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Methods for Computing Free Convolutions

Imperial UCL Numerical Analysis Seminar

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Outline

- Motivation
- Preliminaries in Free Probability
- Inverses of Cauchy transforms
- Recovery of measures
- Computational Examples
- Conclusion

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Motivation: Operations on Random Matrices

Given large hermitian random matrices A and B, what can we say about the eigenvalue statistics of $A + B$?

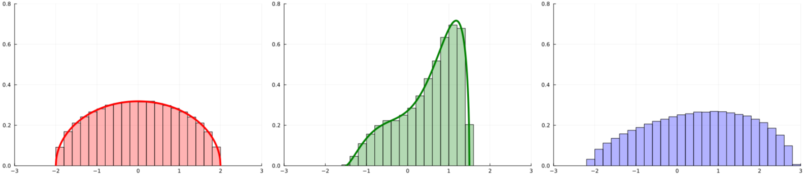

Figure 1: Histogram of eigenvalues of two 4000×4000 matrices A and B, along with the eigenvalues of their sum.

Applicable to many fields with high dimensional statistics *e.g.* signal processing, machine learning. . . Similar questions about multiplication, matrix inverses, polynomials of random matrices. . .

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Eigenvalue Statistics

Definition (Empirical Spectral Distribution, ESD)

For a random $n \times n$ matrix A_n, the **empirical spectral distribution** is defined as

$$\mu_{A_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$$

where λ_i are the eigenvalues of A_n, counting multiplicity.
Thus, the ESD is the counting measure of the eigenvalues of A_n, and the ESD is a random probability measure.

We also say that μ_{A_n} converges almost surely/in probability to a deterministic probability measure μ_A if the random variable $\int_{\mathbb{R}} \phi(x)\mathrm{d}\mu_{A_n}$ converges almost surely/in probability to $\int_{\mathbb{R}} \phi(x)\mathrm{d}\mu_A$ for every test function $\phi \in C_c^\infty(\mathbb{R})$.

In particular, many random matrices have a limiting eigenvalue distribution as the matrix size n goes to infinity.

Moments of random matrices

With the ESD, we may define the moments as

$$\mathbb{E}[\int_R x^k \mathrm{d}\mu_{A_n}] = \frac{1}{n}\mathbb{E}[\mathrm{Tr}A_n^k]$$

Intuition: Trace of A^k is equal to $\lambda_1^k + \cdots + \lambda_n^k$, behaves like the moments of the average eigenvalue.

Thus, we may rephrase the distribution of $A + B$ as purely an operation on probability measures. One can also phrase this problem in terms of the moments of A and B:

Problem

Given $n \times n$ matrices A_n and B_n with limiting eigenvalue distribution μ_A and μ_B, find the limiting eigenvalue distribution of $A_n + B_n$.

Given the limiting moments of A_n and B_n as $n \to \infty$, find the moments of $A_n + B_n$ as $n \to \infty$.

Free probability and independence

Definition (Free probability space)

A *non-commutative* probability space is a pair $(\mathcal{A}, \phi)$ where

- $\mathcal{A}$ is a C^* algebra of elements called random variables
- ϕ is the "expectation", which is linear and normalised to $\phi(1) = 1$

Examples: Space of scalar random variables, space of deterministic matrices, space of random matrices. . .

Definition (Free independence of two random variables)

Random variables a_1 and a_2 are said to be freely independent if for any m polynomials $p_1, p_2, \cdots p_m$ with $m \geq 2$, we have

$$\phi(p_1(a_{i_1})p_2(a_{i_2}) \cdots p_m(a_{i_m})) = 0$$

whenever $\phi(p_j(a_{i_j})) = 0$ and neighbouring indices i_j and i_{j+1} are not equal.

Interpretation: *The alternating product of centered random variables is also centered.*

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Free independence and random matrices

For random matrices, we define $\phi(A)$ as $\frac{1}{n}\mathbb{E}(\mathrm{Tr}(A))$ (Expected value of average eigenvalue)

Many classes of random matrices are *asymptotically free i.e.* Freely independent as $n \to \infty$, where n is the matrix size.

