THE HONG KONG UNIVERSITY OF SCIENCE & TECHNOLOGY

ELEC 3830 & BIEN 3310: Data Science for Neural Engineering

Homework 1 solution

Question 1 (15 points)

A random experiment has sample space $S=\{1,2,3,4\}$ with probabilities $p_1=\frac{1}{2}$, $p_2=\frac{1}{4}$, $p_3=\frac{1}{8}$, $p_4=\frac{1}{8}$.

(a) Describe how this random experiment can be simulated using tosses of a fair coin. *(5 points)*

(b) Describe how this random experiment can be simulated using an urn experiment. *(5 points)*

(c) Describe how this experiment can be simulated using a deck of 52 distinct cards. *(5 points)*

Solution:

(a) We can simulate this experiment by assigning different combinations of tossing the coin for three times.

1. We get a head coin for the first time $p_1=1/2$
2. We get head coins for the first two times $p_2=1/4$
3. We get head coins for all three times $p_3=1/8$
4. We get tail coins for all three times $p_3=1/8$

(b) Let's assume the urn has three balls: four yellow balls, two red balls, one black ball and one white ball. We randomly pick a ball from the urn. The experiment event 1, 2, 3, 4 corresponds to the ball is yellow, red, black, and white respectively.

(c) We pick one card and return for three times in total. Outcome = 1if first time is red. 2 if first two times are red. 3 if all three times are red. 4 if all three times are black.

Question 2 (15 points)

Andy and Bob are playing a basketball shooting game. In each round they will each shoot for one ball. Anyone who makes the goal will get one point. After each round, if one of them has more points than the other, the game will end and the person with higher points wins. If they have the same points after one round, they will have a new round until one wins. Suppose the hit rate for Andy is p_1 and for Bob is p_2. What is the probability that Andy will win the game?

Solution:

Suppose the game ends in the nth round. Then the former n-1 rounds are all tied games and are not our interest. Since the game will end in the nth round, one of them must make the goal and the other will fail to do so, so the probability that Andy wins in any round is

$$P(A=1 \wedge B=0)=p_1(1-p_2)$$

And the probability that Bob wins in any round is

$$P(A=0 \wedge B=1)=(1-p_1)p_2$$

These two cases are the only possible outcomes, so after normalizing, we get the probability that Andy wins the game is

$$P(Andy\ wins)=\frac{p_1(1-p_2)}{p_1(1-p_2)+(1-p_1)\,p_2}$$

Question 3 (15 points)

A binary communication system transmits a signal X that is either a voltage signal +2 or a -2 voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal.

(a) Find the sample space. *(5 points)*

(b) Find the set of outcomes corresponding to the event “transmitted signal was definitely +2” *(5 points)*

(c) Describe in words the event corresponding to the outcome $Y=0$. *(5 points)*

Solution:

(a) {-2 -1 0 1 2}

(b) {1 2}

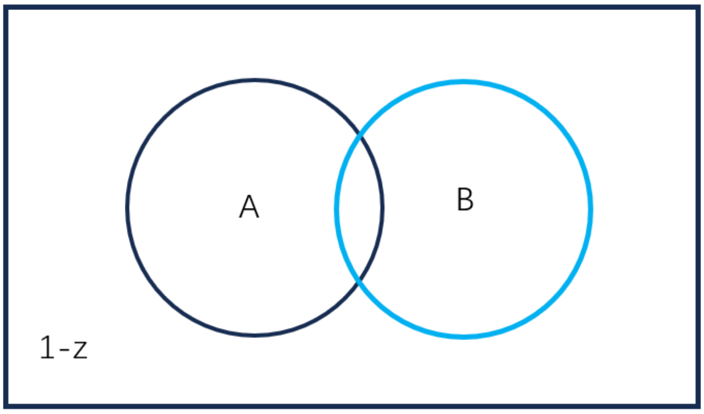
(c) The malicious get two heads, and the magnitude of the signal is reduced to zero.

