ELEC 3210
Introduction to Mobile Robotics
Lecture 7

(Machine Learning and Infomation Processing for Robotics)

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L1 - L6

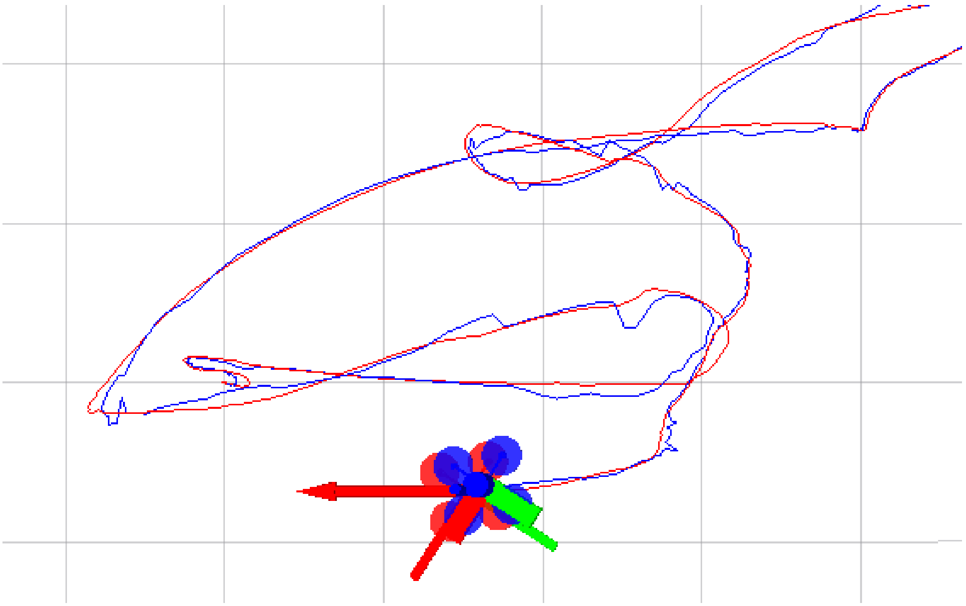
- Navigation, Pose, Odometry, Sensors, ICP, Map etc.
- Project 1 - Iterative Closest Point

Real World

- Noise, Dynamics, Light Change etc.

Why Sensor Fusion?

- Single sensor-only state estimation is too noisy, slow, and delayed for feedback control of agile mobile robots
- To improve robustness with multiple sensors and handle sensor failures
- To estimate quantities that are unobservable using single sensors



Red: Vision+IMU Fusion
Blue: Vision-only

Courtesy: Shaojie Shen

Design Considerations

- Accuracy
- Frequency
- Latency
- Sensor synchronization & timestamp accuracy
- Delayed and out-of-order measurements
- Estimator initialization
- Sensor calibration
- Different measurement models with uncertainties
- Robustness to outliers
- Computational efficiency

