THE HONG KONG UNIVERSITY OF SCIENCE & TECHNOLOGY

ELEC 3810/BIEN 3310A: Data Science in Neural Engineering

Homework 2

Due Date:1pm, 12th Oct 2023

Your answers should be typed, not handwritten. You should submit a Word file or a pdf file.

Submissions are to be made via Canvas. ***Late submission is not allowed (0 point)***

Submitted File Name Format: name_ID_Homework2

Question 1 (15 points)

The input X to a communication channel is +1 or -1 with probability p and $1-p$, respectively. The received signal Y is the sum of X and noise N which has a Gaussian distribution with zero mean and variance $\sigma^2 = 0.16$.

(a) Find the joint probability $P[X = j, Y \leq y]$.

(b) Find the marginal pmf of X and the marginal pdf of Y.

(c) Design a MAP decoder and write the decoding rule.

Question 2 (20 points)

Let X and Y be discrete random variables with joint pmf's:

X/Y	-1	0	1
-1	1/6	1/6	0
0	0	0	1/3
1	1/6	1/6	0

a) Find the minimum mean square error linear estimator for Y given X.

b) Find the minimum mean square error estimator for Y given X.

c) Find the MAP and ML estimators for Y given X.

d) Compare the mean square error of the estimators in parts a, b and c.

Question 3 (10 points)

Let U_1,U_2,U_3 be independent random variables with zero mean and variance 1. Find the linear MMSE estimator of S in terms of Z_1 and Z_2, and the corresponding MSE.

$$\begin{bmatrix} S \\ Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

Question 4 (10 points)

Let X be uniformly distributed in the interval (-1,1) and let $Y = X^3$. Find the best linear estimator for Y in terms of X. Compare its performance to the best estimator.

Question 5 (15 points)

Suppose $X \sim U[1,2]$, and given $X = x$, $Y \sim Exp(\lambda)$ with parameter $\lambda = \frac{1}{x}$.

a) Find the linear MMSE estimate of X given Y.
b) Find the MSE of this estimator.
c) Check that $E[\tilde{X}Y] = 0. (\tilde{X} = X - \hat{X})$

Question 6 (10 points)

Let X be an unobserved random variable with $E(X) = 0, Var(X) = 4$. Assume that we have observed Y_1 and Y_2 given by

$$Y_1 = X + W_1,$$

$$Y_2 = X + W_2,$$

where $E(W_1) = E(W_2) = 0, Var(W_1) = 1,$ and $Var(W_2) = 4$. Assume that $W_1, W_2, and\ X$ are independent random variables. Find the linear MMSE estimator of X, given $Y_1\ and\ Y_2$

Question 7 (20 points)

Let X, Y, Z have joint pdf

$$f_{X,Y,Z}(x, y, z) = k(x + y + z) \quad for\ 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$$

a) Find k
b) Find the minimum mean square error linear estimator for Y given X and Z.
   (Hint: given two observations, you can either calculate the derivative of MSE in terms of every coefficient or list the observations as a vector form.)
c) Find the minimum mean square estimator for Y given X and Z.
d) Find the MAP and ML estimators for Y given X and Z.