Re: Finite Groups With A Prescribed Number Of Cyclic Subgroups

Jamie Chavez Malacara

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1 Outline

After reading Finite Groups With A Prescribed Number Of Cyclic Subgroups (I & II) by Richard Belshoff, Joe Dillstrom and Les Reid I was intrigued to explore the topic further. I decided to begin by considering a slightly different problem: instead of looking at the difference between the order of the group and the number of cyclic subgroups I decided to look at groups where the number of cyclic subgroups is a divisor of the order of the group (ie $|C(G)| = \frac{|G|}{k}$) beyond what's described on the papers (the case $k = 1$). Using GAP I found the groups with order between 1 and 100 with this property, I outline the results below (A:B denotes split extension of A by B, A.B extension of A by B). At a first glance I wasn't able to spot any obvious patterns but I haven't ruled out the possibility that I will upon further inspection.

Afterwards I decided to look at groups where the number of cyclic subgroups is a factor of Euler's totient function for the order of G (ie $|C(G)| = l\phi(|G|)$). Using GAP I found the groups with order between 1 and 100 with this property, I outline the results below for $l = 1$ and $l = 2$ (A:B denotes split extension of A by B, A.B extension of A by B). From a first glance at the results I came up with two conjectures (below), though I haven't had the chance to prove them yet (and hence they might not be true)

Conjecture 1 $|C(G)| = 2\phi(|G|)$ *if*

- $G = C_2^n$ *for* $n = 1, 2, ...$ *(equivalent to* $\Delta(G) = 0$ *case in paper 1)*

- $G = C6 \times C_2^n$ *for* $n = 0, 1, ...$
- $G = A4 \times C_2^n$ *for* $n = 0, 1, ...$
(Note - this is not an exhaustive list of all possible cases as evidenced by the results below - G being one of those groups means $|C(G)| = 2\phi(|G|)$ *but the reverse is not true)*

Attempt at proving conjecture 1:

- Each member of C_2^n generates a different cyclic subgroup so there are $2^n = 2\phi(2^n)$ cyclic subgroups.
- Let $m = |C_6 \times C_2^n| = 3 \times 2^{n+1}$. $|C(C_6 \times C_2^n)| = |C(C_3) \times C(C_2^{n+1})| = |C(C_3)||C(C_2^{n+1})| = 2\phi(2^{n+1}) = 2\phi(2^{n+1})\phi(3) = 2\phi(2^{n+1} \times 3) = 2\phi(m)$
- Note that $A_4 \times C_2^n$ has 1 element of order 1, $2^{n+2} - 1$ elements of order 2, $\frac{4*3*2}{3} = 8$ elements of order 3 and $8(2^n - 1)$ elements of order 6. Hence the number of cyclic subgroups is $1 + 2^{n+2} + 8 + 8(2^n - 1) = 1 + 2^{n+2} + 8 + 8 * 2^n - 8 = 4 * 2^n + 8 * 2^n = 12 * 2^n = 2\phi(|A_4|)$

Conjecture 2 *Let* $|G| = n$. *If* $|C(G)| = 2\phi(n)$ *then* $G = C_2$, $4|n$ *or* $3|n$

Finally to ensure I was fully familiar with the notation and methodology used in paper one (in preparation for attempting to prove the conjectures given at the start) I decided to try to create tables for $\Delta = 6$

2 Computational results Part 1

2.1 $k = 2$

- C8 [8,1]
- C12 [12,2]
- C8 x C2 [16,5]
- C8 : C2 [16,6]
- Q16 [16,9]
- C3 : Q8 [24,4]

- C12 x C2 [24,9]
- (C8 x C2) : C2 [32,5]
- Q8 : C4 [32,10]
- C8 : C4 [32,13]
- C8 : C4 [32,14]
- C8 x C2 x C2 [32,36]
- C2 x (C8 : C2) [32,37]
- (C8 x C2) : C2 [32,38]
- C2 x Q16 [32,41]
- C4 x (C3 : C4) [48,11]
- (C3 : C4) : C4 [48,12]
- C12 : C4 [48,13]
- C3 x ((C4 x C2: C2) [48,21]
- C3 x D16 [48,25]
- C4 x A4 [48,31]
- ((C4 x C2) : C2) : C3 [48,33]
- C2 x (C3 : Q8) [48,34]
- C12 x C2 x C2 [48,44]
- C3 x ((C4 x C2) : C2) [48,47]
- ((C8 x C2) : C2) : C2 [64,4]
- (C4 : C4) : C4 [64,9]
- (C4 x C4) : C4 [64,18]
- (C4 : C4) : C4 [64,20]

