Math 136 | Real Analysis II | Spring 2020
Prof. L. Tu | Final Exam Information

Solutions to Review Problems for Final Exam will be posted on Canvas Sunday, April 26. The Monday, April 27, class will be a problem session for the final.

$\boxed{\text{Final Exam: Friday, May 1, 3:30 p.m. to 5:30 p.m.}}$

The final exam will cover the following sections in Patrick Fitzpatrick's *Advanced Calculus* and Marsden and Hoffman's *Elementary Classical Analysis*, as well as the note on term-by-term integration and differentiation (posted on Canvas):

Patrick Fitzpatrick

§4.1, 4.2 (pp. 99–100 only), 4.3 to p. 106. Differentiation of functions of one variable.

§6.1–6.6. Integration of functions of one variable.

§7.3. The convergence of Darboux and Riemann sums.

§9.4. The uniform limit of functions (Th. 9.32, 9.33, 9.34)

§13.1–13.3. Limits, partial derivatives, mean value theorem, directional derivatives.

§14.1. First-order approximation, tangent planes, and affine functions.

§15.2. The derivative matrix.

§15.3. The chain rule.

§16.1. The inverse function theorem (without proof).

§17.1. Dini's theorem.

§17.2. The implicit function theorem.

§18.1–18.3. Integration of functions of several variables.

§19.1. Fubini's theorem.

§19.2. The change of variables theorem (without proof).

Marsden and Hoffman:

§8.2–8.3. Sets of measure zero, Lebesgue's theorem (without proof).

§10.1–10.2. Inner product spaces, orthogonal families, but omit the best mean approximation theorem 10.2.5.

§10.3. Fourier series. Theorem 10.3.1 (the mean (or L^2) completeness theorem), without proof. Theorem 10.3.2 (pointwise convergence theorem), without proof.

Canvas Notes: Term-by-term integration and differentiation

Definitions:

Review the definitions for Exams 1 and 2, and learn the following new definitions.

Marsden and Hoffman:

characteristic function (p. 451), measure zero (p. 452), complex inner product (p. 546), norm, distance, orthogonal, $L^2 := L^2([a,b],\mathbb{C})$ (the definition in Marsden's Th. 10.1.6, p. 549, is wrong; see Lecture 23 transcript), convergence in mean (i.e., convergence in L^2) (p. 548), Hilbert space (p. 549), ℓ^2, orthogonal family (p. 552), orthonormal family, complete family, Fourier series, Fourier coefficients (p. 553)

Relations among three types of convergence: pointwise, uniform, in mean, examples and counterexamples.

Proofs:

Review the proofs for Exams 1 and 2, and learn the following proofs.

Fitzpatrick:
Th. 9.32, convergence of integrals (statement only)
Th. 9.33, convergence of derivatives.

Marsden and Hoffman:

Th. 8.2.4, A countable union of sets of measure zero has measure zero.
Th. 8.3.1, Lebesgue's theorem (statement only)
Th. 8.3.4(ii), If a nonnegative integrable function has integral 0, then $f = 0$ almost everywhere.
Th. 10.1.2, $C([a,b],\mathbb{C})$, $L^2([a,b],\mathbb{C})$ are complex inner product spaces.
Th. 10.1.3, Cauchy–Schwarz inequality for a complex inner product (statement only).
Th. 10.2.1, Fourier coefficients with respect to an **orthonormal** or an **orthogonal** set.
Th. 10.2.3, Bessel's inequality.
Th. 10.2.4, Parseval's theorem.

Learn the statement of Th. 10.3.1, the mean completeness theorem and the statement of Th. 10.3.2, the pointwise convergence theorem and how to apply it.

Know the exponential orthonormal Fourier system, $\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \mid n \in \mathbb{Z} \right\}$ and the trigonometric orthonormal Fourier system,

$$\left\{ \frac{1}{\sqrt{2\pi}} \right\} \cup \left\{ \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}} \mid n \in \mathbb{N} \right\}.$$

Let $f \in L^2([-\pi, \pi], \mathbb{C})$. Know the formulas for the coefficients a_n and b_n for the *standard Fourier series*

$$f(x) = \frac{c_0}{+} \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right),$$

namely,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx,$$
$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx \text{ for } n \in \mathbb{N}.$$

Problems:

Review all homework problems, Exams 1 and 2, and review problems for final exam.

Review Problems for Final Exam

Integration True-False (adapted from *Marsden*, Chapter 9 Exercises, Problem 24.)

