ELEC3810 HW1

Q1.

Question 1 (15 points)

A random experiment has sample space $S = \{1,2,3,4\}$ with probabilities $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$, $p_3 = \frac{1}{8}$, $p_4 = \frac{1}{8}$.

(a) Describe how this random experiment can be simulated using tosses of a fair coin.

(b) Describe how this random experiment can be simulated using an urn experiment.

(c) Describe how this experiment can be simulated using a deck of 52 distinct cards.

(a)

This can be simulated with a random experiment of tosses with fair coin where we toss the coin three times and record the side it flips. The possible outcomes are: $(H,H,H),(H,H,T),(H,T,H),(T,H,H),(H,T,T),(T,H,T),(T,T,H),(T,T,T)$.

Using sample space “1” as the probability of landing heads at least twice, we get $p_1 = 4/8 = 1/2$.

Using sample space “2” as the probability of landing heads or tails 3 times in a row, we get $p_2 = 2/8 = 1/4$.

Using sample space “3” as the probability of landing heads, heads, then tails in the exact order, we get $p_3 = 1/8$.

Using sample space “4” as the probability of landing heads, tails, then tails in the exact order, we get $p_4 = 1/8$.

(b)

This sample space and probabilities can be simulated with urn experiment by placing and labelling 8 balls in the urn with the numbers 1,2,3,4. We label four balls with the number 1, two balls with the number 2, one ball with number 3 and one ball with number 4. We draw one ball from the urn and look at the number labelled on the ball. The sample space corresponds to the ball’s number drawn from the urn.

Therefore probability of drawing ball with label “1” $= p_1 = 1/2$, probability of drawing ball with label “2” is $p_2 = 1/4$, probability of drawing ball with label “3” is $p_3 = 1/8$, probability of drawing ball with label “4” is $p_4 = 1/8$.

(c)

To simulate this sample space and probability with deck of 52 distinct cards, we define a random experiment where each time we pick a card we will note the colour of the card, place the card back to the deck and re-shuffle the deck. We repeat this 3 times.

Using sample space "1" as the probability of getting at least two cards that are red, we get $p_1 = 1/2$.

Using sample space "2" as the probability of getting the same coloured card three times in a row, we get $p_2 = 1/4$.

Using sample space "3" as the probability of getting red, red, then black in the exact order, we get $p_3 = 1/8$.

Using sample space "4" as the probability of getting black, red, black in the exact order, we get $p_4 = 1/8$.

Q2.

Question 2 (10 points)

Andy and Bob are playing a basketball shooting game. In each round they will each shoot for one ball. Anyone who makes the goal will get one point. After each round, if one of them has more points than the other, the game will end and the person with higher points wins. If they have the same points after one round, they will have a new round until one wins. Suppose the hit rate for Andy is p_1 and for Bob is p_2. What is the probability that Andy will win the game?

Andy win scenarios:

- P(Andy hits and Bob misses) = $p_1 * (1 - p_2)$
- P(First round same points, Andy wins new round) = (P(Both misses) + P(Both hits)) * P(Andy wins) = $((1 - p_1) * (1 - p_2) + p_1 * p_2) * P(\text{Andy wins})$

Therefore,

P(Andy wins) = P(Andy hits and Bob misses) + P(First round same points, Andy wins new round)

P(Andy wins) = $p_1 * (1 - p_2) + (((1 - p_1) * (1 - p_2) + p_1 * p_2) * P(\text{Andy wins}))$

P(Andy wins) = $\frac{p_1 * (1 - p_2)}{(p_2 + p_1 - 2 * p_1 * p_2)}$

Q3.

Question 3 (15 points)

A binary communication system transmits a signal X that is either a voltage signal +2 or a -2 voltage signal. A malicious channel reduces the magnitude of the received signal by the number of heads it counts in two tosses of a coin. Let Y be the resulting signal.

(a) Find the sample space.

(b) Find the set of outcomes corresponding to the event "transmitted signal was definitely +2"

(c) Describe in words the event corresponding to the outcome $Y = 0$.