- (C4 : C4) : C4 [64,21]
- C2 x ((C8 x C2) : C2) [64,87]
- (C2 x (C8 : C2)) : C2 [64,88]
- (C8 x C2 x C2) : C2 [64, 89]
- (C2 x (C8 : C2)) : C2 [64,94]
- C2 x (Q8 : C4) [64,96]
- (Q8 : C4) : C2 [64,100]
- C2 x (C8 : C4) [64,106]
- C2 x (C8 : C4) [64,107]
- (C8 : C4) : C2 [64,108]
- (C8 : C4) : C2 [64,109]
- (C2 x Q16) : C2 [64,132]
- (Q8 : C4) : C2 [64, 148]
- (Q8 : C4) : C2 [64, 151]
- ((C8 : C2) : C2) : C2 [64,152]
- (Q8 : C4) : C2 [64,164]
- (Q8 : C4) : C2 [64,165]
- (Q8 : C4) : C2 [64,166]
- C8 x C2 x C2 x C2 [64,246]
- C2 x C2 x (C8 : C2) [64,247]
- C2 x ((C8 x C2) : C2) [64,248]
- (C2 x (C8 : C2)) : C2 [64,249]
- C2 x C2 x Q16 [64,252]

- C4 x D18 [72,5]
- C3 : ((C4 x C2) : C4) [96,38]
- C3 : ((C4 x C4) : C2) [96,84]
- C3 : ((C2 x Q8) : C2) [96,85]
- C3 : ((C4 x C4) : C2) [96,86]
- C3 : ((C8 x C2) : C2) [96,119]
- C2 x ((C3 : C4) : C4) [96, 130]
- C3 : ((C2 x Q8) : C2) [96,131]
- C2 x (C12 : C4) [96,132]
- C3 : ((C4 x C4) : C2) [96,133]
- C3 : ((C8 x C2) : C2 [96,157]
- C6 x ((C4 x C2) : C2) [96,162]
- C3 x ((C4 x C2 x C2) : C2) [96,168]
- C6 x D16 [96,179]
- (((C4 x C2) : C2) : C3) : C2 [96,192]
- C2 x C4 x A4 [96,196]
- C2 x (((C4 x C2) : C2) : C3) [96,200]
- C2 x C2 x (C3 : Q8) [96,205]
- C12 x C2 x C2 x C2 [96,220]
- C6 x ((C4 x C2) : C2) [96,223]

2.2 $k = 3$

- C9 [9,1]
- C24 [24,2]
- C5 x S3 [30,1]
- (C2 x C2) : C9 [36,3]
- C18 x C2 [36,5]
- C24 x C2 [48,23]
- C3 x (C8 : C2) [48,24]
- C3 x Q16 [48,27]
- C10 x S3 [60,11]
- C2 x ((C2 x C2) : C9) [72,16]
- C18 x C2 x C2 [72,18]
- C4 x (C3 : C8) [96,9]
- C3 : (C8 : C4) [96,10]
- C3 : (C4 : C8) [96,11]
- C3 : (C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2) [96,43]
- C3 x ((C8 x C2) : C2 [96,48]
- C3 x (Q8 : C4) [96,53]
- C3 x (C8 : C4) [96,56]
- C3 x (C8 : C4) [96,57]
- C8 x A4 [96,73]
- ((C8 x C2) : C2) : C3 [96,74]
- C24 x C2 x C2 [96,176]

- C6 x (C8 : C2) [96,177]
- C3 x ((C8 x C2) : C2) [96,180]

2.3 $k = 4$

- C36 [36,2]
- C5 x Q8 [40,11]
- C7 x D8 [56,9]
- C9 : C8 [72,1]
- Q8 : C9 [72,3]
- C36 x C2 [72,9]
- C20 x C4 [80,20]
- C5 x (C4 : C4) [80,22]
- C5 x (C8 : C2) [80,24]
- C3 x (C4 . D8 = C4 . (C4 x C2)) [96,58]

