Chapter 2

Concepts of Random Variables

Overview

- Random variables
- Discrete probability distributions
  - Probability function
- Continuous probability distributions
  - Probability density function
- Cumulative distribution function

Overview (Continued)

- Expectation
  - Mean and variance
  - Expectation of functions of random variables
  - Properties of expectation
- Chebychev's Inequality

2.1 Introduction

- It is frequently the case that, when an experiment is performed, we are mainly interested in some function of the outcome as opposed to the actual outcome itself.
- For instance, in testing 100 electronic components, we are often concerned with the number of defectives that occur.

Introduction (Continued)

- Also, in coin-flipping, we may be interested in the total number of heads that occur and do not care at all about the actual head-tail sequence that result.
- These values are, of course, **random** quantities determined by the outcomes of the experiment.

Example 1

- Let $S = \{HH, HT, TH, TT\}$ be a sample space associated with the experiment of tossing two coins.
- Define the random variable (a function)

  X = number of heads obtained.

  $X : S \to \mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers

  such that $X(HH) = 2, X(HT) = 1, X(TH) = 1$ and $X(TT) = 0$.
- In fact the range space, $\boldsymbol{R_X}$, for the random variable X is

  $\{0, 1, 2\}$.

Example 2

Consider tossing a pair of fair dice.

- Let X be the sum of the upturned faces.

  $S = \{(x, y)|x = 1, 2, \cdots, 6;\ y = 1, 2, \cdots, 6\}.$

  $$X\colon S \to \mathbb{R}$$

  such that $X((x, y)) = x + y$

  for $x = 1, 2, \cdots, 6;\ y = 1, 2, \cdots, 6.$
- e.g. $X\big((1, 1)\big) = 2;\ X\big((1, 2)\big) = 3; X\big((3,6)\big) = 9$
- $\boldsymbol{R_X} = \{2, 3, 4, 5, \cdots, 11, 12\}.$

Example 3

- A coin is thrown until a "Head" occurs.

- Let X be the number of trials required.
  $S = \{H, TH, TTH, TTTH, TTTTH, \cdots\}$.
  $$X: S \to \mathbb{R}$$
  such that $X(H) = 1, X(TH) = 2, X(TTH) = 3$ and so on.

- $\boldsymbol{R_X} = \{1, 2, 3, 4, 5, \cdots\}$, the set of positive integers.

2.1.1 Random Variable

Definition 2.1

- Let S be a sample space associated with the experiment, E.

- A *function* X, which assigns a number to every element $s \in S$, is called a **random variable**.

Random Variable (Continued)

Notes:

1. X is a real-valued function.
2. The range space of X is the set of real numbers

$$\boldsymbol{R_X} = \{x \mid x = X(s), s \in S\}.$$

   Each possible value x of X represents an event that is a subset of the sample space S.

3. If S has elements that are themselves real numbers, we take $X(s) = s$. In this case $\boldsymbol{R_X} = S$.

2.1.2 Equivalent Events

Definition 2.2

- Let E be an experiment and S its sample space.
- Let X be a random variable defined on S and $\boldsymbol{R_X}$ be its range space.

  That is, $X : S \rightarrow \mathbb{R}$
- Let B be an event with respect to $\boldsymbol{R_X}$;

  That is $B \subset \boldsymbol{R_X}$.

Equivalent Events (Continued)

Definition 2.2 (Continued)

- Suppose that A is defined as
$$A = \{s \in S \,|X(s) \in B\}.$$
In words: A consists of all sample points, s, in S for which $X(s) \in B$.

- In this case we say that A and B are **equivalent events** and
$$\Pr(B) = \Pr(A)\,.$$

Example 1 (Continued)

- Consider tossing a coin twice.
- Then $S = \{HH, HT, TH, TT\}$.
- Let X be the number of heads obtained.
- Then the possible values for $X(s)$, (usually we just write X) are 0, 1, 2 hence and $\boldsymbol{R_X} = \{0, 1, 2\}$.

Example 1 (Continued)

- Therefore

  $A_1 = \{HH\}$ is equivalent to $B_1 = \{2\}$

  $A_2 = \{HT, TH\}$ is equivalent to $B_2 = \{1\}$

  $A_3 = \{TT\}$ is equivalent to $B_3 = \{0\}$

  $A_4 = \{HH, HT, TH\}$ is equivalent to $B_4 = \{2, 1\}$

Example 1 (Continued)

$$\Pr(A_1) = \Pr(B_1) = 1/4$$
$$\Pr(A_2) = \Pr(B_2) = 2/4 = 1/2$$
$$\Pr(A_3) = \Pr(B_3) = 1/4$$
$$\Pr(A_4) = \Pr(B_4) = 3/4.$$

Note: Event $\{HH, HT\}$ does not have an equivalent event based on random variable X defined above.

Example 1 (Continued)

We can summarize the probabilities of the random variable X as follows.

Number of heads, x	0	1	2
$\Pr(X = x)$	$1/4$	$1/2$	$1/4$

Example 2

- When a pair of fair dice is tossed, what is the probability that a sum of 3 is obtained?

$$S = \{(x_1, x_2) | x_1 = 1, 2, 3, 4, 5, 6; x_2 = 1, 2, 3, 4, 5, 6\}.$$

- Let $X\big((x_1, x_2)\big) = x_1 + x_2$, then

$$\boldsymbol{R_X} = \{2, 3, 4, \cdots, 12\}.$$

- Hence the event $B = \{3\}$ in $\mathbf{R}_X$ is equivalent to the event $A = \{(1, 2), (2, 1)\}$ in S.

Example 2 (Continued)

- Therefore

$$\begin{aligned}\Pr(X = 3) &= \Pr(\{(1,2),(2,1)\}) \\ &= \Pr(\{(1,2)\}) + \Pr(\{(2,1)\}) \\ &= 1/36 + 1/36 = 1/18.\end{aligned}$$

- The probabilities of all other possible sums can be found in a similar manner. They are given in the table in the next slide.

Example 2 (Continued)

- The probabilities of the random variable X are given in the following table.

x	2	3	4	5	6	7
$\Pr(X = x)$	1/36	2/36	3/36	4/36	5/36	6/36

x	8	9	10	11	12
$\Pr(X = x)$	5/36	4/36	3/36	2/36	1/36

2.2 Discrete Probability Distributions

2.2.1 Discrete Random Variable

Definition 2.3

- Let X be a random variable.

- If the number of possible values of X (i.e., $\boldsymbol{R_X}$, the range space) is **finite or countable infinite**, we call X a **discrete** random variable.

- That is, the possible values of X may be listed as $x_1, x_2, x_3, \cdots$.

2.2.2 Probability Function

- For a discrete random variable, each value of X has a certain probability $f(x)$.

- Such a function $f(x)$ is called the **probability function**, p.f. (or **probability mass function**, p.m.f.).

- The collection of pairs $(x_i, f(x_i))$ is called the **probability distribution** of X.

Probability Function (Continued)

The probability of $X = x_i$ denoted by $f(x_i)$ (i.e. $f(x_i) = \Pr(X = x_i)$), must satisfy the following two conditions.

(1) $f(x_i) \geq 0$ for all x_i.

(2) $\sum_{i=1}^{\infty} f(x_i) = 1$.

