Math 136 HW7 Linyi YE

Part B.

B1. $A \subset I$ of J.C.0, Suppose integrable $f: I \to R$ &
#9. $g: I \to R$ are s.t. $f(x) = g(x)$ for x in $I \setminus A$
P497 Show $\int_I f = \int_I g$

Proof: Define $h(x) = f(x) - g(x)$ for x in $I \setminus A$

$\Rightarrow h(x) = 0$ for x in $I \setminus A$.

f, g is integrable $\Rightarrow f, g$ bdd

$\Rightarrow h(x)$ also bdd

By Lemma 18.29,

$h(x)$ a bdd func is 0 except on A of J.C.0

$\Rightarrow h(x)$ integrable, $\int_I h = 0$

$\Rightarrow \int_I h = \int_I f - g = 0 \Rightarrow \int_I f = \int_I g$ $\square$

B2. $f: I \to R$ integrable, $D := \text{int } I$, show restriction $f: D \to R$
#11 is integrable and $\int_I f = \int_D f$

Proof: Let $\hat{f}$ be the 0 extension of f, $\hat{f}: I \to R$

define $h: I \to R$ by $h(x) = \hat{f}(x) - f(x)$

Then by Lemma 18.29, h is integrable on I, $\int_I h = 0$

By Linearity, $h + f$ is integrable

$\Rightarrow \int_I h + \int_I f = \int_I (h+f) = \int_I \hat{f} \Rightarrow \int_I f = \int_I \hat{f}$

By def, $\int_I \hat{f} = \int_D f$

$\Rightarrow \int_I f = \int_D f$. $\square$

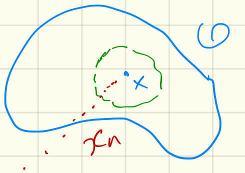
B3. $g: \mathbb{R}^n \to \mathbb{R}$

(a) g is constant on $G \subset \mathbb{R}^n$. Prove $g: \mathbb{R}^n \to \mathbb{R}$ is continuous at $x \in \mathcal{O}$

Proof: $g(x) = c$ for $x \in \mathcal{O} \subset \mathbb{R}^n$, c is constant.

$\{x_n\}$ is a sequence in $\mathbb{R}^n$ that converges to $a \in \mathcal{O}$

Since $\mathcal{O}$ is open, $\{x_n\} \to x \in \mathcal{O}$, eventually $x_n \in \mathcal{O}$



$$\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} g(a) = c.$$

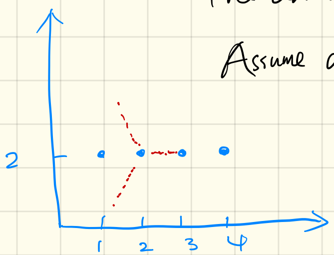
$\Rightarrow g$ is continuous at $x \in \mathcal{O}$

(b) g constant on an arbitrary set $\mathcal{O}$, Is g necessarily continuous for $x \in \mathcal{O}$

No. Consider $g(x) = 2 \quad x \in \mathcal{O}$

The problem becomes if g necessarily continuous

The answer is no.



Assume a seq $\{x_n\}$ in $\mathbb{R} \setminus \mathcal{O}$ cgs to a point in $\mathcal{O}$

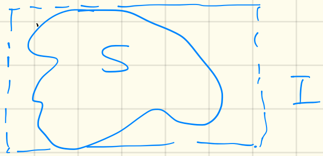
and $\lim_{n\to\infty} g(x_n) = b \neq 2$.

Obviously g is not continuous at $x \in \mathcal{O}$

$\Rightarrow g$ is not necessarily continuous for $x \in \mathcal{O}$

B4. $S \subset \mathbb{R}^n$ has J.c.0

(a) Prove $\mathrm{cl}\, S$ has J.c.0



Proof: $\mathrm{cl} S = S \cup \partial S$.

Let $R_1 \cdots R_n$ be a finite cover of S. with total volume less then ε.

$\partial S \subset \bigcup_i [R_i \cup \partial R_i]$

By definition, R_i is closed so $\partial R_i \subset R_i$

$\Rightarrow \partial S \subset \bigcup_i R_i$

$\Rightarrow \partial S$ has J.c.0.

Thus $\mathrm{cl} S$ has J.c.0.

(b) $\mathrm{int}(S) = \phi$

Proof: If $x \in \mathrm{int}(S)$, then $\exists \delta > 0$ s.t

$$B_\delta(x) \subset S$$

The δ-ball $B_\delta(x)$ has a positive volume V.

Since S has J.c.0

$\forall \varepsilon > 0, \exists R_1 \ldots R_m$ s.t

$$B_\delta(x) \subset S \subset \bigcup_{i=1}^m R_i$$

and

$$V = \mathrm{vol}(B_\delta(x)) \leq \mathrm{Vol}(S) \leq \mathrm{Vol}(\cup R_i) < \varepsilon$$

Choose $\varepsilon = V/2$

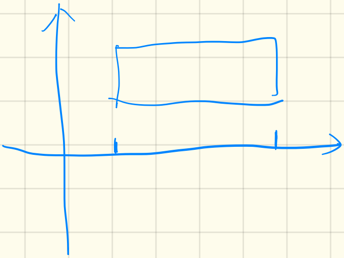
Obviously the above inequality cannot hold.

$\Rightarrow$ No point in $\mathrm{int}(S)$ $\Rightarrow \mathrm{int}(S) = \phi$

B5. A is a bdd subset of $\mathbb{R}^n$

Prove A has J.C. 0 iff A has vol & $\mathrm{vol}(A)=0$

Proof: ① "$\Rightarrow$"



$\mathrm{Vol}(A)=\int_A 1_A$ if $\int_A 1_A$ exist

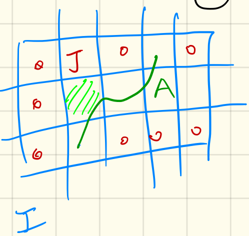
We can define $f: R\to I$ to be the 0 extension of the const function 1 on A.

Since A is bdd $\Rightarrow$ f is bdd

By Th 18.29, f is integrable

and $\int_I f = \int_A^I A = 0$.

② "$\Leftarrow$" if $\mathrm{vol}(A)=0$



by def, $\mathrm{vol}(A)=\int_A 1_A = \int_I \widehat{1_A}$

where $1_A: A\to R$ is a const func with value 1 on A

Let $f=\widehat{1_A}$

By hyp, $\int_I f$ and $\int_I f = 0$

To prove A has J.C.0, let $\varepsilon>0$, since f is integrable on I,

By Riemann's Crit, $\exists$ a part P of I s.t.

$$\mathrm{Osc}(f,P)<\varepsilon$$

$$\Rightarrow \sum_{J \text{ in } P} (\underbrace{M_J(f)-m_J(f)}_{\begin{cases}1-0 & \text{whenever } J \text{ meets } A\\ 0 & \text{whenever } J \text{ does not meet } A\end{cases}})\,\mathrm{Vol}(J)<\varepsilon$$

$$\Rightarrow \sum \mathrm{vol}(J)<\varepsilon$$

$\Rightarrow$ We can cover A with fin many rectangles with total volume $<\varepsilon$ $\Rightarrow$ A has J.C.0