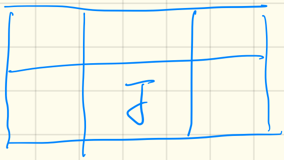
Math 136 HW7 Linyi YE

Part A.

A1. $f \le g \Rightarrow \overline{\int}_A f \le \overline{\int}_A g$.



Proof: Assume f, g are defined on a rectangle I, because if not we can use zero extension of f & g

Let P be a partition of I..

Then $U(f,P) = \sum_{J \text{ in } P} M_J(f) \cdot \mathrm{Vol}(J)$

$U(g,P) = \sum M_J(g)\ \mathrm{Vol}(J)$.

We know $f \le g$,

$\Rightarrow f \le M_J(f) \le g$

Thus $M_J(f)$ is a lower bound for g on J.

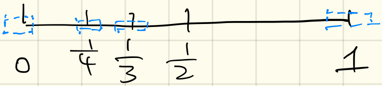
Also, $g \le M_J(g) \Rightarrow M_J(f) \le M_J(g)$.

$$\Rightarrow \inf(U,f,P) = \inf\left(\sum M_J(f) \cdot \mathrm{Vol}(J)\right)$$
$$\le \sum M_J(f) \cdot \mathrm{Vol}(J) \le \inf\left(\sum M_J(g) \cdot \mathrm{Vol}(J)\right)$$
$$\le \sum M_J(g) \cdot \mathrm{Vol}(J)$$

$$\Rightarrow \overline{\int}_I f \le \overline{\int}_I g$$

$$\Rightarrow \overline{\int}_A f \le \overline{\int}_A g$$ (Assume A is I)

A2. show $\{\frac{1}{n} \mid n \in N\}$ has J.C.0



Proof: $\quad A = \{\frac{1}{n} \mid n \in N\}$

In order to show A has J.C.0, we need to cover A with $R_1 \cdots R_n$ s.t $\sum_{i=1}^{n} R_i < \varepsilon$.

Let $\varepsilon > 0$, pick a point a in A, then there is a rectangle $[a - \frac{\varepsilon}{4}, a + \frac{\varepsilon}{4}]$ covering a, and the volume is $\frac{\varepsilon}{2}$.

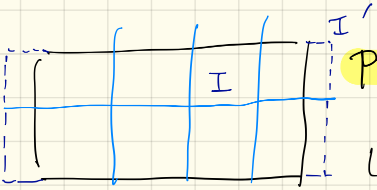
Then we need to cover all other points in A with the sum of length less then $\frac{\varepsilon}{2}$, i.e cover each point (excluding a) in the set with a rectangle of volume $\frac{\varepsilon}{2 \cdot (n-1)}$, then total volume $< \frac{\varepsilon}{2(n-1)} \cdot (n-1)$

$$= \frac{\varepsilon}{2}$$

Here we successfully cover A with rectangles s.t the total volume is less than ε.

$\Rightarrow$ A has J.C.0.

A3. Show bd(I) has J.C.0.



Proof: Let $I = [a_1, b_1] \times \cdots [a_n, b_n]$

Let $I' = [a_1 - \frac{b_1 - a_1}{m}, b_1 + \frac{b_1 - a_1}{m}] \times$

$\cdots \times [a_n - \frac{b_n - a_n}{m}, b_n - \frac{b_n - a_n}{m}]$.

Let P be a regular partition of I' and P divides I' into $m+2$ parts.

This give rises to $(m+2)^n$ rectangles in I'

Notice that we can get the covering set by using $I' - \text{int } I$

Also, P give rises to m^n rectangles in I.

Thus, all we need to do is pick m s.t.

$$[(m+2)^n - m^n] \cdot \text{vol}(I)/m^n < \varepsilon$$

Then $(m+2)^n - m^n$ rectangles cover ∂I.

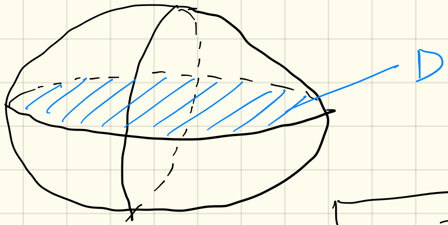
And the total volume is less than ε.

$\Rightarrow \partial I$ has J.C.0

A4. (An ellipsoid)

For positive numbers a, b & c, show that the ellipsoid $\left\{ (x,y,z) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$ has J.C.0.