Example 1

- Consider tossing a coin twice.
- Let X be the number of heads obtained.
- Then we have

x	0	1	2
$f(x) = \Pr(X = x)$	$1/4$	$1/2$	$1/4$

Example 1 (Continued)

- Hence, we can see the condition (1) is satisfied since

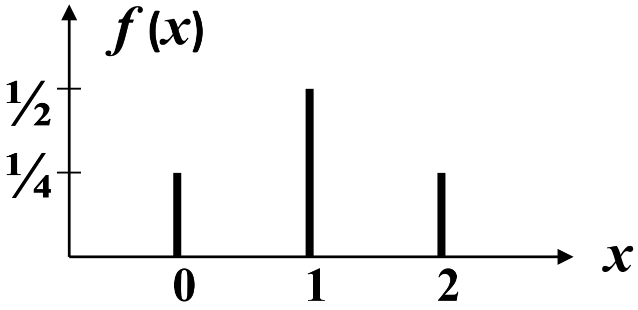
$$f(0) = 1/4 > 0, f(1) = 1/2 > 0 \text{ and } f(2) = 1/4 > 0.$$

- Condition (2) is also satisfied since

$$f(0) + f(1) + f(2) = 1/4 + 1/2 + 1/4 = 1.$$

Example 1 (Continued)

- One might plot the points $(x_i, f(x_i))$ of the Example 1 so that the probability distribution is displayed in a probability line graph as follows.



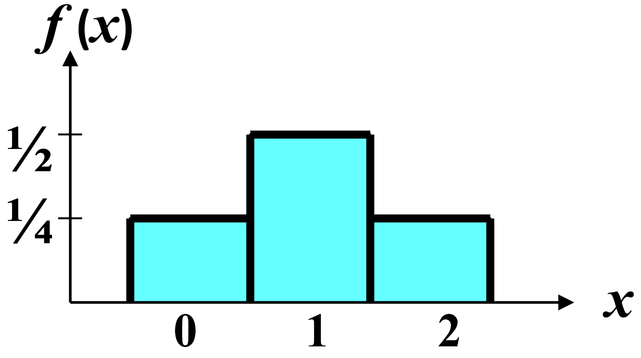
Example 1 (Continued)

- Instead of using lines, we can use rectangles for the plot and the resulting plot is called a **probability histogram**.

- The rectangles are constructed so that their bases of equal width are centered at each value of X, and their heights are equal to the corresponding probabilities.

Example 1 (Continued)

- If the base of the rectangle has <u>unit</u> width, then the probability $\Pr(X = x_i)$ is equal to the area of the rectangle centered at x_i.
- The total area of all the rectangles is 1.



Example 2

- Consider throwing a pair of fair dice.
- Let X be the sum of the two dice. Then we have

x	2	3	4	5	6	7
$\Pr(X = x)$	$1/36$	$2/36$	$3/36$	$4/36$	$5/36$	$6/36$

x	8	9	10	11	12
$\Pr(X = x)$	$5/36$	$4/36$	$3/36$	$2/36$	$1/36$

Example 2 (Continued)

- Notice that **all** $f(x) > 0$ for $x = 2, 3, 4, 5, \cdots, 11, 12$ and
- $f(2) + f(3) + f(4) + \cdots + f(11) + f(12) = 1$.

Example 3

Six lots of components are ready to be shipped by a certain supplier. The number of defective components in each lot is as follows:

Lot	1	2	3	4	5	6
Number of defectives	0	2	0	1	2	0

- One of these lots is to be <u>randomly</u> selected for the shipment to a particular customer.
- Let X be the number of defectives in the selected lot.
- The three possible X values are 0, 1, and 2.

Example 3 (Continued)

- Of the six equally likely choices of one lot, three result in $X = 0$, one in $X = 1$, and the other two in $X = 2$.
- Let $f(x)$ denote the probability that $X = x$ for $x = 0, 1, 2$.
- Then

$$f(0) = \Pr(X = 0) = \Pr(\text{lot 1 or 3 or 6 is selected}) = 3/6.$$

$$f(1) = \Pr(X = 1) = \Pr(\text{lot 4 is selected}) = 1/6.$$

$$f(2) = \Pr(X = 2) = \Pr(\text{lot 2 or 5 is selected}) = 2/6.$$

Example 3 (Continued)

- Therefore, the probability function of X is given by

x	0	1	2
$f(x) = \Pr(X = x)$	1/2	1/6	1/3

- Hence, $f(x_i) > 0$ for $x_1 = 0$, $x_2 = 1$ and $x_3 = 2$.

- Also $f(0) + f(1) + f(2) = 1$.

Example 4

- Find the constant c so that
$$f(x) = cx, \text{ for } x = 1, 2, 3, 4,$$
and 0 otherwise, is a **probability function** of a random variable X.
- Hence compute $\Pr(X \geq 3)$.

Example 4 (Continued)

Solution

- By the property that $\sum_{i=1}^{\infty} f(x_i) = 1$, we have

$$f(x_1) + f(x_2) + f(x_3) + f(x_4) = 1$$

$$c + 2c + 3c + 4c = 1.$$

- Therefore $c = 1/10$.
- Hence

$$\Pr(X \geq 3) = f(3) + f(4) = 3/10 + 4/10 = 7/10.$$

Example 5

- Consider a group of five potential blood donors — A, B, C, D and E — of whom only A and B have type O+ blood.

- Five blood samples, one from each individual, will be typed in random order until an O+ individual is identified.

Example 5 (Continued)

- Let the random variable Y = the number of typing necessary to identify an O+ individual.
- Let O_i and O_i' be the event that an O+ and a non-O+ individual is typed in the i-th typing

$$f(1) = \Pr(Y = 1) = \Pr(O_1) = 2/5 = 0.4.$$

$$f(2) = \Pr(Y = 2)$$

$$= \Pr(O_1')\Pr(O_2|O_1')$$

$$= \left(\frac{3}{5}\right)\left(\frac{2}{4}\right) = \frac{3}{10} = 0.3.$$

Example 5 (Continued)

$$f(3) = \Pr(Y = 3) = \Pr(O_1') \Pr(O_2' | O_1') \Pr(O_3 | O_1' \cap O_2')$$
$$= \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{2}{3}\right) = \frac{1}{5} = 0.2.$$

$$f(4) = \Pr(Y = 4)$$
$$= \Pr(O_1') \Pr(O_2' | O_1') \Pr(O_3' | O_1' \cap O_2') \Pr(O_4 | O_1' \cap O_2' \cap O_3')$$
$$= \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right)\left(\frac{2}{2}\right) = \frac{1}{10} = 0.1.$$

$$f(y) = 0 \text{ if } y \neq 1, 2, 3, 4.$$

Example 5 (Continued)

- Then the probability function of Y is

y	1	2	3	4
$f(y)$	0.4	0.3	0.2	0.1

2.2.3 Another View of Probability Function

It is often to think of a probability function as specifying a mathematical model for a finite population.

Example

- Consider selecting at random a student who is among the 35,000 registered for the current semester in NUS.
- Let X = the number of modules for which the selected student is registered and suppose that X has the probability function.

Another View of Probability Function (Continued)

x	1	2	3	4	5	6	7
$f(x)$	0.01	0.03	0.13	0.25	0.39	0.17	0.02

- One way to view this situation is to think of the population as consisting 35,000 individuals, each having his or her own X value; the proportion with each X value is given by $f(x)$ above.

