MATH60031 Markov Processes

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1 Basics

Definition 1.1. A probability space is a triple $(\Omega, \mathcal{F}, \mathbb{P})$ consisting of a state space Ω, a σ-algebra $\mathcal{F}$ and a probability measure $\mathbb{P}$. A state space $\mathcal{X}$ is assumed to be a complete seperable metric space. Let $\mathcal{B}(\mathcal{X})$ be the Borel σ-algebra.

Definition 1.2. Given the above setup, a Stochastic Process is a collection of random variables/measurable functions from $(\Omega, \mathcal{F})$ to $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$. The indexing of the random variables is a totally ordered set usually.

1.1 Information and Filtrations

Definition 1.3. The information on the stochastic process at time n is the σ-algebra of all possible events at this time.

$$\sigma(X_n) = \sigma\left(\left\{X_n^{-1}(A) : A \in \mathcal{B}(\mathcal{X})\right\}\right) \tag{1.1}$$

This can be thought of as all the information we can find about Ω given X_n. Think about what would happen if X_n was constant: the σ algebra is trivial and you have no information about it.

Definition 1.4. $\sigma(X_0, X_1, \dots, X_n)$ is the smallest σ-algebra such that each random variable is measurable. This can be thought of as the information up to time n.

Definition 1.5. A family of σ-algebras $\{\mathcal{F}_n\}$ satisfying $\mathcal{F}_n \subset \mathcal{F}_m$ whenever $n < m$ is a filtration. This makes sure that we can't see extra information in the future when we haven't reached it yet.

- A stochastic process X_n is said to be adapted to a filtration $\mathcal{F}_n$ if for all n, X_n is measurable with respect to $\mathcal{F}_n$.
- The filtration $\mathcal{F}_n^X = \sigma(X_0, X_1, \dots, X_n)$ is the smallest filtration, and is called the natural filtration.