Math 136 Real Analysis II Spring 2020
Prof. L. Tu

References: Jerrold E. Marsden and Michael J. Hoffman, *Elementary Classical Analysis*, 2nd edition, W. H. Freeman and Company, New York, 1993. Chapters 8 and 10 (Fourier Analysis) only.

Readings for the week of April 13, 2020

Marsden and Hoffman

§10.2 Orthogonal families
§10.2 Bessel's inequality, Parseval's theorem
§10.3 Mean completeness and pointwise convergence theorems (omit the proofs), computation of Fourier series (as done in class)
§10.5 Omit this section from the syllabus.

Problem Set 10B (50 points)

(Due at 3:30 p.m., Thursday, April 23, 2020)

In these problems you may use the following integral formulas:

$$\int u\cos nu\,du = \frac{1}{n^2}\cos nu + \frac{u}{n}\sin nu + C,$$
$$\int u\sin nu\,du = \frac{1}{n^2}\sin u - \frac{u}{n}\cos nu + C,$$
$$\int u^2\sin nu\,du = \frac{2u}{n^2}\sin nu + \left(\frac{2}{n^3} - \frac{u^2}{n}\right)\cos nu + C,$$
$$\int u^2\cos nu\,du = \frac{2u}{n^2}\cos nu + \left(\frac{u^2}{n} - \frac{2}{n^3}\right)\sin nu + C.$$

B1. (15 points) (**Measure 0**) Decide whether each of the following sets has measure 0. Give a short justification.

(a) $A = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

(b) the xy-plane in $\mathbb{R}^3$ (*Hint*: One way is to write the xy-plane as a countable union of sets of measure 0 and apply Th. 8.2.4 in Marsden.)

(c) the interval $[0,1]$ in $\mathbb{R}$

(d) the set of irrational numbers in $[0,1]$

(e) the boundary ∂A of a set A of measure 0 in $\mathbb{R}^n$

B2. (5 points) (**Fourier coefficients**) Let $\{\varphi_0, \varphi_1, \dots\}$ be an infinite orthonormal system in the inner product space V. Let $f \in V$. Prove that $\lim_{n\to\infty} \langle f, \varphi_n \rangle = 0$.

B3. (10 points) (**Integral of a periodic function under translation**) Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π periodic function that is continuous. Use a change of variables that you learned in calculus plus the rules of integration that we proved in class to show $\int_0^{2\pi} f = \int_{-\pi}^{\pi} f$. (This results holds for unbounded functions that are integrable using the definitions we've learned, but the proofs require a bit more bookkeeping because of the limits.)

B4. (10 points) (**Parseval's theorem**)
(a) Compute the trigonometric Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$.
(b) Prove that

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}.$$

B5. (10 points) (**Principal value integral**) In this problem, we will investigate the definitions of integrability and what to do when they fail.

Let $f : [-1,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1/x & x \in [-1,0) \cup (0,1] \\ 0 & x = 0 \end{cases}$.

(a) Explain why $f : [-1,1] \to \mathbb{R}$ is not integrable (see the transcript of Lecture 20 for the definition of an improper integral).
However, there are ways to define integrals that allow one to integrate such functions. Here is one. Assume f is a continuous function on $[-1,1] \setminus \{0\}$. Define the principal value integral (at 0)

$$PV \int_{-1}^{1} f(x)dx = \lim_{\varepsilon \to 0^+} \left(\int_{-1}^{-\varepsilon} f(x)dx + \int_{\varepsilon}^{1} f(x)dx \right) \quad \text{if the limit exists and is finite.} \tag{1}$$

(b) Show that f has a principal value integral on $[-1,1]$ and find it.
This very cool principal value integral allows one to "integrate" functions that are not integrable. It is used in complex analysis when calculating residues if a pole is on the curve of integration.

(End of Problem Set 10B)