ST2334 PRACTICE EXAM QUESTIONS

Question 1

Given two events $E, F \in \mathscr{F}$ show that

(1) $\mathbb{P}(E^c \cap F) = \mathbb{P}(F) - \mathbb{P}(E \cap F)$.
(2) $\mathbb{P}(E \cap F) \geq \mathbb{P}(E) + \mathbb{P}(F) - 1$.

Solution.

(1) As $F = (E \cap F) \cup (E^c \cap F)$ so,

$$\mathbb{P}(F) = \mathbb{P}(E \cap F) + \mathbb{P}(E^c \cap F) \Rightarrow \mathbb{P}(E^c \cap F) = \mathbb{P}(F) - \mathbb{P}(E \cap F).$$

(2) Using results from lectures:

$$\mathbb{P}(E \cap F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cup F) \geq \mathbb{P}(E) + \mathbb{P}(F) - 1.$$

Question 2

Monty Hall Game Show[1]

In a TV game show, a contestent selects one of three doors; behind one of the doors, there is a prize, and behind the other two there are no prizes. After the contestant selects a door, the game show host opens one of the remaining doors, and reveals that there is no prize behind it. The host then asks the contestant whether they want to SWITCH their choice to the unopened door, or STICK to their original choice.

Is it probabilistically advantageous for the contestant to SWITCH doors, or is the probability of winning the same whether they STICK or SWITCH? (You may assume that the host selects a door to open from those *available*, with equal probability). [Hint: consider Bayes Theorem].

Solution. Without loss of generality, let events A, B, C correspond to the prize being behind the selected, opened and remaining door respectively and let H_B denote the event that the host opens door B. We want to compare $\mathbb{P}(A|H_B)$ (STICK) with $\mathbb{P}(C|H_B)$ (SWITCH). Now $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = 1/3$ and we are told that $\mathbb{P}(H_B|A) = 1/2, \mathbb{P}(H_B|B) = 0, \mathbb{P}(H_B|C) = 1$. Then via Bayes theorem:

$$\begin{aligned}
\mathbb{P}(A|H_B) &= \frac{\mathbb{P}(H_B|A)\mathbb{P}(A)}{\mathbb{P}(H_B)} = \frac{\mathbb{P}(H_B|A)\mathbb{P}(A)}{\mathbb{P}(H_B|A)\mathbb{P}(A) + \mathbb{P}(H_B|B)\mathbb{P}(B) + \mathbb{P}(H_B|C)\mathbb{P}(C)} \\
&= \frac{1/2 \times 1/3}{1/2 \times 1/3 + 0 \times 1/3 + 1 \times 1/3} = 1/3.
\end{aligned}$$

Similarly, $\mathbb{P}(C|H_B) = 2/3$. Thus one should SWITCH.

[1]This is a well known problem often asked in job interviews

Question 3

The Prisoners Dilema

Three prisoners A, B, C are in solitary confinement under sentence of death, but each knows that one of them, chosen at random with equal probability, is to be pardoned. Prisoner A begs the governer to tell him whether he, A, is to be pardoned or executed. The governer refuses to answer this, but he does say that B is to be executed. The governer thinks that he is not giving useful information, as A know at least one of B or C is to be executed.

A suddenly feels much happier, as he believes his chances of being pardoned have *risen* from 1/3 to 1/2. The governor, who, if A were actually to be pardoned is equally likely to give C or B's name, is mystified by A's euphoria. Who is correct?

[Hint: Let A, B, C be the events that A, B or C respectively are to be pardoned. Then A, B, C partition the state-space Ω. Now let G_{AB} be the event that the goverener tells A that B is to be executed. You are asked to compute $\mathbb{P}(A|G_{AB})$, so consider the three conditional probabilities of G_{AB}, given A, B, C and then use Bayes theorem.]

What should C feel, if he overhears the governor's reply, but assumes the question was asked by a prison guard? [consider the event G_{PB} that the governor tells a guard that B is to be executed].

Solution. We are given that $\mathbb{P}(A) = \mathbb{P}(B) = \mathbb{P}(C) = 1/3$ and $\mathbb{P}(G_{AB}|A) = 1/2, \mathbb{P}(G_{AB}|B) = 0, \mathbb{P}(G_{AB}|C) = 1$ and hence by using Bayes theorem (as for the previous question): $\mathbb{P}(A|G_{AB}) = 1/3$ and the governor is correct.
For the second part $\mathbb{P}(G_{PB}|A) = 1/2$, $\mathbb{P}(G_{PB}|B) = 0$, but $\mathbb{P}(G_{PB}|C) = 1/2$, so by Bayes theorem $\mathbb{P}(C|G_{PB}) = 1/2$ and hence C should feel happier.