Courtesy: Shaojie Shen

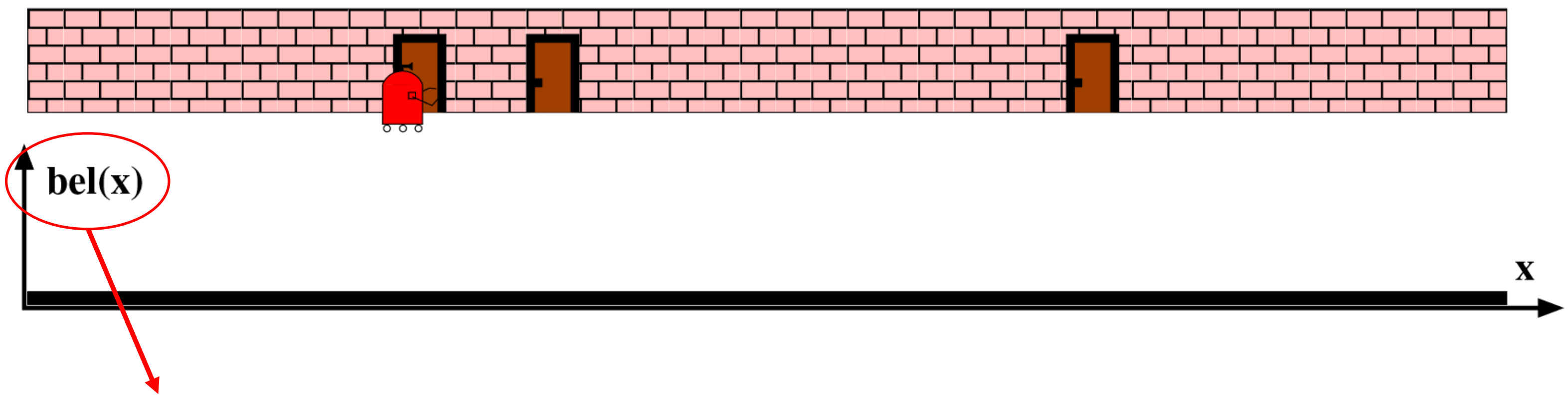
Outline

- Bayes Filter
  - Introduction to probability
  - Bayes Filter

Bayesian Filter

Example Problem

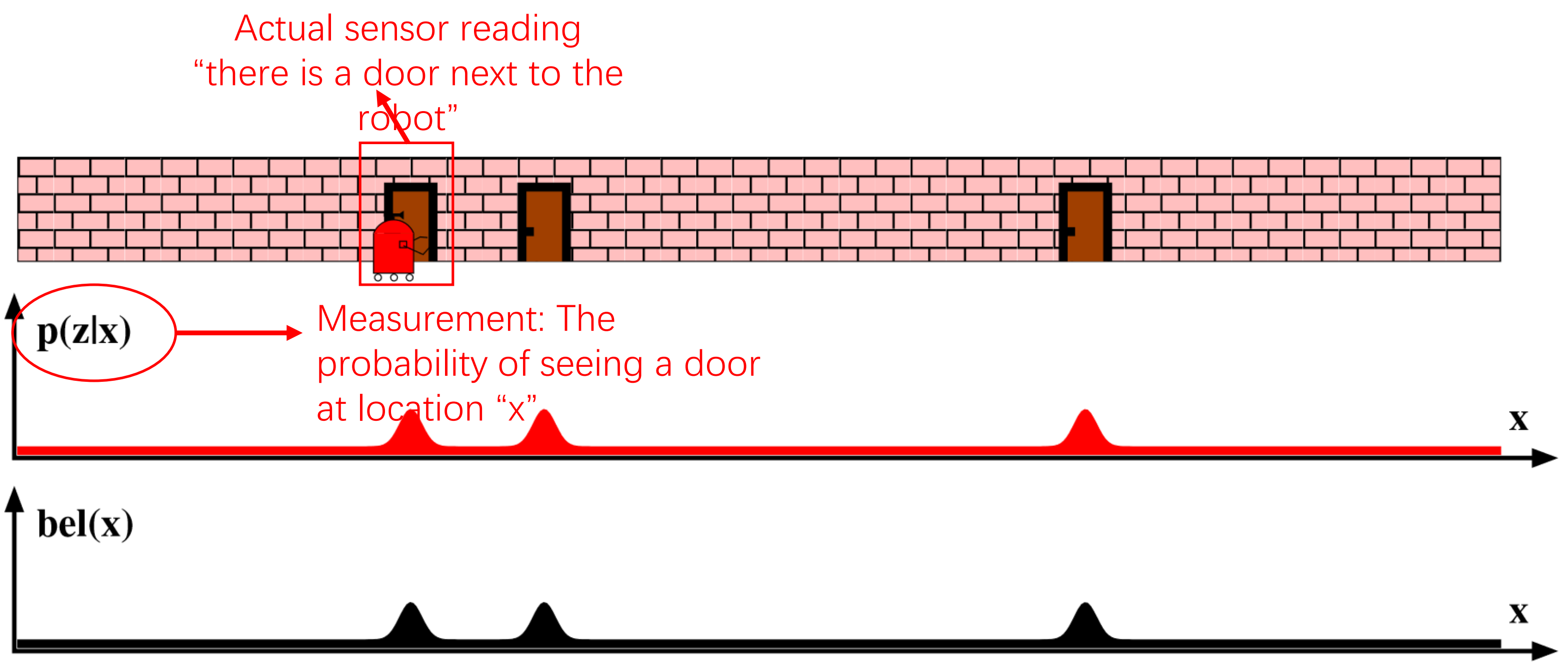
Known map



Use "belief" (or probability distribution) to represent robot state (pose)

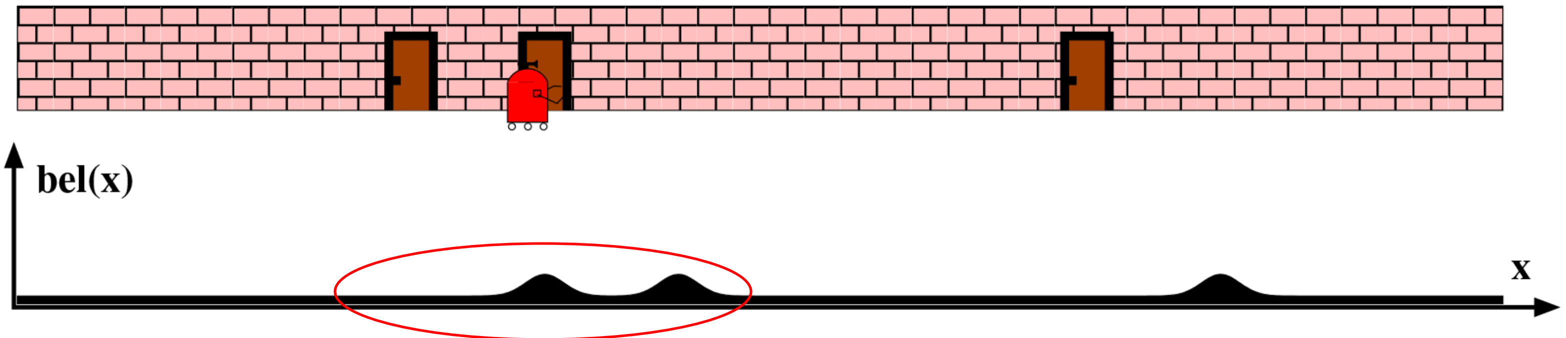
Courtesy: Thrun, Burgard, and Fox. Probabilistic Robotics. 2005

