ELEC 3810 / BIEN 3310 A

HW 2

Q1: (a) $P[X=j,\ Y\le y]$

$= P[X=j,\ X+N\le y]$

$= P[X=j,\ N\le y-X]$

$= P[X=j]\cdot P[N\le y-X \mid X=j]$

$= \begin{cases} p\cdot\int_{-\infty}^{y-1}\frac{1}{0.4\cdot\sqrt{2\pi}}\cdot e^{-\frac{x^2}{0.32}}dx, & x=+1 \\ (1-p)\cdot\int_{-\infty}^{y+1}\frac{1}{0.4\cdot\sqrt{2\pi}}\cdot e^{-\frac{x^2}{0.32}}dx & x=-1\end{cases}$ or $\begin{cases} p\cdot\phi(\frac{y-1}{0.4}), & x=+1 \\ (1-p)\phi(\frac{y+1}{0.4}), & x=-1\end{cases}$

(b) $P[X=j] = \begin{cases} p, & j=+1 \\ 1-p, & j=-1\end{cases}$

$f_Y(y) = p\cdot\frac{1}{0.4\sqrt{2\pi}}\cdot e^{-\frac{1}{2}(\frac{y-1}{0.4})^2} + (1-p)\frac{1}{0.4\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{y+1}{0.4})^2}$

(c) $\hat{X}_{MAP}(y) = \arg\max_{x} P(X=x \mid Y=y)$

$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{P(X=j)\cdot\frac{1}{0.4\sqrt{2\pi}}\cdot e^{-\frac{1}{2}(\frac{y-j}{0.4})^2}}{p\cdot\frac{1}{0.4\sqrt{2\pi}}\cdot e^{-\frac{1}{2}(\frac{y-1}{0.4})^2}+(1-p)\frac{1}{0.4\sqrt{2\pi}}e^{-\frac{1}{2}(\frac{y+1}{0.4})^2}}$

~~$f_{X=1|Y}(x=1|y)$~~

$\frac{f_{X|Y}(x=1|y)}{f_{X|Y}(x=-1|y)} = \frac{p}{1-p}e^{-\frac{1}{0.32}((y+1)^2-(y-1)^2)} = \frac{p}{1-p}e^{\frac{y}{0.08}}$

When $y \ge 0.08\ln\frac{1-p}{p}$, $MAP = f_{X|Y}(x=1|y)$

When $y \le 0.08\ln\frac{1-p}{p}$, $MAP = f_{X|Y}(x=-1|y)$

Q2: (a) $E[X] = 0$, $Var[X] = \frac{2}{3}$

$E[Y] = 0$, $Var[Y] = \frac{2}{3}$

$E[XY] = 0$, $COV[XY] = 0$

$$\hat{Y} = \frac{Cov[X,Y]}{Var(X)}(X - E(X)) + E(Y) = 0$$

(b)

$$\hat{Y} = E[Y \mid X]$$

$$= \begin{cases} -\frac{1}{2}, & X = -1 \\ 1, & X = 0 \\ -\frac{1}{2}, & X = 1 \end{cases}$$

(c)

$$\hat{Y}_{MAP} = \underset{y}{\operatorname{argmax}}\, P_{Y|X}(y|x)$$

$$= \begin{cases} -1 \text{ or } 0, & x = -1 \\ 1, & x = 0 \\ -1 \text{ or } 0, & x = 1 \end{cases}$$

with equal probability

$$\hat{Y}_{MLE} = \underset{y}{\operatorname{argmax}}\, P_{X|Y}(x|y)$$

$$= \begin{cases} -1 \text{ or } 0, & x = -1 \\ 1, & x = 0 \\ -1 \text{ or } 0, & x = 1 \end{cases}$$

(d)

$$MSE_a = E[(Y - Y_{LMMSE})^2] = \frac{1}{3} \times (-1-0)^2 + \frac{1}{3} \times 0 + \frac{1}{3} \times (1-0)^2$$

$$= \frac{2}{3}$$

$$MSE_b = E[(Y - Y_{MMSE})^2] = \frac{1}{3} \times (-1-(-\frac{1}{2}))^2 + \frac{1}{3} \times (0-(-\frac{1}{2}))^2 + \frac{1}{3} \times (1-1)^2$$

$$= \frac{1}{6}$$

$$MSE_{MAP} = E[(Y - Y_{MAP})^2] = 4 \times \frac{1}{6} \times (\frac{1}{2} \times 1^2 + \frac{1}{2} \times 0) + \frac{1}{3} \times (1-1)^2$$

$$= \frac{1}{3}$$

$$MSE_{MLE} = E[(Y - Y_{MLE})^2] = MSE_{MAP} \text{ (in this case)}$$

$$= \frac{1}{3}$$

Q3: $\vec{Z} = [Z_1 \quad Z_2]$, $\hat{S} = \cancel{[Z_1 \ Z_2}\ \vec{Z}\vec{a}^T$, $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 5 & 1 & 2 \end{bmatrix}$

$$E\left[\vec{Z}^T (S - \hat{S})\right] = 0$$

$$E\left[\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}\left([Z_1 \ Z_2]\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - S\right)\right] = 0$$

$$E\left[\begin{bmatrix} Z_1^2 & Z_1 Z_2 \\ Z_2 Z_1 & Z_2^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} S\right] = 0$$

$$\vec{a} = \frac{E[\vec{Z} S]}{E[\vec{Z}^T \cdot \vec{Z}]}$$

$$E\left[\begin{bmatrix} S \\ Z_1 \\ Z_2 \end{bmatrix}[S \ Z_1 \ Z_2]\right] = E[\vec{u}\vec{u}^T] A A^T$$

$$= \begin{bmatrix} S^2 & SZ_1 & SZ_2 \\ SZ_1 & Z_1^2 & Z_1 Z_2 \\ SZ_2 & Z_1 Z_2 & Z_2^2 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 5 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

