ELEC3810 Homework 2

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Q1

Question 1 (15 points)

The input X to a communication channel is +1 or -1 with probability p and $1-p$, respectively. The received signal Y is the sum of X and noise N which has a Gaussian distribution with zero mean and variance $\sigma^2 = 0.16$.

(a) Find the joint probability $P[X=j, Y \leq y]$.

(b) Find the marginal pmf of X and the marginal pdf of Y.

(c) Design a MAP decoder and write the decoding rule.

$Y = X + N$

(a)

When j = 1:

$P[X=j, Y \leq y] = P[X=1, 1+N \leq y] = P[X=1, N \leq y-1]$

$= p * \frac{1}{0.5*\sqrt{2\pi}} * \int_{-\infty}^{y-1} e^{-\frac{x^2}{2*0.25}} dx$

$= p * \frac{1}{0.5*\sqrt{2\pi}} * \int_{-\infty}^{y-1} e^{-2x^2} dx$

When j = -1:

$P[X=j, Y \leq y] = P[X=-1, -1+N \leq y] = P[X=-1, N \leq y+1]$

$= (1-p) * \frac{1}{0.5*\sqrt{2\pi}} * \int_{-\infty}^{y+1} e^{-\frac{x^2}{2*0.25}} dx$

$= (1-p) * \frac{1}{0.5*\sqrt{2\pi}} * \int_{-\infty}^{y+1} e^{-2x^2} dx$

(b)

Marginal pmf of X:

$p_x(1) = p,\ p_x(-1) = 1-p$

Marginal cdf of Y:

$F_y(y) = P[Y \leq y | x=-1] P[x=-1] + P[Y \leq y | x=1] P[x=1]$

$= P[N-1 \leq y | x=-1] * (1-p) + P[N+1 \leq y | x=1] * p$

$= (1-p) * \frac{1}{0.5*\sqrt{2\pi}} * \int_{-\infty}^{y} e^{-2(y+1)^2} dy + (p) * \frac{1}{0.5*\sqrt{2\pi}} * \int_{-\infty}^{y} e^{-2(y-1)^2} dy$

Marginal pdf of Y:

$f_y(y) = \frac{d}{dy} F_y(y)$

$= \frac{(1-p)}{0.5*\sqrt{2\pi}} * e^{-2(y+1)^2} dy + \frac{p}{0.5*\sqrt{2\pi}} * e^{-2(y-1)^2} dy$

(c)

MAP decoder rule:

$\hat{X} = +1$ if $p_{X|Y}(X = +1|Y) > p_{X|Y}(X = -1|Y)$

$\hat{X} = -1$ otherwise

Q2

Question 2 (20 points)

Let X and Y be discrete random variables with joint pmf's:

X/Y	-1	0	1
-1	1/6	1/6	0
0	0	0	1/3
1	1/6	1/6	0

a) Find the minimum mean square error linear estimator for Y given X.
b) Find the minimum mean square error estimator for Y given X.
c) Find the MAP and ML estimators for Y given X.
d) Compare the mean square error of the estimators in parts a, b and c.

$E[X] = 0, E[Y] = 0, E[XY] = 0, E[X^2] = 2/3, E[Y^2] = 2/3$

$Var[X] = E[X^2] - E[X]^2 = 2/3$

$Var[Y] = 2/3$

$Cov(X, Y) = E[XY] - E[X]E[Y] = 0$

(a)

$\hat{Y}_L = \frac{\text{Cov}(X,Y)}{Var(X)}(X - E[X]) + E[X]$

$= 0$

(b)

$\hat{Y}(-1) = E[Y|X = -1] = -1 * 1/2 + 0 * 1/2 + 0$

$= -1/2$

$\hat{Y}(0) = E[Y|X = 0] = 0 + 0 + 1 * 1$

$= 1$

$\hat{Y}(1) = E[Y|X = 1] = -1 * 1/2 + 0 * 1/2 + 0$

$= -1/2$

(c)

MAP:

value of Y that maximizes: $P_Y(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$

$\hat{Y}(-1) = 0$ or -1

$\hat{Y}(-1) = 1$

$\hat{Y}(-1) = 0$ or -1

ML:

value of Y that maximizes: $P_X(X|Y) = \frac{P(Y|X)P(X)}{P(Y)}$

$\hat{Y}(-1) = 0$ or -1

$\hat{Y}(-1) = 1$

$\hat{Y}(-1) = 0$ or -1

(d)

