Lecture 10: Newton's and interpolation-based methods

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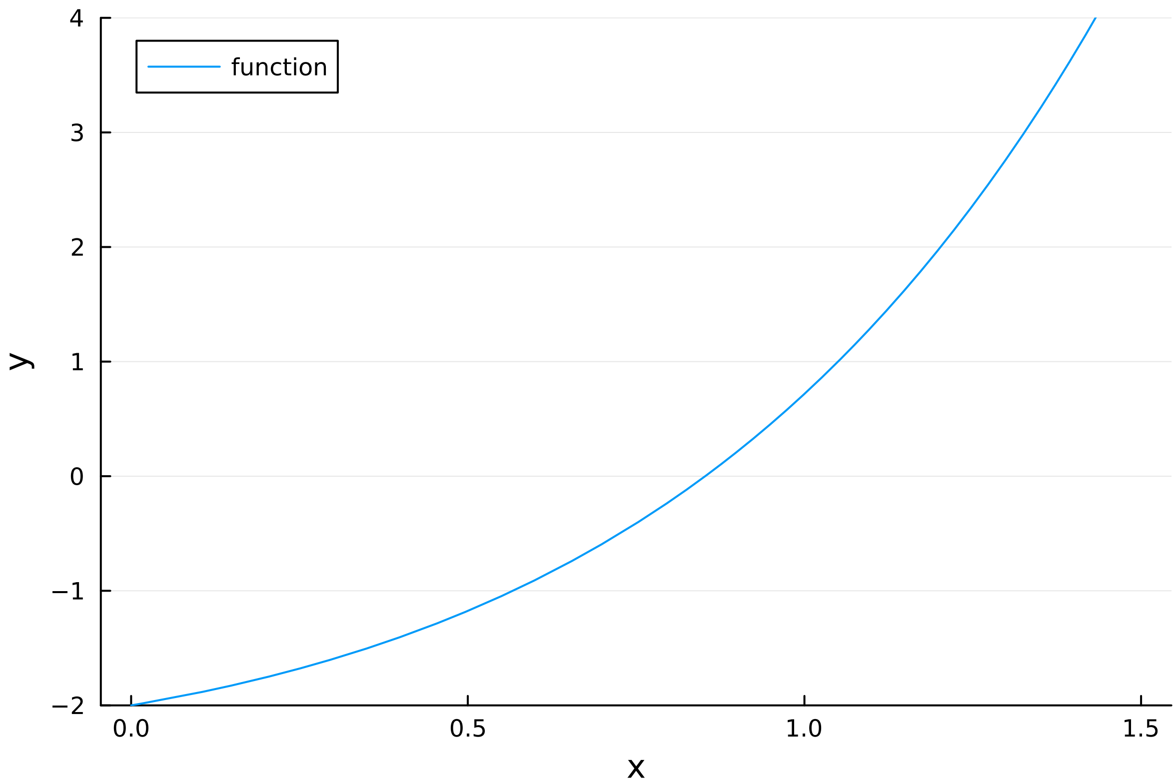
1 Newton's method

Newton's method is one of the most fundamental methods for rootfinding but it also introduces us to some other big ideas in iterative methods – superlinear convergence!