- Gaussian Ensemble is asymptotically free with any other random matrix
- Given any pair of random matrices A and B, the matrix QAQ^* is asymptotically free with B, where Q is a Haar-distributed unitary matrix.

In general, if the matrix entries are classically independent, then the random matrices *tend* to be asymptotically freely independent.

Cauchy transform

Let μ be a finite measure with support Γ, which we assume is compact for simplicity.

$$\text{Cauchy (Stieltjes) transform } G_\mu(z) = \int_\Gamma \frac{1}{z-x}\mathrm{d}\mu$$

- G_μ is analytic on $\mathbb{C} \setminus \Gamma$ and at ∞.
- $G_\mu(z) = \frac{m_0}{z} + \frac{m_1}{z^2} + \frac{m_2}{z^3} + \cdots$ as $z \to \infty$, where

  $$m_n = \int_\Gamma x^n \mathrm{d}\mu$$

  G_μ is univalent (injective) near ∞. In addition, if μ is a probability measure, then $G_\mu(z) = \frac{1}{z} + O(\frac{1}{z^2})$.
- **Plemelj's lemma**: Suppose μ admits a density $\rho(x)$ which is Hölder continuous. Denote $G_\mu^\pm(x) = \lim_{\varepsilon\downarrow 0} G_\mu(x \pm i\varepsilon)$ for $x \in \Gamma$. Then we have

  $$2\mathrm{Re}(G^+(x)) = G^+(x) + G^-(x) = 2\fint_\Gamma \frac{\rho(u)}{x-u}\mathrm{d}u$$
  $$-2\mathrm{Im}(G^+(x))i = G^+(x) - G^-(x) = -2\pi i\rho(x)$$

Additive free convolution

$$\textcolor{green}{R \text{ - transform}}\ R_\mu(z) \text{ satisfies } G_\mu\left(R_\mu(z) + \frac{1}{z}\right) = z$$

- R_μ is analytic on some region of $\mathbb{C}$ and if Γ is compact, then R_μ is analytic on a disc centered around 0.
- Near zero, $R_\mu(z) = G_\mu^{-1}(z) - \frac{1}{z}$.

Theorem (Voiculescu, 1986)

Given two probability measures μ_a and μ_b, there is a probability measure $\mu_{a \boxplus b}$ satisfying

$$R_{\mu_a \boxplus \mu_b}(z) = R_{\mu_a}(z) + R_{\mu_b}(z)$$

The measure $\mu_a \boxplus \mu_b$ is the distribution of $a + b$ when a and b are freely independent.

Interpretation: Eigenvalue distribution of $A + B$ is given by the free convolution of the eigenvalue distributions of A and B, if the matrices are asymptotically freely independent.

The R - transform is analogous to the logarithm of the characteristic function for classical probability theory.

Additive Free Convolution

Rearranging the equation for the R - transform yields the following relation in a neighbourhood of 0:

Corollary (Voiculescu)

Given probability measures μ_a, μ_b and their additive convolution $\mu_{a \boxplus b}$ with their Cauchy transforms defined in a neighbourhood of infinity, we have the following in a neighbourhood of zero.

$$G^{-1}_{\mu_a \boxplus \mu_b}(y) = G^{-1}_{\mu_a}(y) + G^{-1}_{\mu_b}(y) - \frac{1}{y}$$

Moreover, given $y \in G_{\mu_a}(\mathbb{C}) \cap G_{\mu_b}(\mathbb{C})$, the following are equivalent:

- $\mathrm{sgn}(\mathrm{Im}(G^{-1}_{\mu_a \boxplus \mu_b}(y))) \neq \mathrm{sgn}(\mathrm{Im}(y))$
- $y \in G_{\mu_a \boxplus \mu_b}(\mathbb{C})$

Remark: forms the basis of free convolution algorithms in (Nadakuditi, Olver, 2013).