Question 4

A surgical procedure is successful with probability $\theta \in (0, 1)$. The surgery is carried out on five patients, with the success or failure of each operation independent of all other operations. Let X be the discrete random variable corresponding to the number of successful operations.

Find the probability mass function of X and evaluate the probability that:

(1) All five operations are successful, if $\theta = 0.8$,
(2) exactly four operations are successful, if $\theta = 0.6$
(3) fewer than two are successful if $\theta = 0.3$.

Solution. Consider a binary sequence of length $n = 5$ corresponding to the results of the procedures (1=success,0=failure). All such sequences containing x 1's and $n - x$ 0's have a probability

$$\theta^x(1-\theta)^{n-x}.$$

Since there are $\binom{n}{x}$ such sequences:

$$f(x) = \binom{n}{x}\theta^x(1-\theta)^{n-x} \quad x \in \mathsf{X} = \{0, 1, \ldots, n\}.$$

Thus $X \sim \mathcal{B}(n, \theta)$. Then:

(1) $\theta = 0.8$, $\mathbb{P}(X = 5) = 0.3227$
(2) $\theta = 0.6$, $\mathbb{P}(X = 4) = 0.2592$

(3) $\theta = 0.3$, $\mathbb{P}(X < 2) = \mathbb{P}(X = 0) + \mathbb{P}(X = 1) = 0.5282$.

Question 5

An individual repeatedly attempts to pass their driving test. Suppose that the probability that the test is passed is θ, and that the results of successive tests are independent. Let X be the discrete random variable corresponding to the number of tests taken until the individual passes.

Find the probability mass function of X and evaluate the probability that:

(1) the test is passed in three or less tests, if $\theta = 0.25$,
(2) More than five tests are required for a pass to be obtained, if $\theta = 0.7$.

Solution. Experiment: a sequence of independent and indentical binary trials until the first success; for $X = x$, need $x-1$ failures, then a success, and as all successive tests are independent:

$$f(x) = (1-\theta)^{x-1}\theta \quad x \in \mathsf{X} = \{1, 2, \dots\}.$$

The distribution associated with this PMF is called a geometric random variable: $X \sim \mathcal{G}e(\theta)$. Furthermore, one can calculate the associated CDF:

$$F(x) = \sum_{y=1}^{x} (1-\theta)^{y-1}\theta = 1 - (1-\theta)^x \quad x \in \mathsf{X}.$$

Thus

(1) $\theta = 0.25$, $\mathbb{P}(X \le 3) = F(3) = 0.5781$.
(2) $\theta = 0.7$, $\mathbb{P}(X > 5) = 1 - \mathbb{P}(X \le 5) = 1 - F(5) = 0.00243$.

Question 6

A fair coin is flipped repeatedly, with successive flips identical and independent. Let X be the discrete random variable corresponding to the number of flips required to obtain 3 heads (that is, the sequence of flips which concludes once the third head has been seen).

Find the probability mass function of X.

Solution. Experiment: a sequence of independent and identical binary trials until the third success. For $X = x$ we require that we have $n-1$ successes in the first $x-1$ trials, for ehiahc we can calcultae the probability using the $\mathcal{B}(x-1,\theta)$ formula, and then a success on the $x^{th}-$trial. Hence:

$$f(x) = \binom{x-1}{n-1}\theta^{n-1}(1-\theta)^{(x-1)-(n-1)} \times \theta = \binom{x-1}{n-1}\theta^n(1-\theta)^{x-n} \quad x \in \mathsf{X} = \{n, n+1, \dots, \}.$$

This is the negative binomial distribution and we write $X \sim \mathcal{N}e(n,\theta)$. For the example of interest $X \sim \mathcal{N}e(3, 1/2)$.

Question 7

For what values of the constant c do the following functions define a valid probability mass function for the random variable X on the support $\mathsf{X} = \{1, 2, \dots\}$:

(1) $f(x) = c/2^x$.
(2) $f(x) = c2^x/x!$.

In both cases calculate $\mathbb{P}(X > 1)$.

Solution. We need $\sum_{x=1}^{\infty} f(x) = 1$, so:

(1) $c^{-1} = \sum_{x=1}^{\infty} 1/2^x = 1$. We use a Geometric sum.
(2) $c^{-1} = \sum_{x=1}^{\infty} 2^x/x! = e^2 - 1$. We have used the exponential taylor series.

