50/50 nice!

Math 136 Lingyi YE
HW 6 Part B.

B2. An infinite sum as Riemann Sum

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2}$$

Solution: Let $f(x) = \frac{x}{1+x^2}$, $x \in [0,1]$.

f is continuous $\Rightarrow$ f integrable.

~~Let $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n} = 1\}$ be a part of $[0,1]$~~

~~$C_i = \frac{i}{n} \in [\frac{i-1}{n}, \frac{i}{n}]$ for $1 \le i \le n$~~

Then $R(f, P_n, C_i) = \sum f(C_i) \cdot \frac{1}{n}$

$$= \sum \frac{\frac{i}{n}}{1+(\frac{i}{n})^2} \cdot \frac{1}{n} = \sum \frac{i}{n^2+i^2}.$$

Since $\lim_{n\to\infty} Gap(P_n) = \lim_{n\to\infty} \frac{1}{n} = 0$

By Riemann Sum Convergence Theorem, $\lim_{n\to\infty} R(f, P_n, C_i) = \int_a^b f$

Thus $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+k^2} = \lim_{n\to\infty} R(\cdot)$

$$= \int_0^1 \frac{x}{1+x^2}\,dx$$

$$= \int_0^1 \frac{1}{2} \cdot \frac{1}{1+y}\,dy.$$

$$= \frac{1}{2} \ln(1+y)\Big|_0^1 = \frac{1}{2}\ln 2.$$

So $\lim_{n\to\infty} \sum \frac{k}{n^2+k^2} = \frac{1}{2}\ln 2$.

B3. Suppose $f: [a,b] \to R$ is cont and P is a part of $[a,b]$.
Show that $\exists$ $R(f, P, C) = \int_a^b f$.

Proof: Let $P = \{x_0, x_1 \cdots x_n\}$ be a part of $[a,b]$.
f is continuous, then f is continuous on the subinterval $[x_{i-1}, x_i]$ in P.

By Mean Value Theorem for integrals, for every i,

~~$\exists\, C_i \in [x_{i-1}, x_i]$ s.t $\int_{x_{i-1}}^{x_i} f = (x_i - x_{i-1}) \cdot f(C_i)$~~

By addivity, we have $\int_a^b f = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f = \sum (x_i - x_{i-1}) f(C_i)$

$= R(f, P, C)$

B4. For $I = [0,1] \times [0,1]$, $f: I \to \mathbb{R}$ defined by

$f(x,y) = xy$

Use A-R Th to evaluate $\int_I f$.

Solution: Let P_n be a regular part of $[0,1]$.

$$P_n = \{x_0, x_1, \cdots x_n\}$$

Let $P_n' = P_n \times P_n$ be a regular part of $[0,1] \times [0,1]$

Then
$$L(f, P_n) = \sum_{J \text{ in } P_n} m_J \cdot Vol(J)$$
$$= \frac{1}{n^2} \cdot \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{(i-1)(j-1)}{n^2}$$
$$= \frac{1}{n^4} \left[\frac{n(n-1)}{2}\right]^2 = \frac{1}{4}\left(1 - \frac{1}{n}\right)^2$$
$$= \frac{1}{4} - \frac{1}{2n} + \frac{1}{4n^2}$$

$$U(f, P_n) = \sum_{J \text{ in } P_n} M_J \cdot Vol(J)$$
$$= \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{ij}{n^2} = \frac{1}{n^4}\left[\frac{n(n+1)}{2}\right]^2$$
$$= \frac{1}{4} + \frac{1}{2n} + \frac{1}{4n^2}$$

$$\lim_{n \to \infty} Osc(f, P_n) = \lim_{n \to \infty} \frac{2}{2n} = 0$$

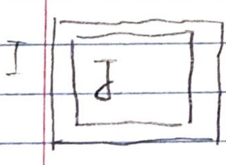
$\Rightarrow$ P_n is Arch., f is integrable.

$$\int_I f = \lim_{n \to \infty} U(f, P_n) = \frac{1}{4}.$$

B5 Let I be a gen rectangle in $\mathbb{R}^n$. $\varepsilon > 0$.

Show that $\exists\, J \subset int(I)$ s.t

$$Vol\, I - Vol\, J < \varepsilon$$



Proof: Let $I = [a_1, b_1] \times \cdots [a_n, b_n]$, $I \subset \mathbb{R}^n$

We want $J \subset I$ s.t $Vol\, I - Vol\, J < \varepsilon$

Let $J = [a_1', b_1'] \times \cdots [a_n', b_n']$

where $[a_i', b_i'] \subset [a_i, b_i]$, $i \in [1, n]$

We want $Vol(I) - Vol(J) = \prod_{i=1}^{n}(b_i - a_i) - \prod_{i=1}^{n}(b_i' - a_i') < \varepsilon$

Let $b_i' - a_i' = m(b_i - a_i)$ s.t $m > \left(1 - \frac{\varepsilon}{Vol(I)}\right)^{\frac{1}{n}}$

Then $Vol(I) - Vol(J) = (1 - m^n) \cdot Vol(I) < \frac{\varepsilon}{Vol(I)} \cdot Vol(I) = \varepsilon$

Thus $\exists\, J \subset Int(I)$ s.t $Vol(I) - Vol(J) < \varepsilon$