Additive Free Convolution

Theorem

Given $y \in \mathbb{C} \setminus \mathbb{R}$, the following are equivalent for all $z \in \mathbb{C} \setminus \mathbb{R}$:

- $$G_{\mu_a \boxplus \mu_b}(z) = y$$
- $\mathrm{sgn}(\mathrm{Im}(z)) \neq \mathrm{sgn}(\mathrm{Im}(y))$ and there exist z_1 and z_2 such that $z_1 + z_2 - \frac{1}{y} = z$ and
$$G_{\mu_a}(z_1) = G_{\mu_b}(z_2) = y$$

Proof follows from existence and uniqueness of subordination functions (Belinschi, Bercovici, 2007).

Finding all solutions to $G_{\mu_a}(z) = y$ and $G_{\mu_b}(z) = y$ allows us to find all solutions to $G_{\mu_a \boxplus \mu_b}(z) = y$, from which the density can be recovered from.

Valid for all probability measures! In particular, no univalency assumption is required for G_{μ_a} or G_{μ_b}.

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Existing algorithms for numerical evaluation of free convolutions

Sheehan Olver and Raj Rao Nadakuditi. *Numerical computation of convolutions in free probability theory.* 2013. arXiv 2305.01819 [math.PR].
The methods described form the basis of our algorithm, which is a generalisation to a wider class of probability measures.

Alice Cortinovis and Lexing Ying. *Computing Free Convolutions via Contour Integrals.* 2023. arXiv: 2305.01819 [math.NA].
Recent development on computing convolutions via evaluation of the R - transform using Cauchy's theorem.

N. Raj Rao and Alan Edelman. *The polynomial method for random matrices* 2007. In: Foundations of Computational Mathematics 8.6 (Dec. 2008), pp. 649–702
Computing free convolutions of measures which have an algebraic Cauchy transform.

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Inverses of Cauchy transforms

Reformulation of free convolution in terms of only the Cauchy transform motivates the following root finding problem.

Problem

Given a probability measure μ supported on Γ and its associated Cauchy transform G_μ, for any $y \in \mathbb{C}$, find **ALL** $z \in \mathbb{C} \setminus \Gamma$ such that $G_\mu(z) = y$.

Specific cases:

- If μ is a discrete measure with finitely many atoms, then the inverse problem reduces to finding the roots of a polynomial.
- If μ is a measure supported on a single interval $[a, b]$ with **square-root** behaviour at the boundary *i.e.* $\rho(x) = r(x)\frac{2\sqrt{b-x}\sqrt{x-a}}{b-a}$ where $r(x)$ is a $C^1([a, b])$ function with derivative that is of bounded variation, then by expanding $r(x)$ in Chebyshev-U polynomials, the inverse problem also reduces to finding the roots of a polynomial (Nadakuditi, Olver, 2013).

Kravanja–Sakurai–Van Barel method for computing zeros of analytic functions

Consider an open set $\Omega \subset \mathbb{C}$ and an analytic function $f : \Omega \to \mathbb{C}$. We wish to find all the roots of f inside a given closed contour γ, denoted by z_i.

Define the moments A_n as follows.

$$A_n = \frac{1}{2\pi i} \oint_\gamma z^n \frac{f'(z)}{f(z)} \mathrm{d}z = \sum_{z_i} z_i^n v_i$$

where in the last equality the generalised argument principle was used (v_i denotes multiplicity).

Theorem (Kravanja, Sakurai, Van Barel, 1999)

Suppose there are N roots of f. Then every root $z_i, i = 1, \dots, N$ is an eigenvalue of the pencil $H_1 - \lambda H_0$ with Hankel matrices

$$H_1 = [A_{k+l}]_{k,l=0}^{N-1} \qquad H_0 = [A_{k+l+1}]_{k,l=0}^{N-1}$$

One may replace the logarithmic derivative $\frac{f'(z)}{f(z)}$ with $\frac{1}{f(z)}$ or replace the standard monomial basis with other polynomials with similar results.

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Kravanja–Sakurai–Van Barel method for computing zeros of analytic functions

Closed contour integrals are numerically evaluated using the trapezium rule, which exhibits exponential convergence with analytic functions.

