Math 136 Solutions to Problem Set 1

A1. Let $f = (f_1, \ldots, f_m)$ and $L = (\ell_1, \ldots, \ell_m)$. Prove that

$$\lim_{x \to x_*} f(x) = L \quad \text{iff} \quad \lim_{x \to x_*} f_j(x) = \ell_j \quad \text{for } j = 1, \ldots, m.$$

Solution. ($\Rightarrow$) Suppose $\lim_{x \to x_*} f(x) = L$. Let x_k be a sequence in $A \setminus \{x_*\}$ that converges to x_*. By the definition of the limit of a function,

$$f(x_k) = \begin{bmatrix} f_1(x_k) \\ \vdots \\ f_m(x_k) \end{bmatrix} \longrightarrow L = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_m \end{bmatrix}.$$

By the componentwise convergence criterion (Th. 10.9), $f_j(x_k) \to \ell_j$ for all $j = 1, \ldots, m$. Thus, $\lim_{x \to x_*} f_j(x) = \ell_j$ for all j.

($\Leftarrow$) Suppose $\lim_{x \to x_*} f_j(x) = \ell_j$ for all j. Let x_k be a sequence in $A \setminus \{x_*\}$ that converges to x_*. By the definition of the limit of a function, $f_j(x_k) \to \ell_j$ for all j. By the componentwise convergence criterion,

$$f(x_k) = \begin{bmatrix} f_1(x_k) \\ \vdots \\ f_m(x_k) \end{bmatrix} \longrightarrow \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_m \end{bmatrix} = L.$$

Thus, $\lim_{x \to x_*} f(x) = L$. $\square$

A2. Use the definition of derivative to compute the derivative of f(x) = 1/(1 + x^3) for any real number x .

Solution. Let a be a real number. By the definition of derivative.

$$f'(a) = \lim_{x \to a} \frac{\frac{1}{1+x^3} - \frac{1}{1+a^3}}{x-a}$$

$$= \lim_{x \to a} \frac{a^3 - x^3}{(1+x^3)(1+a^3)(x-a)}$$

$$= \lim_{x \to a} \frac{(a-x)(a^2+ax+x^2)}{(x-a)(1+x^3)(1+a^3)}$$

$$= \lim_{x \to a} -\frac{(a^2+ax+x^2)}{(1+x^3)(1+a^3)}$$

$$= \frac{-3a^2}{(1+a^3)^2}.$$

Therefore,

$$f'(x) = -\frac{3x^2}{(1+x^3)^2}. \qquad \square$$

A3. Let $f(x) = \sqrt{x}$, $x \geq 0$. Compute $f'(x)$ for $x > 0$ using the definition of the derivative.

Solution.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{(x+h) - x}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}. \qquad \square$$

A4. Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Assume f has k zeros. Prove that f' has at least $k-1$ zeros.

Proof. By Rolle's theorem, between two consecutive zeros of f, there is at least one zero of f'. If f has k zeros, then there are $k-1$ pairs of consecutive zeros, so f' has at least $k-1$ zeros. $\square$

Math 136 **Real Analysis II** **Spring 2020**
Prof. L. Tu

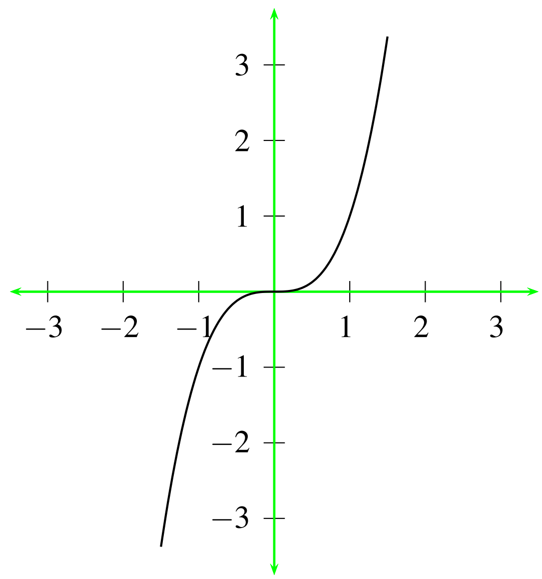
Textbooks: Patrick Fitzpatrick, *Advanced Calculus*, 2nd edition, American Mathematical Society, 2006. (ISBN-10: 0821847910) We will cover Chapters 4, 6, 13–19.

Solutions to Problem Set 1[1]

A5. (10 points) §4.3, p. 108 # 1 acd. For each of the following statements, determine whether it is true or false and justify your answer.

a. If the differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing, then $f'(x) > 0$ for all x.

False, $f(x) = x^3$ is strictly increasing, but $f'(x) = 3x^2 = 0$ at $x = 0$.

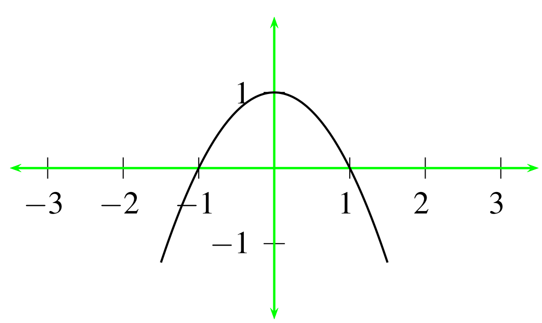


c. If the function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f(x) \le f(0) \quad \text{for all } x \in [-1, 1],$$

then $f'(0) = 0$.

True, because f is differetiable and has a local max at $x = 0$.

