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Computing the eigenvalue distribution of sums of random matrices

Part III Seminar Series

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Motivation

Given independent real valued random variables X and Y, what can we say about the distribution of $X + Y$?

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Large? Independent? Eigenvalue distribution?

Outline

- Free probability spaces
- Free convolutions and Cauchy transforms
- Finding roots of analytic functions and Cauchy transforms
- Recovering measures and orthogonal polynomials
- Numerical examples

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Free probability spaces

Classical probability space: $(\Omega, \mathcal{F}, \mathbb{P})$, random variables $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which are measurable, expectation $\mathbb{E}(\cdot)\dots$

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Definition (Free probability space)

A *non-commutative* probability space is a pair $(\mathcal{A}, \phi)$ where

- $\mathcal{A}$ is a $*$-algebra of elements called random variables
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Examples: Space of scalar random variables, space of deterministic matrices, space of random matrices...

Free independence

Classical independence: X and Y independent if for all polynomials p,q such that $\mathbb{E}(p(X)) = \mathbb{E}(q(Y)) = 0$, then

$$\mathbb{E}(p(X)q(Y)) = 0$$

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Definition (Free independence)

Random variables $a_1, a_2, \cdots a_n$ are said to be freely independent if for any m polynomials $p_1, p_2, \cdots p_m$ with $m \geq 2$, we have

$$\phi(p_1(a_{i_1})p_2(a_{i_2}) \cdots p_m(a_{i_m})) = 0$$

whenever $i_1 \neq i_2, i_2 \neq i_3, \cdots i_{m-1} \neq i_m$ and $\phi(p_j(a_{i_j})) = 0$ for all j.

Interpretation: *The alternating product of centered random variables is also centered.*

Free independence and random matrices

For an $n \times n$ Hermitian random matrix A, we define

$$\phi(A) = \frac{1}{n}\mathbb{E}(\mathrm{Tr}(A))$$

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- Gaussian Ensemble is asymptotically free with any other random matrix.
- Given any pair of random matrices A and B, the matrix QAQ^* is asymptotically free with B, where Q is a Haar-distributed unitary matrix.

Cauchy transform

Let μ be a (non-zero) finite measure with support Γ, which we assume is compact for simplicity.

$$\text{Cauchy (Stieltjes) transform } G_\mu(z) = \int_\Gamma \frac{1}{z-x}\,\mathrm{d}\mu$$

- G_μ is analytic on $\mathbb{C} \setminus \Gamma$ and at ∞.

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- G_μ is analytic on $\mathbb{C} \setminus \Gamma$ and at ∞.
- $G_\mu(z) = \frac{m_0}{z} + \frac{m_1}{z^2} + \frac{m_2}{z^3} + \cdots$ as $z \to \infty$, where

$$m_n = \int_\Gamma x^n\,\mathrm{d}\mu$$

Hence G_μ is univalent near ∞.

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  $$m_n = \int_\Gamma x^n \,\mathrm{d}\mu$$

  Hence G_μ is univalent near ∞.
- **Plemelj's lemma**: Suppose μ admits a density $\rho(x)$ which is Hölder continuous. Denote $G_\mu^\pm(x) = \lim_{\varepsilon\downarrow 0} G_\mu(x \pm i\varepsilon)$ for $x \in \Gamma$. Then we have

  $$G^+(x) + G^-(x) = 2 \fint_\Gamma \frac{\rho(u)}{x-u}\,\mathrm{d}u$$
  $$G^+(x) - G^-(x) = -2\pi i \rho(x)$$

Additive Free Convolution

$$R\text{ - transform } R_\mu(z) \text{ satisfies } G_\mu\left(R_\mu(z) + \frac{1}{z}\right) = z$$

Define the distribution of a random variable as a probability measure μ_a with moments $\phi(a^n)$.

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Theorem (Voiculescu, 1986)

Let μ_a and μ_b be the distributions of freely independent random variables a and b, the there is a unique probability measure $\mu_b \boxplus \mu_b$ satisfying

$$R_{\mu_a \boxplus \mu_b}(z) = R_{\mu_a}(z) + R_{\mu_b}(z)$$

$\mu_a \boxplus \mu_b$ is the distribution of $a + b$ when a and b are freely independent.

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Interpretation: Eigenvalue distribution of $A + B$ is given by the free convolution of the eigenvalue distributions of A and B, if the matrices are asymptotically freely independent.

Additive Free Convolution

Theorem

Given $y \in \mathbb{C} \setminus \mathbb{R}$, the following are equivalent for all $z \in \mathbb{C} \setminus \mathbb{R}$:

- $$G_{\mu_a \boxplus \mu_b}(z) = y$$
- $\mathrm{sgn}(\mathrm{Im}(z)) \neq \mathrm{sgn}(\mathrm{Im}(y))$ and there exist z_1 and z_2 such that $z_1 + z_2 - \frac{1}{y} = z$ and
$$G_{\mu_a}(z_1) = G_{\mu_b}(z_2) = y$$

Finding all solutions to $G_{\mu_a}(z) = y$ and $G_{\mu_b}(z) = y$ allows us to find all solutions to $G_{\mu_a \boxplus \mu_b}(z) = y$.