Another View of Probability Function (Continued)

- An alternative viewpoint is to forget about the students and think of the population itself as consisting of the X values.
- There are some 1's in the population, some 2's, $\cdots$ and finally some 7's.
- The population then consists of the numbers $1, 2, \cdots, 7$ (so are discrete values), and $f(x)$ gives a model for the distribution of population values.
- Once we have such a mathematical model for a population, we will use it to compute values of population characteristics (such as the mean).

2.3 Continuous Probability Distributions

2.3.1 Continuous Random Variable

Definition 2.4

- Suppose that $\boldsymbol{R_X}$, the range space of a random variable, X, is an **interval or a collection of intervals**.
- Then we say that X is a **continuous random variable.**

2.3.2 Probability Density Function

Definition 2.5

- Let X be a continuous random variable.
- The **probability density function (p.d.f.)** $\boldsymbol{f(x)}$, is a function, $f(x)$, satisfying the following conditions:
  1. $f(x) \geq 0$ for all $x \in \boldsymbol{R_X}$,
  2. $\int_{\boldsymbol{R_X}} f(x)\, dx = 1$ or $\int_{-\infty}^{\infty} f(x)\, dx = 1$

     since $f(x) = 0$ for x not in $\boldsymbol{R_X}$.

Probability Density Function (Continued)

Definition 2.5 (Continued)

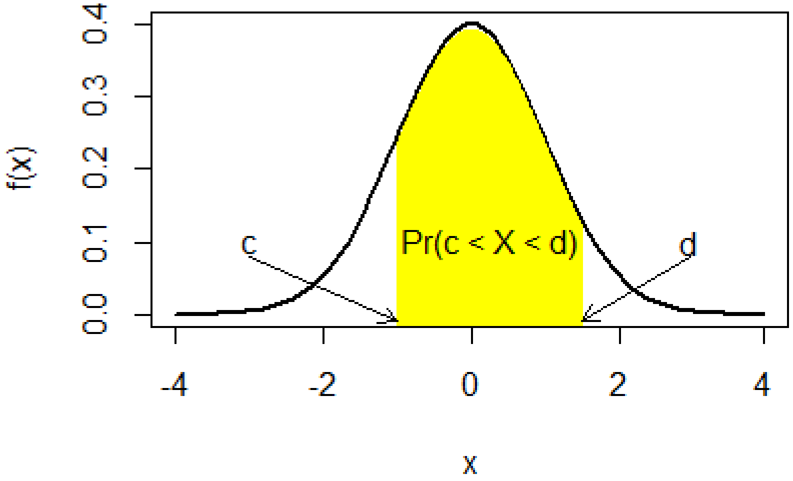
3. For any c and d such that $c < d$, (i.e. $(c, d) \subset \boldsymbol{R_X}$),

$$\Pr(c \leq X \leq d) = \int_c^d f(x)dx$$

Remarks

1. $\Pr(c \leq X \leq d) = \int_c^d f(x)dx$

   represents the area under the graph of the p.d.f. $f(x)$ between $x = c$ and $x = d$.



Remarks (Continued)

2. For any specified value of X, say x_0, we have

$$\Pr(X = x_0) = \int_{x_0}^{x_0} f(x)dx = 0$$

Hence in the **continuous** case, the probability of X equals to a fixed value is 0 and

$\Pr(c \leq X \leq d) = \Pr(c \leq X < d) = \Pr(c < X \leq d) = \Pr(c < X < d)$.

Therefore in the continuous case, $\leq$ **and** $<$ can be used **interchangeably** in a probability statement.

Remarks (Continued)

3. $\Pr(A) = 0$ does **not** necessary imply $A = \emptyset$.

4. If X assumes values only in some interval $[a, b]$, we may simply set $f(x) = 0$ for all X outside $[a, b]$.

Example 1

- Suppose that the random variable X is continuous.
- Let the p.d.f. $f(x)$ be given by

$$f(x) = \begin{cases} cx, & \text{for } 0 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Example 1 (Continued)

(a) Find the value c.

(b) Find $\Pr(X \leq 1/2)$.

(c) Find $\Pr(1/3 \leq X \leq 2/3)$.

(d) Find $\Pr(X \geq 3/4)$.

Solution to Example 1

(a)

$$\int_0^1 cx\,dx = c\left[\frac{x^2}{2}\right]_0^1 = \frac{c}{2}$$

- Hence $\int_0^1 cx\,dx = 1$ implies that $c/2 = 1$.
- Therefore $c = 2$.

Solution to Example 1 (Continued)

(b)

$$\Pr\left(X \le \frac{1}{2}\right) = \int_0^{1/2} f(x)dx$$

$$= \int_0^{1/2} 2x\, dx = [x^2]_0^{1/2}$$

$$= \frac{1}{4}$$

Solution to Example 1 (Continued)

(c)

$$\Pr\left(\frac{1}{3} \le X \le \frac{2}{3}\right) = \int_{1/3}^{2/3} 2x\,dx$$

$$= [x^2]_{1/3}^{2/3}$$

$$= \left(\frac{4}{9} - \frac{1}{9}\right) = \frac{1}{3}$$

Solution to Example 1 (Continued)

(d)

$$\Pr\left(X \geq \frac{3}{4}\right) = \int_{3/4}^{1} 2x\, dx$$

$$= [x^2]_{3/4}^{1}$$

$$= 1 - \frac{9}{16} = \frac{7}{16}$$

Example 2

- “Time headway” in traffic flow is the elapsed time between the time that one car finishes passing a fixed point and the instant that the next car begins to pass that point.
- Let X = the time headway for two randomly chosen consecutive cars on a highway during a period of heavy flow.

Example 2 (Continued)

- The following p.d.f. of X was suggested:

$$f(x) = \begin{cases} 0.15e^{-0.15(x-0.5)}, & \text{for } x \geq 0.5; \\ 0, & \text{otherwise.} \end{cases}$$

Note: $f(x) \geq 0$ for all x.

Example 2 (Continued)

- Clearly, $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0.5} 0\, dx = \int_{0.5}^{\infty} 0.15\, e^{-0.15(x-0.5)}\, dx$$

$$= 0.15e^{0.075} \int_{0.5}^{\infty} e^{-0.15x}\, dx$$

$$= 0.15e^{0.075} \left[-\frac{1}{0.15} e^{-0.15x}\right]_{0.5}^{\infty}$$

$$= 0.15e^{0.075} \left(0 - \left(-\frac{1}{0.15} \mathrm{e}^{-0.15(0.5)}\right)\right) = 1$$

Example 2 (Continued)

- Hence the given function $f(x)$ is a legitimate probability density function.

- What is the probability that headway time is **at most** 5 sec?

  i.e. What is $\Pr(X \leq 5)$?

Solution to Example 2

$$\Pr(X \leq 5) = \int_{-\infty}^{5} f(x)dx$$

$$= \int_{-\infty}^{0.5} 0\, dx + \int_{0.5}^{5} 0.15\, e^{-0.15(x-0.5)}\, dx$$

$$= 0.15\, e^{0.075} \int_{0.5}^{5} e^{-0.15x}\, dx$$

$$= 0.15\, e^{0.075} \left[-\frac{1}{0.15} e^{-0.15x}\right]_{0.5}^{5}$$

$$= e^{0.075}(-e^{-0.75} + e^{-0.075}) = 0.4908.$$

2.4 Cumulative Distribution Function

Definition 2.6

- Let X be a random variable, discrete or continuous.

