The number of cyclic subgroups of a group: a brief introduction

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Group theory fundamentals

What is a group?

A group is a set G combined with an operation $\circ$ with the following properties:

- G is closed under $\circ$ ie if g_1 and g_2 are in G then so is $g_1 \circ g_2$.
- * is associative: for all $g_1, g_2, g_3 \in G, g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$.
- G has an identity element e such that for all $g \in G, g \circ e = e \circ g = g$.
- G is closed under inverses: for all $g \in G$ there is some $g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$.

H is a subgroup of G if it a subset of G that forms a group under the same operation as G.

A few special kinds of groups

- A group is abelian if for all $g_1, g_2 \in G, g_1 \circ g_2 = g_2 = g_1$.
- A group is cyclic if $G = \{g^n | n \in \mathbb{N}\}$ This is denoted as $\langle g \rangle$ and we say that G is the group generated by g. The cyclic group of order n is denoted C_n.
- A group G is simple if G $\neq \{e\}$ and its only subgroups are $\{e\}$ and G.

Other Relevant Definitions

- The order of a group G is the number of elements in a group, which will be denoted as $|G|$.
- The order of an element g is the smallest natural number such that $g^n = e$ and will be denoted as $|g|$.
- The cross product of the groups G and H denoted $G \times H := \{(g, h) | g \in G, h \in H\}$.

A few examples

Permutation Groups

Permutation groups, denoted S_n are the groups of possible permutations of the numbers $\{1, .., n\}$. The order of S_n is $n!$. Its elements can be represented with "cycle notation" as follows: (a) is a 1-cycle, where a is sent to a (fixed), 1-cycles are often omitted. (a b) is a 2-cycle in which a is sent to b and b is sent to a. (a b c) is a 3 cycle in which a is sent to b, b is sent to c and c is sent to a. The group operation is composition.

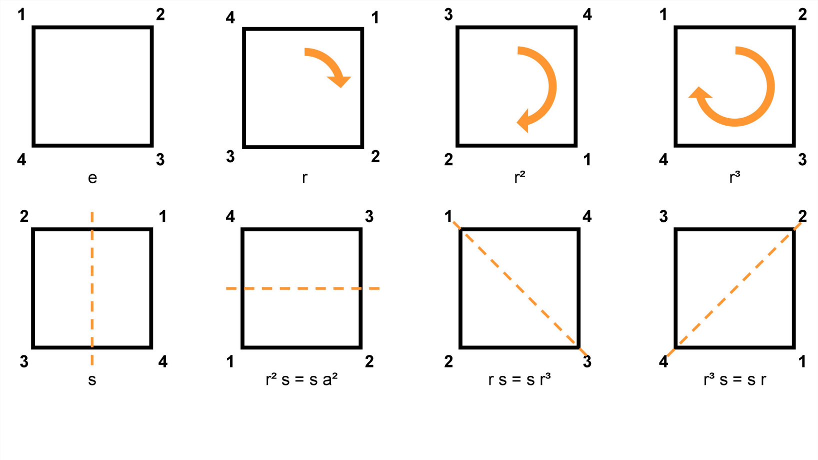
S_3: the group of permutations of $\{1, 2, 3\}$

The order of S_3 is $3! = 6$. It has the following elements: {(1), (1 2), (1 3), (2 3), (1 2 3), (1 3 2)} where (1) is the identity element. The order of the elements of S_n is the least common multiple of the length of the cycles hence the elements have order 1, 2, 2, 2, 3, 3 respectively. Remember the definition of a cyclic group: $G = \{g^n | n \in \mathbb{N}\}$ so each element generates a cyclic group that has the same order as the element itself. The cyclic subgroups of S_3 are therefore: {(1)}, {(1),(1 2)}, {(1), (1 3)}, {(1), (2 3)} and {(1), (1 2 3), (1 3 2) } $= \langle(123)\rangle = \langle(132)\rangle$

Dihedral Groups

Dihedral groups, denoted D_{2n} are the groups of symmetries of a regular n-gon. D_{2n} has order $2n$ and can be generated by the following relations: $\langle r, s | r^n = s^2 = e, rs = sr^{-1} \rangle$. All of the powers of r represent rotations and the elements containing an s represent reflections. The group operation is composition. From the last relationship it can be seen that D_{2n} is not abelian in general.

D_8: the group of symmetries of the square



The cyclic subgroups of D_8 are $\{e\}, \langle r\rangle, \langle s\rangle, \langle rs\rangle, \langle r^2 s\rangle, \langle r^3 s\rangle, \langle r^2\rangle$

Classifying groups by their number of cyclic subgroups

A few more definitions and basic results

- We define $\alpha(G)$ to be the number of cyclic subgroups of G divided by the order of G.
- $\alpha(G) = \alpha(G \times C_2^n)$
- for all finite G, $0 < \alpha(G) \leq 1$
- the order of $(g, h) \in G \times H$ is the least common multiple of the order of g in G and the order of h in H.