(a)

Sample space of Y is {-2,-1,0,1,2}.

When 0 total heads: $(X=-2, Y=-2), (X=+2, Y=+2)$

When 1 total heads: $(X=-2, Y=-1), (X=+2, Y=+1)$

When 2 total heads: $(X=-2, Y=0), (X=+2, Y=0)$

(b)

Set of outcomes with event "transmitted signal was definitely +2" (ie. X=+2): $\{Y=+2,\ Y=+1\}$

(c)

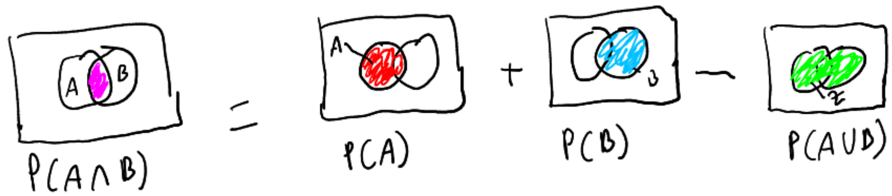
To get the outcome Y = 0, we must obtain 2 heads in the two tosses of the coin. The input X cannot be determined as X can either be +2 or -2.

Q4.

Question 4 (10 points)

Let the events A and B have $P[A]=x, P[B]=y, and\ P[A \cup B]=z$. Use Venn diagrams to find $P[A \cap B],\ P[A^c \cap B^c],\ P[A^c \cup B^c],\ P[A \cap B^c],\ P[A^c \cap B]$

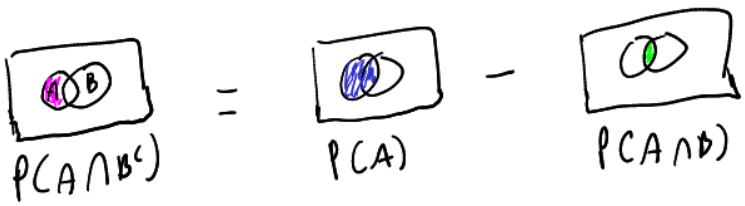
$P(A \cap B)$ = x + y - z



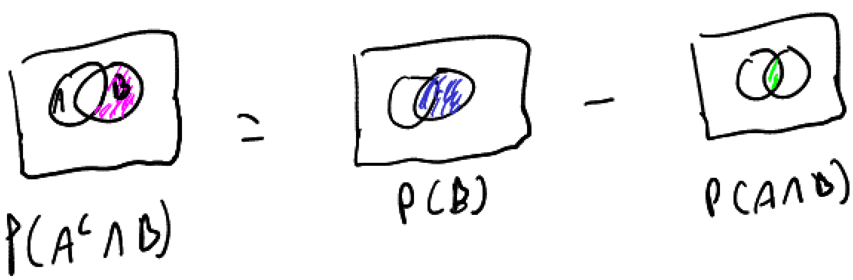
$P(A^c \cap B^c)$ = 1 - z

$P(A^c \cup B^c)$ = 1 - $P(A \cap B)$ = 1 - (x + y - z) = 1 + z - x - y

$P(A \cap B^c) = P(A)$ - $P(A \cap B)$ = x - (x + y - z) = z - y



$P(A^c \cap B) = P(B)$ - $P(A \cap B)$ = y - (x + y - z) = z-x



Q5.

Question 5 (10 points)

Assume we have a random number generator that can generate a random number uniformly distributed in the range [0, 1]. If now we want to use this generator to generate a random number X that has a negative exponential distribution $f_X(x) = 3e^{-3x}(x \geq 0)$. Suppose the random number generator generates 0.75. What should be the value for X.

CDF of $f_x(x)$ is $F_x(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{\infty} 3e^{-3x}dx$

$= (-e^{-3t})|_0^x$

therefore $F_x(x) = 1 - e^{-3x}$, to get value for X that will generate 0.75:

$0.75 = 1 - e^{-3x}$

$x = -ln(0.25)/3$

x = 0.462

Q6.