Example Problem



Courtesy: Thrun, Burgard, and Fox. Probabilistic Robotics. 2005

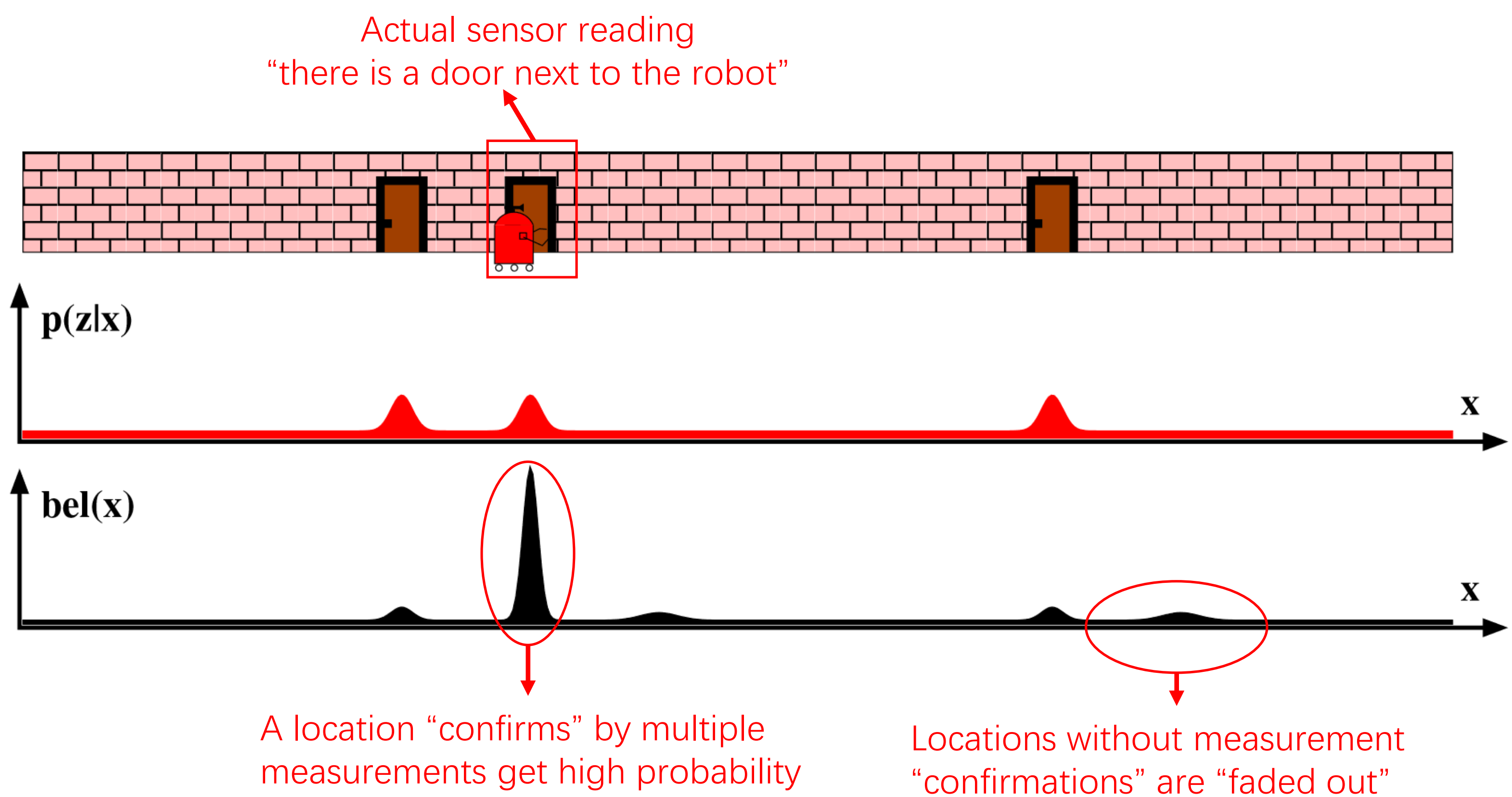
Example Problem



Robot movement “shifts”
and “flatten” the belief

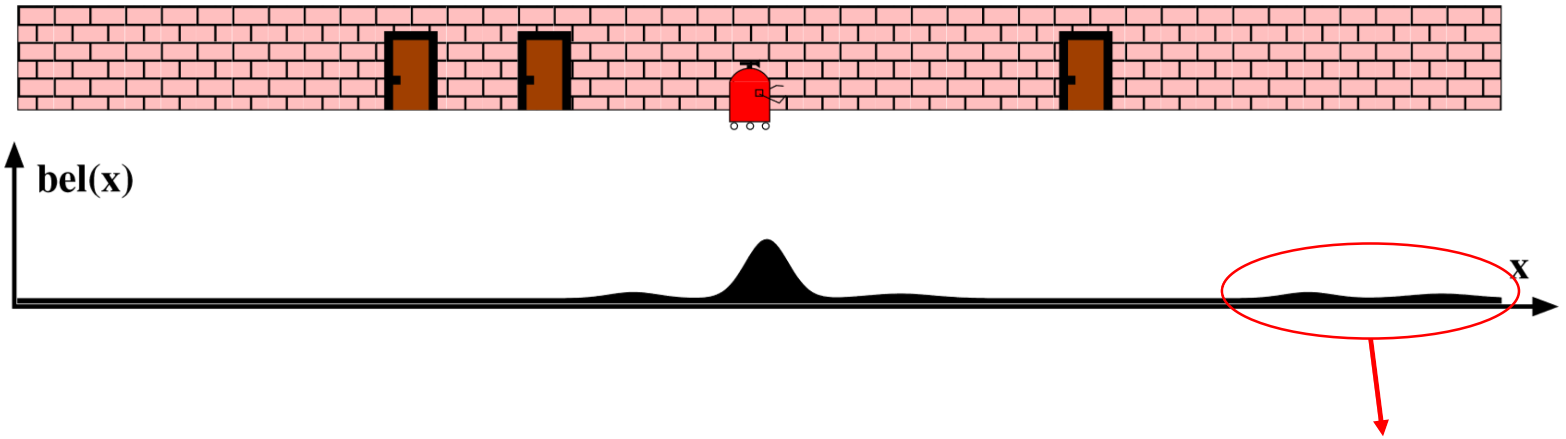
Courtesy: Thrun, Burgard, and Fox. Probabilistic Robotics. 2005