Some groups with $\alpha > 3/4$

- As we saw before the cyclic subgroups of S_3 are {(1)}, {(1),(1 2)}, {(1), (1 3)}, {(1), (2 3)} and {(1), (1 2 3), (1 3 2) } $= \langle(123)\rangle = \langle(132)\rangle$ so as $|S_3| = 6$ it follows that $\alpha(S_3) = 5/6$. all groups of the form $S_3 \times C_2^n$ for $n \geq 0$ have $\alpha = 5/6$.
- The cyclic subgroups of D_8 are $\{e\}, \langle r\rangle, \langle s\rangle, \langle rs\rangle, \langle r^2s\rangle, \langle r^3s\rangle, \langle r^2\rangle$ so $\alpha(D_8) = 7/8$ and all groups of the form $D_8 \times C_2^n$ for $n \geq 0$ have $\alpha = 7/8$.
- if $\alpha(G) = 1$ then G is C_2^n for $n \geq 1$.

Other groups with known α

- Let $C_3 = \{1, x, x^2\}$. Then the cyclic subgroups of C_3 are $\{1\}$ and $C_3 = \langle x \rangle = \langle x^2 \rangle$. All groups of the form $C_3 \times C_2^n$ for $n \geq 0$ have $\alpha = 2/3$.
- Up to isomorphism all finite abelian 2 groups with $\alpha = 1/2$ are of the form $C_8 \times C_2^n$ for $n \in \mathbb{N}$.
- $\alpha(S_4) = 17/24$. All groups of the form $S_4 \times C_2^n$ for $n \geq 0$ have $\alpha = 17/24$.
- $\alpha(S_5) = 67/120$. All groups of the form $S_5 \times C_2^n$ for $n \geq 0$ have $\alpha = 67/120$.

Using GAP to check α

We can use the following code to check that $\alpha(S_5) = 67/120$

```
G := SmallGroup(120,34);
s := AllSubgroups(G);
sc := Filtered(s, g->IsCyclic(g)=true);
alpha := Size(sc)/Order(G);
```

The same code can be used to check $\alpha(S_4) = 17/24$ by using the right group ID, namely SmallGroup(24,12), in place of SmallGroup(120,34).

Our project

More terminology

- An involution is an element of order 2
- Nilpotency is a generalization of the concept of abelian groups.
- In what follows $\tau(n)$ denotes the number of divisors of n, including 1 and n. For example, the divisors of 2 are {1,2} so $\tau(2) = 2$; the divisors of 4 are {1,2,4} so $\tau(4) = 3$; the divisors of 6 are {1,2,3,6} so $\tau(6) = 4$.

Existing work

- All groups with $\alpha > 3/4$ have been classified by Garonzi and Lima (2018), using results from a paper titled ”On groups consisting mostly of involutions” by Wall (1970).
- A partial classification of nilpotent groups with $\alpha = 3/4$ was published by Tarnauceanu and Lazorec (2018).

Therefore we tried to find a complete classification of groups with $\alpha = 3/4$, starting with a computational analysis using GAP. What follows are the conjectures we formed based on the results.

Lemma: The only dihedral groups with $\alpha = 3/4$ are D_{16} and D_{24}

The number of cyclic subgroups of the dihedral group D_{2n} is $\tau(n) + n$. Therefore $\alpha = \frac{3}{4} \Rightarrow \frac{\tau(n)+n}{2n} = \frac{3}{4} \Rightarrow \tau(n) = \frac{n}{2}$. We know that n and $\frac{n}{2}$ are divisors of n (the latter as n is even). The next largest possible divisor is $\frac{n}{3}$ so we can bound $\tau(n)$ by $\frac{n}{3} + 2$ (with equality when every number between 1 and $\frac{n}{3}$ divides n). Hence $\tau(n) = \frac{n}{2} \leq \frac{n}{3} + 2 \Rightarrow 3n \leq 2n + 12 \Rightarrow n \leq 12$ so all possible values of n are: 2, 4, 6, 8 or 12. As $\tau(2) = 2, \tau(4) = 3, \tau(6) = 4, \tau(8) = 4, \tau(10) = 4, \tau(12) = 6$; it follows that $\tau(n) = \frac{n}{2} \Rightarrow n \in \{8, 12\}$

Note - $\alpha(D_{16}) = 3/4$ was a known result.

Conjectures

- Up to isomorphism the only non-nilpotent groups with $\alpha = 3/4$ are $D_{24} \times C_2^n$ for $n \geq 0$.
- If G is nilpotent, has $\alpha = 3/4$ and contains elements of order 8 it is of the form $D_{16} \times C_2^n$ for $n \geq 0$.
- If G is nilpotent, has $\alpha = 3/4$ and all its elements have order less than 8 then it belongs to one of the families of groups with $|G|/2 - 1$ involutions classified by Miller (1919).