Theorem (Kravanja, Sakurai, Van Barel, 2003)

Suppose we have an analytic function $f(z)$ on the unit disk. Let H_0, H_1 be the $N \times N$ Hankel matrices from before, and H_0^*, H_1^* be the same matrices obtained via quadrature. Then the quadrature error with K sampling points in the moments A_i decays exponentially

$$H_1^* - \lambda H_0^* = H_1^{P_N} - \lambda H_0^{P_N} + O(\rho^{2N-K})$$

where $\rho > 1$ denotes the largest region $|z| < \rho$ such that $f(z)$ is analytic on and $H_1^{P_N} - \lambda H_0^{P_N}$ denotes a matrix pencil which has the same eigenvalues as $H_1 - \lambda H_0$. Furthermore, the computed roots converge exponentially to the true roots to the same degree.

Application of the Kravanja–Sakurai–Van Barel method

Assumption: Γ is compact and is a finite union of disjoint closed intervals $[A_j^-, A_j^+], j = 1, \ldots, N$. No assumptions on the Lebesgue decomposition of μ are made.

Given a $y \in \mathbb{C}$, consider $f : \mathbb{D} \to \mathbb{C}$ where $f(z) = G(\zeta(z)) - y$ where ζ is a suitably chosen conformal mapping.
Define the following maps:

$$J : \mathbb{D} \to \mathbb{C} \setminus [-1, 1] \qquad J(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \text{(Joukowski map)}$$

$$M_{a,b}(z) = \frac{b+a}{2} + \frac{b-a}{2} z \qquad \text{(Affine map from } [-1,1] \text{ to } [a,b])$$

Application of the Kravanja–Sakurai–Van Barel method

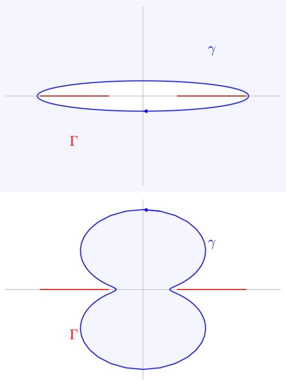

Figure 2: Red: support of μ. Blue: region where inverses can be recovered.

Locating zeros on $\mathbb{C} \setminus I$ where $I = [a, b] = [\min(\Gamma), \max(\Gamma)]$, the convex hull of Γ

Choose $\zeta(z) = M_{a,b}(J(z))$.

Then the region where inverses can be recovered is the outside of an ellipse whose size depends on r.

As $r \to 1$, the ellipse shrinks towards I. However, G_μ generally has singularities on Γ so the size of the annulus on which the integrand of A_n is analytic decreases, slowing convergence.

Locating zeros on $(c, d) \subset I \setminus \Gamma$

Choose $\zeta(z) = M_{c,d}(\frac{1}{J(z)})$

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Bounding the number of inverses

Given μ and its associated Cauchy transform, upper bounds on the number of solutions are necessary for stopping criteria for root-finding algorithms.

Theorem

Let $\rho(x)$ be a density function such that $\mathrm{d}\mu = \rho(x)\mathrm{d}x$, with support on a finite union of intervals $[A_i^-, A_i^+]$ and power law behaviour

$$\rho(x) = \phi_i(x)(x - A_i^-)^{\alpha_i}(A_i^+ - x)^{\beta_i}$$

on the interval $[A_i^-, A_i^+]$, where $\alpha_i, \beta_i > 0$ and $\phi_i(x)$ is Hölder continuous.
Then the number of solutions to $G(z) = y$ is bounded above by half the number of solutions to $\rho(x) = \frac{|\mathrm{Im}(y)|}{\pi}$, rounded down.

Bounding the number of inverses

Sketch of proof: Define the image of $\Gamma = \mathrm{supp}(\mu)$ under G by the union of the upper and lower limits $G^{+}(\Gamma)$ and $G^{-}(\Gamma)$.