Example Problem



Courtesy: Thrun, Burgard, and Fox. Probabilistic Robotics. 2005

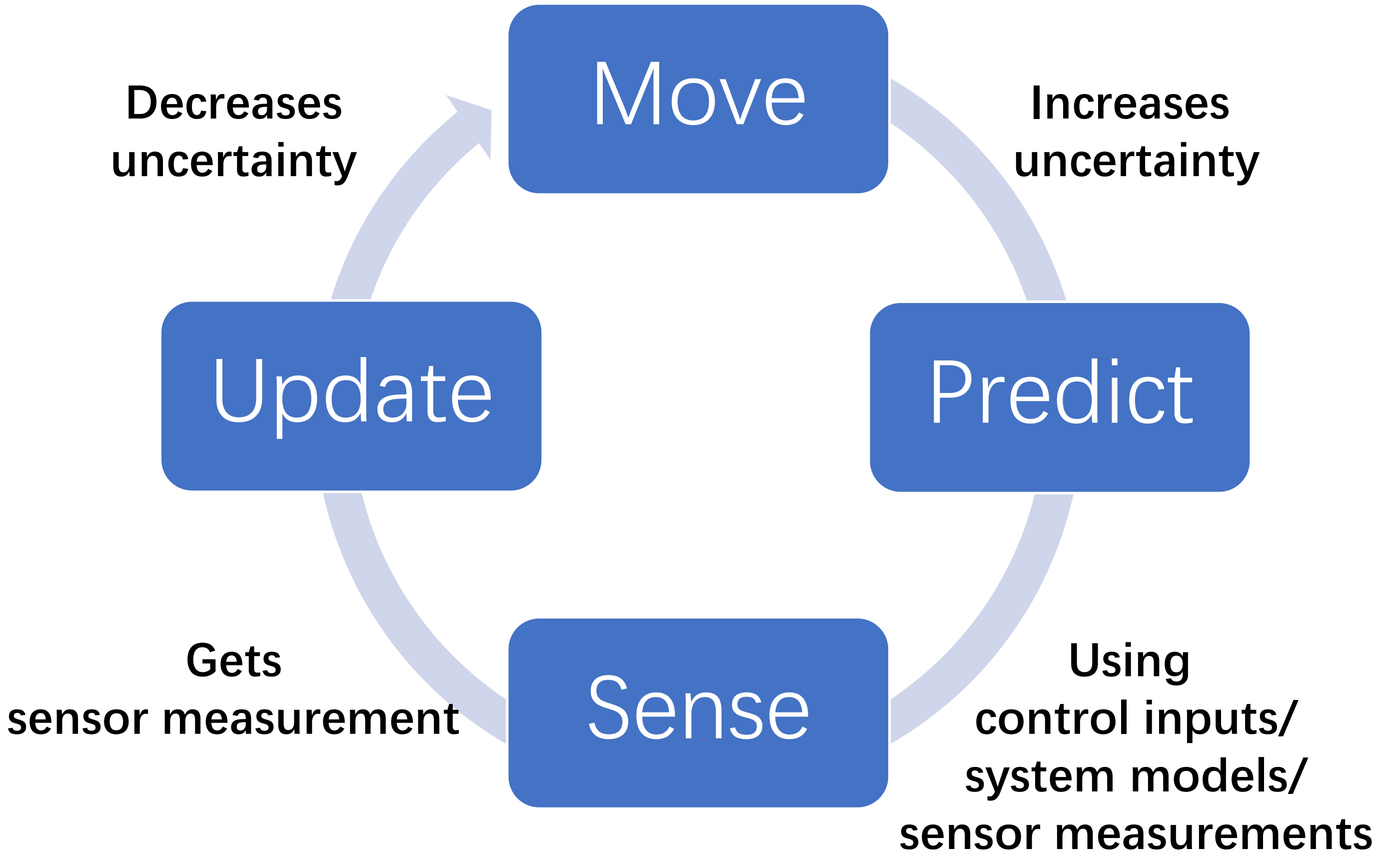
Example Problem



Locations without measurement "confirmations" are "faded out"

Courtesy: Thrun, Burgard, and Fox. Probabilistic Robotics. 2005

Problem Overview



Courtesy: Thrun, Burgard, and Fox. Probabilistic Robotics. 2005

Questions

- What are sources of uncertainty?
- How do we mathematically represent uncertainty in the system?
- How do we use collected evidence to update our belief?

Courtesy: Shaojie Shen

Introduction to Probability

Random Variables

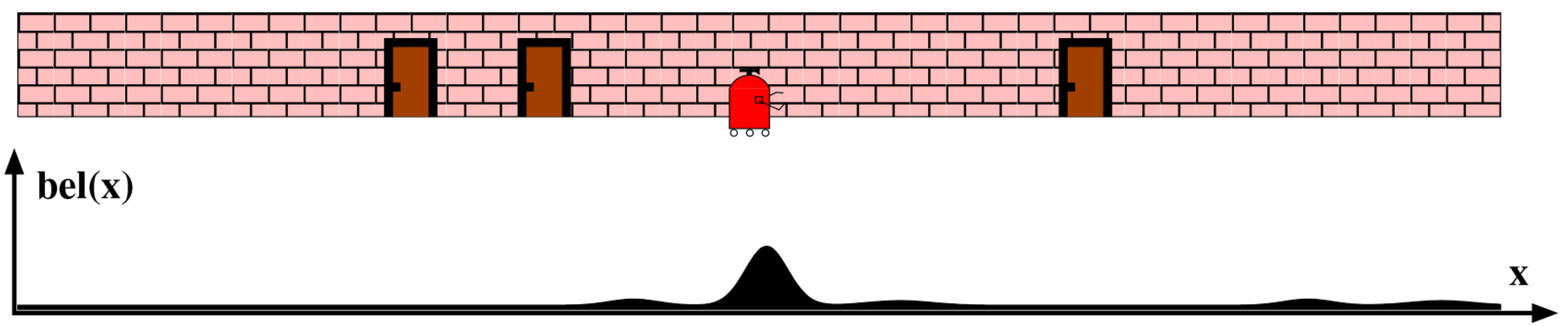
- **Definition:**

  Variable whose value is subject to change due to randomness or chance

- **Properties:**
  - Can be continuous (e.g., position in 3D) or discrete (e.g., roll of a die)
  - Observed values of random variables are called ***realizations***
- **Example:**
  - Pose of a robot, $p(X = x)$, or value of a rolled die, $p(D = d)$

Probability Density Function

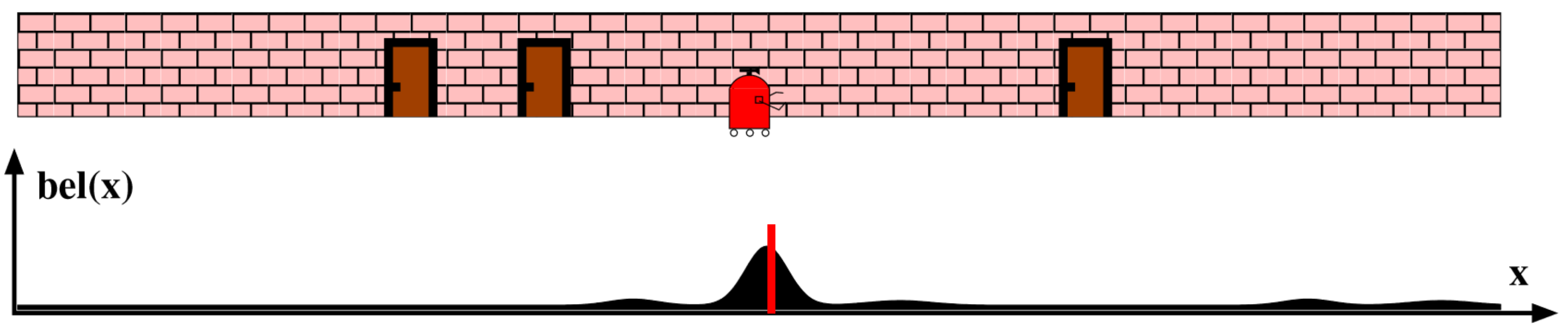
- **Definition:** Function describing the likelihood that a random variable X will take on a particular value x
- **Properties:**
  - Total probability is 1, $\int p(X = x)dx = 1$, $\sum_x p(X = x) = 1$
  - Non-negative, $p(X = x) \geq 0$



Courtesy: Shaojie Shen

Expected Value

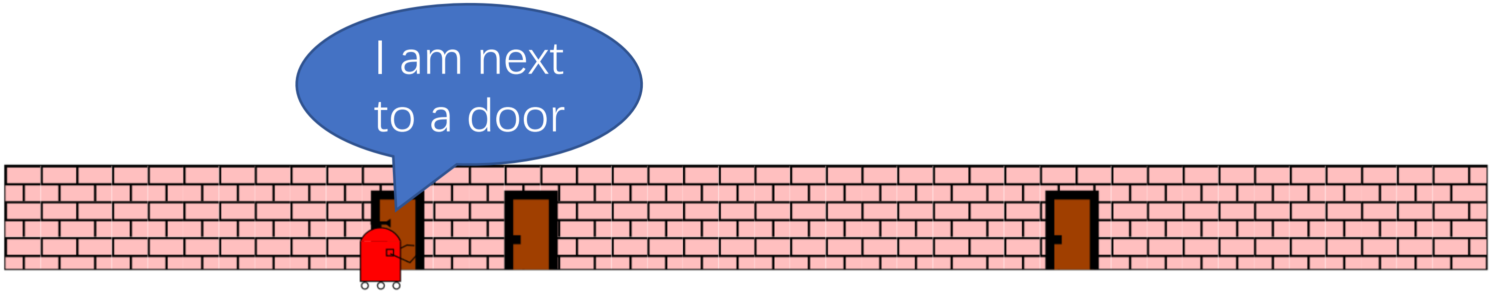
- **Definition:** Probability-weighted average value
  - $E[X] = \int p(X = x)\, x\, dx$
- **Intuition:** “Center of mass” of the probability distribution