(a) An integrable function on $[0,1]$ must be continuous on $[0,1]$.

(b) If f and g are integrable functions on $[a,b]$, then $f-2g$ is integrable on $[a,b]$.

(c) If f is a continuous real-valued function on $[0,1]$ such that $f(x) \geq 0$ for all $x \in [0,1]$ and $\int_0^1 f(x)\,dx = 0$, then $f(x) = 0$ for all $x \in [0,1]$.

(d) If the sequence of functions $\{f_k\}$ converges uniformly to f on $[a,b]$ and if f_k is integrable on $[a,b]$ for each $k = 1,2,\ldots$, then f is integrable on $[a,b]$.

(e) If f is a continuous real-valued function on (a,b), then there exists a differentiable function F on (a,b) such that $f = F'$ on (a,b).

Additional Review Problems:

1. Let $f\colon [-\pi,\pi] \to \mathbb{C}$ be the function defined by

$$f(x) = \begin{cases} 1, & -\pi/2 \leq x \leq \pi/2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Compute the trigonometric Fourier series of f.

   (b) Use part (a) to show that
$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

2. (a) Give the definition of an antiderivative of a function $f\colon \mathbb{R} \to \mathbb{R}$.

   (b) State the First Fundamental Theorem of Calculus (the one about integrating derivatives, Theorem 6.22 on p. 161), including all of its assumptions.

   (c) Prove the formula for integration by substitution: if $g\colon [a,b] \to [c,d]$ is a continuous function with bounded continuous derivative $g'(x)$ on $[a,b]$ and $f\colon [c,d] \to \mathbb{R}$ is continuous, then
$$\int_a^b f(g(t))g'(t)dt = \int_{g(a)}^{g(b)} f(x)dx$$

3. Define $f\colon [0,1]^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x = \frac{1}{2} \text{ and } y \in \mathbb{Q} \cap [0,1], \\ 0 & \text{otherwise.} \end{cases}$$

This problem continues on the other side of this page.

   (a) Prove that f is integrable on $[0,1]^2$.

(b) Compute $\int_{[0,1]^2} f$ using the definition of the integral.

(c) Can we compute $\int_{[0,1]^2} f$ using Fubini's Theorem?

4. Let $f(x) = \sum_{n=0}^{\infty} \frac{e^{-nx^2}}{n!}$.

   (a) Prove that the series for f converges uniformly to f on $(-1,1)$.

   (b) Prove that f' exists for all $x \in (-1,1)$ and find a series expansion for f'.

5. Prove both of the following results

   (a) The volume of a bounded set $A \subset \mathbb{R}^n$ is defined if and only if the boundary of A has measure zero.

   (b) Let $\Phi = \{\varphi_k \mid k = 0,1,2,\dots\}$ be an orthonormal system in an inner-product space, V. Prove Bessel's Inequality. That is, prove that for every $f \in V$, the series $\sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2$ converges and $\sum_{k=0}^{\infty} |\langle f, \varphi_k \rangle|^2 \leq \|f\|^2$.

6. (a) Let $A \subset \mathbb{R}^n$ be a bounded set. State the definition that A has ***Jordan content zero***.

   (b) Let$A \subset \mathbb{R}^n$ be a bounded set. State the definition that A has ***volume zero.***

   (c) Are the concepts *volume zero* and *Jordan content zero* equivalent? If so, explain why and if not, provide a counterexample.

   (d) Let $A \subset \mathbb{R}^n$ be bounded or unbounded. State the definition that A has ***measure zero*** and list one fact that can be true about sets of measure zero but not about sets of volume zero.

7. Prove that the x-axis, $A = \{(x,0) \mid x \in \mathbb{R}\}$, is a set of measure zero in $\mathbb{R}^2$.

8. Let $(V, \langle\ ,\ \rangle)$ be an inner product space and let $\Phi = \{\varphi_k \mid k = 0,1,2,\dots\}$ be a ***complete*** orthonormal system.

   (a) Define what it means for Φ to be complete.

   (b) Let $f \in V$ and assume f is orthogonal to all φ_k in Φ. Prove $f = 0$.

   (c) In general, if V has a complete orthonormal system, is V a complete inner product space (i.e., is V a Hilbert space)?

