Homework #1

ELEC/IEDA3180 - Data Driven Portfolio Optimization Spring 2022/2023

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The HKUST Academic Honor Code applies. This assignment is to be done individually. Cheating **won't** be tolerated. Total marks: 100.

Problem 1

In Modern Portfolio Theory, one is interested in allocating a certain amount of money into a set of N stocks. The portfolio $\mathbf{w} \in \mathbb{R}^N$ is the normalized money invested in each stock such that the sum of this N dimensional vector is 1.

This allocation process is formulated as the following optimization problem:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \\ \text{subject to} & \boldsymbol{\mu}^\top \mathbf{w} \geq \beta \\ & \mathbf{w} \geq \mathbf{0} \\ & \mathbf{w}^\top \mathbf{1} = 1, \end{array}$$

where $\boldsymbol{\mu}$ is the given vector of expected stock returns, and $\boldsymbol{\Sigma}$ is the given covariance matrix of the stock returns.

In the above formulation, the objective $\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}$ is portfolio variance, and the first component of the constraints $\mathbf{w}^\top \boldsymbol{\mu}$ is portfolio expected return, and β is the parameter that controls the lower bound of expected return. Intuitively, our goal is to minimize the risk we take under the certain level of money we earn. The solution to the above problem, $\mathbf{w}^\star$, is often called the "Markowitz portfolio" or the "Mean-variance portfolio."

Question (1)

Argue whether this problem is convex or not.

Question (2)

Write a piece of code in `Python` and use CVXPY package[1] to solve the above problem given the following values of
$\beta = \{0, 0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05, 0.055, 0.06$
with the following values of $\mathbf{\Sigma}$ and $\boldsymbol{\mu}$:

$$\mathbf{\Sigma} = \begin{bmatrix} 1.0 & -0.003 & 0.02 \\ -0.003 & 1.0 & -0.15 \\ 0.02 & -0.15 & 1.0 \end{bmatrix}$$
$$\boldsymbol{\mu} = [0.02, 0.003, 0.07]^\top.$$

For each solution $\mathbf{w}_\beta^\star$, **compute** the following quantities

$$\mathsf{PortfolioExpectedReturn}(\beta) = {\mathbf{w}_\beta^\star}^\top \boldsymbol{\mu}$$

$$\mathsf{PortfolioVolatility}(\beta) = \sqrt{{\mathbf{w}_\beta^\star}^\top \mathbf{\Sigma} \mathbf{w}_\beta^\star}$$

Create a curve plot $\mathsf{PortfolioExpectedReturn}$ versus $\mathsf{PortfolioVolatility}$.

Question (3)

If $\boldsymbol{\mu}^\top \mathbf{w}_\beta^\star = \beta$, we say the inequality constraint $\boldsymbol{\mu}^\top \mathbf{w} \geq \beta$ is active at $\mathbf{w}_\beta^\star$. Otherwise, we say it is inactive.

Create a curve plot $\boldsymbol{\mu}^\top \mathbf{w}_\beta^\star$ versus β.

Find for which β the inequality constraint $\boldsymbol{\mu}^\top \mathbf{w} \geq \beta$ is active.

Explain what will happen if $\beta \geq 0.07$.

Hint: Consider the constraint $\boldsymbol{\mu}^\top \mathbf{w} \geq \beta$.

Question (4)

Evaluating a set of portfolios with different β is necessary to choose the optimal hyperparameter (also called hyperparameter tuning). There are two main methods to conduct this repetitive evaluation. Method 1: Reconstruct the entire problem and plug in a new β repeatedly, i.e., each problem is independent. Method 2: Use the CVXPY parameters function and only define the problem once[2].

Please finish the following tasks:

Write code of both methods for Question (2) (you might have already implemented one). **Print** the $\mathbf{w}_\beta^*$ with $\beta = 0.01$ of both methods and ensure that they are the same.

Compare their speeds: How many times faster is the second method than the first method in this example?

Question (5)

Assume the constraint $\boldsymbol{\mu}^\top \mathbf{w} \geq \beta$ changes to $\boldsymbol{\mu}^\top \mathbf{w} = \beta$. Then the allocation process is formulated as the following optimization problem:

$$\begin{array}{ll} \underset{\mathbf{w}}{\text{minimize}} & \mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w} \\ \text{subject to} & \boldsymbol{\mu}^\top \mathbf{w} = \beta \\ & \mathbf{w} \geq \mathbf{0} \\ & \mathbf{w}^\top \mathbf{1} = 1, \end{array}$$

Please solve the above problem given the following values of $\beta = \{0.005, 0.01, 0.015, 0.02, 0.025, 0.03, 0.035, 0.04, 0.045, 0.05, 0.055, 0.06\}$ ($\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ remain the same as in Question (2)).

Then finish the following tasks:

Create a curve plot $\text{PortfolioExpectedReturn}$ versus $\text{PortfolioVolatility}$.

Explain how and why it is different from the curve in Question (2).

Observation #1: Make sure to write your code in a modular, readable way. Code organization will be taken into account for the grading. Use meaningful names for variables, create functions to organize your code as much as possible. In case of doubt on coding best practices, take a look at the Google's style guide. Note that you don't have to strictly follow that style, you can develop your own, but make sure it is understandable and you use it consistently.

Observation #2: Submit your code via canvas in zipped file (`.zip`) containing a Jupyter notebook (`.ipynb`) and also its exported version in the HTML (`.html`) format.

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1. If you are still confused about the proper use of CVXPY, check the official document.↩

2. See parameters for details.↩