Courtesy: Shaojie Shen

Joint Probability Distribution

- **Definition:** The probability density function of a set of two or more random variables
- Also called a ***multivariate distribution***
- **Example:**
  - $p(X = x, Z = z) =$ a robot having a pose x and receiving a measurement z



Courtesy: Shaojie Shen

Covariance

- **Definition:** A measure of how two random variables change together
  - $\sigma(X,Y) = E[(X - E[X])(Y - E[Y])]$

- The ***variance*** is a special case where the two random variables are identical
  - $\sigma^2(X) = \sigma(X,X)$

- **Intuition:** The "moment of inertia" of the probability distribution

Courtesy: Shaojie Shen

Covariance Matrix

- For a multivariate distribution over $\boldsymbol{X} = [X_1, X_2, \dots, X_n]^T$ we define the ***covariance matrix*** to be

- $\boldsymbol{\Sigma} = E[(\boldsymbol{X} - E[\boldsymbol{X}])(\boldsymbol{X} - E[\boldsymbol{X}])^T]$

$$= \begin{bmatrix} \sigma^2(X_1) & \sigma(X_1, X_2) & \cdots & \sigma(X_1, X_n) \\ \sigma(X_2, X_1) & \sigma^2(X_2) & \cdots & \sigma(X_2, X_n) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma(X_n, X_1) & \sigma(X_n, X_2) & \cdots & \sigma^2(X_n) \end{bmatrix}$$

- The covariance matrix is symmetric and positive semi-definite
  - Proof: https://statproofbook.github.io/P/covmat-psd.html

Courtesy: Shaojie Shen

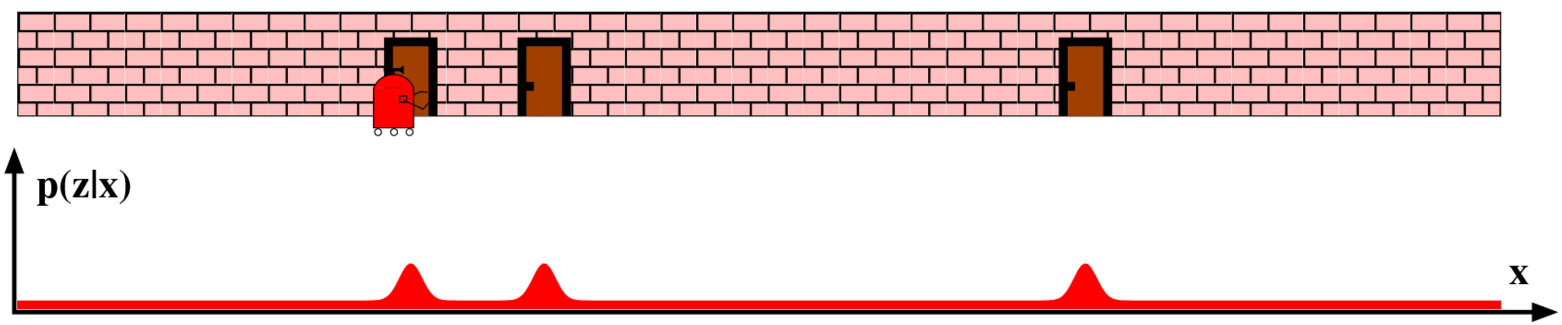
Independence

- **Definition:** Two random variables are independent if the outcome of one has *no effect* on the outcome of the other
- $p(x, z) = p(x)\, p(z)$
- **Example:**
  - If X, Z are the outcomes of two dice rolls
- **Properties:**
  - Independent random variables are ***uncorrelated***, $\sigma(X, Z) = 0$
  - Uncorrelated random variables are ***not*** necessarily independent
- Example:
  - X=U[-1,1] (uniform distribution between -1 and 1)
  - $Y = X^2$
  - X and Y are uncorrelated but clearly dependent

Courtesy: Shaojie Shen

Conditional Probability

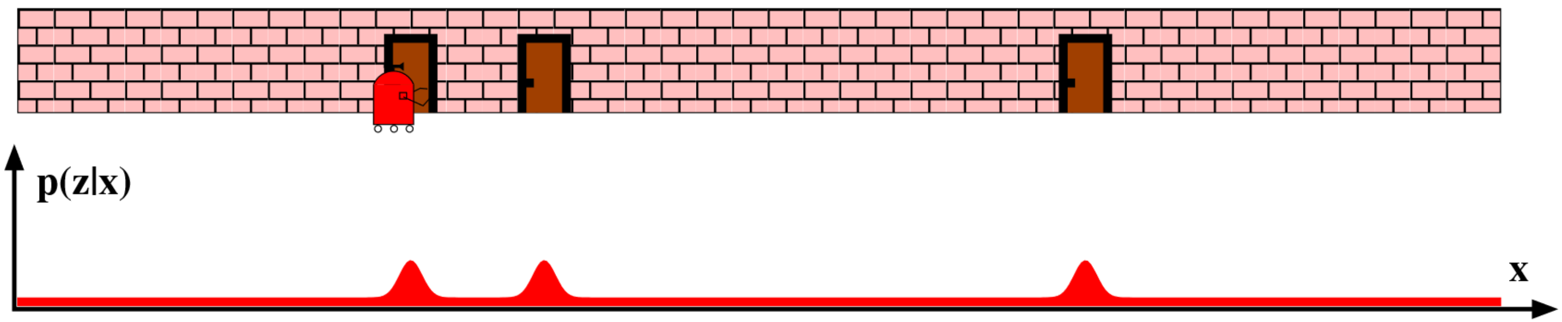
- **Definition:** Probability of an event z occurring conditioned on another event x occurring
- $p(z \mid x) = \frac{p(x,z)}{p(x)} \quad \Leftrightarrow \quad p(x, z) = p(z \mid x)\, p(x)$
- **Example:**
  - z = {there is a door next to the robot}