1.1 Demo

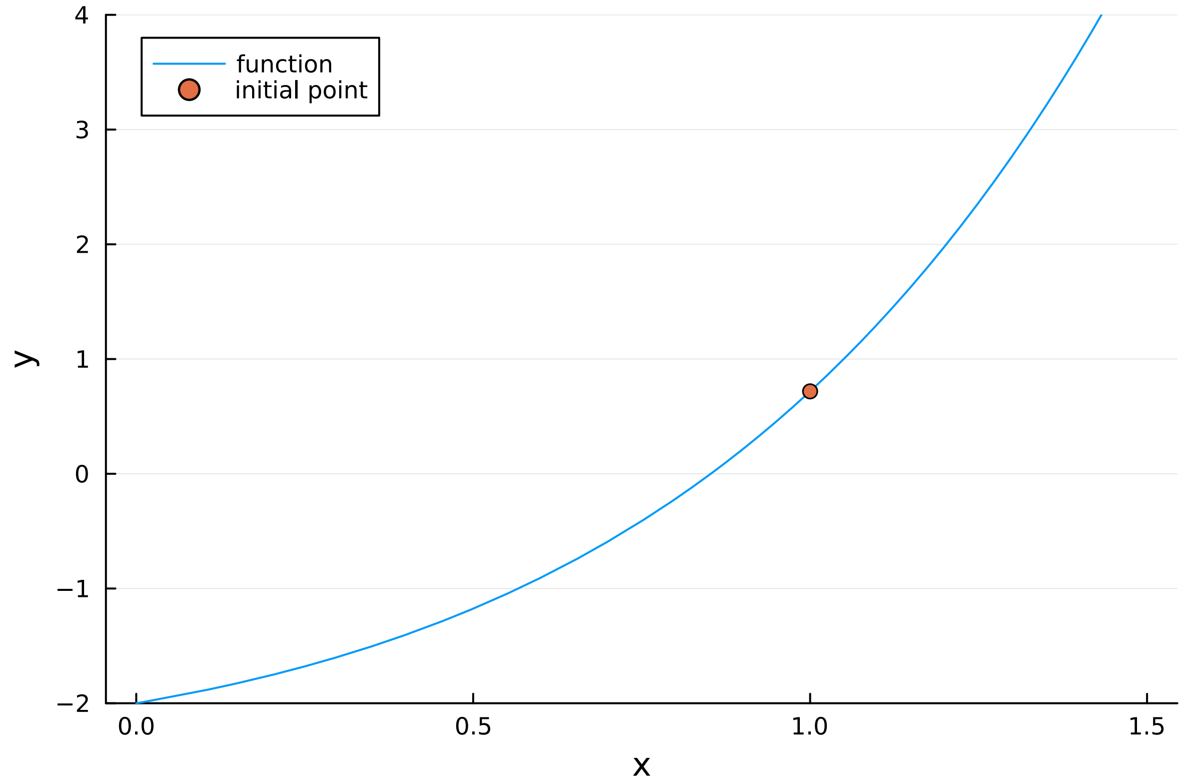
```
f = x -> x*exp(x) - 2   #function defining the rootfinding problem

plot(f,0,1.5,label="function",grid=:y,ylim=[-2,4],xlabel="x",ylabel="y",legend=:topleft)
```



We can see that there is a root near $x = 1$. This will be our initial guess, x_1.

```
x1 = 1
y1 = f(x1)
scatter!([x1],[y1],label="initial point")
```