Question 4 (10 points)

Let the events A and B have $P[A]=x, P[B]=y, \wedge P[A \cup B]=z$. Use Venn diagrams to find

$P[A\cap B]$, $P[A^c\cap B^c]$, $P[A^c \cup B^c]$, $P[A\cap B^c]$' $P[A^c\cap B]$

Solution:



$$P[A\cap B]=x+y-z$$
$P[A^c\cap B^c]$= 1-z
$P[A^c\cup B^c]$=1-(x+y-z)
$$P[A\cap B^c]=z-y$$
$$P[A^c\cap B]=z-x$$

Question 5 (15 points)

Assume we have a random number generator that can generate a random number uniformly distributed in the range [0, 1]. If now we want to use this generator to generate a random number X that has a negative exponential distribution $f_X(x)=3e^{-3x}(x\geq 0)$. Suppose the random number generator generates 0.75. What should be the value for X.

Solution:

For uniform distributed RV U, its PDF

$$F_U(u)=\begin{cases} 0 & u<0 \\ u & 0\leq u<1 \\ 1 & u\geq 1 \end{cases}$$

For random variable X, its PDF

$$F_X(x) = \begin{cases} 1 - e^{-3x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Since RV X is generated by RV U, assume X = g(U)

$$F_X(x) = P(X \leq x), x > 0 \text{¿} P(g(U) \leq x) \text{¿} \begin{cases} P(U \leq g^{-1}(x))\, g(x)\, is\, monotonically\, increasing \\ P(U \geq g^{-1}(x))\, g(x)\, is\, monotonically\, decreasing \end{cases}$$

$$\text{¿} \begin{cases} g^{-1}(x)\, g(x)\, is\, monotonically\, increasing \\ 1 - g^{-1}(x)\, g(x)\, is\, monotonically\, decreasing \end{cases}$$

When the g(x) is monotonically increasing, we obtain

$$g^{-1}(x) = 1 - e^{-3x}$$

Therefore $X = \frac{-1}{3} \ln(1 - U \text{¿}) \text{¿}$

If U equals 0.75, X is equal to 0.4621.

When the g(x) is monotonically decreasing, we get

$$1 - g^{-1}(x) = 1 - e^{-3x}$$

Therefore $X = \frac{-1}{3} \ln(U \text{¿}) \text{¿}$

When U equals 0.75, X is equal to 0.0959

Question 6 (15 points)

Let X be a random variable with pmf $p_k = c/k^2$ for $k = 1, 2, \ldots$

(a) Estimate the value of c numerically. *(5 points)*

(b) Find $P[X > 4]$. *(5 points)*

(c) Find $P[6 \leq X \leq 8]$. *(5 points)*

Solution:

(a) We need to ensure $\sum p_k = 1$. So we obtain $\frac{c * \pi^2}{6} = 1$, $c = \frac{6}{\pi^2}$

(b) $P[x > 4] = 1 - P[x \leq 4] = 0.135$

(c) $P[x = 6] + P[x = 7] + P[x = 8] = 0.039$

Question 7 (15 points)

A radio transmitter sends a signal $s > 0$ to a receiver using three paths. The signals that arrive at the receiver along each path are:

$$X_1 = s + N_1, X_2 = s + N_2, X_3 = s + N_3$$

Where N_1, N_2 and N_3 are independent Gaussian random variables with zero mean and unit variance.

(a) Find the joint pdf of $X = (X_1, X_2, X_3)$. Are X_1, X_2 and X_3 independent random variables? *(5 points)*

(b) Find the probability that the minimum of all three signals is positive. *(5 points)*

(c) Find the probability that a majority of the signals are positive. *(5 points)*

Solution:

(a) $f_{X_1}(x)=f_{X_2}(x)=f_{X_3}(x)=\frac{1}{(2\pi)^{1/2}}\exp\left(\frac{-1}{2}(x-s)^2\right)$

X_1, X_2 and X_3 are independent.

Joint pdf: $f_{X_1,X_2,X_3}(x_1,x_2,x_3)=\frac{1}{(2\pi)^{1/2}}\exp\left(\frac{-1}{2}\{(x_1-s)^2+(x_2-s)^2+(x_3-s)^2\}\right)$

(b) $P(\textit{minimum of three}>0)=P(x_1>0)P(x_2>0)P(x_3>0)=\Phi^3(s)$

(c) $P(\textit{a majortiy}>0)=\Phi^3(s)+3\Phi^2(s)(1-\Phi(s))=3\Phi^2(s)-2\Phi^3(s)$