- $G(\Gamma)$ is a bounded continuous curve due to the non-negative power law behaviour of $\rho(x)$.
- For y not on $G(\Gamma)$, the number of solutions to $G(z) = y$ is given by the winding number of $G(\Gamma)$ around y due to the argument principle and boundedness of the curve. Denote the number of solutions as N.
- Plemelj's lemma: parametrise $G(\Gamma)$ into real and imaginary parts, which are the Hilbert transform up to some constant scaling factor and the density $\frac{\rho(x)}{\pi}$.
- By considering the logarithm with the branch cut at $(\infty, 0)$, one can show that $G(\Gamma)$ must cross the ray $\{y + x : x \in (-\infty, 0)\}$ at least N times. Similar for $\{y + x : x \in (0, \infty)\}$.
- For y on $G(\Gamma)$, the inverse function theorem yields the same bound.

Remark: We expect similar bounds to hold for arbitrary Jacobi measures and even mixtures of pure points and absolutely continuous measures. However, the image of the support is in general not bounded.

Example 1: Jacobi measure

Absolutely continuous measure $\mathrm{d}\mu = \rho(x)\mathrm{d}x$ where

$$\rho(x) = \frac{15}{32}(1 + 7x^2)(1 - x^2)^2 \mathbf{1}_{[-1,1]}$$

Consider points $y_1 = G(1 + 2i)$ and $y_2 = G(0.14 + 0.06i)$. $G(z) = y_1$ has one solution, whilst $G(z) = y_2$ has two solutions.

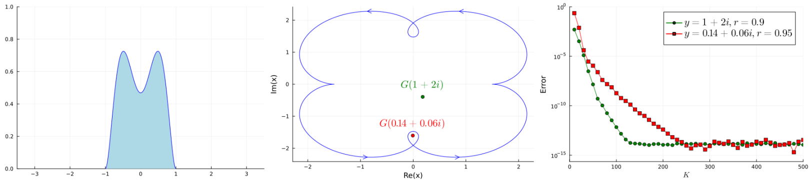


Figure 3: Left: density of μ with Jacobi weights. Middle: image of G_μ. Right: Error of computed inverses plotted against number of quadrature points. Error is measured $\max_i |G(z_i) - y|$ where z_i are the computed roots.

Example 2: Chebyshev supported on disjoint intervals

Absolutely continuous measure $\mathrm{d}\mu = \rho(x)\mathrm{d}x$ where

$$\rho(x) = \frac{1}{\pi}\left(\sqrt{1-(x+2)^2}\mathbf{1}_{[-3,-1]} + \sqrt{1-(x-2)^2}\mathbf{1}_{[1,3]}\right)$$

Consider points $y_1 = G(0.6-0.4i)$ and $y_2 = G(0.8)$, which both have two inverses. In the case of y_2, we cannot recover the inverse lying in $(-1,1)$.

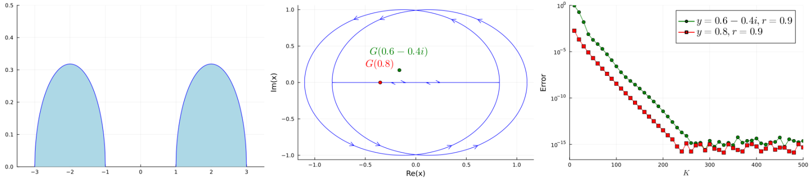
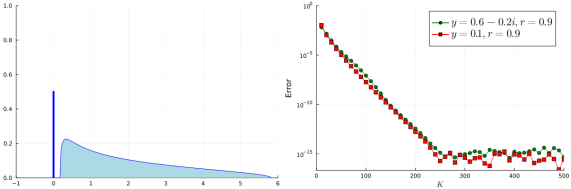


Figure 4: Left: density of μ with Jacobi weights. Middle: image of G_μ. Right: Error of computed inverses plotted against number of quadrature points. Error is measured $\max_i |G(z_i) - y|$ where z_i are the computed roots.