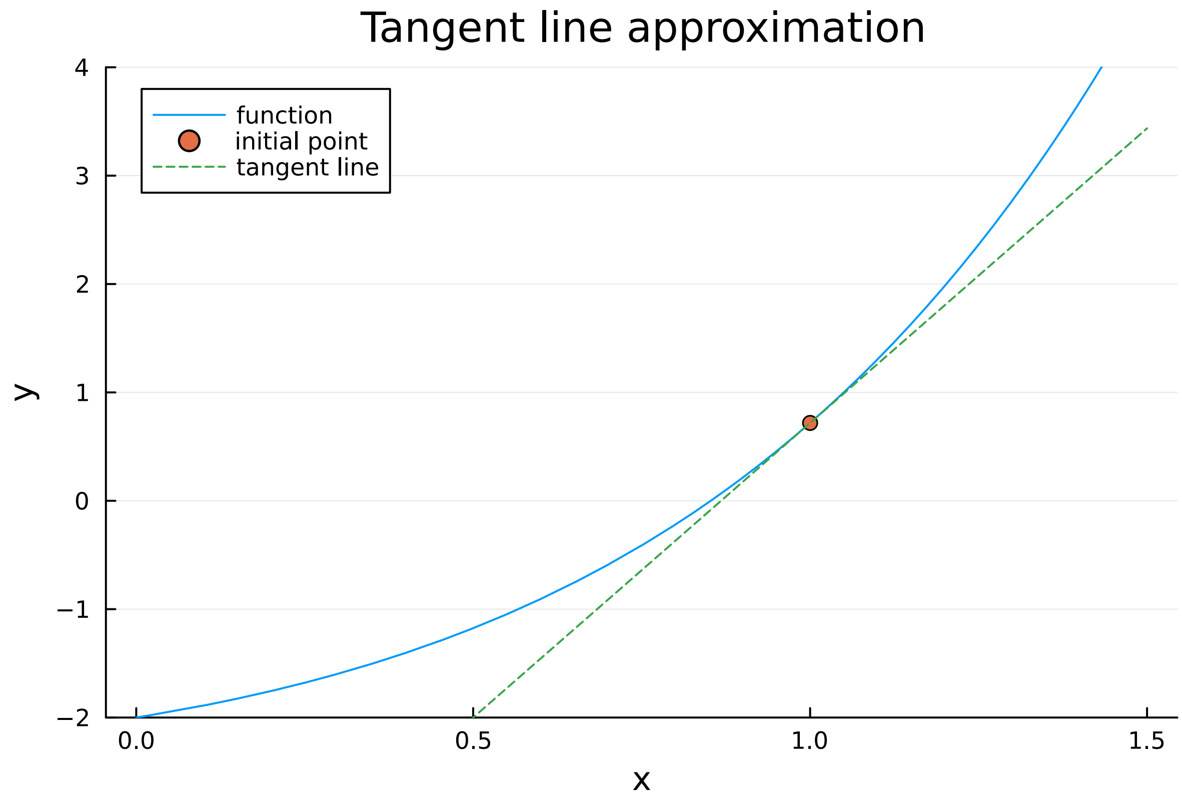


Next, we compute the tangent line at the point $(x_1, f(x_1))$ using the derivative.

```
dfdx = x ->exp(x)*(x+1)
m1 = dfdx(x1)
tangent = x -> y1 + m1*(x-x1)
```

```
#7 (generic function with 1 method)
```

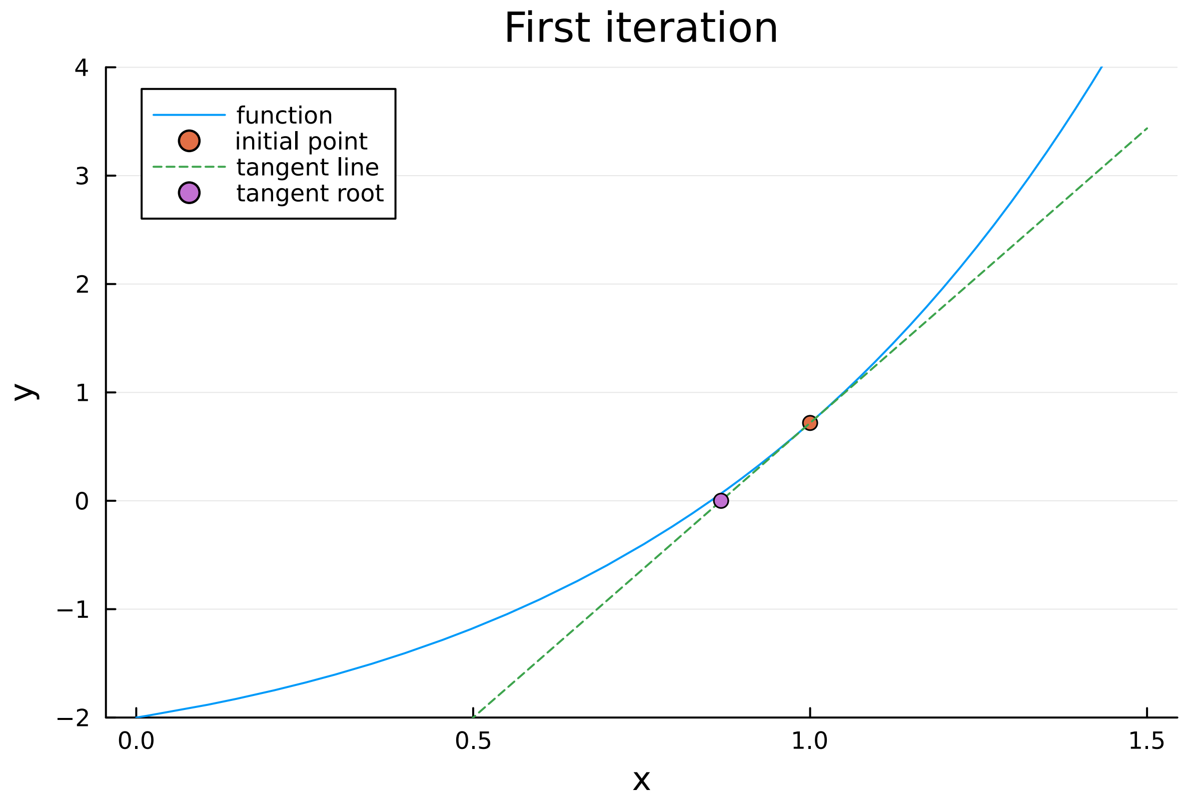
```
plot!(tangent,0,1.5,l=:dash,label="tangent line", title="Tangent line approximation")
```



Rather than finding the root of f itself, we settle for finding the root of the tangent line and let this be our next approximation x_2.

```
@show x2 = x1 - y1/m1
scatter!([x2],[0],label="tangent root",title="First iteration")
```

```
x2 = x1 - y1 / m1 = 0.8678794411714423
```




```
y2 = f(x2)
```

