

NATIONAL UNIVERSITY OF SINGAPORE
Department of Statistics and Applied Probability

(2021/22) Semester 1

ST2334 Probability and Statistics

Tutorial 7

1. Suppose that X and Y are random variables having the joint probability distribution below.

$f_{X,Y}(x, y)$		x	
		2	4
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

- Determine whether or not X and Y are independent.
 - Find $E(Y|X = 2)$.
 - Find $E(X|Y = 3)$.
 - Find $E(2X - 3Y)$.
 - Find $E(XY)$.
 - Find $V(X)$ and $V(Y)$.
 - Find $\sigma_{X,Y}$ and $\rho_{X,Y}$.
2. Consider a ferry that can carry both buses and cars on a trip across a waterway. Each trip costs the owner approximately \$10. The fee for cars is \$3 and the fee for buses is \$8. Let X and Y denote the number of buses and cars, respectively, carried on a given trip. The joint distribution of X and Y is given below.

$f_{X,Y}(x, y)$		x		
		0	1	2
y	0	0.01	0.01	0.03
	1	0.03	0.08	0.07
	2	0.03	0.06	0.06
	3	0.07	0.07	0.13
	4	0.12	0.04	0.03
	5	0.08	0.06	0.02

Compute the expected value and variance of profit for the ferry trip.

3. A store operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y respectively, be the proportions of times that the drive-in and walk-in facilities are in use, and suppose that the joint density function of these random variables is given below.

$$f_{X,Y}(x, y) = \begin{cases} \frac{2}{3}(x + 2y), & 0 \leq x \leq 1, 0 \leq y \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

- Determine whether X and Y are independent.
- Find σ_X^2
- Find σ_Y^2
- Find $\sigma_{X,Y}$

4. A service facility operates with two service lines. On a randomly selected day, let X be the proportion of time that the first line is in use whereas Y is the proportion of time that the second line is in use. Suppose that the joint probability density function for (X, Y) is given below.

$$f_{X,Y}(x, y) = \begin{cases} \frac{3}{2}(x^2 + y^2), & 0 \leq x \leq 1, 0 \leq y \leq 1; \\ 0, & \text{otherwise} \end{cases}$$

- (a) Determine whether X and Y are independent.
 - (b) Find the mean and variance of X .
 - (c) Find the mean and variance of Y .
 - (d) Find the covariance of X and Y .
 - (e) Find the mean and variance of $X + Y$.
5. The random variables X and Y have the joint probability density function below.

$$f_{X,Y}(x, y) = \begin{cases} x + y, & 0 \leq x \leq 1, 0 \leq y \leq 1; \\ 0, & \text{otherwise} . \end{cases}$$

Find

- (a) $\sigma_{X,Y}$
 - (b) $E(Y|X = 0.2)$
 - (c) $E(X|Y = 0.5)$
6. Given that $Var(X) = 5$ and $Var(Y) = 3$, and Z is defined as $Z = -2X + 4Y - 3$.
- (a) Find the variance of Z if X and Y are independent.
 - (b) If $Cov(X, Y) = 1$, find the variance of Z .
 - (c) If $Cov(X, Y) = 1$, compute the correlation of X and Y .
7. An employee is selected from a staff of 10 to supervise a certain project by selecting a tag at random from a box containing 10 tags numbered from 1 to 10.
- (a) Find the formula for the probability distribution of X representing the number on the tag that is drawn
 - (b) What is the probability that the number drawn is less than 4?
 - (c) Find the mean and variance of X .

Answers to selected problems

1. (a) $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ for all (x, y) . X and Y are independent
(b) 3
(c) 3.2
(d) -2.6
(e) 9.6
(f) 0.96; 2
(g) 0; 0
2. 6.55; 44.4275
3. (a) $f_X(x) = \frac{2}{3}(x + 1)$, $0 \leq x \leq 1$. $f_Y(y) = \frac{1}{3}(4y + 1)$, $0 \leq y \leq 1$. $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$. Therefore, X and Y are dependent.
(b) 0.08024 ($= 13/162$)
(c) 0.07099 ($= 23/324$)
(d) -0.00617 ($= -1/162$)
4. (a) $f_X(x) = \frac{3}{2}\left(x^2 + \frac{1}{3}\right)$, $0 \leq x \leq 1$. $f_Y(y) = \frac{3}{2}\left(\frac{1}{3} + y^2\right)$, $0 \leq y \leq 1$. $f_{X,Y}(x, y) \neq f_X(x)f_Y(y)$. Therefore, X and Y are dependent.
(b) 0.625 ($= 5/8$); 0.07604 ($= 73/960$)
(c) 0.625 ($= 5/8$); 0.07604 ($= 73/960$)
(d) -0.01563 ($= -1/64$)
(e) $E(X + Y) = 1.25$ ($= 5/4$); $V(X + Y) = V(X) + V(Y) + 2Cov(X, Y) = 0.12083$ ($= 29/240$)
5. (a) $f_X(x) = x + 1/2$, for $0 \leq x \leq 1$, $f_Y(y) = y + 1/2$, for $0 \leq y \leq 1$. $E(XY) = 1/3$, $Cov(X, Y) = -0.00694$ ($= -1/144$)
(b) $f_{Y|X}(y|0.2) = (2 + 10y)/7$, for $0 \leq y \leq 1$. $E(Y|X = 0.2) = 0.61905$ ($= 13/21$)
(c) $f_{X|Y}(x|0.5) = x + 1/2$, for $0 \leq x \leq 1$. $E(X|Y = 0.5) = 0.58333$ ($= 7/12$)
6. (a) 68
(b) 52
(c) 0.2582
7. (a) $f_{X,Y}(x, y) = \begin{cases} \frac{1}{10}, & x = 1, 2, \dots, 10; \\ 0, & \text{otherwise.} \end{cases}$
(b) 0.3
(c) $E(X) = 5.5$; $V(X) = 8.25$