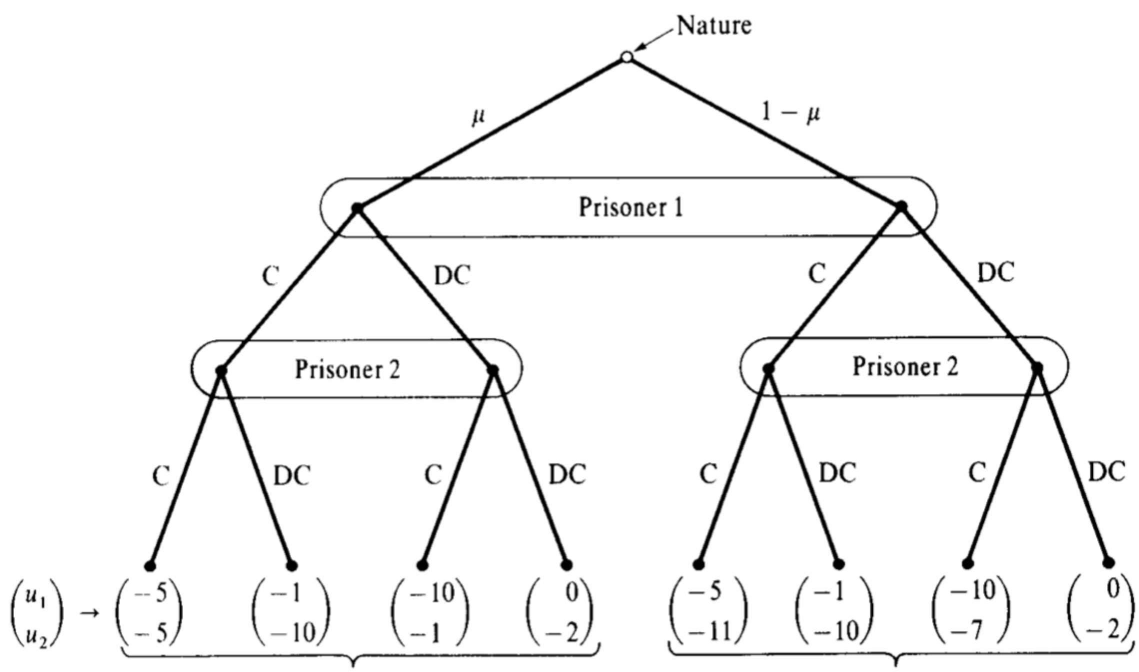
EC 204 Micro II Note Week 3   L2   20/01/29

1. Games of Incomplete Information: Bayesian Nash Equilibrium

- Example: A modification of DA's Brother game in 8.B.3
  - Setting, with probability μ, prisoner 2 has the preference (type I), with 1- μ, prisoner 2 is type II (hates to rat on his accomplice), there is why we have (-10, -1) vs (-10, -7) because there is a psychological penalty. (Notice that the only difference is "-7", player 1's payoffs are the same regardless of prison 2's type.



Prisoner 2 (type I)

Prisoner 1	DC	C
DC	0, −2	−10, −1
C	−1, −10	−5, −5

Prisoner 2 (type II)

Prisoner 1	DC	C
DC	0, −2	−10, −7
C	−1, −10	−5, −11

Note that prisoner 2 has four possible pure strategies:

(confess if type I, confess if type II)

(confess if type I, don't confess if type II)

(don't confess if type I, confess if type II)

(don't confess if type I, don't confess if type II)

Player 1 play DC iff

E(playing DC) >= E(playing C)

- ⇨ 10* mu + 0*(1-mu) >= -5mu-(1-mu). => mu <= 1/6

- Bayesian Game

Formally, in a Bayesian game, each player i has a payoff function $u_i(s_i, s_{-i}, \theta_i)$, where $\theta_i \in \Theta_i$ is a random variable chosen by nature that is observed only by player i. The joint probability distribution of the θ_i's is given by $F(\theta_1, \ldots, \theta_I)$, which is assumed to be common knowledge among the players. Letting $\Theta = \Theta_1 \times \cdots \times \Theta_I$, a Bayesian game is summarized by the data $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$.

- **Bayesian Nash Equilibrium for the Bayesian Game**

Definition 8.E.1: A (pure strategy) *Bayesian Nash equilibrium* for the Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ is a profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ that constitutes a Nash equilibrium of game $\Gamma_N = [I, \{\mathscr{S}_i\}, \{\tilde{u}_i(\cdot)\}]$. That is, for every $i = 1, \ldots, I$,

$$\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot)) \geq \tilde{u}_i(s_i'(\cdot), s_{-i}(\cdot))$$

for all $s_i'(\cdot) \in \mathscr{S}_i$, where $\tilde{u}_i(s_i(\cdot), s_{-i}(\cdot))$ is defined as in (8.E.1).

