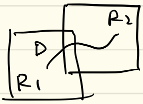
Week 9 Mon Mar 9.

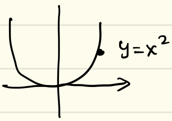
– Review

Jordan Cont 0 : A set $D \subset \mathbb{R}^n$ has J.C.0 if $\forall \varepsilon > 0$ $\exists$ fin many closed rectangles $R_1 \cdots R_m$ s.t $D \subset \bigcup R_i$ and $\sum vol(R_i) < \varepsilon$



Def. Graph of $f: A \to \mathbb{R}$

$$Graph(f) = \{(x, f(x)) \mid x \in A,\ f(x) \in \mathbb{R}\}$$



$$Graph(f) = \{(x, f(x)) \subset \mathbb{R} \times \mathbb{R}\}$$

Th. The graph of an integrable $f: I \to \mathbb{R}$, where I is a rectangle, has J.C.0.

Proof: Let $\varepsilon > 0$ since f integrable
$\exists$ a P of I s.t

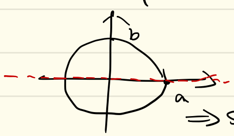
$$Osc(f, P) = U(\cdot) - L(\cdot) < \varepsilon$$

$$= \sum (M_J - m_J)\, vol(J) = \sum_J \underbrace{vol(J) \times len[m_J, M_J]}_{\text{rectangle}}.$$

$< \varepsilon$

So we can cover graph of f with $J \times [m_J, M_J]$ for J in P. These are fin many with total vol $< \varepsilon$.

Ex. Ellipse

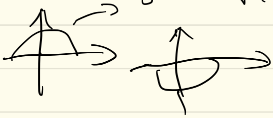


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Show that $\{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$

$\Rightarrow$ split the graph into upper / lower half

$\Rightarrow \quad y = \pm b \cdot \sqrt{1 - \frac{x^2}{a^2}}$



$\to y = b \cdot \sqrt{1 - \frac{x^2}{a^2}}$

$\to y = -b \cdot \sqrt{1 - \frac{x^2}{a^2}}$

continuity $\to$ integrable $\to$ graph has J.C.0.

($\cup$ two sets has J.C.0 has J.C.0).

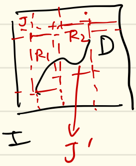
- General Th. Integrable

  Let f be bdd. $-M \le f \le M$

  if $f: I \to \mathbb{R}$ is cont except on a set has J.C.0, then it is integrable.

Pf. let $\varepsilon > 0$

$\exists$ fin many Rectangles R_i s.t $D \subset \bigcup_i R_i$ & $\sum \text{Vol}(R_i) < \frac{\varepsilon}{4M}$



Use end points of R_i as partitition points to get a P of I

The subrectangle in R_i we call J', those not in R_i we call J.

Since f is cont on the int(J) & bdd.

$\Rightarrow f$ integrable on J

Last time we showed $\quad Osc(f, P_{R_1}) = \sum_{J' \subset R_1} (M_{J'} - m_{J'}) \, \text{Vol}(J')$

$\le 2M \cdot \sum \text{Vol}(J') < 2M \cdot \frac{\varepsilon}{2M} = \varepsilon$

(Cannot say f integrable on R_1. Need a second ε for R_1 since ε is actually fixed.)

The contribution of all R_i to the $Osc(f, P)$ is

$$\sum_i \sum_{J' \subset R_i} (M_{J'} - m_{J'}) \cdot \text{Vol}(J').$$

$$\le \sum_i 2M \cdot \text{Vol}(R_i) = 2M \cdot \sum_i \text{Vol}(R_i) < 2M \cdot \frac{\varepsilon}{4M} = \frac{\varepsilon}{2}$$

Since f integrable on all other Js, choose P_i of J_i s.t

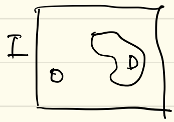
$Osc(f, P_i) < \varepsilon / 2m$. Let $P^* = \bigcup_i^m P_i$.

$$\Rightarrow \ Osc(f, P^*) \le Osc(f, P^*|_{\cup R_i}) + Osc(f, P^*|_{J_i})$$
$$< Osc(f, P|_{\cup R_i}) + \frac{\varepsilon}{2m} \cdot m$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

- Integral on a General Region

Def. Let D be a bounded set in R^n and $f: D \to R$ is bdd. Define zero extension of f.

$\hat{f}: I \to R$ where I is a rectangle $\supset D$



$$\hat{f}(x) = \begin{cases} f(x) & x \in D \\ 0 & x \notin D \end{cases}$$

Def. $\int_D f = \int_I \hat{f}$ if $\int_I \hat{f}$ exists

Ex. Let $D = \mathbb{Q} \cap [0,1]$

Def $f(x) = 1$ for $x \in D$

Then $\hat{f}(x)$ on $[0,1]$ is the Dirichlet fun (not integrable).

- Properties.

- Monotonicity.

Th. If $f \le g$ on D, f, g integrable on D

then $\int_D f \le \int_D g$.

Pf: $f(x) \le g(x)$ on D.

$\hat{f}(x) \le \hat{g}(x)$ on I.

By Monotonicity on I, $\int_I \hat{f} \le \int_I \hat{g}$

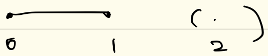
$$\Rightarrow \int_D f \le \int_D g$$

- Jordan Domain (set has small boundary).

$bd(D) = \partial D = \text{boundary of } D \subset \mathbb{R}^n$

def. x is a boundary pt if every neighborhood of x contains a pt in D & a pt not in D.

Ex.

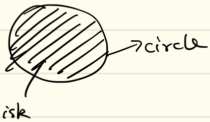


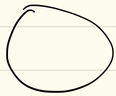
$\Rightarrow bd(\bullet\!\!-\!\!\bullet \cup \{2\}) = \{0, 1, 2\}$

Ex. $\partial(\mathbb{Q} \cap [0,1]) = [0,1]$

- Def. A <u>Jordan Domain</u> in $\mathbb{R}^n$ is a bd set D in $\mathbb{R}^n$ s.t ∂D has J.C.O.

ex. $D = \mathbb{Q} \cap [0,1]$ is not a Jordan Domain since $\partial D = [0,1]$ which does not have J.C.O ($vol(\bigcup R_i) \geq 1$)

$D =$ 

$\partial D =$  has J.C.O

so D is a J. domain

From 135

- $\mathbb{R}^n = \underbrace{\text{int} D \cup \partial D}_{cl D} \cup \text{ext} D$
- 3 sets are disjoint
- $\text{int} D$, $\text{ext} D$ are open, $cl D$ is closed
- $cl D = \text{int} D \cup \partial D = D \cup bd D$
- $cl D$ is the smallest closed set containing D. i.e if F is any set containing D, then $cl D \subseteq F$