$\therefore \quad a_1 = \frac{3}{7}, \quad a_2 = \frac{3}{7}$

$$MSE_{\hat{S}} = E[S^2] + E[\hat{S}^2] - 2E[S \cdot \hat{S}]$$

$E[S] = 0$, $Var[S] = 3$, $E[S \cdot \hat{S}] = \frac{18}{7}$

$E[S^2] = 3$, $E[\hat{S}^2] = E\left[\left(\frac{3}{7}Z_1 + \frac{3}{7}Z_2\right)^2\right] = \frac{18}{7}$

$$MSE_{\hat{S}} = \frac{3}{7}$$

Q4: $E[XY] = \frac{1}{5}$, $E[X] = 0$, $E[Y] = 0$

$VAR[X] = \frac{1}{3}$, $VAR[Y] = \frac{1}{7}$

$\hat{Y}_{LMMSE} = \frac{3}{5}X$, $MSE_{LMMSE} = \frac{4}{175}$

$\hat{Y}_{MMSE} = X^3$, $MSE_{best} = 0$

Q5: (a) $E[Y] = \cancel{E[Y|} \; E\left[E[Y|X]\right] = \frac{3}{2}$

$E[Y^2] = E\left[E[Y^2|X]\right] = \frac{14}{3}$

$Var[Y] = \frac{29}{12}$ $\quad E[XY] = \frac{7}{3}$

$Cov[XY] = \frac{1}{12}$,

$\hat{X} = \frac{1}{29}Y + \frac{42}{29}$

(b) $MSE = (1-\rho^2)\,Var(X) = \frac{7}{87}$

(c) $E\left[(X-\hat{X})Y\right] = E\left[XY - \frac{1}{29}Y^2 - \frac{42}{29}Y\right]$

~~Q6:~~ $= 0$

Q6: $\begin{bmatrix} E[Y_1^2] & E[Y_1Y_2] \\ E[Y_1Y_2] & E[Y_2^2] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} - \begin{bmatrix} E[Y_1X] \\ E[Y_2X] \end{bmatrix} = 0$

$a_1 = \frac{2}{3}$, $a_2 = \frac{1}{6}$

$\therefore \hat{X} = \frac{2}{3}Y_1 + \frac{1}{6}Y_2$

Q7:

(a) $\int_0^1\int_0^1\int_0^1 f(x,y,z)\,dx\,dy\,dz = 1$

$k = \frac{2}{3}$

(b) $f_X(x) = \frac{2}{3}(x+1)$, $f_Y(y) = \frac{2}{3}(y+1)$, $f_Z(z) = \frac{2}{3}(z+1)$

$f(x,y) = \frac{1}{3}(2x+2y+1)$, Similarly $f(x,z)$, $f(y,z)$

$E[X] = E[Y] = E[Z] = \frac{5}{9}$

$Var[X] = Var[Y] = Var[Z] = E[X^2] - E^2[X] = \frac{13}{162}$

$\cancel{E[XY]}$ $Cov[X,Y] = Cov[X,Z] = Cov[Y,Z]$

$= E[XY] - E[X]E[Y] = -\frac{1}{324}$

$\hat{Y} = a^T \begin{bmatrix} X - E[X] \\ Z - E[Z] \end{bmatrix} + E[Y]$

$a = \begin{bmatrix} Var[X] & Cov[X,Z] \\ Cov[X,Z] & Var(Z) \end{bmatrix}^{-1} \begin{bmatrix} Cov[X,Y] \\ Cov[Y,Z] \end{bmatrix}$

$= \begin{bmatrix} -\frac{1}{25} \\ -\frac{1}{25} \end{bmatrix}$

$\therefore \hat{Y} = -\frac{1}{25}(X+Z) + \frac{3}{5}$

(c) $f_Y(y|x,z) = \frac{x+y+z}{x+z+\frac{1}{2}}$, $\hat{Y} = E[Y|X,Z] = \frac{3(x+z)+2}{6(x+z)+3}$

(d) MAP: $\hat{Y}_{MAP} = \arg\max_Y f_Y(y|x,z) = \cancel{\frac{2(x+z)}{6(x+z)+3}}\ \frac{x+y+z}{x+z+\frac{1}{2}}$

$\therefore \hat{Y}_{MAP} = 1$

MLE: $\hat{Y}_{MLE} = \arg\max_Y f_{X,Z}(x,z|y) = \frac{x+y+z}{y+1}$

$\therefore \hat{Y}_{MLE} = \begin{cases} 1, & x+z < 1 \\ 0, & x+z > 1 \end{cases}$