Math 136 **Real Analysis II** **Spring 2020**
Prof. L. Tu **Exam 1 Review**

Exam 1 is scheduled for **Monday, February 24**, from **noon to 1:20 p.m.** in a room **TBA**. We will have class that day.

There will be a review session on Sunday, February 23, from 3 to 4:15 in our regular classroom BP-2. Solutions to the Review Problems for Exam 1 will be posted on the Math 136 Canvas website by Friday evening, February 21. You should make every effort to do the problems on your own or with friends before consulting the solutions.

The exam will cover the first four problem sets up to and including Section 17.2. Some questions will ask you to reproduce definitions, statements of theorems, or proofs. Other questions will be variations of problems or proofs that you have done before. Still others will be examples or applications of theorems.

Learn the definitions of terms introduced in class. Study the statements of all the propositions, theorems, and corollaries; pay particular attention to the hypotheses.

In particular, learn the following definitions:

> Derivative of real valued functions (p. 88), limit point (p. 348. What Fitzpatrick calls a limit point is normally called an **accumulation point**), limit of a function (p. 348), partial derivative (p. 355), continuously differentiable (also known as C^1) (p. 358), directional derivative (p. 366), gradient (p. 367), kth-order approximation (p. 373), tangent plane (p. 374), affine function (p. 377), derivative matrix (p. 408), neighborhood (p. 423), locally invertible (as given in class), the notation for the derivative submatrices on p. 450.

Learn the statements of the following theorems:

> Th. 13.10 (equality of mixed partials)
> Thm. 15.39 (chain rule for general mappings–how the proof uses Th. 15.34)
> Inverse function theorem in the plane (Thm. 16.2, p. 423) <u>and</u> the general version given in class for $\mathbb{R}^n$
> Implicit function theorem

Learn the statements and proofs of the following theorems:

> Prop. 4.5 (differentiability implies continuity)
> Lemma 4.16 (derivative at a local extremum)
> Theorems 4.17 and 4.18, Rolle's theorem and the mean value theorem for $f\colon [a,b] \to \mathbb{R}$
> Lemma 4.19 (zero derivative implies constant function)
> Cor. 4.21 (criterion for strict monotonicity).
> Th. 13.10 (equality of mixed partials)
> Th. 13.16 (directional derivative theorem)
> Th. 13.17 (mean value theorem)
> Th. 13.20 (continuously differentiable implies continuous)
> Th. 15.31 (first-order approximation theorem for C^1 functions)
> Th. (uniqueness of tangent plane) Done in class.
> Th. 15.32 (affine first-order approximation)

Th. 15.34 (chain rule: proof using the mean value theorem given in class)
Inverse function theorem $\Rightarrow$ Dini's theorem (implicit function theorem for functions of two variables)

Review all homework problems. All solutions will be posted on Trunk by Friday evening, February 21.

Review Problems for Exam 1

1. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$. State the definition of $f'(a)$.
   (b) Use the definition of the derivative to compute $f'(a)$ for $f(x) = x^3$.

2. Suppose $f(x, y, z) = x^2 + yz$, $g(x, y) = y^3 + xy$, $h(x) = \sin x$, and
$$G(x, y, z) = h\big(f(x, y, z)g(x, y)\big).$$
Compute $\partial G/\partial x$.

3. Let f be a function defined on an open subset $\mathcal{O}$ of $\mathbb{R}^2$, and suppose that $\partial f/\partial x$ and $\partial f/\partial y$ are defined and bounded everywhere on $\mathcal{O}$. Show that f is continuous on $\mathcal{O}$. *Hint*: The proof is very similar to how we proved "continuously differentiable $\Rightarrow$ continuous" in class using the Mean Value Theorem on the partial derivatives of f. In particular, start by writing
$$f(a + h, b + k) - f(a, b) = f(a + h, b + k) - f(a + h, b) + f(a + h, b) - f(a, b).$$

4. Define $g(x, y) = \begin{cases} \dfrac{x^2y^4}{x^2 + y^2} & \text{for } (x, y) \neq (0, 0), \\ 0 & \text{for } (x, y) = (0, 0). \end{cases}$ Prove that the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ has first-order partial derivatives. Is the function $g\colon \mathbb{R}^2 \to \mathbb{R}$ continuously differentiable?