Example 3: Pure point + AC mixture

Consider the random matrix $Y_n = \frac{1}{n} X X^T$ where X is a $m \times n$ matrix of i.i.d standard normal random variables. Then as $m, n \to \infty$ with $\frac{m}{n} \to \lambda > 0$, the limit eigenvalue distribution is the **Marchenko-Pastur** distribution with parameter λ. In the case of $\lambda > 1$ we have

$$\mu = \left(1 - \frac{1}{\lambda}\right)\delta_0 + \nu \qquad \mathrm{d}\nu = \frac{\sqrt{(\lambda_+ - x)(x - \lambda_-)}}{2\pi\lambda x}\mathbf{1}_{[\lambda_-, \lambda_+]}\mathrm{d}x \qquad \lambda_\pm = (1 \pm \sqrt{\lambda})^2$$



We set $\lambda = 2$ as an example and recover inverses of $y_1 = G(0.6 - 0.2i)$ and $y_2 = G(0.1)$.

Figure 5: Left: density of μ with Jacobi weights. Right: Error of computed inverses plotted against number of quadrature points. Error is measured $\max_i |G(z_i) - y|$ where z_i are the computed roots.

Recovery of measures: Regularity

Free convolution has remarkably strong regularisation properties:

Theorem (Bao, Erdös, Schnelli, 2018)

Let μ_a and μ_b be measures supported on single intervals with power-law behaviour at the boundary of the supports, such that all exponents are in $(-1, 1)$. Then $\mu_a \boxplus \mu_b$ is an absolutely continuous measure supported on a single interval. The density $\rho(x)$ vanishes only at the boundary and at the boundary, $\rho(x)$ behaves like a square-root.

Theorem (Morellion, Schnelli, 2023)

Let μ_a and μ_b be measures supported on multiple intervals with power-law behaviour at the boundary of the supports, such that all exponents are in $(-1, 1)$. Then $\mu_a \boxplus \mu_b$ is an absolutely continuous measure supported multiple intervals. Let $\rho(x)$ be the density.

- At the boundary points, $\rho(x)$ vanishes and has square-root singularities.
- At the interior points where $\rho(x)$ vanishes, the singularity is a double-sided cubic singularity.

In general, the convolution of measures with "nice" enough behaviour usually has square-root behaviour at the boundary points.

Cauchy transforms of weighted orthogonal polynomials

Let μ be an absolutely continuous measure with density $\rho(x)$ and support on an infinite compact set Γ. Assume that $\rho(x) = r(x)w(x)$, where $r(x)$ is a non-negative function and $w(x)$ is the weight of some sequence of orthogonal polynomials $p_n(x)$.

Define the Cauchy transforms of the weighted orthogonal polynomials, which are all analytic functions off Γ.

$$q_n(z) = \int_\Gamma \frac{p_n(x)w(x)}{z-x}\mathrm{d}x$$

Proposition (Trogdon, Olver, 2015)

Suppose $r(x) = \sum_{n=0}^\infty \phi_n p_n(x)$, where $\phi \in \ell^1$ and $\|p_n\|_{L^\infty(\Gamma)}$ is uniformly bounded over n by a constant M. Then for $z \in \mathbb{C} \setminus \Gamma$.

$$G_\mu(z) = \sum_{n=0}^\infty \phi_n q_n(z) \text{ and } \left| G_\mu(z) - \sum_{n=0}^m \phi_n q_n(z) \right| \leq \|\phi - \phi_{m+1}\|_{\ell^1} \sup_k |q_k(z)|$$

Thus, for fixed z, the convergence rate of the series for G_μ is exactly the same as the convergence rate for $r(x)$.

Recovery of measures

Assuming that we know the support of $\mu = \mu_a \boxplus \mu_b$ (Nadakuditi, Olver, 2013), we consider the Chebyshev-U expansion of G_μ as follows, for the single interval $[a, b]$ case:

$$\rho(x) = r(x)\frac{2\sqrt{b-x}\sqrt{x-a}}{b-a}\mathbf{1}_{[a,b]}$$

$$G_\mu(z) = \sum_{n=0}^{\infty} \phi_n q_n(z) = \pi \sum_{n=0}^{\infty} \phi_n J_+^{-1}(M_{ab}^{-1}(z))^{n+1}$$

Similarly for multiple intervals $\cup_j [a_j, b_j]$

$$\rho(x) = \sum_j \rho_j(x) = \sum_j r_j(x)\frac{2\sqrt{b_j-x}\sqrt{x-a_j}}{b_j-a_j}\mathbf{1}_{[a_j,b_j]}$$

$$G_\mu(z) = \sum_j \sum_{n=0}^{\infty} \phi_{n,j} q_{n,j}(z) = \pi \sum_j \sum_{n=0}^{\infty} \phi_{n,j} J_+^{-1}(M_{a_j b_j}^{-1}(z))^{n+1}$$

Here, $J_+^{-1}(z) = z - \sqrt{z-1}\sqrt{1+z}$ denotes an inverse of the Joukowski transform sending $\mathbb{C} \setminus [-1, 1]$ to the unit disk, with $J_+^{-1}(\infty) = 0$.