9. Is it possible to solve

$$\begin{aligned} xy^2 + xzu + yv^2 &= 3, \\ u^3yz + 2xv - u^2v^2 &= 2 \end{aligned}$$

   for $u(x,y,z)$, $v(x,y,z)$ near $(x,y,z) = (1,1,1)$, $(u,v) = (1,1)$? Compute $\partial v / \partial y$ at $(1,1,1)$.

10. (a) Suppose that $\{f_k\}$ is a sequence of integrable functions on the interval $[a,b]$ which converges uniformly to a function f on $[a,b]$. You do not need to prove f is integrable on $[a,b]$. Prove that $f_k \to f$ in the mean (i.e., in L^2).

    (b) Find functions f_k and f in $L^2([0,1],\mathbb{R})$ such that $f_k \to f$ pointwise but not in the mean.

11. Let $\mathbb{I}$ be a generalized rectangle in $\mathbb{R}^n$ and let f and g be bounded integrable functions from $\mathbb{I}$ to $\mathbb{R}$. Use the following fact: for any partition $\boldsymbol{P}$ of $\mathbb{I}$

$$L(f,\boldsymbol{P}) + L(g,\boldsymbol{P}) \leq L(f+g,\boldsymbol{P}) \leq U(f+g,\boldsymbol{P}) \leq U(f,\boldsymbol{P}) + U(g,\boldsymbol{P})$$

to prove $f+g$ is integrable and $\int_{\mathbb{I}} f + \int_{\mathbb{I}} g = \int_{\mathbb{I}}(f+g)$.

12. Let V be an inner product space and assume $\Phi = \{\varphi_n \mid n \in \mathbb{N}\}$ is an infinite dimensional orthonormal system. Define the *Fourier map* $F : V \to \ell^2$ by

$$F(v) = (\langle v, \varphi_1 \rangle, \langle v, \varphi_2 \rangle, \langle v, \varphi_3 \rangle, \ldots, \langle v, \varphi_1 \rangle, \ldots) \quad \text{for } v \in V.$$

   (a) Prove that V is an infinite dimensional vector space.
   (b) Prove that F is linear.
   (c) Prove that F is continuous. HINT: use Bessel's inequality.
   (d) Now assume that Φ is a *complete orthonormal system*. Prove for each $f \in V$ that $\|f\|_V = \|F(f)\|_{\ell^2}$ where $\|\ \|_V$ is the norm in V and $\|\ \|_{\ell^2}$ is the norm in ℓ^2.

13. Let $f : [a,b] \to \mathbb{R}$ be continuous. Prove the Second Fundamental Theorem of Calculus. That is, let $f : [a,b] \to \mathbb{R}$ be continuous and define $F(x) = \int_a^x f(t)\,dt$.

   (a) Explain why F is defined for $x \in [a,b]$.
   Let $x \in (a,b)$.
   (b) Let $h > 0$ such that $x + h \leq b$. Prove that there is a $c \in (x, x+h)$ such that

$$f(c) = \frac{F(x+h) - F(x)}{h} \tag{1}$$

   (c) Now, assume $h < 0$ and $a \leq x + h$. Prove that there is a $c \in (x+h, x)$ such that (1) holds.
   (d) Now prove that F is differentiable at x and $F'(x) = f(x)$.

14. Give an example of a function $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 |f|^2 < \infty$ but f is not in $\mathcal{L}^2([0,1],\mathbb{R})$. Explain why f is not in $\mathcal{L}^2([0,1],\mathbb{C})$.

15. Let $f \colon \mathcal{O} \to \mathbb{R}$ be continously differentiable on an open subset $\mathcal{O}$ of $\mathbb{R}^n$. If $\partial f / \partial x^i = 0$ for all $i = 1, \ldots, n$ and all $x \in \mathcal{O}$, prove that f is constant on $\mathcal{O}$.

16. (**The volume of a parallelopiped**) The ***volume*** of a bounded set $A \subset \mathbb{R}^n$ is defined to be

$$\operatorname{vol}(A) = \int_A dx^1 \cdots dx^n,$$

if the integral exists. Let A be the parallelopiped spanned by three linearly independent vectors v_1, v_2, v_3 in $\mathbb{R}^3$, i.e.,

$$A = \left\{ \sum_{i=1}^{3} t_i v_i \mid 0 \leq t_i \leq 1 \text{ for } i = 1,2,3 \right\}.$$

Compute $\operatorname{vol}(A)$. (*Hint*: Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation that takes the standard basis e_1, e_2, e_3 of $\mathbb{R}^3$ to v_1, v_2, v_3 respectively. Note that T is a smooth change of variables.)