Question 6 (15 points)

Let X be a random variable with pmf $p_k = c/k^2$ for $k = 1,2,\ldots$

(a) Estimate the value of c numerically.

(b) Find $P[X > 4]$.

(c) Find $P[6 \leq X \leq 8]$.

(a)

$\sum_{k=1}^{\infty} p_k = 1$

$\sum_{k=1}^{\infty} \frac{c}{k^2} = 1$

$c * \sum_{k=1}^{\infty} \frac{1}{k^2} = 1$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, therefore:

$c * \frac{\pi^2}{6} = 1$

$c = \frac{6}{\pi^2}$ or $c \approx 0.6079$

(b)

$P(X > 4) = 1 - P(X <= 4) = 1 - c * \sum_{k=1}^{4} \frac{1}{k^2} = 1 - \frac{6}{\pi^2}(1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16})$

$= 0.1345$

(c)

$P(6 \leq X \leq 8) = P(6) + P(7) + P(8) = c * \sum_{k=6}^{8} \frac{1}{k^2} = \frac{6}{\pi^2}(\frac{1}{36} + \frac{1}{49} + \frac{1}{64})$

$= 0.0388$

Q7.

Question 7 (15 points)

A radio transmitter sends a signal $s > 0$ to a receiver using three paths. The signals that arrive at the receiver along each path are:

$$X_1 = s + N_1, X_2 = s + N_2, X_3 = s + N_3$$

Where N_1, N_2 and N_3 are independent Gaussian random variables with zero mean and unit variance.

(a) Find the joint pdf of $X = (X_1, X_2, X_3)$. Are X_1, X_2 and X_3 independent random variables?

(b) Find the probability that the minimum of all three signals is positive.

(c) Find the probability that a majority of the signals are positive.

(a)

$F_{X_1,X_2,X_3}(x_1,x_2,x_3) = P(N_1 \le x_1 - s)P(N_2 \le x_2 - s)P(N_3 \le x_3 - s) = F_{N_1}(x_1 - s)F_{N_2}(x_2 - s)F_{N_3}(x_3 - s)$

$f_{X_1,X_2,X_3}(x_1,x_2,x_3) = f_{n_1}(x_1 - s) * f_{n_2}(x_2 - s) * f_{n_3}(x_3 - s)$

$= \frac{1}{2\pi}(e^{-\frac{(x_1-s)^2}{2}} * e^{-\frac{(x_2-s)^2}{2}} * e^{-\frac{(x_3-s)^2}{2}})$

Yes, X_1, X_2, X_3 are independent random variables.

(b)

$P(x_1 > 0, x_2 > 0, x_3 > 0) = 1 - P(x_1 \le 0, x_2 \le 0, x_3 \le 0)$

$= 1 - ((1 - P(x_1 \le 0)) * (1 - P(x_2 \le 0)) * (1 - P(x_3 \le 0)))$

$= 1 - ((1 - F_{N_1}(-s)) * (1 - F_{N_2}(-s)) * (1 - F_{N_3}(-s)))$

$= 1 - (1 - F_N(-s))^3$

(c)

P(Majority Positve) = P(All Postive) + P(X1, X2 Positive) + P(X1, X3 Positive)
+ P(X2, X3 Positive)

$= P(X_1 > 0, X_2 > 0, X_3 > 0) + P(X_1 > 0, X_2 > 0, X_3 \le 0) +$

$P(X_1 > 0, X_2 \le 0, X_3 > 0) + P(X_1 \le 0, X_2 > 0, X_3 > 0)$

$= (1 - F_N(-s))^3 + (1 - F_N(-s))^2 * F_N(-s) + (1 - F_N(-s))^2 * F_N(-s) + (1 - F_N(-s))^2 * F_N(-s)$

$= (1 - F_N(-s))^3 + 3 * ((1 - F_N(-s))^2 * F_N(-s))$