A very useful point to note is that in a (pure strategy) Bayesian Nash equilibrium each player must be playing a best response to the conditional distribution of his opponents' strategies *for each type that he might end up having*. Proposition 8.E.1 provides a more formal statement of this point.

Proposition 8.E.1: A profile of decision rules $(s_1(\cdot), \ldots, s_I(\cdot))$ is a Bayesian Nash equilibrium in Bayesian game $[I, \{S_i\}, \{u_i(\cdot)\}, \Theta, F(\cdot)]$ if and only if, for all i and all $\bar{\theta}_i \in \Theta_i$ occurring with positive probability[8]

$$E_{\theta_{-i}}[u_i(s_i(\bar{\theta}_i), s_{-i}(\theta_{-i}), \bar{\theta}_i) | \bar{\theta}_i] \geq E_{\theta_{-i}}[u_i(s_i', s_{-i}(\theta_{-i}), \bar{\theta}_i) | \bar{\theta}_i] \tag{8.E.2}$$

for all $s_i' \in S_i$, where the expectation is taken over realizations of the other players' random variables conditional on player i's realization of his signal $\bar{\theta}_i$.

Proof: For necessity, note that if (8.E.2) did not hold for some player i for some $\bar{\theta}_i \in \Theta_i$ that occurs with positive probability, then player i could do better by changing his strategy choice in the event he gets realization $\bar{\theta}_i$, contradicting $(s_1(\cdot), \ldots, s_I(\cdot))$ being a Bayesian Nash equilibrium. In the other direction, if condition (8.E.2) holds for all $\bar{\theta}_i \in \Theta_i$ occurring with positive probability, then player i cannot improve on the payoff he receives by playing strategy $s_i(\cdot)$. ■

Proposition 8.E.1 tells us that, in essence, we can think of each type of player i as being a separate player who maximizes his payoff given his conditional probability distribution over the strategy choices of his rivals.

2. Alphabeta Consortium

- Setting:
  - Cost of new invention $c \in (0,1)$
  - Firm of type *i*, $\theta_i \in [0,1]$
  - Benefit of invention to *i*: θ_i^2
  - $s_i(\theta_i) = 1, \mathit{firm\ i\ develops\ the\ Zigger}, 0\ \mathit{otherwise}$

- Firm i decide to develop the Zigger iff

$$\theta_i \geq \left[\frac{c}{1 - \text{Prob}\,(s_j(\theta_j) = 1)}\right]^{1/2}.$$

(after some algebra)

Suppose then that $\hat{\theta}_1, \hat{\theta}_2 \in (0, 1)$ are the cutoff values for firms 1 and 2 respectively in a Bayesian Nash equilibrium (it can be shown that $0 < \hat{\theta}_i < 1$ for $i = 1, 2$ in any Bayesian Nash equilibrium of this game). If so, then using the fact that $\text{Prob}(s_j(\theta_j) = 1) = 1 - \hat{\theta}_j$, condition (8.E.3) applied first for $i = 1$ and then for $i = 2$ tells us that we must have

$$(\hat{\theta}_1)^2 \hat{\theta}_2 = c$$

and

$$(\hat{\theta}_2)^2 \hat{\theta}_1 = c.$$

Because $(\hat{\theta}_1)^2\hat{\theta}_2 = (\hat{\theta}_2)^2\hat{\theta}_1$ implies that $\hat{\theta}_1 = \hat{\theta}_2$, we see that any Bayesian Nash equilibrium of this game involves an identical cutoff value for the two firms, $\theta^* = (c)^{1/3}$. In this equilibrium, the probability that neither firm develops the Zigger is $(\theta^*)^2$, the probability that exactly one firm develops it is $2\theta^*(1 - \theta^*)$, and the probability that both do is $(1 - \theta^*)^2$. ■

3. Dynamic Games

- Principle of Sequential Rationality: A player's strategy should specify optimal actions at every point in the game tree.

- Backward Induction in Finite Games of Perfect Information

Proposition 9.B.1: (*Zermelo's Theorem*) Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no player has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.

Proof: First, note that in finite games of perfect information, the backward induction procedure is well defined: The player who moves at each decision node has a finite number of possible choices, so optimal actions necessarily exist at each stage of the procedure (if a player is indifferent, we can choose any of her optimal actions). Moreover, the procedure fully specifies all of the players' strategies after a finite number of stages. Second, note that if no player has the same payoffs at any two terminal nodes, then the optimal actions must be *unique* at every stage of the procedure, and so in this case the backward induction procedure identifies a unique strategy profile for the game.