Mean Square error of (a) = 1/3

Mean Square error of (b) = 1/6

Mean Square error of (c) (ML and MAP same) = 1/3

Q3

Question 3 (10 points)

Let U_1,U_2,U_3 be independent random variables with zero mean and variance 1. Find the linear MMSE estimator of S in terms of Z_1 and Z_2, and the corresponding MSE.

$$\begin{bmatrix} S \\ Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

Q4

Question 4 (10 points)

Let X be uniformly distributed in the interval (-1,1) and let $Y = X^3$. Find the best linear estimator for Y in terms of X. Compare its performance to the best estimator.

f(x) = 1/2 in interval (-1,1), 0 otherwise

$E[X] = \int_{-1}^{1} x * f(x)dx = \int_{-1}^{1} \frac{x}{2}dx = 0$

$E[Y] = E[X^3] = \int_{-1}^{1} \frac{x^3}{2} dx = 0$

$E[XY] = E[X * X^3] = \int_{-1}^{1} \frac{x^4}{2} dx = 0$

$Cov(X, Y) = E[XY] - E[X]E[Y] = 0$

The best linear estimator for Y is:

$\hat{Y} = \frac{\text{Cov}(X,Y)}{\text{Var}(X)} + E[Y]$

$= E[Y] = 0$

the mean square error is Var[Y]

$Var[Y] = E[Y^2] - E[Y]^2$

$E[Y^2] = E[X^6] = \int_{-1}^{1} \frac{x^6}{2} dx = 1/7$

$Var[Y] = 1/7$ which is the mean square error for the best linear estimator

The best estimator for Y is:

$E[Y|X = x] = E[X^3|X = x] = x^3$

with mean square error of 0, meaning that the best estimator performs better than best linear estimator.

Q5

Question 5 (15 points)

Suppose $X \sim U[1,2]$, and given $X = x$, $Y \sim Exp(\lambda)$ with parameter $\lambda = \frac{1}{x}$.

a) Find the linear MMSE estimate of X given Y.
b) Find the MSE of this estimator.
c) Check that $E[\tilde{X}Y] = 0. (\tilde{X} = X - \hat{X})$

$E[X] = \int_1^2 x * 1dx = 3/2$

$E[Y] = E[E[Y|X]] = E[X] = 3/2$

$E[Y^2] = E[E[Y^2|X]] = E[2X^2] = \int_1^2 2x^2 * 1dx = 14/3$

$Var[Y] = E[Y^2] - E[Y]^2 = 29/12$

$E[XY] = \int_1^2 x^2 * 1dx = 7/3$

$Cov(X, Y) = E[XY] - E[X]E[Y] = 1/12$

(a)

$\hat{X}_L = \frac{\text{Cov}(X,Y)}{\text{Var}(Y)}(Y - E[Y]) + E[X].$

$= \frac{1}{29}\left(Y - \frac{3}{2}\right) + \frac{3}{2}$

$= \frac{Y}{29} + \frac{42}{29}$

(b)

$\rho^2 = \frac{\text{Cov}^2(X,Y)}{\text{Var}(X)\text{Var}(Y)} = \frac{1}{29}$

$MSE = (1 - \rho^2)\text{Var}(X)$

$= (1-1/29)*1/12 $

$= \frac{7}{87}$

(c)

$\tilde{X} = X - \hat{X}_L$

$= X - \frac{Y}{29} - \frac{42}{29}$

$E[\tilde{X}Y] = E\left[\left(X - \frac{Y}{29} - \frac{42}{29}\right)Y\right]$
$= E[XY] - \frac{EY^2}{29} - \frac{42}{29}EY$
$= 0$

Q6

Question 6 (10 points)

Let X be an unobserved random variable with $E(X) = 0, Var(X) = 4$. Assume that we have observed Y_1 and Y_2 given by

$$Y_1 = X + W_1,$$

$$Y_2 = X + W_2,$$

where $E(W_1) = E(W_2) = 0, Var(W_1) = 1,$ and $Var(W_2) = 4$. Assume that $W_1, W_2, and\ X$ are independent random variables. Find the linear MMSE estimator of X, given $Y_1\ and\ Y_2$