Recovery of measures

We recover the first n coefficients of ϕ by solving a least-squares system $G_\mu(G_\mu^{-1}(y)) = y$, in the single interval case. Given that we can calculate $G_\mu^{-1}(y)$, we consider pairs (z_m, y_m) where $G_\mu(z_m) = y_m$. Then we have

$$\pi \sum_{k=1}^{n} \phi_{k-1} \mathrm{Re}\left(J_+^{-1}(M_{ab}^{-1}(z_m))^k\right) \approx \mathrm{Re}\,(y_m)$$
$$\pi \sum_{k=1}^{n} \phi_{k-1} \mathrm{Im}\left(J_+^{-1}(M_{ab}^{-1}(z_m))^k\right) \approx \mathrm{Im}(y_m)$$

Similar, for the multiple interval case, we have

$$\pi \sum_{j} \sum_{k=1}^{n_j} \phi_{k-1,j} \mathrm{Re}\left(J_+^{-1}(M_{a_j b_j}^{-1}(z_m))^k\right) \approx \mathrm{Re}\,(y_m)$$
$$\pi \sum_{j} \sum_{k=1}^{n_j} \phi_{k-1,j} \mathrm{Im}\left(J_+^{-1}(M_{a_j b_j}^{-1}(z_m))^k\right) \approx \mathrm{Im}\,(y_m)$$

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Recovery of measures

Theorem (Nadakuditi, Olver, 2013)

Suppose μ is a square root measure supported on a single interval $[a, b]$. If $\{y_m\}$ is a sequence of sets of points which cover the region $G_\mu(\mathbb{C} \setminus \Gamma) \cap \mathbb{C}^+$ as $m \to \infty$ at a sufficiently fast rate, then there exists m sufficiently large depending on n such that the calculated density function $r_n(x)$ satisfies

$$\| r_n\left(M_{ab}(J(z))\right) - r\left(M_{ab}(J(z))\right) \| \to 0$$

as $n \to \infty$. Here, the norm is the L^2 norm on the unit disk.

Theorem

Suppose μ is a square root measure supported on multiple intervals $[a_j, b_j]$. Denote the convex hull of Γ as an interval $[a, b]$. Then the same statement in the above theorem holds *i.e.*

$$\| \sum_j r_{n,j}\left(M_{ab}(J(z))\right) - \sum_j r_j\left(M_{ab}(J(z))\right) \| \to 0$$

Summary of algorithm

Given measures μ_a, μ_b and their convolution $\mu_a \boxplus \mu_b$:

- Solve $G_{\mu_a}(z) = y$ and $G_{\mu_b}(z) = y$ for a suitable cloud of points y_m to obtain **ALL** solutions to $G_{\mu_a \boxplus \mu_b}(z) = y$
- Calculate the support of $\mu_a \boxplus \mu_b$ using methods from (Nadakuditi, Olver, 2013), which we assume is square-root decaying at the boundary due to regularity.
- Solve a least-squares system to recover the first few coefficients in a weighted orthogonal polynomial expansion of the density

Time to run some code!

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Conclusion

- Describe algorithms to compute convolutions for a wide class of measures, including measures supported on multiple intervals and measures which are not absolutely continuous.
- Future work: Further investigation of free multiplicative convolution, polynomials of free random variables, rational functions of free random variables...
- Repository for Julia code is on Github at: https://github.com/dlfivefifty/NumericalFreeProbability.jl
- Special thanks to Sheehan Olver for supervising me throughout this summer

Thank you!