2.4 $k = 5$

- C40 [40,2]
- C60 [60,4]
- C7 x D10 [70,1]
- C80 [80,2]
- C40 x C2 [80,23]
- C5 x (C8 : C2) [80,24]
- C5 x Q16 [80,27]

2.5 $k = 6$

- C72 [72,2]
- C7 x (C3 : C4) [84,3]

2.6 $k = 7$

- C56 [56,2]
- C84 [84,6]

2.7 $k = 8$

- C80 [80,2]
- C96 [96,2]

2.8 $k = 11$

- C88 [88,2]

3 Computational Results Part 2

3.1 $l = 1$

- 1 [1,1]
- C3 [3,1]
- C8 [8,1]
- C10 [10,2]
- C8 x C2 [16,5]
- C8 : C2 [16,6]
- Q16 [16,9]
- C18 [18,2]

- C10 x C2 [20,5]
- C24 [24,2]
- (C8 x C2) : C2 [32,5]
- Q8 : C4 [32,10]
- C8 : C4 [32,13]
- C8 : C4 [32,14]
- C8 x C2 x C2 [32,36]
- C2 x (C8 : C2) [32,37]
- (C8 x C2) : C2 [32,38]
- C2 x Q16 [32,41]
- (C2 x C2) : C9 [36,3]
- C18 x C2 [36,5]
- C5 : Q8 [40,4]
- C10 x C2 x C2 [40,14]
- C24 x C2 [48,23]
- C3 x (C8 : C2) [48,24]
- C5 x A4 [60,9]
- C30 x C2 [60,13]
- ((C8 x C2) : C2) : C2 [64,4]
- (C4 : C4) : C4 [64,9]
- (C4 x C4) : C4 [64,18]
- (C4 : C4) : C4 [64,20]
- (C4 : C4) : C4 [64,21]

- C2 x ((C8 x C2) : C2) [64,87]
- (C2 x (C8 : C2)) : C2 [64,88]
- (C8 x C2 x C2) : C2 [64, 89]
- (C2 x (C8 : C2)) : C2 [64,94]
- C2 x (Q8 : C4) [64,96]
- (Q8 : C4) : C2 [64,100]
- C2 x (C8 : C4) [64,106]
- C2 x (C8 : C4) [64,107]
- (C8 : C4) : C2 [64,108]
- (C8 : C4) : C2 [64,109]
- (C2 x Q16) : C2 [64,132]
- (Q8 : C4) : C2 [64, 148]
- (Q8 : C4) : C2 [64, 151]
- ((C8 : C2) : C2) : C2 [64,152]
- (Q8 : C4) : C2 [64,164]
- (Q8 : C4) : C2 [64,165]
- (Q8 : C4) : C2 [64,166]
- C8 x C2 x C2 x C2 [64,246]
- C2 x C2 x (C8 : C2) [64,247]
- C2 x ((C8 x C2) : C2) [64,248]
- (C2 x (C8 : C2)) : C2 [64,249]
- C2 x C2 x Q16 [64,252]
- (C3 x C3) : C8 [72,13]

- C2 x ((C2 x C2) : C9) [72,16]
- C18 x C2 x C2 [72,18]
- C2 x (C5 : Q8) [80,35]
- (C2 x C2 x C2 x C2) : C5 [80,49]
- C10 x C2 x C2 x C2 [80,52]
- C5 x D18 [90,1]
- C3 : ((C4 x C2) : C4) [96,38]
- C3 : ((C4 x C4) : C2) [96,84]
- C3 : ((C2 x Q8) : C2) [96,85]
- C3 : ((C4 x C4) : C2) [96,86]
- C3 : ((C8 x C2) : C2) [96,119]
- C2 x ((C3 : C4) : C4) [96, 130]
- C3 : ((C2 x Q8) : C2) [96,131]
- C2 x (C12 : C4) [96,132]
- C3 : ((C4 x C4) : C2) [96,133]
- C3 : ((C8 x C2) : C2 [96,157]
- C6 x ((C4 x C2) : C2) [96,162]
- C3 x ((C4 x C2 x C2) : C2) [96,168]
- C6 x D16 [96,179]
- (((C4 x C2) : C2) : C3) : C2 [96,192]
- C2 x C4 x A4 [96,196]
- C2 x (((C4 x C2) : C2) : C3) [96,200]
- C2 x C2 x (C3 : Q8) [96,205]