Inverses of Cauchy transforms

Computing free convolution motivates finding inverses of G_μ

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Problem

Given a probability measure μ supported on Γ and its associated Cauchy transform G_μ, for any $y \in \mathbb{C}$, find **all** $z \in \mathbb{C} \setminus \Gamma$ such that $G_\mu(z) = y$.

$$G_\mu(z) = \int_\Gamma \frac{1}{z - x} \, \mathrm{d}\mu$$

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All inverses important! Needs upper bound of some kind!

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Bounding the number of inverses

Given μ and the associated Cauchy transform G_μ, upper bounds on the number of solutions are necessary for stopping criteria for root-finding algorithms.

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Theorem

Let $\rho(x)$ be a density function such that $\mathrm{d}\mu = \rho(x)\,\mathrm{d}x$, with compact support and $\rho(x) \in W^{1,p}(\mathbb{R})$ for some $p \in (1, \infty)$
Then the number of solutions to $G_\mu(z) = y$ is bounded above by

$$N = \frac{1}{2} \sup_{v>0} C\left(\{x \in \mathbb{R} : \rho(x) = v\}\right)$$

where $C : \mathcal{P}(\mathbb{R}) \to \mathbb{N} \cup \{\infty\}$

$$C(A) = \text{The number of connected components in the set } A$$

Bounding the number of inverses

Sketch of proof: Define the image of $\Gamma = \mathrm{supp}(\mu)$ under G by the union of the upper and lower limits $G^+(\Gamma)$ and $G^-(\Gamma)$.

- $G(\Gamma)$ is a bounded continuous curve.

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- Plemelj's lemma: parametrise $G(\Gamma)$ into real and imaginary parts, which are the Hilbert transform up to some constant scaling factor and the density $\frac{\rho(x)}{\pi}$.
- The curve $G(\Gamma)$ must cross the ray $\{y + x : x \in (-\infty, 0)\}$ at least N times. Similar for $\{y + x : x \in (0, \infty)\}$.

Example: Jacobi measure

Absolutely continuous measure $\mathrm{d}\mu = \rho(x)\,\mathrm{d}x$ where

$$\rho(x) = \frac{15}{32}(1 + 7x^2)(1 - x^2)^2 \mathbf{1}_{[-1,1]}$$

Consider points $y_1 = G(1 + 2i)$ and $y_2 = G(0.14 + 0.06i)$. $G(z) = y_1$ has one solution, whilst $G(z) = y_2$ has two solutions.

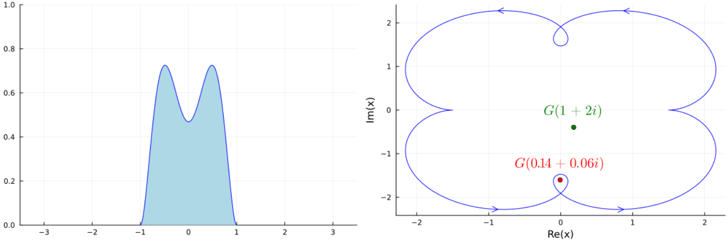

Figure 1: Left: density of μ with Jacobi weights. Right: image of G_μ^+ and G_μ^-.

Computing all zeros of analytic functions

Consider an open set $\Omega \subset \mathbb{C}$ and an analytic function $f : \Omega \to \mathbb{C}$.

Goal: find all the roots of f inside a given closed contour γ, denoted by z_i.

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Define the moments s_n as follows.

$$s_n = \frac{1}{2\pi i} \oint_\gamma z^n \frac{f'(z)}{f(z)} \, \mathrm{d}z = \sum_{z_i} z_i^n v_i$$

where in the last equality the generalised argument principle was used (v_i denotes multiplicity).

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Theorem (Kravanja, Sakurai, Van Barel, 1999)

Suppose there are N roots of f. Then every root $z_i, i = 1, \ldots, N$ is an eigenvalue of the pencil $H_1 - \lambda H_0$ with Hankel matrices

$$H_0 = [s_{k+l}]_{k,l=0}^{N-1} \qquad H_1 = [s_{k+l+1}]_{k,l=0}^{N-1}$$

Application to Cauchy transforms

Assumption: Γ is compact and is a finite union of disjoint closed intervals $[A_j^-, A_j^+], j = 1, \ldots, N$.

Given a $y \in \mathbb{C}$, consider $f : \mathbb{D} \to \mathbb{C}$ where $f(z) = G(\zeta(z)) - y$ where ζ is a suitably chosen conformal mapping.