- We define $F(x)$ to be the **cumulative distribution function** of the random variable X (abbreviated as c.d.f.) where

$$F(x) = \Pr(X \leq x).$$

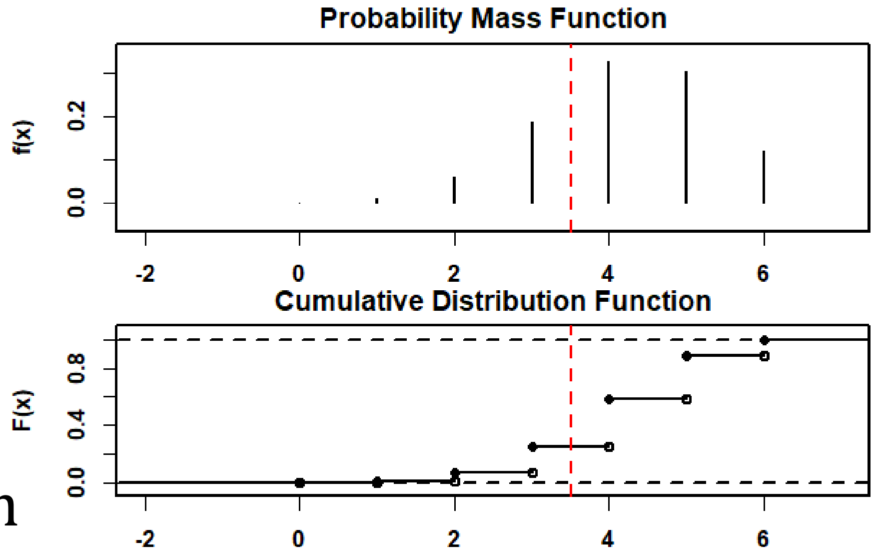
2.4.1 CDF for Discrete Random Variables

- If X is a **discrete random variable**, then

$$F(x) = \sum_{t \leq x} f(t)$$

$$= \sum_{t \leq x} \Pr(X = t)$$

- The c.d.f. of a discrete random variable is a step function.



CDF for Discrete Random Variables (Continued)

- For any two numbers a and b with $a \leq b$.

$$\begin{aligned}\Pr(a \leq X \leq b) &= \Pr(X \leq b) - \Pr(X < a) \\ &= F(b) - F(a^-)\end{aligned}$$

where "a^-" represents the largest possible value of X value that is strictly less than a.

CDF for Discrete Random Variables (Continued)

- In particular, if the only possible values are **integers** and if a and b are integers, then

$$\Pr(a \le X \le b) = \Pr(X = a \text{ or } a+1 \text{ or } \cdots \text{ or } b)$$

Also $\Pr(a \le X \le b) = F(b) - F(a-1)$

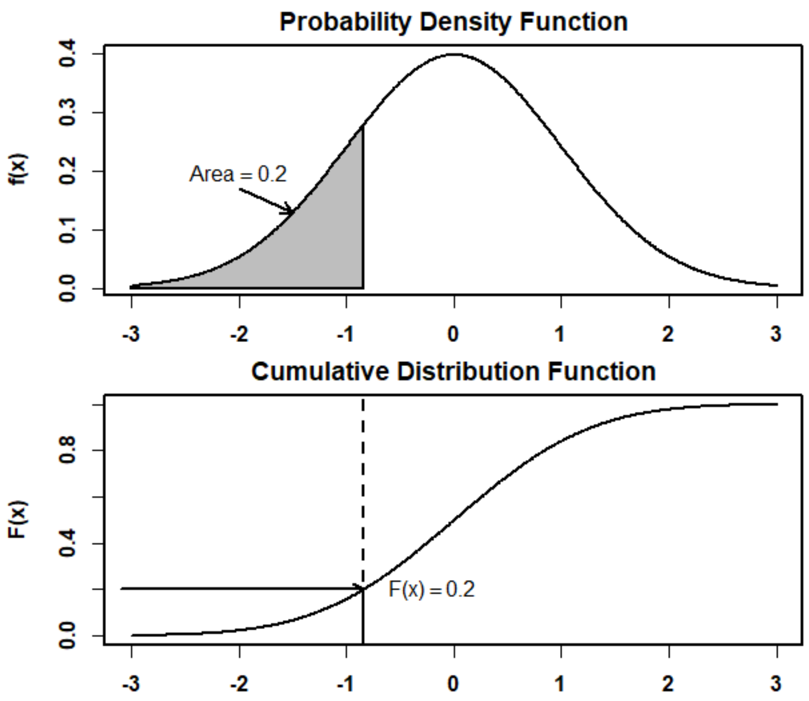
- Taking $a = b$ yields

$$\Pr(X = a) = F(a) - F(a-1).$$

2.4.2 CDF for Continuous Random Variables

- If X is a **continuous random variable**, then

$$F(x) = \int_{-\infty}^{x} f(t)dt$$



CDF for Continuous Random Variables (Continued)

- For a **continuous random variable** X,

$$f(x) = \frac{d\,F(x)}{dx}$$

if the derivative exists.

- Also,

$$\begin{aligned}\Pr(a \le X \le b) &= \Pr(a < X \le b) \\ &= F(b) - F(a).\end{aligned}$$

CDF for Continuous Random Variables (Continued)

Remarks:

- $F(x)$ is a **non-decreasing** function.

  That is, if $x_1 < x_2$, then $F(x_1) \leq F(x_2)$.

- $0 \leq F(x) \leq 1$.

Example 1

The p.f. of X is given as follows

$$f(x) = \begin{cases} (1-p)^{x-1}p, & \text{if } x = 1, 2, 3, \cdots; \\ 0, & \text{otherwise.} \end{cases}$$

Example 1 (Continued)

For any positive integers x,

$$F(x) = \sum_{t \le x} f(t) = \sum_{t=1}^{x} (1-p)^{t-1} p$$

$$= p \sum_{s=0}^{x-1} (1-p)^s, \qquad \text{where } s = t-1$$

$$= \frac{p(1-(1-p)^x)}{1-(1-p)}$$

$$= 1-(1-p)^x, \qquad \text{for } x = 1, 2, 3, \cdots$$

Example 1 (Continued)

Remark

- Since $f(x) = 0$ between positive integers, hence $F(x)$ is constant between positive integers and

$$F(x) = \begin{cases} 0, & \text{if } x < 1, \\ 1 - (1-p)^{[x]}, & \text{if } x \geq 1 \end{cases}$$

where $[x]$ is the largest integer $\leq x$.

(e.g., $[2.7] = 2$, $[3] = 3$).

Example 2

- Let X = the number of days of sick leave taken by a randomly selected employee of a large company during a particular year.
- If the maximum number of allowable sick leave days per year is 14, possible values of X are $0, 1, 2, \cdots, 14$.
- Suppose it is given that $F(0) = 0.58, F(1) = 0.72, F(2) = 0.76$, $F(3) = 0.81, F(4) = 0.88$, and $F(5) = 0.94$.

Example 2 (Continued)

- Then

$$\begin{aligned}\Pr(2 \le X \le 5) &= F(5) - F(2^-)\\ &= F(5) - F(1)\\ &= 0.94 - 0.72 = 0.22.\end{aligned}$$

and

$$\begin{aligned}\Pr(X = 3) &= F(3) - F(3^-)\\ &= F(3) - F(2)\\ &= 0.81 - 0.76 = 0.05.\end{aligned}$$

Example 3

- Many manufacturers have quality control programs that include inspection of incoming materials for defects.
- Suppose a computer manufacturer receives computer boards in lots of five.
- Two boards are selected from each lot for inspection.