Courtesy: Shaojie Shen

Conditional Independence

- **Definition:** Two random variables are ***conditionally independent*** if the outcome of one has ***no effect*** on the outcome of the other when conditioned on the outcome of a third random variable
- $p(z_1, z_2 \mid x) = p(z_1 \mid x)\, p(z_2 \mid x)$
- **Example:**
  - Let $\mathrm{Z}_1, \mathrm{Z}_2$ be two measurements taken from the same place
  - $\mathrm{Z}_1, \mathrm{Z}_2$ are conditionally independent given X

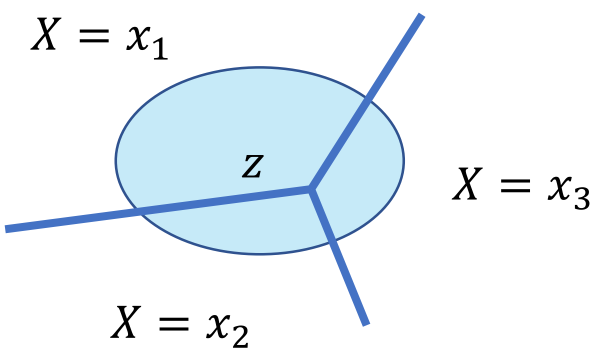


- **Question:** Are $\mathrm{Z}_1, \mathrm{Z}_2$ are independent?

Courtesy: Shaojie Shen

Marginal Distribution

- **Definition:** The probability distribution of the subset of a collection of random variables
- $p(z) = \int p(x, z)\, dx$
- Also known as the ***Law of Total Probability***



$$p(z) = \sum_{i=1}^{3} p(z \mid X = x_i)\, p(X = x_i)$$

Courtesy: Shaojie Shen

Bayes Filter

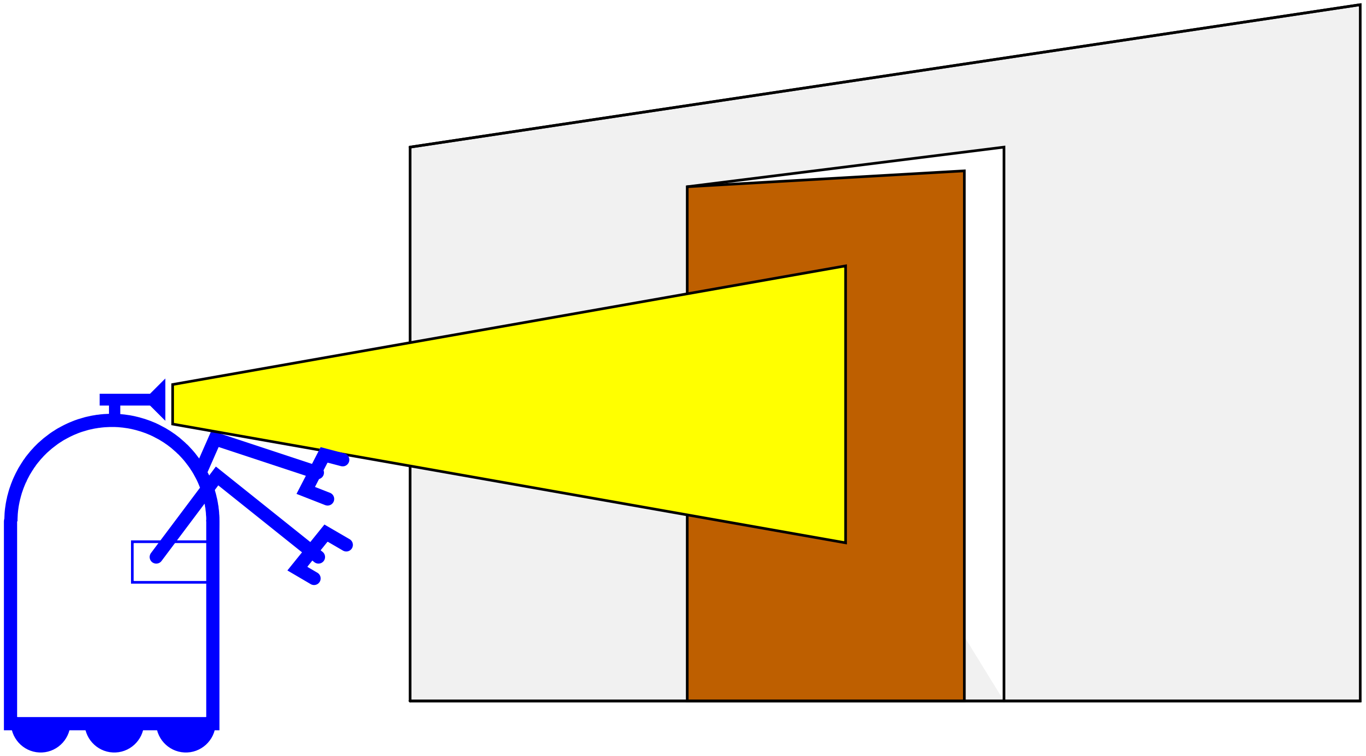
Bayes Theorem

- $p(x, z) = p(z \mid x)\, p(x) = p(x \mid z) p(z)$
- $p(x \mid z) = \dfrac{p(z \mid x)\, p(x)}{p(z)} = \dfrac{liklihood * prior}{evidence}$
- **Intuition:** Describes how the belief about a random variable X should change to account for the collected evidence (measurement) z

Thomas Bayes
1701-1761

Simple Example

- Suppose a robot obtains measurement *z (e.g. brightness)*
- What is *P(open|z)?*



Courtesy: Sebastian Thrun, Wolfram Burgard and Dieter Fox

Causal vs. Diagnostic Reasoning

- $P(open|z)$ is diagnostic.
- $P(z|open)$ is causal
  - Light sensor: If the door is open, what's the likelihood that the sensor receives this amount of light
  - Robot localization: Given a map, what's the likelihood that the sensor (camera/laser/etc.) gets this measurement
- Often causal knowledge is easier to obtain.
- Bayes rule allows us to use causal knowledge:

$$P(open \mid z) = \frac{P(z \mid open) P(open)}{P(z)}$$

Courtesy: Sebastian Thrun, Wolfram Burgard and Dieter Fox

Example

- $P(z|open) = 0.6 \qquad P(z|\neg open) = 0.3$ Likelihood
- $P(open) = P(\neg open) = 0.5$ Prior

$$P(open \mid z) = \frac{P(z \mid open)P(open)}{\boxed{P(z \mid open)\,p(open) + P(z \mid \neg open)\,p(\neg open)}}$$