- C12 x C2 x C2 x C2 [96,220]
- C6 x ((C4 x C2) : C2) [96,223]

3.2 $l = 2$

- C2 [2,1]
- C2 x C2 [4,2]
- C6 [6,2]
- C2 x C2 x C2 [8,5]
- A4 [12,3]
- C6 x C2 [12,5]
- C2 x C2 x C2 x C2 [16,14]
- D18 [18,1]
- C2 x A4 [24,13]
- C6 x C2 x C2 [24,15]
- C2 x C2 x C2 x C2 x C2 [32,51]
- D36 [36,4]
- D48 [48,7]
- C2 x C2 x A4 [48,49]
- (c2 x C2 x C2 x C2) : C3 [48,50]
- (C2 x C2 x C2 x C2) : C3 [48,52]
- A5 [60,5]
- C2 x C2 x C2 x C2 x C2 x C2 [64,267]
- C2 x C2 x D18 [72,17]

- C2 x D48 [96,110]
- C3 : ((C2 x C2 x C2 x C2) : C2) [96,144]
- C3 : ((C4 x C4) : C2) [96,147]
- C3 : ((C2 x C2 x C2) : (C2 x C2)) [96,211]
- C2 x C2 x C2 x A4 [96,228]
- C2 x ((C2 x C2 x C2 x C2) : C3) [96,229]
- C6 x C2 x C2 x C2 x C2 [96,231]

4 $\Delta = 6$ tables

Partition		$\sigma(G)$
6	$6 \cdot 1$	(3,3,3,3,3,3), (4,4,4,4,4,4), (6,6,6,6,6,6)
5+1	$1 \cdot 5{+}1$	(7,3), (7,4), (7,6), (9,3), (9,4), (9,6), (14,3),(14,4),(14,6),(18,3),(18,4),(18,6)
	$5 \cdot 1{+}1$	(3,3,3,3,3,4), (3,3,3,3,3,6), (4,4,4,4,4,3), (4,4,4,4,4,6), (6,6,6,6,6,3), (6,6,6,6,6,4)
4+2		(3,3,3,3,4,4), (3,3,3,3,6,6), (4,4,4,4,3,3), (4,4,4,4,6,6), (6,6,6,6,3,3), (6,6,6,6,4,4)
3+3	$3 \cdot 1 + 3 \cdot 1$	(3,3,3,4,4,4), (3,3,3,6,6,6), (4,4,4,6,6,6)
	$1 \cdot 3 + 3 \cdot 1$	(5,3,3,3), (5,4,4,4), (5,6,6,6), (8,3,3,3), (8,4,4,4), (8,6,6,6), (10,3,3,3), (10,4,4,4), (10,6,6,6), (12,3,3,3), (12,4,4,4), (12,6,6,6)
	$1 \cdot 3 + 1 \cdot 3$	(5,8), (5,10), (5,12), (8,10), (8,12)
4+1+1		(3,3,3,3,4), (3,3,3,3,6), (4,4,4,4,3), (4,4,4,4,6), (6,6,6,6,3), (6,6,6,6,4)
3+2+1	$1 \cdot 3 + 2 + 1$	(5,3,3,4), (5,3,3,6), (5,4,4,3), (5,4,4,6), (5,6,6,3), (5,6,6,4), (8,3,3,4), (8,3,3,6), (8,4,4,3), (8,4,4,6), (8,6,6,3), (8,6,6,4), (10,3,3,4), (10,3,3,6), (10,4,4,3), (10,4,4,6), (10,6,6,3), (10,6,6,4), (12,3,3,4), (12,3,3,6), (12,4,4,3), (12,4,4,6), (12,6,6,3), (12,6,6,4)
	$3 \cdot 1 + 2 + 1$	(3,3,3,4,4,6), (3,3,3,6,6,4), (4,4,4,3,3,6), (4,4,4,6,6,3), (6,6,6,3,3,4), (6,6,6,4,4,3)
2+2+2		(3,3,4,4,6,6)
3+1+1+1		(5,3,4,6), (8,3,4,6), (10,3,4,6), (12,3,4,6)
2+2+1+1		none
2+1+1+1+1		none
1+1+1+1+1+1		none