Example 3 (Continued)

(a) List all possible inspections.

(b) Suppose that boards 1 and 2 are the only defectives in a lot of five.

Two boards are chosen randomly.

Define X to be the **number of defective boards** observed among those inspected.

Find the probability distribution of X.

(c) Let $F(x)$ denote the c.d.f. of X. Obtain $F(x)$ for **<u>all</u>** x.

Solution to Example 3

(a) $\#(S) = {}_5C_2 = 5!/(2!\,3!) = 10.$

The possible selections are

$\{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}.$

(b) X takes values 0, 1, and 2.

$f(0) = \Pr(X = 0) = \Pr(\{(3,4),(3,5),(4,5)\}) = 3/10,$

$f(2) = \Pr(X = 2) = \Pr(\{(1,2)\}) = 1/10,$

$f(1) = \Pr(X = 1) = 1 - (f(0) + f(2)) = 6/10,$

and $f(x) = 0$ for $x \neq 0, 1, 2$.

Solution to Example 3 (Continued)

(c)

$$F(0) = \Pr(X \le 0) = \Pr(X = 0) = 0.3,$$

$$\begin{aligned} F(1) = \Pr(X \le 1) &= \Pr(X = 0 \text{ or } 1) \\ &= \Pr(X = 0) + \Pr(X = 1) \\ &= 0.3 + 0.6 = 0.9, \end{aligned}$$

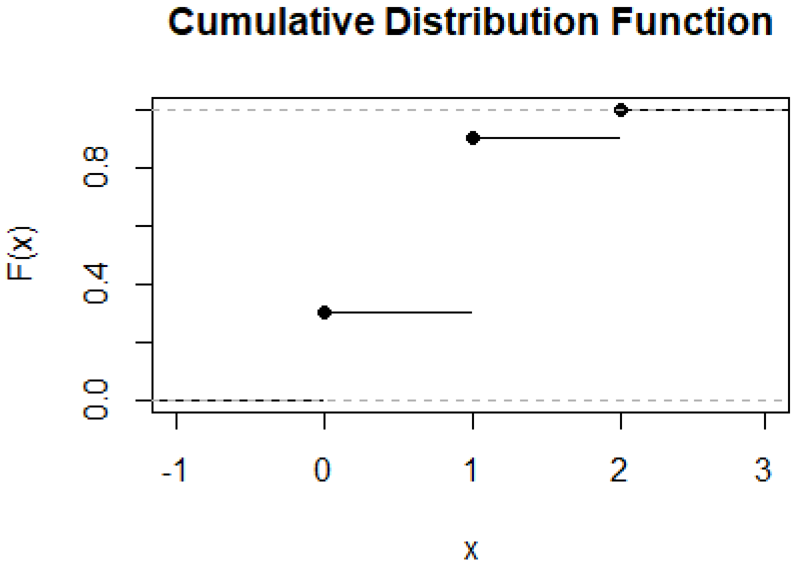
$$F(2) = \Pr(X \le 2) = 1.$$

Solution to Example 3 (Continued)

(c)

The c.d.f. is thus given by

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ 0.3, & \text{if } 0 \leq x < 1, \\ 0.9, & \text{if } 1 \leq x < 2, \\ 1, & \text{if } 2 \leq x. \end{cases}$$



Example 4

- The p.d.f. of a random variable X is given by

$$f(x) = \begin{cases} 2x, & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the c.d.f of X.

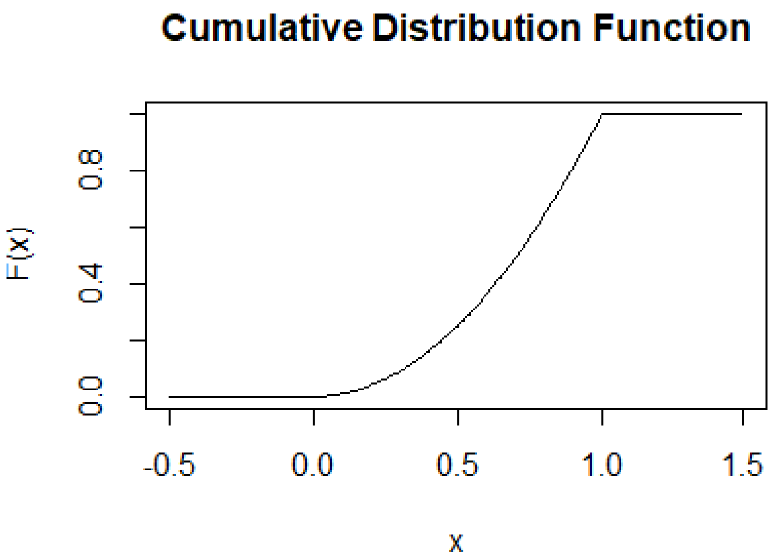
Solution to Example 4

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

$$= \begin{cases} \int_{-\infty}^{x} 0\, dt, & \text{for } x < 0, \\ \int_{-\infty}^{0} 0\, dt + \int_{0}^{x} 2t\, dt, & \text{for } 0 \le x < 1, \\ \int_{-\infty}^{0} 0\, dt + \int_{0}^{1} 2t\, dt + \int_{1}^{x} 0 dt, & \text{for } x \ge 1. \end{cases}$$

Solution to Example 4 (Continued)

$$F(x) = \int_{-\infty}^{x} f(t)dt$$
$$= \cdots$$
$$= \begin{cases} 0, & \text{for } x < 0, \\ x^2, & \text{for } 0 \leq x < 1, \\ 1, & \text{for } x \geq 1. \end{cases}$$



Example 5

- Let X denote the vibratory stress (psi) on a wind turbine blade at a particular wind tunnel.

- The following p.d.f. for X is proposed

$$f(x;\theta) = \begin{cases} \dfrac{x}{\theta^2} e^{-x^2/(2\theta^2)}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Example 5 (Continued)

(a) Verify that $f(x;\theta)$ is a legitimate p.d.f.

(b) Suppose that $\theta = 100$.

What is the probability that X is at most 200? Less than 200?

At least 200?

(c) Give an expression for $\Pr(X \leq x)$.

(d) What is the probability that X is between 100 and 200?

Solution to Example 5

(a) It is obvious that $f(x) > 0$ for $x > 0$.

$$\int_{-\infty}^{\infty} f(x;\theta)\,dx = \int_0^{\infty} \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}}\,dx$$

$$= -\int_0^{\infty} d\left(e^{-\frac{x^2}{2\theta^2}}\right)$$

$$= \left[-e^{-\frac{x^2}{2\theta^2}}\right]_0^{\infty}$$

$$= 0 - (-1) = 1.$$

Solution to Example 5 (Continued)

(b)

$$\Pr(X \leq 200; \theta = 100) = \int_{-\infty}^{200} f(x;\theta)dx$$

$$= \int_{0}^{200} \frac{x}{100^2} e^{-\left(\frac{x^2}{2(100)^2}\right)} dx$$

$$= \left[-e^{-\left(\frac{x^2}{20000}\right)}\right]_0^{200} = -e^{-2} + e^{0}$$

$$= -0.1353 + 1 = 0.8647.$$

Solution to Example 5 (Continued)

(b)

- $\Pr(X < 200) = \Pr(X \leq 200) = 0.8647$ since X is a continuous random variable.

- $\Pr(X \geq 200) = 1 - \Pr(X \leq 200) = 0.1353$.