What remains is to show that a strategy profile identified in this way, say $\sigma = (\sigma_1, \ldots, \sigma_I)$, is necessarily a Nash equilibrium of Γ_E. Suppose that it is not. Then there is some player i who has a deviation, say to strategy $\hat{\sigma}_i$, that strictly increases her payoff given that the other players continue to play strategies σ_{-i}. That is, letting $u_i(\sigma_i, \sigma_{-i})$ be player i's payoff function,[3]

$$u_i(\hat{\sigma}_i, \sigma_{-i}) > u_i(\sigma_i, \sigma_{-i}). \tag{9.B.1}$$

We argue that this cannot be. The proof is inductive. We shall say that decision node x has *distance* n if, among the various paths that continue from it to the terminal nodes, the maximal number of decision nodes lying between it and a terminal node is n. We let N denote the maximum distance of any decision node in the game; since Γ_E is a finite game, N is a finite number. Define $\hat{\sigma}_i(n)$ to be the strategy that plays in accordance with strategy σ_i at all nodes with distances $0, \ldots, n$, and plays in accordance with strategy $\hat{\sigma}_i$ at all nodes with distances greater than n.

By the construction of σ through the backward induction procedure, $u_i(\hat{\sigma}_i(0), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. That is, player i can do at least as well as she does with strategy $\hat{\sigma}_i$ by instead playing the moves specified in strategy σ_i at all nodes with distance 0 (i.e., at the final decision nodes in the game) and following strategy $\hat{\sigma}_i$ elsewhere.

We now argue that if $u_i(\hat{\sigma}_i(n-1), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$, then $u_i(\hat{\sigma}_i(n), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. This is straightforward. The only difference between strategy $\hat{\sigma}_i(n)$ and strategy $\hat{\sigma}_i(n-1)$ is in player i's moves at nodes with distance n. In both strategies, player i plays according to σ_i at all decision nodes that follow the distance-n nodes and in accordance with strategy $\hat{\sigma}_i$ before them. But given that all players are playing in accordance with strategy profile σ after the distance-n nodes, the moves derived for the distance-n decision nodes through backward induction, namely those in σ_i, must be optimal choices for player i at these nodes. Hence, $u_i(\hat{\sigma}_i(n), \sigma_{-i}) \geq u_i(\hat{\sigma}_i(n-1), \sigma_{-i})$.

Applying induction, we therefore have $u_i(\hat{\sigma}_i(N), \sigma_{-i}) \geq u_i(\hat{\sigma}_i, \sigma_{-i})$. But $\hat{\sigma}_i(N) = \sigma_i$, and so we have a contradiction to (9.B.1). Strategy profile σ must therefore constitute a Nash equilibrium of Γ_E. ■

4. Subgame

- A subgame: game begins with decision node and contains all successor nodes and none other. If decision node in subgame, then all information set containing that subgame are.

Insert graph from note here.

- SPNE

Definition 9.B.2: A profile of strategies $\sigma = (\sigma_1, \ldots, \sigma_I)$ in an I-player extensive form game Γ_E is a *subgame perfect Nash equilibrium* (SPNE) if it induces a Nash equilibrium in every subgame of Γ_E.

- The centipede game

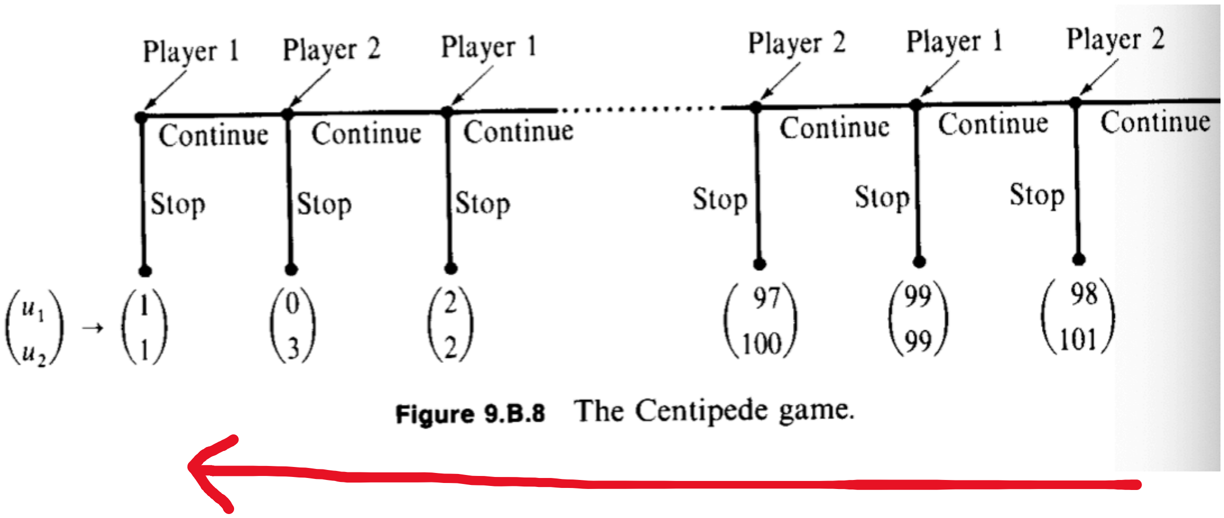


Figure 9.B.8 The Centipede game.