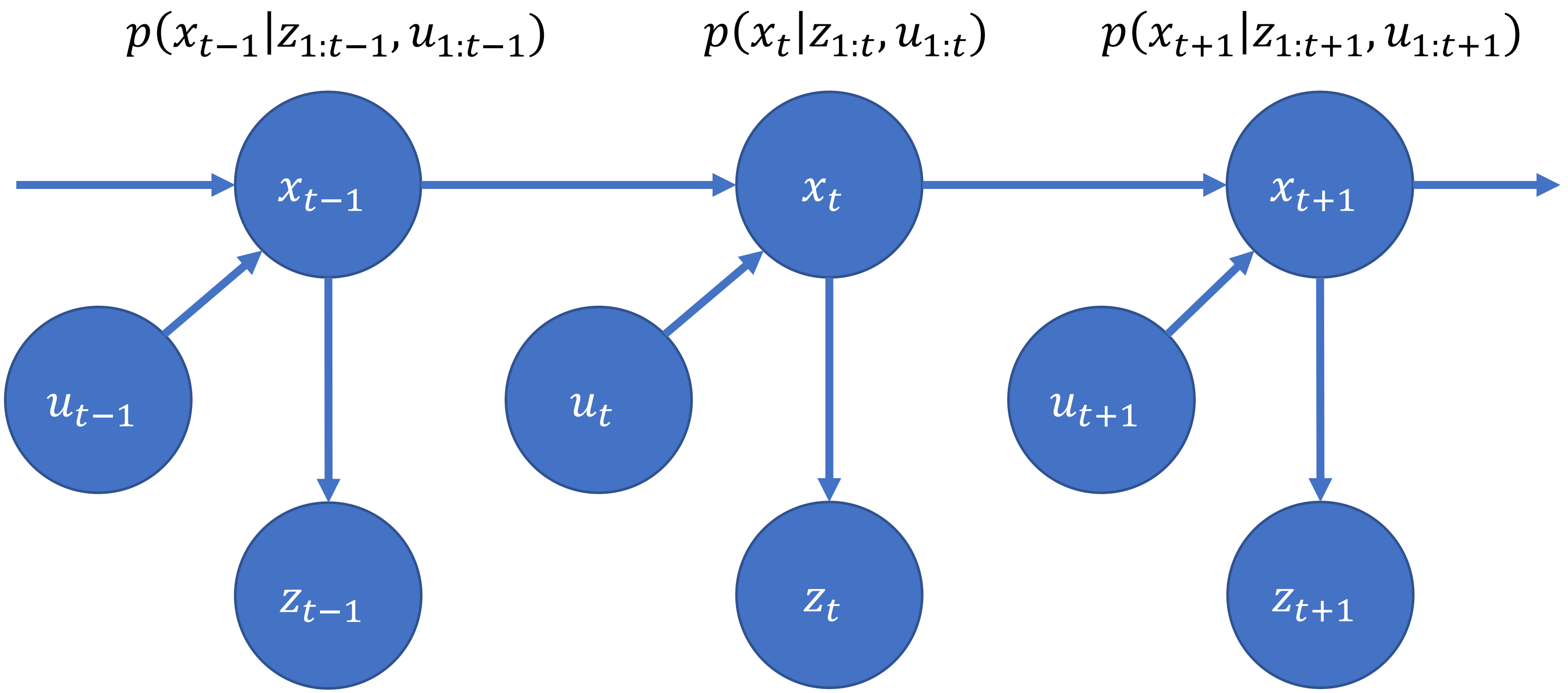
Law of Total Probability

$$P(open \mid z) = \frac{0.6 \cdot 0.5}{0.6 \cdot 0.5 + 0.3 \cdot 0.5} = \frac{2}{3} = 0.67$$

- *z* raises the probability that the door is open.

Bayesian Filter

- x: state
- u: control signal
- z: measurement



Markov Property

- **Definition:** The future state of the system is conditionally independent of the past states given the current state
  - $p(x_{t+1}|x_{0:t}) = p(x_{t+1}|x_t)$
  - $p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$
  - $p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$
- **Question:**
  - Which of the following satisfy the Markov assumption?
    - A first order system with $x = [\text{position}], u = [\text{velocity}]$
    - A second order system with $x = [\text{position}], u = [\text{acceleration}]$
      - How about with $x = [\text{position, velocity}], u = [\text{acceleration}]$

Courtesy: Shaojie Shen

Bayes Filter Derivation

- **Goal:** Update the probability distribution of the robot pose using the realizations of the control input and measurement
- **Note:** Bayes Rule and decompose measurement
- $p(x_t \mid z_{1:t}, u_{1:t}) = \dfrac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t})\, p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})}$
- How?

Bayes Filter Derivation

- **Goal:** Update the probability distribution of the robot pose using the realizations of the control input and measurement
- **Note:** Bayes Rule and decompose measurement
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- **Note:** The measurement is *conditionally independent* of the past measurements and control inputs given the current state of the robot
- $p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$
- **Note:** The denominator can be found as a *marginal distribution* of the numerator
- $p(z_t \mid z_{1:t-1},\ u_{1:t}) = \int p(x_t, z_t \mid z_{1:t-1}, u_{1:t})\, dx_t = \dfrac{1}{\eta}$

Process Model

- Also known as the ***transition model*** or ***motion model***
- $p(x_t \mid z_{1:t-1}, u_{1:t})$
- **Note:** Can find the current pose via marginalization
- $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int p(x_t, x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}$
- $\quad = \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t})\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t})\, dx_{t-1}$
- **Note:** The future state is *conditionally independent* of the past measurements and control inputs given the current state and input
- $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$

Prediction | Process model | Prior

Courtesy: Shaojie Shen

Bayes Filter

- **Prior:** $p(x_0)$
- **Process model:** $p(x_t \mid x_{t-1}, u_t)$

- **Prediction step:**
- $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$

Bayes Filter

- **Prior:** $p(x_0)$
- **Process model:** $p(x_t \mid x_{t-1}, u_t)$
- **Measurement model:** $p(z_t \mid x_t)$

- **Prediction step:**
- $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$
- **Update step:**
- $p(x_t \mid z_{1:t}, u_{1:t}) = \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t})$

Bayes Filter

- **Prior:** $p(x_0)$
- **Process model:** $p(x_t \mid x_{t-1}, u_t)$
- **Measurement model:** $p(z_t \mid x_t)$

- **Prediction step:**
- $p(x_t \mid z_{1:t-1}, u_{1:t}) = \int p(x_t \mid x_{t-1}, u_t)\, p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1})\, dx_{t-1}$
- **Update step:**
- $p(x_t \mid z_{1:t}, u_{1:t}) = \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t})$

