Math 136 | Real Analysis II | Spring 2020
Prof. L. Tu

Exam 2 Review

Exam 2 is scheduled for Wednesday, April 8, 2020, from noon to 1:20 p.m. EDT. After the exam has ended, you have 10 minutes to scan and submit your work. Starting at 1:30 p.m., one point will be deducted for each minute of late submission.

In lieu of the usual problem session, there will be a review session for the exam from 7:30 p.m. to 8:30 p.m. EDT on Monday, April 6.

Solutions to the review problems will be posted by Friday evening, April 2. You should make every effort to do the problems on your own or with friends before consulting the solutions.

The exam will cover Problem Sets 4 through 8A, comprising the following sections from Patrick Fitzpatrick:

§6.1, §6.2, §6.3, §6.4, §6.5, §6.6

§7.3

§18.1, §18.2, §18.3

§19.1

Learn the following definitions:

Over an interval: partition of an interval (p. 136), partition point, partition interval, upper Darboux sum, lower Darboux sum, refinement of a partition, upper integral, lower integral, integrable (p. 142), regular partition, gap of a partition, antiderivative (p. 165), Riemann sum (p. 187),

Over a rectangle: generalized rectangle (p. 471), volume of a rectangle, partition of a rectangle (p. 471), upper Darboux sum, lower Darboux sum, refinement of a partition, upper integral, lower integral, integrable (p. 142), diameter of a bounded subset (p. 480), gap of a partition of a rectangle (p. 481),

Jordan content 0, zero extension (p. 489), Jordan domain (p. 492), volume of a set (p. 495)

Learn the statements of the following theorems:

Theorem 18.14 (The Darboux Sum Convergence Criterion)

Theorem 18.20 (If $f\colon I \to \mathbb{R}$ is bounded and $\operatorname{Discont}(f,I) = \{x \in I \mid f \text{ is not continuous at } x\}$ has Jordan content zero, then f is integrable on I.)

Learn the following proofs:

Lemma 6.2 (Refinement lemma)

Lemma 6.3 (Lower sum always less than or equal to upper sum)

Lemma 6.4 (Lower integral always less than or equal to upper integral)

Th. 6.8 (Archimedes–Riemann theorem)

Example 6.9 (Integrability of monotone functions)

Th. 6.12 and Exercise 6.3, #6 (Additivity): This theorem is an "if and only if".

Th. 6.13 (Monotonicity of the integral)

Th. 6.15 (linearity for addition: $\int_a^b f+g = \int_a^b f + \int_a^b g$)
First prove $m_i(f)+m_i(g) \le m_i(f+g)$.

Cor. 6.16 ("Triangle inequality" for the integral)

Th. 6.18 (Continuity implies integrability)

Th. 6.19 (Integrability of a function that is continuous except at endpoints)

Th. 6.26 (Mean value theorem for integrals)

Th. 6.29 and 6.22 (Fundamental theorem of calculus): The proof given in Canvas notes. It is easier to prove Th. 6.29 (Derivative of an integral) first, and then deduce Th. 6.22 (Integral of a derivative) by the identity criterion.

Th. 7.13 (Riemann sum convergence theorem)

Th. 18.10 (Riemann's integrability criterion: the ε-criterion for integrability)

Th. 18.25 (Graph of an integrable function over a rectangle has Jordan content 0)

Cor. 18.26 (Graph of an integrable function over a Jordan domain has Jordan content 0)

Lemma 18.29 (A bounded function $f\colon I \to \mathbb{R}$ that is zero except on a set of Jordan content 0 is integrable and the integral is 0)

Th. 18.30 (Additivity of integration on bounded sets D_1 and D_2 in $\mathbb{R}^n$)

Th. 19.1 (Fubini's theorem in the plane)

Review all homework problems. Solutions will be posted on the Math 136 website soon after the final due date of each assignment.

Exam 2 Review Problems

1. Prove that in the definition of Jordan content zero (p. 486), one could use open rectangles instead of closed rectangles, i.e. a bounded set S in $\mathbb{R}^n$ has Jordan content 0 if and only if for any $\varepsilon > 0$, there are finitely many open rectangles $U_1, \ldots, U_m$ that cover S with total volume $\sum_{i=1}^m \mathrm{vol}(U_i) < \varepsilon$.