Proof:



for the upper half, $z = c\sqrt{1 - \frac{y^2}{b^2} - \frac{x^2}{a^2}}$, let $z = f(x,y)$

f is obviously continuous therefore it is integrable on the disk D

Let $\hat{f}$ be the zero extension of z to a rectangle I

By Th in the book, Graph$(\hat{f})$ has J.C.0

$\Rightarrow$ Graph(f) = the upper hemisphere $\subset$ Graph$(\hat{f})$

$\Rightarrow$ the upper hemisphere has J.C.0

Similarly the lower hemisphere has J.C.0

$\Rightarrow$ The ellipsoid has J.C.0 ($\cup$ two sets has J.C.0 has J.C.0)

A5. (a) $\partial(D_1 \cup D_2) \subseteq \partial D_1 \cup \partial D_2$.

(b) $\bigcup_{\text{two Jordan domains}}$ is a Jordan domain.

Proof. (a). Need to prove $x \in \partial(D_1 \cup D_2)$

$\Rightarrow x \in \partial D_1 \cup \partial D_2$. ☆

Prove by contradiction

Suppose $x \notin \partial D_1 \cup \partial D_2$

then $x \notin \partial D_1$ and $x \notin \partial D_2$

Since $x \notin \partial D_1 \Rightarrow x \in \text{int } D_1$ or $x \in \text{ext} D_1$

$x \notin \partial D_2 \Rightarrow x \in \text{int} D_2$ or $x \in \text{int } D_2$

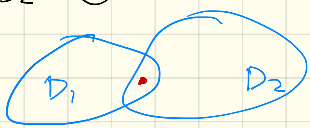
So four cases.

$x \in \text{int } D_1$ — $x \in \text{int} D_2$ ①

$x \in \text{int } D_1$ — $x \in \text{ext} D_2$ ②

$x \in \text{ext} D_1$ — $x \in \text{int} D_2$ ③

$x \in \text{ext} D_1$ — $x \in \text{ext} D_2$ ④

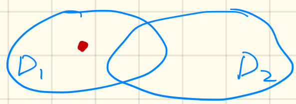


①: By def of $\text{int} D_1$

$\exists \varepsilon_1 > 0$ s.t $B_{\varepsilon_1}(x) \subseteq D_1$

Then $B_{\varepsilon_1}(x) \subseteq D_1 \subseteq D_1 \cup D_2$

so $x \in \text{int}(D_1 \cup D_2) \Rightarrow x \notin \partial(D_1 \cup D_2)$



②: By def of $\text{int} D_1$

$\exists \varepsilon_1 > 0$ s.t $B_{\varepsilon_1}(x) \subseteq D_1$

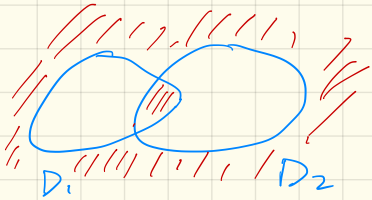
by def of $\text{ext} D_2$,

$B_{\varepsilon_1}(x) \nsubseteq cl(D_2) \Rightarrow B_{\varepsilon_1}(x) \subseteq D_1 \setminus (D_1 \cap D_2)$

$\Rightarrow x \in \text{int}(D_1 \setminus (D_1 \cap D_2)) \Rightarrow x \notin \partial(D_1 \cup D_2)$

③ This is similar to ②.

④



By def of ext D_1

$\Rightarrow \exists \varepsilon_1 > 0$ s.t $B_{\varepsilon_1}(x) \subset D_1^c \setminus \partial D_1$

Similarly

$\exists \varepsilon_2 > 0$ s.t $B_{\varepsilon_2}(x) \subset D_2^c \setminus \partial D_2$.

$\Rightarrow x \in (D_1^c \setminus \partial D_1) \cap (D_2^c \setminus \partial D_2)$

$x \in \text{ext}(D_1 \cup D_2) \cup \text{int}(D_1 \cap D_2)$

$\Rightarrow x \notin \partial(D_1 \cup D_2)$

In conclusion,

$x \notin \partial D_1 \cup \partial D_2 \Rightarrow x \notin \partial(D_1 \cup D_2)$

This is equivalent to $*$.

(b) A, B are two Jordan domains.

By definition, ∂A and ∂B has J.C.O.

Since we prove $\partial(D_1 \cup D_2) \subseteq \partial D_1 \cup \partial D_2$

and $\partial D_1 \cup \partial D_2$ has J.C.O

Then $\partial(D_1 \cup D_2)$ has J.C.O

This is because infinitely many rectangles R_i with $\sum R_i < \varepsilon$ covering $(\partial D_1 \cup \partial D_2)$ must also cover $\partial(D_1 \cup D_2)$