Math 136 Lingi YE

HW6 Part A

50

P1

A1. Calculate

a. $\frac{d}{dx}\int_0^x x^2t^2\,dt$

$= 2x\int_0^x t^2\,dt + x^2\cdot x^2$

$= \frac{2}{3}x^4 + x^4 = \frac{5}{3}x^4$

b. $\frac{d}{dx}\int_0^{e^x} \ln t\,dt \quad (*)$

Set $F(x) = \int_1^x \ln t\,dt$, $x \geq 1$

Thus $* = \frac{d}{dx}F(e^x)$

$= F'(e^x)\cdot e^x$

$= \ln e^x \cdot e^x = xe^x$

c. $\frac{d}{dx}\int_{-x}^x e^{t^2}\,dt \quad (*)$

Set $F(x) = \int_0^x e^{t^2}\,dt$

$* = \frac{d}{dx}\left(F(x) - F(-x)\right)$

$= F'(x) - F'(-x)\cdot(-1) = 2e^{x^2}$

d. $\frac{d}{dx}\int_1^x \cos(x+t)\,dt$.

$\cos(x+y) = \cos x\cos y - \sin x \sin y$ ~~cos x cos~~

$\sin(x+y) = \sin\cos + \cos\sin$

$= \frac{d}{dx}\int_1^x(\cos x\cos t - \sin x\sin t)\,dt$

$= -\sin x\int_1^x \cos t\,dt - \cos x\int_1^x \sin t\,dt$
$+ \cos^2 x - \sin^2 x$

$= -\sin x(\sin x - \sin 1) - \cos x(\cos 1 - \cos x) + \cos^2 x - \sin^2 x$

$= 2(\cos^2 x - \sin^2 x) - (\cos x\cos 1 - \sin x\sin 1)$

$= 2\cos 2x - \cos(x+1)$

A2. Suppose $f: \mathbb{R}\to\mathbb{R}$ differentiable.

$H: \mathbb{R}\to\mathbb{R}$ by $H(x) = \int_{-x}^x [f(t) + f(-t)]\,dt$ for x.

Find $H''(x)$

S: $H(x) = 2\int_0^x [f(t)+f(-t)]\,dt = 2\int_0^x f(t)\,dt + 2\int_0^x f(-t)\,dt$

$H'(x) = 2f(x) + 2f(-t)$

Since f is diff, by linearity & additivity, $H'(x)$ is diff $\Rightarrow H''(x) = 2f'(x) - 2f'(-x)$

A3 $f: R \to R$ has continuous second derivative

Prove $f(x) = f(0) + f'(0)x + \int_0^x (x-t) f''(t)\,dt$

Proof Let $h(x) = f(0) + f'(0)x + \int_0^x (x-t) f''(t)\,dt$.

$h(x)$ is differentiable since f is C2.

$$\Rightarrow h'(x) = f'(0) + \frac{d}{dx}\int_0^x (x-t) f''(t)\,dt$$

$$= f'(0) + \frac{d}{dx}\left(\int_0^x x \cdot f''(t)\,dt - \int_0^x t\, f''(t)\,dt\right)$$

$$= f'(0) + \int_0^x f''(t)\,dt + x \cdot f''(x) - x f''(x)$$

$$= f'(0) + f'(x) - f'(0) + 0$$

$$= f'(x)$$

Also, notice $h(0) = f(0) + f'(0) \cdot 0 + \int_0^0 t\, f''(t)\,dt$

$$= f(0)$$

Thus, by Identity Criterion, $h(x) = f(x)$

$$\Rightarrow f(x) = f(0) + f'(0)x + \int_0^x (x-t) f''(t)\,dt$$

A4. Suppose $f: R \to R$ continuous.

$$G(x) = \int_0^x (x-t) f(t)\,dt$$

prove $G''(x) = f(x)$

Proof: since f continuous $\Rightarrow$ f differentiable $\Rightarrow$ G is differentiable.

$$G(x) = x \cdot \int_0^x f(t)\,dt - \int_0^x t\, f(t)\,dt$$

$$\Rightarrow G'(x) = \int_0^x f(t)\,dt + x \cdot f(x) - x f(x)$$

$$= \int_0^x f(t)\,dt$$

$$\Rightarrow G''(x) = f'(x)$$

Math 136 Liuyi YE
HW 6 Part A

A5. Suppose $f, g: [a,b] \to R$ are two bdd functions and P is a part of $[a,b]$.

(a) Prove
$$Osc(fg, P) \le Osc(g,P) \cdot M(|f|) + Osc(f,P) \cdot M(|g|).$$

Proof: Let $[x_{i-1}, x_i]$ be ith subinterval of P.
For $x, y \in [x_{i-1}, x_i]$ we have,

$$|f(x)g(x) - f(y)g(y)|$$
$$= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(x)|$$
$$= |f(x)(g(x)-g(y)) + g(y)\cdot(f(x)-f(y))|$$
$$\le |f(x)||g(x)-g(y)| + |g(y)| \cdot |f(x)-f(y)|$$
$$\le |f(x)| \cdot (M_i(g) - m_i(g)) + |g(y)| \cdot (M_i(f) - m_i(f))$$
$$\le |f(x)| \cdot Osc(g,P) + |g(y)| \cdot Osc(f,P).$$

Since f & g is bdd, then $|f| \le M(|f|)$, $|g| \le M(|g|)$.

$$\Rightarrow Max|f(x)g(x) - f(y)g(y)| \le M(|f|) \cdot Osc(g,P) + M(|g|) \cdot Osc(f,P)$$

Since $Osc(fg) = U(fg, P) - L(fg,P)$
$$= |U(fg,P) - L(fg,P)|$$
$$\le Max|f(x)g(x) - f(y)g(y)|$$
$$= M(|f|) \cdot Osc(g,P) + M(|g|) \cdot Osc(f,P)$$

(b) f, g integrable $\Rightarrow$ f, g integrable.

Proof: if f integrable, then we have

$$Osc(f^2, P) = \Sigma(M_i^2 - m_i^2) \cdot \Delta x_i$$
$$= \Sigma(M_i + m_i)(M_i - m_i) \cdot \Delta x_i$$
$$< 2M \cdot (\Sigma M_i - \cancel{M_i}\, m_i) \cdot \cancel{\Delta x_i} \cdot \Delta x_i$$
$$= 2M \cdot Osc(f,P)$$

Thus, since f integrable, $Osc(f,P) < \frac{\varepsilon}{2M}$, $M = \sup f$.
Then $Osc(f^2, P) = 2M \cdot Osc(f,P) < 2M \cdot \varepsilon/2M = \varepsilon$.
$\Rightarrow f^2$ is integrable.
Then by additivity $\frac{1}{2}[(f+g)^2 - f^2 - g^2] = fg$
$\Rightarrow fg$ is integrable