2. Let D_1 and D_2 be Jordan domains in $\mathbb{R}^n$.

   (a) Prove that $\mathrm{bd}(D_1 \cap D_2) \subset \mathrm{bd}(D_1) \cup \mathrm{bd}(D_2)$. Therefore, $D_1 \cap D_2$ is a Jordan domain. (Note: Fitzpatrick sometimes writes bd( ) as $\partial(\ )$.)

   (b) Prove that
   $$\mathrm{vol}(D_1 \cup D_2) = \mathrm{vol}(D_1) + \mathrm{vol}(D_2) - \mathrm{vol}(D_1 \cap D_2).$$

3. Let $f\colon [a,b] \to \mathbb{R}$ be integrable and bounded by $M > 0$; that is, $|f(x)| \le M$ for all $x \in [a,b]$. Define the function $F(x) = \int_a^x f$ for $x \in [a,b]$.

   (a) Prove that F is continuous on $[a,b]$.

   (b) Using properties of the integral, prove that $|F(y) - F(x)| \le M|y-x|$ for all x and y in $[a,b]$. This shows that F is Lipschitz on $[a,b]$.

4. Let $f\colon [0,1] \to \mathbb{R}$ be defined by $f(x) = x^3 + 2$. Find $\int_0^1 f$ using the Archimedes–Riemann theorem. Use the regular partition P_n given by
   $$0 = x_0 < x_1 < \cdots < x_n = 1,$$
   where $x_j = j/n$ for $j = 0, 1, 2, \ldots, n$. You may use the fact that $\sum_{j=1}^n j^3 = (n(n+1)/2)^2$.

5. Let f be continuous on $[a,b]$ and let g be differentiable on (c,d), where $g\big((c,d)\big) \subset [a,b]$. For $x \in [c,d]$, define
   $$h(x) = \int_a^{g(x)} f(t)dt.$$
   prove that h is differentiable and that
   $$h'(x) = \big(f(g(x)\big)g'(x).$$
   (Hint. Set $u = g(x)$ and $F(u) = \int_a^u f$. Apply the chain rule to $h(x) = F(g(x))$.)

6. If
$$h(x) = \int_x^{x^2} \sqrt{1+t^2}\,dt,$$
compute $h'(x)$. (Hint. $\int_x^{x^2} = \int_a^{x^2} - \int_a^x$.)

7. Prove that if f is integrable on $[a,b]$ and $g(x) = f(x)$ for all but finitely many x in $[a,b]$, then g is integrable on $[a,b]$.

8. Compute
$$\lim_{n\to\infty} \frac{\sqrt{2}+\sqrt{4}+\cdots+\sqrt{2n}}{n^{3/2}}.$$

9. For positive numbers a and b, show that the ellipse
$$E = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$$
has Jordan content 0.

10. (a) Does a set S of Jordan content 0 necessarily have volume 0? Prove or disprove.
    (b) Is the open ball $B = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1\}$ a Jordan domain? Prove or disprove.

11. Define $f\colon I = [0,1]\times[0,1] \to \mathbb{R}$ by
$$f(x,y) = \begin{cases} 0 & \text{if } y = x^2, \\ 1 & \text{if } y \neq x^2. \end{cases}$$
Is f integrable? If so, compute the integral $\int_I f$.

12. True–False.
    (a) ______________ A function $f\colon [0,1] \to \mathbb{R}$ that is continuous except at the infinitely many points $1/n$, $n = 1,2,3,\ldots$, is integrable.
    (b) ______________ Suppose that $f\colon [a,b] \to \mathbb{R}$ is integrable. If (P_n) is an Archimedean sequence of partitions for f on $[a,b]$, then $\lim_{n\to\infty} \operatorname{gap} P_n = 0$.

13. Compute the integral
$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy.$$

14. Let $f\colon I = [0,1]\times[0,1] \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} 1 & \text{for } x = 0, \\ 2 & \text{for } x \in (0,1]. \end{cases}$$
Compute the upper and lower sums of f over the regular partition P_n with $\Delta x = \Delta y = 1/n$. Use this information to decide if f is integrable on I and, if so, find $\int_I f$.