Now clearly $\mathbb{P}(X > 1) = 1 - \mathbb{P}(X = 1)$, so

(1) $\mathbb{P}(X > 1) = 1/2$
(2) $\mathbb{P}(X > 1) = (e^2 - 3)/(e^2 - 1)$.

Question 8

Suppose $X \sim \mathcal{G}e(\theta)$, that is

$$f(x) = (1-\theta)^{x-1}\theta \quad x \in \mathsf{X} = \{1, 2, \dots\}.$$

Show that for $n, k \in \{1, 2, \dots, \}$

$$\mathbb{P}(X = n + k | X > n) = \mathbb{P}(X = k).$$

This is called the *lack of memory* property.

Solution. Recall that

$$F(x) = 1 - (1-\theta)^x.$$

Thus

$$\mathbb{P}(X > n) = 1 - \mathbb{P}(X \leq n) = (1-\theta)^n.$$

Now:

$$\mathbb{P}(X = n + k | X > n) = \frac{\mathbb{P}(X = n + k \cap X > n)}{\mathbb{P}(X > n)}$$

As $\mathbb{P}(X = n + k \cap X > n) = \mathbb{P}(X = n + k)$

$$\frac{\mathbb{P}(X = n + k \cap X > n)}{\mathbb{P}(X > n)} = \frac{\mathbb{P}(X = n + k)}{\mathbb{P}(X > n)} = \frac{(1-\theta)^{n+k-1}\theta}{(1-\theta)^n} = (1-\theta)^{k-1}\theta = \mathbb{P}(X = k).$$

Question 9

Suppose (X, Y) are jointly defined discrete random variables on $\mathsf{Z} = \{x : x \in \{0, 1, \dots\} \times \{y : y \in \{0, 1, \dots\}$ with joint PMF:

$$f(x, y) = \frac{c2^{x+y}}{x!y!} \quad (x, y) \in \mathsf{Z}$$

for some constant c. Find c and the marginal PMFs of X and Y. Show that X and Y are independent random variables.

Solution. We have $\sum_{(x,y)\in\mathsf{Z}} f(x, y) = 1$, thus

$$c^{-1} = \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} \frac{2^{x+y}}{x!y!} = \sum_{x=0}^{\infty} \frac{2^x}{x!} \sum_{y=0}^{\infty} \frac{2^y}{y!} = e^2 e^2.$$

Thus $c = e^{-4}$. One can complete the question by recognizing that independently $X \sim \mathcal{P}(2)$ and $Y \sim \mathcal{P}(2)$. Alternatively, for $x \in \{0, 1, \dots\}$:

$$f(x) = \sum_{y=0}^{\infty} e^{-4} \frac{2^{x+y}}{x!y!} = e^{-2}\frac{2^x}{x!} \sum_{y=0}^{\infty} e^{-2}\frac{2^y}{y!} = e^{-2}\frac{2^x}{x!}.$$

Similarly for $y \in \{0, 1, \dots\}$:

$$f(y) = e^{-2}\frac{2^y}{y!}.$$

Clearly $f(x,y) = f(x)f(y)$.

Question 10

Let $X_1 \sim \mathcal{B}(1,p_1)$ and independently $X_2 \sim \mathcal{B}(1,p_2)$ and let $Z = X_1 + X_2$. Find $\mathbb{E}[Z]$ and $\mathbb{V}\mathrm{ar}[Z]$.

Solution. We know from lectures that

$$\mathbb{E}[Z] = \mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = p_1 + p_2.$$

Similarly as X_1 and X_2 are independent:

$$\mathbb{V}\mathrm{ar}[X_1 + X_2] = \mathbb{V}\mathrm{ar}[X_1] + \mathbb{V}\mathrm{ar}[X_2] = p_1(1-p_1) + p_2(1-p_2).$$

Note that if $X \sim \mathcal{B}(1,p)$ then:

$$\mathbb{E}[X] = \sum_{x=0}^{1} x p^x (1-p)^{1-x} p = \mathbb{E}[X^2].$$

So

$$\mathbb{V}\mathrm{ar}[X] = p - p^2 = p(1-p).$$

Question 11

A continuous random variable $X \in \mathsf{X} = [0,1]$ has CDF given by

$$F(x) = c(\alpha x^\beta - \beta x^\alpha) \quad x \in \mathsf{X}$$

for constants $1 \le \beta < \alpha$. Find the value of constant c, and evaluate $\mathbb{E}[X]$.