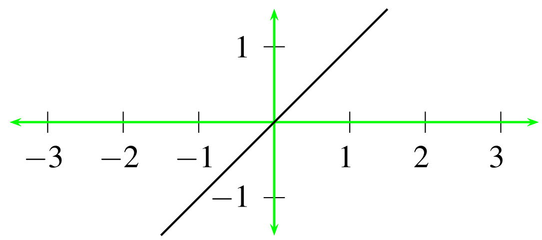


d. If the function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and

$$f(x) \leq f(1) \quad \text{for all } x \in [-1,1],$$

then $f'(1) = 0$.

False, $f(x) = x$ satisfies the hypotheses, but $f'(1) = 1 \neq 0$. The difference from part (c) is that in part (d), the point 1 is a boundary point of the interval $[-1,1]$, not an interior point. Lemma 4.16 is true only for a point in an open interval, i.e., an interior point.



B1. §4.1, p. 95, #10. For real numbers a and b, define

$$g(x) = \begin{cases} 3x^2 & \text{if } x \leq 1, \\ a + bx & \text{if } x > 1. \end{cases}$$

For what values of a and b is the function $g\colon \mathbb{R} \to \mathbb{R}$ differentiable at $x = 1$?

Solution. Since $g(x)$ is differentiable at $x = 1$, it is continuous there. Hence

$$\lim_{x \to 1} g(x) = a + b = g(1) = 3.$$

Moreover, at $x = 1$ the slope of the tangent line from the right must equal to the slope of the tangent line from the left, so

$$g'(1) = 6x|_{x=1} = b \;\Rightarrow\; b = 6.$$

Since $a + b = 3$ and $b = 6$, we must have $a = -3$. □

B2. §4.2, p. 101, #5. Let I be a neighborhood of x_0 and let $f\colon I \to \mathbb{R}$ be continuous, strictly monotone, and differentiable at x_0. Assume that $f'(x_0) = 0$. Use the characteristic property of inverses,

$$f^{-1}\big(f(x)\big) = x \quad \text{for } x \in I,$$

and the chain rule to prove that the inverse function $f^{-1}\colon f(I) \to \mathbb{R}$ is not differentiable at $f(x_0)$. Thus, the assumption in Theorem 4.11 that $f'(x_0) \neq 0$ is necessary.

Solution. Suppose $f^{-1}\colon f(I) \to \mathbb{R}$ is differentiable at $f(x_0)$. Differentiating

$$f^{-1}\big(f(x)\big) = x$$

by the chain rule gives

$$(f^{-1})'\big(f(x_0)\big) f'(x_0) = 1.$$

If $f'(x_0) = 0$, then this gives $0 = 1$, a contradiction. This contradiction proves that if $f'(x_0) = 0$, then f^{-1} is not differentiable at $f(x_0)$. Therefore, in Theorem 4.11, which computes the deriative of f^{-1}, we must assume $f'(x_0) \neq 0$. □

B3. §4.2, p. 101, #9. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *even* if

$$f(-x) = f(x) \qquad \text{for all } x,$$

and $f\colon \mathbb{R} \to \mathbb{R}$ is called *odd* if

$$f(-x) = -f(x) \qquad \text{for all } x.$$

Prove that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and odd, then $f'\colon \mathbb{R} \to \mathbb{R}$ is even.

Solution. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable and odd. By definition,

$$f(-x) = -f(x).$$

Now differentiate both sides with respect to x:

$$\begin{aligned} f'(-x)\frac{d}{dx}(-x) &= -f'(x), \\ -f'(-x) &= -f'(x), \\ \text{or} \quad f'(-x) &= f'(x). \end{aligned}$$

This proves that f' is even. $\square$

B4. §4.3, p. 108, #6. Prove that the following equation has exactly two solutions:

$$x^4 + 2x^2 - 6x + 2 = 0, \quad x \in \mathbb{R}.$$

Solution. Let $f(x) = x^4 + 2x^2 - 6x + 2$. Then

$$f'(x) = 4x^3 + 4x - 6,$$

and

$$f"(x) = 12x^2 + 4.$$

Since $f"$ is always positive, f' is strictly increasing. Since $f'(x)$ is negative for $x << 0$ and $f'(x)$ is positive for $x >> 0$, by the intermediate value theorem, f' has exactly one zero. By Rolle's theorem, f can have at most two zeros.

Next we will show that f has at least two zeros. Because

$$f(0) = 2, \quad f(1) = -1, \quad f(2) = 14,$$

by the intermediate value theorem, f has a zero between 0 and 1 and a zero between 1 and 2. Therefore, f has at least two zeros.

These two paragraphs prove that f has exactly two zeros. $\square$

B5. Let D be the set of nonzero real numbers. Suppose that the functions $g\colon D \to \mathbb{R}$ and $h\colon D \to \mathbb{R}$ are differentiable and that

$$g'(x) = h'(x) \qquad \text{for all } x \in D.$$

Do the functions $g\colon D \to \mathbb{R}$ amd $h\colon D \to \mathbb{R}$ differ by a constant? (*Hint*: Is D and interval?)

Solution. If D were an open interval, then g and h would differ by a constant by the identity criterion. However, $D = (-\infty, 0) \cup (0, \infty)$ is the disjoint union of two open intervals. On each open interval, g and h differ by a constant, but the two constants over the two open intervals don't have to be the same. So g and h do not have to differ by a constant; they could differ by two constants, one over each open interval. For example, h could be the identically zero function, while $g = 1$ on $(-\infty, 0)$ and 2 on $(0, \infty)$. $\square$