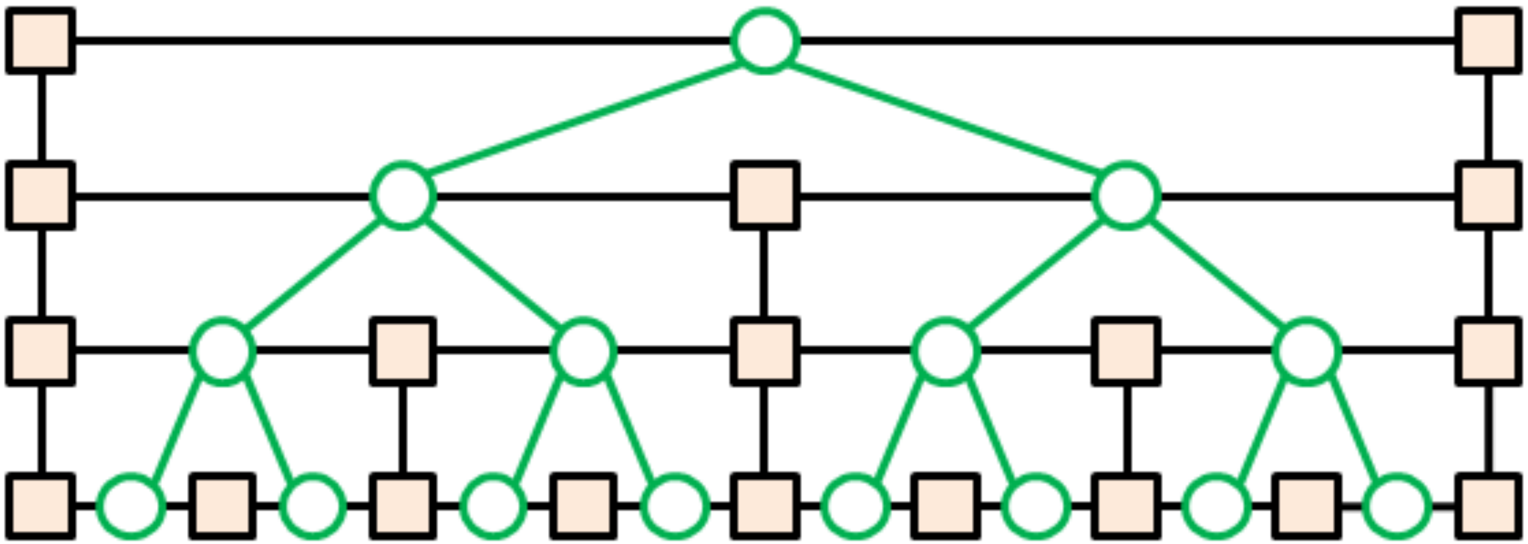
Red-Black trees

- More loosely balanced
- Rebalance using rotations on insert/delete
- O(1) rotations for all operations.
- Java TreeSet implementation
- Faster (than AVL) for insert/delete
- Slower (than AVL) for search

Balanced Search Trees

Skip Lists and Treaps

- Randomized data structures
- Random insertions => balanced tree
- Use randomness on insertion to maintain balance



Plan of the Day

Trees

- Terminology
- Traversals
- Operations

Balanced Trees

- Height-balanced binary search trees
- AVL trees
- Rotations