Solution to Example 5 (Continued)

(c) For $x > 0$,

$$F(x;\theta) = \Pr(X \le x;\theta) = \int_{-\infty}^{x} f(t;\theta)dt$$

$$= \int_0^x \frac{t}{\theta^2} e^{-\left(\frac{t^2}{2\theta^2}\right)} dt$$

$$= \left[-e^{-\left(\frac{t^2}{2\theta^2}\right)}\right]_0^x$$

$$= 1 - e^{-\left(\frac{x^2}{2\theta^2}\right)}.$$

Solution to Example 5 (Continued)

(d)

$$\begin{aligned}
\Pr(100 \le X \le 200) &= F(200) - F(100) \\
&= 1 - \exp(-(200)^2/20000) \\
&\quad - [1 - \exp(-(100)^2/20000)] \\
&= e^{-1/2} - e^{-2} \\
&= 0.4712.
\end{aligned}$$

Solution to Example 5 (Continued)

(d)

Alternatively,

$$\Pr(100 \leq X \leq 200) = \int_{100}^{200} \frac{x}{100^2} e^{-\left(\frac{x^2}{2(100)^2}\right)} dx$$

$$= \left[-e^{-\left(\frac{x^2}{20000}\right)}\right]_{100}^{200} = 0.4712.$$

2.5 Mean and Variance of a Random Variable

2.5.1 Expected Values

Definition 2.7a

- If X is a **discrete** random variable taking on values $x_1, x_2, \cdots$ with probability function $f_X(x)$,
- then the **mean** or **expected value** (or **mathematical expectation**) of X, denoted by $\boldsymbol{E(X)}$ as well as by μ_X, is defined by

$$\mu_X = E(X) = \sum_i x_i\, f_X(x_i) = \sum_x x\, f_X(x)$$

Mean and Variance of a Random Variable (Continued)

2.5.1 Expected Values (Continued)

Note : The expected value is not necessarily a possible value of the random variable X.

Expected Values (Continued)

Definition 2.7b

- If X is a **continuous** random variable with probability density function $f_X(x)$, the mean of X is defined by

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx\,.$$

- Roughly speaking, the mathematical expectation is an "average" (or more precisely, a "weighted average").

Remarks

1. The expected value exists provided the sum or the integral in the above definitions exists.

2. In the discrete case, if $f_X(x) = 1/N$ for each of the N values of x, hence the mean,

$$E(X) = \sum_i x_i \, f(x_i) = \frac{1}{N} \sum_i x_i \, ,$$

becomes the average of the N items.

Example 1

- In a gambling game, a man gains 5 if he gets all heads or all tails in tossing a fair coin 3 times,
- and he pays out 3 if either 1 or 2 heads show.
- What is his expected gain?

Solution to Example 1

- Let X be the amount he can gain.
- Then $X = 5$ or -3 with the following probabilities:

  $\Pr(X = 5) = \Pr(\{HHH, TTT\}) = 1/8 + 1/8 = 1/4$

  $\Pr(X = -3) = 1 - \Pr(X = 5) = 3/4.$
- Therefore $E(X) = 5\left(\frac{1}{4}\right) + (-3)\left(\frac{3}{4}\right) = -1.$
- Hence, he will lose 1 per toss **in a long run**.

Example 2

- Suppose a game consists of rolling a balanced die.
- We pay c to play the game and we get i if number i occurs.
- How much should we pay if the game is fair?

  (We say a game is "fair" if $E(\text{gain}) = 0$.)

Solution to Example 2

- Let X denote the amount that one gets when rolling a die.
- Then clearly $\Pr(X = 1) = \cdots = \Pr(X = 6) = 1/6$.
- $E(X) = (1 + 2 + \cdots + 6)\left(\frac{1}{6}\right) = 3.5$.
- To be a fair game, E(paying) = E(getting), and hence for a fair game, the admission fee should be $c = E(X) = 3.5$.

Solution to Example 2 (Continued)

<u>Alternate Solution</u>,

- Let Y denote the amount that one gains when rolling a die.
- Then $Y = i - c$, where $i = 1, 2, \cdots, 6$
- Then clearly $\Pr(Y = 1 - c\ ) = \cdots = \Pr(Y = 6 - c) = 1/6$.
- $E(Y) = \left((1 - c) + \cdots + (6 - c)\right)\left(\frac{1}{6}\right) = 3.5\ - c$.
- To be a fair game, $E(Y) = 0$, and hence the admission fee, c, satisfies

$$3.5 - c = 0 \quad \text{or} \quad c = 3.5$$

Example 3

- A private pilot wishes to insure his airline for 1,000,000.
- The insurance company estimates that a total loss may occur with probability 0.0002, a 50% loss with probability 0.001, and a 25% loss with probability 0.01, and a 10% loss with probability 0.01.
- Ignoring all other partial losses, what premium should the insurance company charge each year to realize an average profit of 5,000?

Solution to Example 3

- The expected loss is given by

  $1000000(0.0002) + 500000(0.001) + 250000(0.01) + 100000(0.01) + 0(1 - 0.0002 - 0.001 - 0.01 - 0.01) = 4200$.

- Therefore, the insurance company should charge the premium 9200 so as to realize an average profit of 5000.

Example 4

The p.d.f. of weekly gravel sales X is

$$f_X(x) = \begin{cases} \frac{3}{2}(1 - x^2), & \text{for } 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find $E(X)$.

Solution to Example 4

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x) dx = \int_0^1 x \frac{3}{2}(1 - x^2)\, dx$$

$$= \frac{3}{2}\int_0^1 (x - x^3) dx = \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{8}.$$

Example 5

- The probability density function of a continuous random variable X, the total number of hours, in units of 100 hours, that a family runs a vacuum cleaner over a period of one year, was given as follows.

$$f_X(x) = \begin{cases} x, & \text{for } 0 < x < 1, \\ 2 - x, & \text{for } 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the average number of hours per year that families run their vacuum cleaners.

Solution to Example 5

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

$$= \int_{-\infty}^{0} x(0)\, dx + \int_{0}^{1} x^2\, dx + \int_{1}^{2} x(2-x) dx + \int_{2}^{\infty} x(0) dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

Solution to Example 5 (Continued)

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \cdots$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \left[\frac{1}{3} - 0\right] + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = 1.$$

- Families run their vacuum cleaners 100 hours per year on average.

2.5.2 Expectation of a Function of a RV

Definition 2.8

For any function $g(X)$ of a random variable X with p.f. (or p.d.f.) $f_X(x)$,

(a) $E[g(X)] = \sum_x g(x) f_X(x)$

if X is a **discrete** r.v. providing the sum exists; and

(b) $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

if X is a **continuous** r.v. providing the integral exists.

Some Special Cases

1. $g(x) = (x - \mu_X)^2$.

   This leads to the definition of variance of a given random variable X.

Some Special Cases (Continued)

Definition 2.9

- Let X be a random variable with p.f. (or p.d.f.) $f(x)$, then the **variance** of X is defined as

$$\sigma_X^2 = V(X) = E[(X - \mu_X)^2]$$

$$= \begin{cases} \displaystyle\sum_x (x - \mu_X)^2 f_X(x), & \text{if } X \text{ is discrete,} \\ \displaystyle\int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x)\, dx, & \text{if } X \text{ is continuous.} \end{cases}$$

Some Special Cases (Continued)

Remarks:

(a) $V(X) \geq 0$.

(b) $V(X) = E(X^2) - [E(X)]^2$.