5. Suppose $\mathbf{F}(x, y) = (u, v) = (e^{x+y}, e^{x-y})$ for $(x, y) \in \mathbb{R}^2$. Is $\mathbf{F}$ locally invertible near $(x, y) = (0, 0)$? If so, calculate $\partial x/\partial u(1, 1)$. Note that $(1, 1) = \mathbf{F}(0, 0)$. Justify your answers.

6. True-False.
   (a) If it exists, a first-order approximation to a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at a point $(a, b) \in \mathbb{R}^2$ is unique.
   (b) If it exists, an affine first-order approximation to a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ at a point $(a, b) \in \mathbb{R}^2$ is unique.

7. Show that the equations
$$x^2 - y^2 - u^3 + v^2 + 4 = 0,$$
$$2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$
determine functions $u(x, y)$ and $v(x, y)$ near $x = 2$, $y = -1$ such that $u(2, -1) = 2$, $v(2, -1) = 1$ (i.e., near $(x, y, u, v) = (2, -1, 2, 1)$) and compute $\frac{\partial v}{\partial x}(2, -1)$.

8. Let a and b be real numbers. The equations
$$z^2 + xy = a$$
$$z^2 + x^2 - y^2 = b$$
describe a set of points in $\mathbb{R}^3$. Assume this set is nonempty. Let (x_0, y_0, z_0) be a point on this set.
   (a) Under what sufficient conditions may the part of the set near (x_0, y_0, z_0) be represented in the form $x = f(z)$ and $y = g(z)$?
   (b) Let $\mathbf{F}(x, y, z) = \begin{pmatrix} z^2 + xy \\ z^2 + x^2 - y^2 \end{pmatrix}$, $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^2$. Under the conditions in part (a) on (x_0, y_0, z_0) explain why there are an infinite number of solutions to $\mathbf{F}(x, y, z) = \begin{pmatrix} a \\ b \end{pmatrix}$

9. You will prove a special case of Theorem 15.32. Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $\vec{\mathbf{x}} \in \mathbb{R}^n$. Assume there is a vector $\vec{\mathbf{b}} \in \mathbb{R}^n$ such that
$$\lim_{\vec{\mathbf{h}} \to \mathbf{0}} \frac{\left| f(\vec{\mathbf{x}} + \vec{\mathbf{h}}) - \left[f(\vec{\mathbf{x}}) + \left\langle \vec{\mathbf{b}}, \vec{\mathbf{h}} \right\rangle \right] \right|}{\|\vec{\mathbf{h}}\|} = 0.$$
Let $\vec{\mathbf{p}} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$. Prove that the directional derivative $\partial f / \partial \vec{\mathbf{p}}(\vec{\mathbf{x}})$ exists and is equal to $\left\langle \vec{\mathbf{b}}, \vec{\mathbf{p}} \right\rangle$.

10. let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be continuously differentiable (C^1). Assume $\mathbf{F}(1, 2, 3) = (4, 5, 6)$ and
$$\mathbf{DF}(1, 2, 3) = A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \text{ You may assume that } A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$
   (a) Does $\mathbf{F}$ satisfy the inverse function theorem at $(1, 2, 3)$? Justify your answer. Assume the domain of $\mathbf{F}$ is given coordinates (x, y, z) and the target has coordinates (u, v, w).
   (b) Find $\frac{\partial u}{\partial x}(1, 2, 3)$.
   (c) Find $\frac{\partial x}{\partial u}(4, 5, 6)$. Note that $\mathbf{F}(1, 2, 3) = (4, 5, 6)$.

11. Let $f : (a, b) \to \mathbb{R}$ be differentiable at $x_0 \in (a, b)$. Prove that f is continuous at x_0.

12. Let $\mathcal{O} \in \mathbb{R}^n$ be open and let $f : \mathcal{O} \to \mathbb{R}$ be continuously differentiable. Prove that f is continuous on $\mathcal{O}$.

13. Use the mean value theorem to establish the following inequalities:
   (a) $|\cos x - \cos a| \leq |x - a|$ for $x, a \in \mathbb{R}$.
   (b) $e^x > 1 + x$ for $x > 0$.