ISYE 6644 Simulation Note
Module 3 Hand Simulation

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Overview

1. Solving a differential equation
2. Monte Carlo integration
3. Making some pi
4. Single-server queue
5. (s,S) inventory system
6. Simulating random variables
7. Spreadsheet simulation

1. Differential Equation

Recall: If $f(x)$ is continuous, then it has the *derivative*

$$\frac{d}{dx}f(x) \equiv f'(x) \equiv \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

if the limit exists and is well-defined for any given x. Think of the derivative as the slope of the function.

Then for small h,

$$f'(x) \approx \frac{f(x+h)-f(x)}{h}$$

and

$$f(x+h) \approx f(x)+hf'(x).$$

- Euler's method

Example: Suppose you have a differential equation of a population growth model, $f'(x) = 2f(x)$ with $f(0) = 10$. Let's "solve" this using a fixed-increment time approach with $h = 0.01$. (This is known as *Euler's method*.) By (1), we have

$$f(x+h) \approx f(x)+hf'(x) = f(x)+2hf(x) = (1+2h)f(x).$$

Similarly,

$$f(x+2h) = f((x+h)+h) \approx (1+2h)f(x+h) \approx (1+2h)^2 f(x).$$

$$f(x+ih) \approx (1+2h)^i f(x), \quad i = 0,1,2,\ldots,$$

though the approximation may deteriorate as i gets large.

Plugging in $f(0) = 10$ and $h = 0.01$, we have

$$f(0.01i) \approx 10(1.02)^i, \quad i = 0, 1, 2, \ldots.$$

In any case, let's see how well the approximation does....

$x = ih = 0.01i$	0	0.01	0.02	0.03	0.04	$\cdots$	0.10
approx $f(x) \approx 10(1.02)^i$	10	10.20	10.40	10.61	10.82	$\cdots$	12.19
true $f(x) = 10e^{2x}$	10	10.20	10.41	10.62	10.83	$\cdots$	12.21

A very good approximation (Recall nth order approximation in Real Analysis class)

2. Monte Carlo Integration (physics, finance…)
   - Integration definition

Integration

Definition The function $F(x)$ having derivative $f(x)$ is called the *antiderivative*. The antiderivative is denoted $F(x) = \int f(x)\,dx$; and this is also called the *indefinite integral* of $f(x)$.

Fundamental Theorem of Calculus: If $f(x)$ is continuous, then the area under the curve for $x \in [a, b]$ is denoted and given by the *definite integral* [3]

$$\int_a^b f(x)\,dx \equiv F(x)\Big|_a^b \equiv F(b) - F(a).$$

Suppose $U_1, U_2, \ldots$ are iid Unif(0,1), and define

$$I_i \equiv (b-a)g(a + (b-a)U_i) \quad \text{for } i = 1, 2, \ldots, n.$$

We can use the sample average as an estimator for I:

$$\bar{I}_n \equiv \frac{1}{n}\sum_{i=1}^n I_i = \frac{b-a}{n}\sum_{i=1}^n g(a + (b-a)U_i).$$

by the Law of the Unconscious Statistician, notice that

$$\begin{aligned} \mathrm{E}[\bar{I}_n] &= (b-a)\mathrm{E}[g(a + (b-a)U_i)] \\ &= (b-a)\int_{\mathbb{R}} g(a + (b-a)u)f(u)\,du \\ &\quad \text{(where } f(u) \text{ is the Unif(0,1) pdf)} \\ &= (b-a)\int_0^1 g(a + (b-a)u)\,du = I. \end{aligned}$$

- **Proof**

So $\bar{I}_n$ is unbiased for I.

Since it can also be shown that $\text{Var}(\bar{I}_n) = O(1/n)$, the LLN implies $\bar{I}_n \to I$ as $n \to \infty$.

Approximate Confidence Interval for I:

By the CLT, we have

$$\bar{I}_n \approx \text{Nor}\big(\text{E}[\bar{I}_n], \text{Var}(\bar{I}_n)\big) \sim \text{Nor}\big(I, \text{Var}(I_i)/n\big).$$

This suggests that a reasonable $100(1-\alpha)\%$ confidence interval for I is

$$I \in \bar{I}_n \pm z_{\alpha/2}\sqrt{S_I^2/n}, \tag{3}$$

where $z_{\alpha/2}$ is the usual standard normal quantile, and $S_I^2 \equiv \frac{1}{n-1}\sum_{i=1}^{n}(I_i - \bar{I}_n)^2$ is the sample variance of the I_i's.

Example: Suppose $I = \int_0^1 \sin(\pi x)\,dx$ (and pretend we don't know the actual answer, $2/\pi \doteq 0.6366$).

Let's take $n = 4$ Unif(0,1) observations:

$$U_1 = 0.79 \quad U_2 = 0.11 \quad U_3 = 0.68 \quad U_4 = 0.31$$

Since

$$I_i = (b-a)g(a+(b-a)U_i) = g(U_i) = \sin(\pi U_i),$$

we obtain

$$\bar{I}_n = \frac{1}{4}\sum_{i=1}^{4} I_i = \frac{1}{4}\sum_{i=1}^{4}\sin(\pi U_i) = 0.656,$$

Note that n=4 is small.

3. Making Some Pi
   Considering a unit square and toss darts randomly to estimate Pi

How would we actually conduct such an experiment?