- The positive square root of the variance is called the **standard deviation** of X. That is

$$\sigma_X = \sqrt{V(X)}$$

Some Special Cases (Continued)

2. $g(x) = x^k$.

   Then $E(X^k)$ is called the ***k*-th moment of *X***.

Example 1

Let the p.d.f. of X be given by:

x	-1	0	1	2
$f_X(x)$	$1/8$	$2/8$	$1/8$	$4/8$

(a) Find $E(X)$ and $V(X)$.

(b) Define $Y = X^2 + 2$. Find $E(Y)$ and $V(Y)$.

Solution to Example 1

(a)

$$E(X) = \sum_{x} x\, f_X(x)$$

$$= (-1)\left(\frac{1}{8}\right) + 0\left(\frac{2}{8}\right) + 1\left(\frac{1}{8}\right) + 2\left(\frac{4}{8}\right) = 1.$$

Solution to Example 1 (Continued)

(a) (Continued)

$$V(X) = \sum_x (x-1)^2 f_X(x), \qquad \text{since } \mu_X = 1$$

$$= (-1-1)^2\left(\frac{1}{8}\right) + (0-1)^2\left(\frac{2}{8}\right)$$

$$+(1-1)^2\left(\frac{1}{8}\right) + (2-1)^2\left(\frac{4}{8}\right) = \frac{5}{4}.$$

Solution to Example 1 (Continued)

(a) (Continued)

Alternatively, we can find $E(X^2)$ first and then obtain $V(X)$ using the formula $V(X) = E(X^2) - [E(X)]^2$.

$$E(X^2) = \sum_x x^2 f_X(x)$$

$$= (-1)^2\left(\frac{1}{8}\right) + 0^2\left(\frac{2}{8}\right) + 1^2\left(\frac{1}{8}\right) + 2^2\left(\frac{4}{8}\right) = \frac{9}{4}.$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{9}{4} - 1^2 = \frac{5}{4}.$$

Solution to Example 1 (Continued)

(b)

$$E(Y) = E(X^2 + 2) = \sum_x (x^2 + 2) f_X(x)$$

$$= ((-1)^2 + 2)\left(\frac{1}{8}\right) + (0^2 + 2)\left(\frac{2}{8}\right) + (1^2 + 2)\left(\frac{1}{8}\right)$$

$$+(2^2 + 2)\left(\frac{4}{8}\right) = \frac{17}{4}.$$

Alternatively, $E(Y) = E(X^2) + 2 = \frac{9}{4} + 2 = \frac{17}{4}$.

Solution to Example 1 (Continued)

(b)

$$V(Y) = \sum_x \left(x^2 + 2 - \frac{17}{4}\right)^2 f_X(x)$$

$$= \left(3 - \frac{17}{4}\right)^2 \left(\frac{1}{8}\right) + \left(2 - \frac{17}{4}\right)^2 \left(\frac{2}{8}\right)$$

$$+ \left(3 - \frac{17}{4}\right)^2 \left(\frac{1}{8}\right) + \left(6 - \frac{17}{4}\right)^2 \left(\frac{4}{8}\right) = \frac{51}{16}.$$

Solution to Example 1 (Continued)

(b) (Continued)

Alternatively, we can find $E(Y^2)$ first and then obtain $V(Y)$ using the formula

$$V(Y) = E(Y^2) - [E(Y)]^2,$$

where

$$E(Y^2) = 3^2\left(\frac{1}{8}\right) + 2^2\left(\frac{2}{8}\right) + 3^2\left(\frac{1}{8}\right) + 6^2\left(\frac{4}{8}\right) = \frac{170}{8}$$

$$V(Y) = \frac{170}{8} - \left(\frac{17}{4}\right)^2 = \frac{51}{16}.$$

Example 2

- Suppose that the random variable X is continuous with the following p.d.f.:

$$f_X(x) = \begin{cases} \dfrac{x}{225}, & \text{for } 0 < x < 15, \\ \dfrac{30-x}{225}, & \text{for } 15 \leq x \leq 30, \\ 0, & \text{otherwise.} \end{cases}$$

- Find $E(X)$ and $V(X)$.

Solution to Example 2

$$E(X) = \int_0^{15} x\left(\frac{x}{225}\right) dx + \int_{15}^{30} x\left(\frac{30-x}{225}\right) dx$$

$$= \frac{1}{225}\left\{\left[\frac{x^3}{3}\right]_0^{15} + \left[15x^2 - \frac{x^3}{3}\right]_{15}^{30}\right\}$$

$$= \frac{1}{225}\left\{\frac{15^3}{3} + \left[15(30)^2 - \frac{30^3}{3} - 15(15)^2 + \frac{15^3}{3}\right]\right\}$$

$$= 15.$$

Solution to Example 2 (Continued)

$$E(X^2) = \int_0^{15} x^2 \left(\frac{x}{225}\right) dx + \int_{15}^{30} x^2 \left(\frac{30-x}{225}\right) dx$$

$$= \frac{1}{225}\left\{\left[\frac{x^4}{4}\right]_0^{15} + \left[10x^3 - \frac{x^4}{4}\right]_{15}^{30}\right\} = \frac{525}{2} = 262.5.$$

Therefore,

$$V(X) = E(X^2) - [E(X)^2]$$

$$= 262.5 - 15^2 = 37.5.$$

Example 3

- Let X denote the amount of time for which a book on 2-hour reserve at the science library is checked out by a randomly selected student and suppose X has the probability density function

$$f_X(x) = \begin{cases} \dfrac{x}{2}, & \text{for } 0 \le x < 2, \\ 0, & \text{otherwise.} \end{cases}$$

Example 3 (Continued)

(a) Compute $E(X)$.

(b) Compute $V(X)$ and σ_X.

(c) If the borrower is charged an amount $h(X) = X^2$ when checkout duration is X, compute the expected charge $E[h(X)]$.

Solution to Example 3

(a)

$$E(X) = \int_0^2 x\left(\frac{x}{2}\right)\,dx$$

$$= \left[\frac{x^3}{6}\right]_0^2 = \frac{4}{3}.$$

Solution to Example 3 (Continued)

(b)

$$E(X^2) = \int_0^2 x^2 \left(\frac{x}{2}\right) dx = \left[\frac{x^4}{8}\right]_0^2 = 2.$$

$$V(X) = E(X^2) - [E(X)^2] = 2 - \left(\frac{4}{3}\right)^2 = \frac{2}{9}.$$

$$\sigma_X = \sqrt{2/9}$$

(c) $E[h(X)] = E[X^2] = 2.$

2.5.3 Properties of Expectation

Property 1

$$E(aX + b) = a\,E(X) + b,$$

where a and b are constants.

Properties of Expectation (Continued)

Property 1 (Continued)

- Proof: For discrete case,

$$E(aX+b)=\sum_x (ax+b)f_X(x)$$

$$=\sum_x ax\, f_X(x)+\sum_x b\, f_X(x)$$

$$=a\left[\sum_x x\, f_X(x)\right]+b\left[\sum_x f_X(x)\right]=aE(X)+b.$$

Properties of Expectation (Continued)

Two special cases:

(a) Put $b = 0$, we have $E(aX) = a\,E(X)$.

(b) Put $a = 1$, we have $E(X + b) = E(X) + b$.

In general,

$$E[a_1 g_1(X) + a_2 g_2(X) + \cdots + a_k g_k(X)] = a_1 E[g_1(X)] + a_2 E[g_2(X)] + \cdots + a_k E[g_k(X)]$$

where $a_1, a_2, \cdots, a_k$ are constants.

