Math 136 HW9 Part A

Linyi YE

A1. $U = \{(x,y) \mid x>0,\ y>0\}$

$\phi : U \to R^2$ by $\phi(x,y) = (x^2-y^2,\ xy)$.

Show that $\phi: U \to R^2$ is a smooth change of variables.

proof: ① obviously $\phi$ is a cont diff mapping. To see this, the derivative matrix is given by.

$$\begin{pmatrix} 2x & y \\ -2y & x \end{pmatrix}$$

They are all continuous functions, So $\phi$ is continuous differentiable.

② $D\phi = 2x^2+2y^2 > 0 \Rightarrow D\phi(x,y)$ is invertible

③ $\phi$ is 1-1:

let $s = x^2-y^2$, $t = xy$.

We need to show that each $(s,t)$ gives a unique $(x,y)$.

This means solving for $x, y$ in terms of $s, t$.

of finding a unique sol.

$$\begin{cases} x^2 - y^2 = s & (1) \\ xy = t & (2) \end{cases}$$

$(1) \Rightarrow \quad y^2 = x^2 - s$

$$y = \sqrt{x^2 - s} \qquad (y > 0)$$

$(2) \Rightarrow \quad xy = x \cdot \sqrt{x^2 - s} = t.$

$$x^4 - x^2 \cdot s - t^2 = 0.$$

$$x^2 = \frac{s \pm \sqrt{s^2 + 4t^2}}{2}$$

obviously $x^2 = \dfrac{s + \sqrt{s^2 + 4t^2}}{2}.$ $(x^2 > 0)$

And $x = \sqrt{\dfrac{s + \sqrt{s^2 + 4t^2}}{2}}$ $(x > 0)$

Thus $x$ is unique.

$y = \frac{t}{x}$ is also unique.

So we get a unique $(x, y)$

This means $\phi: U \to \mathbb{R}^2$ is $1-1$.

By the definition and ① ② ③, we prove that

$\phi: U \to \mathbb{R}^2$ is a smooth change of variables

A2. $D = \{(x,y) \mid x>0,\ y>0,\ 1<x^2-y^2<9,\ 2<xy<4\}$

For a cont func $f: D \to \mathbb{R}$, Use hyperbolic coordinates from A1. to show

$$\int_D |x^2+y^2| \, dx\, dy = 8.$$

Proof: From A1. We know.

$\phi: U \to \mathbb{R}$ is a smooth c-o-v.

$\phi(x,y) = (x^2-y^2,\ xy)$

Make $u = x^2-y^2$, $t = xy$

$x = \frac{t}{y}$. $\Rightarrow u = \left(\frac{t}{y}\right)^2 - y^2 \Rightarrow y^2(u+1) = t^2$. $y = \frac{t}{\sqrt{u+1}}$

$\Rightarrow x = \sqrt{u+1}$ $y = \frac{t}{\sqrt{u+1}}$

By Th. 19.9 C-o-V th and Fubini's theorem

$$\int_{\phi(D)} f(x,y)\, dx dy = \int f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du dv$$

$$x^2+y^2 = \sqrt{(x^2-y^2)^2 + 4(xy)^2}$$
$$= \sqrt{u^2+4t^2}$$

$$\det\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\det\left|\frac{\partial(u,v)}{\partial(x,y)}\right|} = \frac{1}{\begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}}$$

$$= \frac{1}{2x^2+2y^2} = \frac{1}{2\sqrt{u^2+4v^2}}$$

The integral becomes.

$$\int_{\phi(D)} \frac{1}{2}\,du\,dv = \int_1^9 \int_2^4 \frac{1}{2}\,du\,dv = 8$$

A3. $f_n(x)=\sqrt{x^2+\frac{1}{n}}$ , $x\in(-1,1)$.

§9.4 P254

#1 define $f(x)=|x|$

① Prove $\{f_n\} \xrightarrow{\text{uniform}} f$ on $(1,1)$.

② Check if $f_n$ is $C1$, ( $\lim f_n$ not diff at $x=0$ )

③ Does this contradicts Th. 9.33 ?