Solution. We must have $F(1) = 1$, so

$$c(\alpha - \beta) = 1$$

thus $c = 1/(\alpha - \beta)$. Now for $x \in \mathsf{X}$

$$f(x) = \frac{dF(x)}{dx} = \frac{\alpha\beta}{\alpha - \beta}(x^{\beta-1} - x^{\alpha-1}).$$

Thus

$$\mathbb{E}[X] = \frac{\alpha\beta}{\alpha-\beta}\int_0^1 (x^\beta - x^\alpha)dx = \frac{\alpha\beta}{\alpha-\beta}\frac{\alpha-\beta}{(\alpha+1)(\beta+1)} = \frac{\alpha\beta}{(\alpha+1)(\beta+1)}.$$

Question 12

Continuous random variables X and Y $(X,Y) \in \mathsf{Z} = \mathbb{R}_+ \times \mathbb{R}$ have joint CDF

$$F(x,y) = (1-e^{-x})\left(\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(y)\right) \quad (x,y) \in \mathsf{Z}.$$

Find the joint PDF of X and Y. Are X and Y independent?

Solution. The joint PDF of (X,Y) is, for $(x,y)\in \mathsf{Z}$:

$$f(x,y)=\frac{\partial^2 F(x,y)}{\partial x\partial y}.$$

Clearly

$$\frac{\partial F(x,y)}{\partial x}=e^{-x}\left(\frac{1}{2}+\frac{1}{\pi}\tan^{-1}(y)\right)$$

and then

$$\frac{\partial^2 F(x,y)}{\partial x\partial y}=e^{-x}\frac{1}{\pi(1+y^2)}.$$

Now

$$1=\int_0^\infty e^{-x}dx=\int_{-\infty}^\infty \frac{1}{\pi(1+y^2)}dy.$$

So $f(x)=e^{-x}$ and $f(y)=\frac{1}{\pi(1+y^2)}$ and $f(x,y)=f(x)f(y)$; so X and Y are independent.

Question 13

Continuous random variables X and Y $(X,Y)\in \mathsf{Z}=[0,1]^2$ have joint PDF

$$f(x,y)=cx(1-y)\quad (x,y)\in\mathsf{Z}$$

for some constant c. Find the value of c. Are X and Y independent? Let

$$A=\{(x,y)\in\mathsf{Z}:0<x<y<1\}.$$

Calculate $\mathbb{P}((X,Y)\in A)$.

Solution. We have

$$\begin{aligned}
c^{-1} &= \int_0^1\int_0^1 x(1-y)dxdy\\
&= \int_0^1 \left[\frac{x^2}{2}\right]_0^1(1-y)dy\\
&= \frac{1}{2}\left[y-\frac{y^2}{2}\right]_0^1\\
&= \frac{1}{4}
\end{aligned}$$

Thus $c=4$. We have

$$f(x)=4x\int_0^1(1-y)dy=4x\left[y-\frac{y^2}{2}\right]_0^1=2x$$

and similiarly

$$f(y)=2(1-y).$$

Thus $f(x,y) = f(x)f(y)$ so X and Y are independent.
For the next part, we have, by considering the integration region:

$$\begin{aligned}
\mathbb{P}((X,Y) \in A) &= \int_0^1 \int_x^1 4x(1-y)dydx \\
&= \int_0^1 4x\Big[y - \frac{y^2}{2}\Big]_x^1 dx \\
&= \int_0^1 2x - 4x^2 + 2x^3 dx \\
&= \Big[x^2 - \frac{4}{3}x^3 - \frac{1}{2}x^4\Big]_0^1 \\
&= \frac{1}{6}.
\end{aligned}$$

Question 14

Continuous random variables X and Y $(X,Y) \in \mathsf{Z} = \{(x,y) \in \mathbb{R}_+^2 : x+y < 1\}$ have joint PDF

$$f(x,y) = 24xy \quad (x,y) \in \mathsf{Z}.$$

Find the marginal PDF of X.