Properties of Expectation (Continued)

Property 2

$$V(X) = E(X^2) - [E(X)]^2.$$

Properties of Expectation (Continued)

Property 2 (Continued)

Proof:

$$V(X) = E[(X-\mu_X)^2]$$
$$= E[X^2 - 2X\mu_X + \mu_X^2]$$
$$= E(X^2) - E(2X\mu_X) + E(\mu_X^2)$$
$$= E(X^2) - 2\mu_X E(X) + (\mu_X^2)$$
$$= E(X^2) - 2\mu_X^2 + \mu_X^2 = E(X^2) - \mu_X^2$$

Notice that $\mu_X = E(X)$ is a constant.

Properties of Expectation (Continued)

Property 3

$$V(aX + b) = a^2 V(X),$$

where a and b are constants

Properties of Expectation (Continued)

Property 3 (Continued)

Proof:

$$
\begin{aligned}
V(aX+b) &= E[(aX+b)^2] - [E(aX+b)]^2 \\
&= E(a^2X^2 + 2abX + b^2) - (a\mu_X + b)^2 \\
&= a^2E(X^2) + 2ab\,E(X) + b^2 - (a^2\mu_X^2 + 2ab\mu_X + b^2) \\
&= a^2E(X^2) - a^2\mu_X^2 \\
&= a^2[E(X^2) - \mu_X^2] \\
&= a^2V(X).
\end{aligned}
$$

Example 4

- A jewelry shop purchased three necklaces of a certain type at \$500 a piece.
- It will sell them for \$1000 a piece. The designer has agreed to repurchase any necklace still unsold after a specified period at \$200 a piece.
- Let X denote the number of necklaces sold and suppose X follows the following probability distribution.

x	0	1	2	3
$f_X(x)$	0.1	0.2	0.3	0.4

- Find the expected gain and the variance of the gain.

Solution to Example 4

- With $g(X) = \text{revenue} - \text{cost} = 1000X + 200(3 - X) - 3(500) = 800X - 900$.
- $E\left(g(X)\right) = g(0)f_X(0) + g(1)f_X(1) + g(2)f_X(2) + g(3)f_X(3)$
  $= (-900)(0.1) + (-100)(0.2) + (700)(0.3) + 1500(0.4)$
  $= 700.$
- Hence the expected profit is \$700.

Solution to Example 4 (Continued)

- $E\left[\left(g(X)\right)^2\right] = (-900)^2(0.1) + (-100)^2(0.2) + (700)^2(0.3) + 1500^2(0.4) = 1130000.$

- Hence

$$V\left(g(X)\right) = 1130000 - 700^2 = 640000$$

and

$$\sqrt{V\left(g(X)\right)} = \sqrt{640000} = 800$$

Example 5

- The distribution of the amount of gravel (in tons) sold by a particular construction supply company in a given week is a continuous random variable X with p.d.f.

$$f_X(x) = \begin{cases} \frac{3}{2}(1 - x^2), & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

- Find $E(X)$ and $V(X)$.

Solution to Example 5

$$E(X) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_0^1 x\frac{3}{2}(1-x^2)dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = \frac{3}{2}\left(\frac{1}{4}\right) = \frac{3}{8}.$$

Solution to Example 5 (Continued)

$$E(X^2) = \int_{-\infty}^{\infty} x^2\, f_X(x)\, dx = \int_0^1 x^2 \frac{3}{2}(1 - x^2)dx$$

$$= \frac{3}{2}\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{3}{2}\left(\frac{2}{15}\right) = \frac{1}{5}.$$

$$V(X) = \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320} = 0.0594.$$

Example 6

- The hospital period, in days, for patients following treatment for a certain type of kidney disorder is a random variable $Y = X + 4$, where X has the probability density function

$$f_X(x) = \begin{cases} \dfrac{32}{(x+4)^3}, & \text{for } x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Find the average number of days that a person is hospitalized following treatment for this disorder.

Solution to Example 6

$$E(Y) = \int_0^\infty (x+4)\frac{32}{(x+4)^3}\,dx$$

$$= \int_0^\infty \frac{32}{(x+4)^2}\,dx$$

$$= \left[-\frac{32}{x+4}\right]_0^\infty = 8.$$

2.6 Chebyshev's Inequality

- If we know the probability distribution of a random variable X, we may then compute $E(X)$ and $V(X)$.
- However, the converse is not true. From the knowledge of $E(X)$ and $V(X)$ we cannot reconstruct the probability distribution of X and
- hence, we cannot compute quantities such as

$$\Pr(|X - E(X)| \leq c),$$

where c is a positive constant.

[Note: $\Pr(|Y| \leq a)$ is equivalent to $\Pr(-a \leq Y \leq a)$.

Chebyshev's Inequality (Continued)

- Nevertheless, the Russian mathematician Chebyshev gave a very useful upper (or lower) **bound** to such probability.

- This result is known as **Chebyshev's inequality**.

Chebyshev's Inequality (Continued)

- Let X be a random variable (discrete or continuous) with $E(X) = \mu$ and $V(X) = \sigma^2$.
- Then for any positive number k we have

$$\Pr(|X - \mu| \geq k\sigma) \leq 1/k^2.$$

- That is, the probability that the value of X lies at least k standard deviation from its mean is at most $1/k^2$.
- Alternatively,

$$\Pr(|X - \mu| < k\sigma) \geq 1 - 1/k^2.$$

Remarks

1. The quantity k in Chebyshev's Inequality can be any positive number.

2. This inequality is true for **all** distributions with finite mean and variance.

3. The theorem gives a **lower bound** on the probability that $|X - \mu| < k\sigma$. No guarantee that this lower bound is close to the exact probability.

Example 1

- Number of telephone calls X in a day has $\mu_X = 14, \sigma_X = 3.5$.
- What can you say about $\Pr(7 < X < 21)$?

Solution to Example 1

$$\begin{aligned}\Pr(7 < X < 21) &= \Pr\big(14 - 2(3.5) < X < 14 + 2(3.5)\big)\\ &= \Pr\big(-2(3.5) < X - 14 < 2(3.5)\big)\\ &= \Pr\big(|X - 14| < 2(3.5)\big)\\ &\geq 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}.\end{aligned}$$

where we apply the Chebyshev in quality with $k = 2$

Example 2

The p.f. X is given as follows.

x	0	1	2	3	4	5	6
$f_X(x)$	0.1	0.15	0.2	0.25	0.2	0.06	0.04

It can be shown that $E(X) = 2.64$ and $\sigma^2 = 2.37$.

Hence $\sigma = 1.54$.

Solution to Example 2

By **Chebyshev's Inequality**,

$$\Pr(|X-\mu| \geq 2\sigma) \leq 1/2^2 = 0.25.$$

But

$$\begin{aligned} &\Pr(|X-\mu| \geq 2\sigma) \\ &= \Pr(X-\mu \leq -2\sigma \text{ or } X-\mu \geq 2\sigma) \\ &= \Pr(X \leq \mu - 2\sigma \text{ or } X \geq \mu + 2\sigma) \\ &= \Pr(X \leq -0.44 \text{ or } X \geq 5.72) \\ &= \Pr(X = 6) = 0.04. \quad \text{(From the given p.f.)} \end{aligned}$$

Chebyshev's bound of 0.25 is too conservative.