MATH60029 Functional Analysis Coursework 1

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1 Problem set I, exercise I.1.4

It is not a linear space. The function $f(z) = -z^2 - 2z$ is analytic (it's a polynomial) and is also a solution to the differential equation since

$$f'(z) = -2z - 2 \tag{1.1}$$
$$f''(z) = -2 \tag{1.2}$$

But since we are over $\mathbb{R}$ or $\mathbb{C}$, we can multiply by the scalar 2. However, $2f(z)$ is not a solution to the differential equation, hence it is not closed under scalar multiplication, and is not a Linear space.

2 Problem set II, exercise II.1(i.a)

Firstly, we need $s \leq 1$ since otherwise the triangle inequality is violated.

$$\rho_s(0,1) + \rho_s(1,2) = 2 \tag{2.1}$$
$$\rho_s(0,2) = 2^s \tag{2.2}$$

For $0 \leq s \leq 1$, ρ_s is a metric. It is clearly symmetric and positive definite. For the triangle inequality, note that x^s is both monotone increasing and concave, so that

$$|x-z| \leq |x-y| + |y-z| \tag{2.3}$$
$$\implies |x-y|^s \leq (|x-y| + |y-z|)^s \leq |x-y|^s + |y-z|^s \tag{2.4}$$

hence, ρ_s is a metric.

We now investigate whether addition is continuous with respect to the product topology on $\mathbb{R} \times \mathbb{R}$. Note that ρ_s is a translation invariant metric *i.e.* $\rho_s(x+z, y+z) = \rho_s(x,y)$.

Fix $\varepsilon > 0$. Choose $\delta_1 = \frac{\varepsilon}{2}$ and $\delta_2 = \frac{\varepsilon}{2}$. Then, for any x_1, x_2, y_1, y_2 such that we have

$$\rho_s(x_1, x_2) < \delta_1 \qquad \rho_s(y_1, y_2) < \delta_2 \tag{2.5}$$

then

$$\rho_s(x_1 + y_1, x_2 + y_2) = \rho_s(x_1 - x_2, y_2 - y_1) \tag{2.6}$$
$$\leq \rho_s(x_1 - x_2, 0) + \rho_s(0, y_2 - y_1) = \rho_s(x_1, x_2) + \rho_s(y_1, y_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{2.7}$$

Thus, addition is continuous.

Now we turn to scalar multiplication. Under $s = 0$, it is not continuous since if we pick $\lambda_n = \frac{1}{n}$ and any non-zero element $x \in V$, then $(\lambda_n, x) \to (0, x)$.

However $\rho_0(\lambda_n x, 0) = 1$ for any n does not go to zero, hence we lose continuity.

If $s > 0$, then scalar multiplication is continuous. For any point $(\lambda, x) \in \mathbb{R} \times \mathbb{R}$, we fix $\varepsilon > 0$. Choose $\delta_1 = \min\left(\frac{\varepsilon}{2|\lambda|^s}, 1\right)$ and $\delta_2 = \left(\frac{\varepsilon}{2}\frac{1}{|x|^s+1}\right)^{\frac{1}{s}}$. Then for any $\sigma \in \mathbb{R}$ and $y \in \mathbb{R}$ such that

$$\rho_s(x,y) = |x-y|^s < \delta_1 \qquad |\lambda - \sigma| < \delta_2 \tag{2.8}$$

we have

$$\begin{aligned}
\rho_s(\lambda x, \sigma y) &\le \rho_s(\lambda x, \lambda y) + \rho_s(\lambda y, \sigma y) && (2.9)\\
&= |\lambda x - \lambda y|^s + |\lambda y - \sigma y|^s && (2.10)\\
&= |\lambda|^s |x-y|^s + |\lambda - \sigma|^s |y|^s && (2.11)\\
&< |\lambda|^s \delta_1 + \delta_2^s(|x|^s + |y-x|^s) && (2.12)\\
&< |\lambda|^s \delta_1 + \delta_2^s(|x|^s + \delta_1) && (2.13)\\
&\le |\lambda|^s \frac{\varepsilon}{2|\lambda|^s} + \frac{\varepsilon}{2}\left(\frac{1}{|x|^s+1}\right)(|x|^s + \delta_1) \le \varepsilon && (2.14)
\end{aligned}$$

Hence, scalar multiplication is continuous.

Thus, $V, \mathbb{R}$ is a linear space when $s \in (0,1]$.

3 Problem set III.5

3.1 (iii)

We use x_j^n to denote the jth element of the nth term of the sequence. Let $A = \{x \in \ell_p : x_j = 0\}$ for some fixed $j \in \mathbb{N}$ and let x^n be a sequence in A that converges to $x \in \ell_p$.

Suppose $x_j \neq 0$. Then

$$\begin{aligned}
\|x^n - x\|_{\ell_p}^p = \sum_{i=0}^{\infty} |x_i^n - x_i|^p &\ge |x_j^n - x_j|^p && (3.1)\\
&= |x_j|^p > 0 && (3.2)
\end{aligned}$$

Hence, $\|x^n - x\|_{\ell_p}$ does not go to 0, a contradiction. Therefore, $x_j = 0$ and A is closed.

We now show there is no open ball around any point $x \in A$. Consider the sequence x^n defined by

$$x_i^n = x_i + \begin{cases} \frac{1}{n} & i = j \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

Then no element of x^n is in A, and x^n converges to x. To see this last part, note that

$$\|x^n - x\|_{\ell_p} = \frac{1}{n} \to 0 \tag{3.4}$$

Thus, there is no open ball around x since we can approximate x arbitrarily accurately with elements not in A.

3.2 (iv)

Let $B = \{x \in \ell_p : \forall j \in \mathbb{N}, |x_j| \le Cj^{-2/p}\}$ for some $C > 0$. Again, let x^n be a sequence in B that converges to $x \in \ell_p$. Suppose for contradiction, there exists j such that $|x_j| = a > Cj^{-2/p}$.

Then, we have

$$\|x^n - x\|_{\ell_p}^p = \sum_{i=0}^{\infty} |x_i^n - x_i|^p \ge |x_j^n - x_j|^p \ge \big||x_j^n| - a\big|^p \ge |a - Cj^{-2/p}|^p > 0 \tag{3.5}$$

Hence, $\|x^n - x\|_{\ell_p}$ remains bounded away from 0 contradicting the convergence. Thus, $x \in B$ and B is closed.