To simulate a dart toss, suppose U_1 and U_2 are iid Unif(0,1). Then (U_1, U_2) represents the random position of the dart on the unit square. The dart lands in the circle if

$$\left(U_1 - \frac{1}{2}\right)^2 + \left(U_2 - \frac{1}{2}\right)^2 \le \frac{1}{4}.$$

Generate n such pairs of uniforms and count up how many of them fall in the circle. Then plug into $\hat{\pi}_n$. □

4. A single server queue
   - Discrete event simulation

Interarrival time between customers $i-1$ and i is I_i

Customer i's arrival time is $A_i = \sum_{j=1}^{i} I_j$

Customer i starts service at time $T_i = \max(A_i, D_{i-1})$

Customer i's waiting time is $W_i^Q = T_i - A_i$

Customer i's time in the system is $W_i = D_i - A_i$

Customer i's service time is S_i

Customer i's departure time is $D_i = T_i + S_i$

5. An sS Inventory System

A store sells a product at \$d/unit, out inventory policy is to have at least s units in stock at the start of each day. If the stock slips to less than s by the end of day, we place an order with our supplier to push the stock back up to S by the beginning of the next day.

Let I_i denote the inventory at the *end* of day i, and let Z_i denote the order that we place at the end of day i. Clearly,

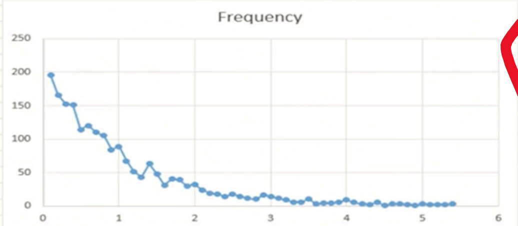
$$Z_i = \begin{cases} S - I_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases} .$$

If an order is placed to the supplier at the end of day i, it costs the store $K + cZ_i$. It costs \$$h$/unit for the store to hold unsold inventory overnight, and a penalty cost of \$$p$/unit if demand can't be met. No backlogs are allowed. Demand on day i is D_i.

$$\begin{aligned}
&\text{Total} \\
&= \text{Sales} - \text{Ordering Cost} - \text{Holding Cost} - \text{Penalty Cost} \\
&= d \min(D_i, \text{inventory at beginning of day } i) \\
&\qquad - \begin{cases} K + cZ_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases} \\
&\qquad - hI_i - p \max(0, D_i - \text{inventory at beginning of day } i) \\
&= d \min(D_i, I_{i-1} + Z_{i-1}) \\
&\qquad - \begin{cases} K + cZ_i & \text{if } I_i < s \\ 0 & \text{otherwise} \end{cases} \\
&\qquad - hI_i - p \max(0, D_i - (I_{i-1} + Z_{i-1})).
\end{aligned}$$

6. Simulating Random Variables
   Review Inverse Transform Theorem.

1				
2	Exp(1) RV Generation			
3				
4	unif(0,1)	Exp(1)	Bins	Frequency
5	0.289364	1.24007		
6	0.016365	4.112615	0.1	196
7	0.563956	0.572778	0.2	166
8	0.924651	0.078339	0.3	153
9	0.242508	1.416722	0.4	151
10	0.193475	1.642604	0.5	114
11	0.563096	0.574305	0.6	120
12	0.184734	1.688839	0.7	110
13	0.601359	0.508563	0.8	105
14	0.795769	0.228447	0.9	84
15	0.802272	0.220307	1	89
16	0.240576	1.424718	1.1	67
17	0.134575	2.005637	1.2	51
18	0.699986	0.356695	1.3	43
19	0.153743	1.87247	1.4	63
20	0.77377	0.25648	1.5	48
21	0.194306	1.638321	1.6	31
22	0.485511	0.722554	1.7	40
23	0.582822	0.539873	1.8	39



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7. Spreadsheet Simulation (even in certain discrete event scenarios)

Let's simulate a fake stock portfolio consisting of 6 stocks from different sectors, as laid out in my Excel file Spreadsheet Stock Portfolio. We start out with \$5000 worth of each stock, and each increases or decreases in value each year according to

$$\text{Previous Value} * \max\Big[\,0,\ \text{Nor}(\mu_i,\,\sigma_i^2) * \text{Nor}\big(1,(0.2)^2\big)\Big]\,,$$

where the first normal term denotes the natural stock fluctuation for stock i, and the second normal denotes natural market conditions (that affect all stocks).

The market on average stays the same, high volatility about 20% a year

NORM.INV(RAND(), mu, sigma)

RAND() is Unif(0,1), NORM.INV use inverse transform method.

	mean	sd	year0	year1	year2	year3	year4	year5
general year performance (so stocks are correlated)				1.18	0.92	0.96	1.11	0.85
energy	0.05	0.30	5000	7295	6636	9095	9509	11482
pharmaceuticals	0.06	0.20	5000	7015	6851	5312	5604	4287
entertainment	0.04	0.10	5000	4702	5997	5822	5630	5534
insurance	0.07	0.05	5000	5941	6004	6505	7368	6666
banking	0.06	0.30	5000	5349	6652	6915	8771	8552
computer technology	0.18	0.50	5000	9814	15901	30057	39026	48933
Totals			30000	40115.8	48040.1	63707.2	75907.7	85454