① uniform convergence.

Obviously $f_n(x) \xrightarrow{\text{pointwise}} f(x)$.

$$\lim_{n\to\infty} f_n(x) = \sqrt{x^2+0} = |x|.$$

$$\lim_{n\to\infty}[f_n(x)-f(x)] = \lim_{n\to\infty}\left(\sqrt{x^2+\frac{1}{n}}-|x|\right)$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n}}{\sqrt{x^2+\frac{1}{n}}+\sqrt{x^2}}$$

$$\leq \lim_{n\to\infty} \frac{1}{n}\Big/\sqrt{\frac{1}{n}}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$$

Thus $\{f_n\} \xrightarrow{\text{uniform}} f$

② $\frac{d f_n(x)}{dx} = \left(\sqrt{x^2+\frac{1}{n}}\right)'$

$$= \frac{1}{2} \cdot \frac{2x}{\sqrt{x^2+\frac{1}{n}}}$$

$$= \frac{x}{\sqrt{x^2+\frac{1}{n}}}$$

Obviously $f_n'(x)$ is a continuous func

$\Longrightarrow$ $f_n(x)$ is continously differentiable

③ No. Since. $f(x) = |x|$ is not

differentiable at $x$.

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} |x| = 0.$$

A4. $f_n(x) = nxe^{-nx^2}$, $x \in [0,1]$

§9.4 P254. #2.

① Prove $\{f_n\} \xrightarrow{\text{Pointwise}} f = 0$.

② But $\{\int_0^1 f_n\} \not\rightarrow 0$

③ Does this contradicts Th. 9.32 ?

Proof: ① when $x = 0$, $f_n(x) = 0$.

when $x \neq 0$. Notice that $e^{nx^2} > \dfrac{n^2x^4}{500}$.

Then $0 < nxe^{-nx^2} < \dfrac{500}{nx^3}$ $\quad \lim\limits_{n\to\infty} \dfrac{500}{nx^3} = 0$.

$\Rightarrow \lim\limits_{n\to\infty} nxe^{-nx^2} = 0$ by sandwich

$\Rightarrow f_n(x) \to 0$ Pointwise.

② For $n \geq 1$, we have

$$\int_0^1 f_n(x) = -\frac{1}{2} e^{-nx^2}\Big|_0^1 = \frac{1}{2}(1 - e^{-n})$$

$$\Rightarrow \lim_{n\to\infty} \int_0^1 f_n(x) = \frac{1}{2}$$

③ $$|f_n(x) - 0| = |nxe^{-nx^2}|$$

$$\overset{x=\frac{1}{\sqrt{n}}}{=} |\sqrt{n} \cdot e^{-1}|$$

$\lim\limits_{n\to\infty} |f_n(x)| = \infty$. $\Rightarrow f_n(x) \overset{\text{uniform}}{\not\Rightarrow} 0$

Hence, this does not contradicts Th. 9.32

A5. Express $\sum_{n=1}^{\infty} x^n / n^2$ as an integral

$$\frac{x^n}{n^2} = \int_0^x \frac{t^{n-1}}{n}\, dt.$$

so
$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \sum_{n=1}^{\infty} \int_0^x \frac{t^{n-1}}{n}\, dt$$

$$= \int_0^x \left( \sum_{n=1}^{\infty} \frac{t^{n-1}}{n} \right) dt.$$

Since $\frac{x^n}{n} = \int_0^x t^{n-1}\, dt$

$$\sum \frac{x^n}{n} = \sum \int_0^x t^{n-1}\, dt$$

$$= \int_0^x \sum t^{n-1}\, dt$$

$$= \int_0^x \frac{1}{1-t}\, dt$$

$$= -\ln(1-t)$$

$$\Rightarrow \sum \frac{t^{n-1}}{n} = -\frac{1}{t} \ln(1-t)$$

$$\Rightarrow \frac{x^n}{n^2} = \int_0^x -\frac{1}{t} \ln(1-t)\, dt$$