

Math 136  
Prof. L. Tu

Real Analysis II

Spring 2020

**Textbook:** Patrick Fitzpatrick, *Advanced Calculus*, 2nd edition, American Mathematical Society, 2006. (ISBN-10: 0821847910) We will cover Chapters 4, 6, 13–19.

**Prof. Tu's office hours:** Mon 4:20–5:20 p.m., Wed 11–11:50 a.m. in BP-206

**Prof. Tu's problem session:** Mon 7:30–8:30 p.m. in BP-3

**TA's office hour:** Wed 4:20–5:20 p.m. in BP-202

**ARC Math 136 tutors office hours:** Make appointments through Tutor Finder ([go.tufts.edu/tutorfinder](http://go.tufts.edu/tutorfinder)).

### Readings for the weeks of February 3, 2020

§14.1 First-order approximation, tangent planes, and affine functions

§15.2 The derivative matrix and the differential

§15.3 The chain rule in several variables

§15.1 is a review of linear algebra. We will assume you know the material in this section and not discuss it. If you are rusty in linear algebra, you should read through it.

### Readings for the weeks of February 10, 2020

§16.1 The inverse function theorem

§17.1 Dini's theorem

§17.2 The implicit function theorem

### Problem Set 3 (100 points)

(Due in class Wednesday, February 12, 2020)

**Part A: Please put the following problems in Homework Folder A.**

A1. (10 points) §14.1, p. 378: # 6 (**Affine first-order approximation**). Define

$$f(x, y, z) = x^2 + y^2 + z \quad \text{for } (x, y, z) \in \mathbb{R}^3.$$

Find the affine function that is a first-order approximation to the function  $f: \mathbb{R}^3 \rightarrow \mathbb{R}$  at the point  $(0, 0, 0)$ .

A2. (10 points) (**First-order approximations**)

Suppose the functions  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $g: \mathbb{R}^2 \rightarrow \mathbb{R}$  are continuously differentiable. Find necessary and sufficient conditions for these functions to be first-order approximations of each other at the point  $(0, 0)$ . Your conditions should be in terms of the values of  $f$ ,  $g$ ,  $f_x$ ,  $f_y$ ,  $g_x$ , and  $g_y$  at  $(0, 0)$ . (*Hint:* Use the first-order approximation theorem and §14.1, # 14.)

A3. (10 points) §14.1, p. 379: # 17

(Turn over for more problems.)

**A4. (10 points) (Tangent plane and directional derivatives)**

Suppose that the continuous function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  has a tangent plane at the point  $(0,0, f(0,0))$ . Prove that the function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  has directional derivatives in all directions at the point  $(0,0)$ . (*Hint:* The tangent plane at  $(0,0, f(0,0))$  has the form  $z = ax + by + f(0,0)$ . Let  $\mathbf{p}$  be a unit vector. The directional derivative of  $f$  in the direction  $\mathbf{p} = \langle c, d \rangle$  at  $(0,0)$  is

$$\frac{\partial f}{\partial \mathbf{p}} = \lim_{t \rightarrow 0} \frac{f(tc, td) - f(0,0)}{t}.$$

Write down the definition of the tangent plane (14.7) and compare.)

**A5. (10 points) §13.3, p. 371, # 10 (There are six implications.)**

**Part B. Please put the following problems in your Homework Folder B.**

**B1. (10 points) §15.2 p. 412 # 3**

**B2. (10 points) §15.2 p. 413 # 5**

**B3. (10 points) §15.3 p. 419 # 2 (calculate only  $D_2\eta = \frac{\partial \eta}{\partial v}$ ).**

**B4. (10 points) (Differentiable maps)** In this problem, we introduce an important definition.

**Definition 1.** Let  $\mathcal{O} \subset \mathbb{R}^n$  be open and let  $F: \mathcal{O} \rightarrow \mathbb{R}^m$ . Let  $\mathbf{x} \in \mathcal{O}$ . Then,  $F$  is **differentiable** at  $\mathbf{x}$  if there is an  $m \times n$  matrix  $A$  such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|F(\mathbf{x} + \mathbf{h}) - [F(\mathbf{x}) + A\mathbf{h}]\|}{\|\mathbf{h}\|} = \lim_{\mathbf{u} \rightarrow \mathbf{x}} \frac{\|F(\mathbf{u}) - [F(\mathbf{x}) + A(\mathbf{u} - \mathbf{x})]\|}{\|\mathbf{u} - \mathbf{x}\|} = 0.$$

Use this definition to show that if  $F$  is differentiable at  $\mathbf{x}$  then  $F$  is continuous at  $\mathbf{x}$ .

**Cultural notes:** Theorem 15.32 shows that, if  $F$  is differentiable at  $\mathbf{x}$ , then  $F$  has all first order partial derivatives at  $\mathbf{x}$  and  $A$  is the derivative matrix  $\mathbb{D}F(\mathbf{x})$ . Furthermore, the function  $G(\mathbf{u}) = F(\mathbf{x}) + A(\mathbf{u} - \mathbf{x})$  is an affine first-order approximation to  $F$  at  $\mathbf{x}$ . Note that if  $f: \mathcal{O} \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ , then this definition is equivalent to  $f$  having a tangent plane at  $\mathbf{x}$  (and the equation of the tangent plane is  $z = G(u, v)$  where  $G$  is the affine function give above).

**B5. (10 points)** Let  $g$  and  $h$  be continuously differentiable functions and let  $c > 0$ . Define  $u(x, t) = g(x - ct) + h(x + ct)$ . Prove that  $u$  satisfies the wave equation:  $c^2 u_{xx} = u_{tt}$ .

**Cultural note:** If  $s$  is position on a line and  $t$  is time, then a function of the form  $g(s - ct)$  is a travelling wave to the right as  $t$  increases with speed  $c$ , and  $h(s + ct)$  is a traveling wave to the left with speed  $c$ .