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Given a $y \in \mathbb{C}$, consider $f : \mathbb{D} \to \mathbb{C}$ where $f(z) = G(\zeta(z)) - y$ where ζ is a suitably chosen conformal mapping.
Define the following maps:

$$J : \mathbb{D} \to \mathbb{C} \setminus [-1, 1] \qquad J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{(Joukowski map)}$$

$$M_{a,b}(z) = \frac{b+a}{2} + \frac{b-a}{2} z \qquad \text{(Affine map from } [-1,1] \text{ to } [a,b])$$

Conformal maps

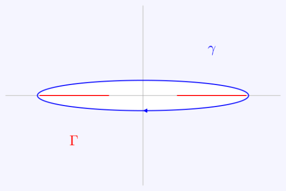

Figure 2: Red: support of μ. Blue: region where inverses can be recovered.

Locating zeros on $\mathbb{C} \setminus I$ where $I = [a, b] = [\min(\Gamma), \max(\Gamma)]$, the convex hull of Γ

Choose $\zeta(z) = M_{a,b}(J(z))$.

Then the region where inverses can be recovered is the outside of an ellipse whose size depends on r.

As $r \to 1$, the ellipse shrinks towards I. However, G_μ generally has singularities on Γ so the size of the annulus on which the integrand of A_n is analytic decreases, slowing convergence.

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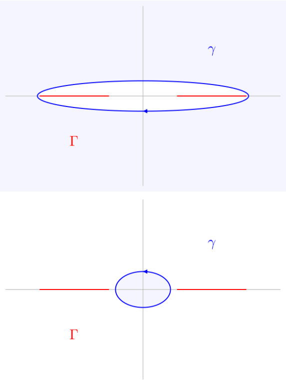

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Locating zeros on $(c, d) \subset I \setminus \Gamma$

Choose $\zeta(z) = M_{c,d}(z)$

Recovery of measures: Regularity

We can now compute all solutions of $G_{\mu_a \boxplus \mu_b}(z) = y$ for fixed y. How to recover?

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Theorem (Bao, Erdös, Schnelli, 2018)

Let μ_a and μ_b be measures supported on single intervals with power-law behaviour at the boundary of the supports, such that all exponents are in $(-1, 1)$. Then $\mu_a \boxplus \mu_b$ is a measure supported on a single interval with square-root behaviour at the boundary.

Recovery of measures: Orthogonal polynomials

Let μ have density $\rho(x)$ and support on a compact set Γ. Assume that $\rho(x) = r(x)w(x)$, where $r(x)$ is a non-negative function and $w(x)$ is the weight of some sequence of orthogonal polynomials $p_n(x)$.

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q_n satisfy a similar three-term recurrence relation to the orthogonal polynomials.

$$\begin{aligned} zq_0(z) &= a_0 q_0(z) + b_0 q_1(z) + \int_\Gamma p_0(x)w(x)\,\mathrm{d}x \\ zq_n(z) &= c_{n-1}q_{n-1}(z) + a_n q_n(z) + b_n q_{n+1}(z), \qquad n \geq 1, \end{aligned}$$

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$$r(x) = \sum_{n=0}^{\infty} \phi_n p_n(x) \implies G_\mu(z) = \sum_{n=0}^{\infty} \phi_n q_n(z)$$

Recovery of measures: Least squares

Assuming that we know the support of $\mu = \mu_a \boxplus \mu_b$, we consider the Chebyshev-U expansion of G_μ as follows, for the single interval $[a, b]$ case:

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$$\rho(x) = r(x)\frac{2\sqrt{b-x}\sqrt{x-a}}{b-a}\mathbf{1}_{[a,b]}$$

$$G_\mu(z) = \sum_{n=0}^{\infty} \phi_n q_n(z) = \pi \sum_{n=0}^{\infty} \phi_n J_+^{-1}(M_{ab}^{-1}(z))^{n+1}$$

Here, $J_+^{-1}(z) = z - \sqrt{z-1}\sqrt{1+z}$ denotes an inverse of the Joukowski transform sending $\mathbb{C} \setminus [-1, 1]$ to the unit disk, with $J_+^{-1}(\infty) = 0$.

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Recovery of measures

We recover the first n coefficients of ϕ by solving a least-squares system $G_\mu(G_\mu^{-1}(y)) = y$, in the single interval case. Given that we can calculate $G_\mu^{-1}(y)$, we consider pairs (z_m, y_m) where $G_\mu(z_m) = y_m$.

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$$\pi \sum_{k=1}^{n} \phi_{k-1} \mathrm{Re}\left(J_+^{-1}(M_{ab}^{-1}(z_m))^k\right) = \mathrm{Re}\,(y_m)$$

$$\pi \sum_{k=1}^{n} \phi_{k-1} \mathrm{Im}\left(J_+^{-1}(M_{ab}^{-1}(z_m))^k\right) = \mathrm{Im}(y_m)$$

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Hence, we recover the coefficients of the density of $\mu_{a\boxplus b}$!

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Conclusion

- Describe algorithms to compute free convolutions for probability measures and use them in practice.
- Further reading:
    - *Free Probability and Random Matrices*, Mingo and Speicher.
    - *Computing the Zeros of Analytic Functions*, Kravanja and Barel
- Repository for Julia code is on Github at: `https://github.com/dlfivefifty/NumericalFreeProbability.jl`
- Special thanks to Sheehan Olver.

Thank you!