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

Derivation

$$\boxed{Bel(x_t)} = P(x_t \mid u_1, z_1 \ldots, u_t, z_t)$$

Bayes $$= \eta \; P(z_t \mid x_t, u_1, z_1, \ldots, u_t) \; P(x_t \mid u_1, z_1, \ldots, u_t)$$

Markov $$= \eta \; P(z_t \mid x_t) \; P(x_t \mid u_1, z_1, \ldots, u_t)$$

Total prob. $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_1, z_1, \ldots, u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \; dx_{t-1}$$

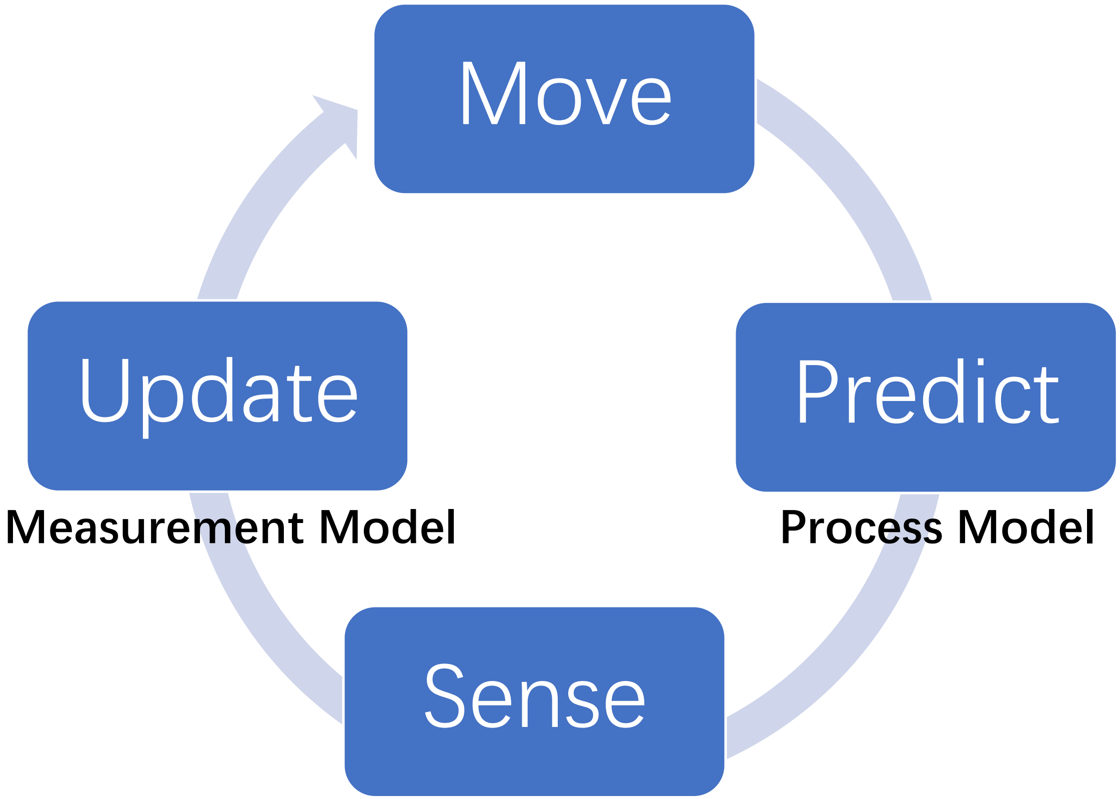
Markov $$= \eta \; P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; P(x_{t-1} \mid u_1, z_1, \ldots, u_t) \; dx_{t-1}$$

Markov $$= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) P(x_{t-1} \mid u_1, z_1, \ldots, z_{t-1}) \; dx_{t-1}$$

$$\boxed{= \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) \; Bel(x_{t-1}) \; dx_{t-1}}$$

Courtesy: Wolfram Burgard

Loop



Courtesy: Thrun, Burgard, and Fox. Probabilistic Robotics. 2005

Bayes Filters are Familiar

$$Bel(x_t) = \eta P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Kalman filters
- Particle filters
- Hidden Markov models
- Dynamic Bayesian networks
- Partially Observable Markov Decision Processes (POMDPs)

Applications

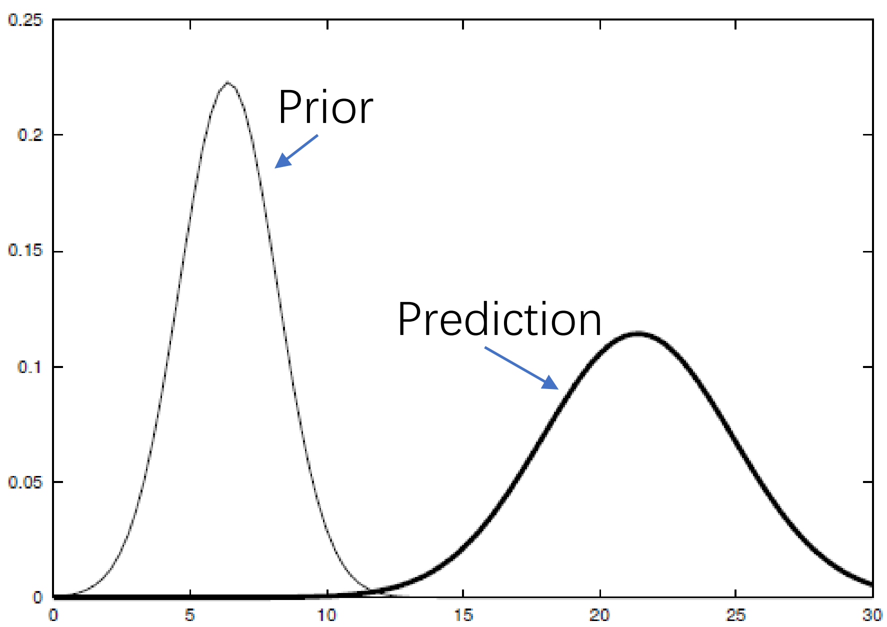
- Can use Bayesian filtering in many other domains
  - Robot Localization
  - Simultaneous localization and mapping (SLAM)
  - Feature tracking
  - Pose estimation
  - Target tracking
  - etc.

Resources

- Probabilistic Robotics: Ch. 2 and Ch. 3
- Bayes Filter by Prof. Cyrill Stachniss

Back to Questions

- What are sources of uncertainty?
- How do we mathematically represent uncertainty in the system?
- How do we use collected evidence to update our belief?



Courtesy: Shaojie Shen

Next Lecture

- October 9th Kalman Filter
  - Kalman Filter is Bayes Filter with Gaussians

- Project 1 out by October 12th
- Homework 1 will be released this week

- Enjoy holiday and study break ☺