$E[\tilde{X}] = -aE[Y_1] - bE[Y_2] - c = -a \cdot 0 - b \cdot 0 - c = -c.$

Since $E[X] = 0, c = 0$

$\text{Cov}(\hat{X}_L, Y_1) = \text{Cov}(aY_1 + bY_2, Y_1)$
$= a\text{Cov}(X + W_1, X + W_1) + b\text{Cov}(X + W_1, X + W_2)$
$= a(\text{Var}(X) + \text{Var}(W_1)) + b\text{Var}(X)$
$= 5a + 4b$

$\text{Cov}(\hat{X}_L, Y_2) = \text{Cov}(aY_1 + bY_2, Y_2)$
$= a\text{Var}(X) + b(\text{Var}(X) + \text{Var}(W_2))$
$= 4a + 8b$

Since $Cov(X, Y_1) = Cov(X, Y_2) = Var(X) = 4$:

$5a + 4b = 4$ and $4a + 8b = 4$, solving both equations give a = 2/3 and b = 1/6

The linear MMSE of X:

$\hat{X}_L = aY_1 + bY_2 + c$

$= \hat{X}_L = \frac{2}{3}Y_1 + \frac{1}{6}Y_2$

Q7

Question 7 (20 points)

Let X, Y, Z have joint pdf

$$f_{X,Y,Z}(x, y, z) = k(x + y + z) \quad for\ 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$$

a) Find k
b) Find the minimum mean square error linear estimator for Y given X and Z.
(Hint: given two observations, you can either calculate the derivative of MSE in terms of every coefficient or list the observations as a vector form.)
c) Find the minimum mean square estimator for Y given X and Z.
d) Find the MAP and ML estimators for Y given X and Z.

(a)

$1 = \int_0^1 \int_0^1 \int_0^1 k(x + y + z)dxdydz$

$1 = 1.5k$

$k = 2/3$

(b)

$f(x) = \frac{2}{3}(x + 1), f(x, y) = \frac{2}{3}(x + y + \frac{1}{2})$

$E[X] = E[Y] = E[Z] = \int_0^1 f(x) * x\ dx = \int_0^1 \frac{2}{3}x(x + 1)dx = \frac{5}{9}$

$E[XY] = \int_0^1 \frac{2}{3}(x + y + \frac{1}{2}) * x * y\ dxdy = \frac{11}{36}$

$Var[X] = Var[Y] = Var[Z] = E[X^2] - E[X] = \frac{13}{162}$

$Cov(X, Y) = Cov(Y, Z) = Cov(X, Z) = E[XY] - E[X]E[Y] = \frac{11}{36} - \frac{5}{9} * \frac{5}{9} = \frac{-1}{324}$

Linear Estimator:

$a = R_{XZ}^{-1} * E[XZ]$

$$(a_1, a_2)^T = \begin{pmatrix} Var(x) & Cov(X,Z) \\ Cov(X,Z) & Var(Z) \end{pmatrix}^{-1} \begin{pmatrix} Cov(Z,X \\ Cov(Y,Z) \end{pmatrix}$$

$$= \begin{pmatrix} 13/162 & -1/324 \\ -1/324 & 13/162 \end{pmatrix}^{-1} \begin{pmatrix} -1/324 \\ -1/324 \end{pmatrix}$$

$$= \begin{pmatrix} -27/705 \\ -27/705 \end{pmatrix}$$

$$\hat{Y} = (a_1, a_2) \begin{pmatrix} X - E[X] \\ Z - E[Z] \end{pmatrix} + E[Y]$$

$$= (-27/705, -27/705) \begin{pmatrix} X - 5/9 \\ Z - 5/9 \end{pmatrix} + 5/9$$

$= \frac{-27}{705}(x+z) + \frac{217}{423}$

Mean Square Error:

$$= Var[X] - a^T \begin{pmatrix} Cov(X,Y) \\ Cov(Y,Z) \end{pmatrix}$$

$= \frac{13}{162} - \frac{1}{1410}$

$= \frac{1514}{19035}$

(c)

$f(y|x,z) = \frac{(x+y+z)}{(x+y+0.5)}$, $0 < y < 1$

$E[Y|X,Z] = \int_0^1 \frac{(x+y+z)*y}{x+y+0.5} dy = \frac{0.5(x+z)+\frac{1}{3}}{x+y+0.5}$

MMSE Estimator for Y given X and Z:

$\hat{Y} = \frac{0.5(x+z)+\frac{1}{3}}{x+y+0.5}$

(d)

ML Estimator:

$\hat{Y} = 1$ if $x + z < 1$

$\hat{Y} = 0$ if $x + z > 1$