$\sigma(G)$	Excluded by	$\sigma(G)$	Excluded by
(3,3,3,3,3,3)	Sylow	(3,4,6,10)	No C5
(6,6,6,6,6,6)	No C3	(3,4,6,8)	Prop 3
(3,7)	Prop 3	(3,4,5,6)	Prop 3
(4,7)	Prop 3	(3,3,4,4,6,6)	Sylow
(6,7)	Prop 3	(3,4,4,6,6,6)	Prop 3
(4,9)	Prop 3	(3,3,4,6,6,6)	Sylow
(6,9)	No C3	(3,4,4,4,6,6)	Prop 3
(3,14)	Prop 3	(3,3,4,4,4,6)	Sylow
(4,14)	No C7	(3,3,3,4,6,6)	Sylow
(6,14)	No C3	(3,3,3,4,4,6)	Sylow
(3,18)	No C6	(4,6,6,12)	No C3
(4,18)	No C3	(3,6,6,12)	No C4
(6,18)	No C3	(4,4,6,12)	No C3
(3,3,3,3,3,4)	Sylow	(3,4,4,12)	No C6
(3,3,3,3,3,6)	Sylow	(3,3,6,12)	No C4
(3,4,4,4,4,4)	Prop 3	(3,3,4,12)	Sylow
(4,4,4,4,4,6)	No C3	(4,6,6,10)	No C5
(4,6,6,6,6,6)	No C3	(3,6,6,10)	No C5
(3,3,3,3,4,4)	Prop 3	(4,4,6,10)	No C5
(3,3,4,4,4,4)	Sylow	(3,4,4,10)	No C5
(4,4,4,4,6,6)	No C3	(3,3,6,10)	Sylow
(3,3,6,6,6,6)	Sylow	(3,3,4,10)	Sylow
(4,4,6,6,6,6)	No C3	(4,6,6,8)	No C3
(3,3,3,4,4,4)	Sylow	(3,6,6,8)	Prop 3
(3,3,3,6,6,6)	Sylow	(4,4,6,8)	No C3
(4,4,4,6,6,6)	No C3	(3,4,4,8)	Prop 3
(3,3,3,5)	Sylow	(3,3,6,8)	Sylow
(4,4,4,5)	Prop 3	(3,3,4,8)	Sylow
(5,6,6,6)	No C3	(4,5,6,6)	No C3
(3,3,3,8)	Sylow	(3,5,6,6)	Prop 3
(6,6,6,8)	No C3	(4,4,5,6)	No C3
(3,3,3,10)	Sylow	(3,4,4,5)	Prop 3
(4,4,4,10)	No C5	(3,3,5,6)	Prop 3
(6,6,6,10)	No C3	(3,3,4,5)	Sylow
(3,3,3,12)	No C4	(4,6,6,6,6)	No C3
(4,4,4,12)	No C3	(4,4,4,4,6)	No C3
(6,6,6,12)	No C3	(3,4,4,4,4)	Prop 3
(5,8)	Prop 3	(3,3,3,3,4)	Prop 3
(5,12)	Prop 3	(8,12)	No C4
(3,3,3,3,6)	Lemma 9	(3,6,6,6,6)	Lemma 6
(8,10)	No C4		

Partition	$\sigma(G)$	Groups
6	(4,4,4,4,4,4)	C4 x C4, C2 x Q8, C4 : C4
5+1	(3,9)	C9, D18
3+3	(5,10)	C10, D20
	(4,4,4,8)	QD16, (C2 x D8) : C2
3+1+1+1	(3,4,6,12)	C12, D24

I am still missing (3,6,6,6,6,6) and (3,3,3,3,3,6) on the table as well as (C2 x C2 x C2) : C2, (C4 x C4) : C2 and (C2 x D8) : C2.