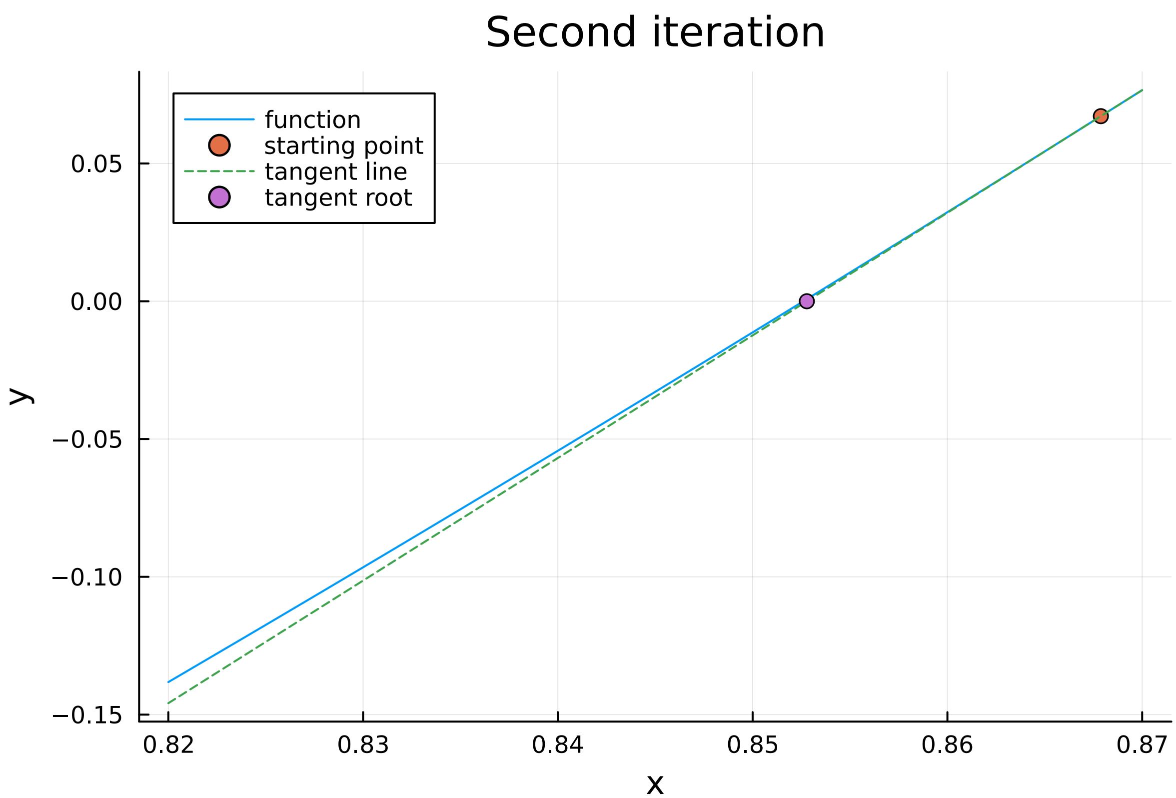
```
0.06716266657572145
```

The residual (i.e., value of f at x_2) is smaller than before but not zero. Thus, we repeat!

```
plot(f,0.82,0.87,label="function",legend=:topleft,xlabel="x",ylabel="y",title="Second iteration")
scatter!([x2],[y2],label="starting point")
m2 = dfdx(x2)
tangent = x -> y2 + m2*(x-x2)
plot!(tangent,0.82,0.87,l=:dash,label="tangent line")

@show x3 = x2-y2/m2
scatter!([x3],[0],label="tangent root")
```

```
x3 = x2 - y2 / m2 = 0.8527833734164099
```




```
y3 = f(x3)
```

```
0.0007730906446230534
```

The residual is decreasing quickly, so we appear to be getting much closer to the root!

1.2 Newton's method

Given a function f, its derivative f', and an initial value x_1, iteratively define

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \qquad k = 1, 2, \cdots.$$

First, note that this is a special case of the fixed-point iteration! Define $g(x) = x - \frac{f(x)}{f'(x)}$. When we identify a root, x, where $f(x) = 0$, we have a fixed point of g, which is the function defining the Newton update.

The previous example also suggests why Newton's method might converge to a root – as we zoom in on the function f, the tangent line and the graph of the differentiable function f must become more and more similar. However, we don't yet know that it will converge or how quickly!

1.3 Convergence

Assume that the sequence x_k converges to limit r which is a root, $f(r) = 0$. Define again the error $\epsilon_k = x_k - r$ for $k = 1, 2, \cdots$.

We can rewrite the update in terms of the ϵs as

$$\epsilon_{k+1} + r = \epsilon_k + r - \frac{f(r + \epsilon_k)}{f'(r + \epsilon_k)}.$$