Solution. We have that for $x \in [0,1]$ (you should sketch the region of integration, to confirm the integration limits):

$$\begin{aligned}
f(x) &= \int_0^{1-x} 24xydy \\
&= 24x\Big[\frac{y^2}{2}\Big]_0^{1-x} \\
&= 24x\frac{(1-x)^2}{2}.
\end{aligned}$$

Question 15

Let $X \in \mathsf{X} = \mathbb{R}_+$ be a continuous random variable with support $\mathsf{X} = \mathbb{R}^+$, with PDF f and CDF F. By writing the expectation in its integral definition form on the left hand side, and changing the order of integration show that

$$\mathbb{E}[X] = \int_0^\infty [1 - F(x)]dx.$$

Solution. We have

$$\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty xf(x)dx \\
&= \int_0^\infty \int_0^x dyf(x)dx \\
&= \int_0^\infty \int_y^\infty f(x)dxdy \\
&= \int_0^\infty [1-F(y)]dy = \int_0^\infty [1-F(x)]dx.
\end{aligned}$$

We have reversed the order of integration on the third line.

Question 16

The annual profit (in millions of USD) of a manufacturing company is a function of product demand. If X is the continuous random variable corresponding to the demand in a given year then the annual profit Y is modelled as:

$$Y = 2(1 - e^{-2X}).$$

If $X \sim \mathcal{E}(6)$ (exponential distribution with parameter 6) find the expected annual profit.

Solution. Perhaps the easiset way to solve this problem is as follows:

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[2(1 - e^{-2X})] \\
&= 2\int_0^\infty (1 - e^{-2x})6e^{-6x}dx \\
&= 12\int_0^\infty e^{-6x} - e^{-8x}dx \\
&= 12\Big[-\frac{1}{6}e^{-6x} + \frac{1}{8}e^{-8x}\Big]_0^\infty \\
&= \frac{1}{2}.
\end{aligned}$$

So the expected profit is 500k USD.

Question 17

Let f_1, f_2 be PDFs on support X. Let

$$g(x) = \pi f_1(x) + (1 - \pi)f_2(x) \quad x \in \mathsf{X}$$

where we assume $\pi \in (0, 1)$ is known.
(1) Show that $g(x)$ is a PDF on X. In addition, denoting $\mathbb{E}_g[X] = \int_{\mathsf{X}} xg(x)dx$, show that

$$\mathbb{E}_g[X] = \pi\mathbb{E}_{f_1}[X] + (1 - \pi)\mathbb{E}_{f_2}[X]$$

where, for $i \in \{1, 2\}$, $\mathbb{E}_{f_i}[X] = \int_{\mathsf{X}} xf_i(x)dx$.
(2) The working life-times of two batteries labelled 'regular' (Z_1) and 'long-life' (Z_2) are modeled as, $Z_1 \sim \mathcal{G}(4, 2.5)$, $Z_2 \sim \mathcal{G}(11, 4)$, where $\mathcal{G}(a, b)$ is the Gamma distribution. A battery is selected at random from a mixed box containing 80% regular and 20% long-life batteries and its life-time X is measured. Find the expected life-time of X.

Solution. (1) We first note that $\pi > 0$ and $(1 - \pi) > 0$ and $f_i(x) \geq 0$ for each $x \in \mathsf{X}$ hence $g(x) \geq 0$ for each $x \in \mathsf{X}$. Now,

$$\int_{\mathsf{X}} g(x)dx = \int_{\mathsf{X}}[\pi f_1(x)+(1-\pi)f_2(x)]dx = \pi\int_{\mathsf{X}} f_1(x)dx+(1-\pi)\int_{\mathsf{X}} f_2(x)dx = \pi+(1-\pi) = 1.$$

Hence, $g(x)$ is a PDF. Now

$$\begin{aligned}
\mathbb{E}_g[X] &= \int_{\mathsf{X}} x g(x) dx \\
&= \int_{\mathsf{X}} x[\pi f_1(x) + (1-\pi) f_2(x)] dx \\
&= \pi \int_{\mathsf{X}} x f_1(x) dx + (1-\pi) \int_{\mathsf{X}} x f_2(x) dx \\
&= \pi \mathbb{E}_{f_1}[X] + (1-\pi)\mathbb{E}_{f_2}[X].
\end{aligned}$$

(2) The situation we have is

$$g(x) = 0.8 \frac{2.5^4}{\Gamma(4)} x^3 e^{-2.5x} + 0.2 \frac{4^{11}}{\Gamma(11)} x^{10} e^{-4x} \quad x \in \mathbb{R}_+.$$

Here we just use (1) and the fact that for $Y \sim \mathcal{G}(a,b)$, $\mathbb{E}[Y] = a/b$. Thus,

$$\mathbb{E}[X] = 0.8(4/2.5) + 0.2(11/4) = 1.83.$$