Lecture 3: Polynomial interpolation, Linear systems

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1 Polynomial Interpolation

Often we encounter data to which we hope to fit a function – see e.g., most of machine learning! One of the most fundamental such problems is to find a polynomial function that passes through all data points. This problem is known as **polynomial interpolation**.

Definition: Polynomial interpolation

Given n points $(t_1, y_1), \cdots, (t_n, y_n)$, where the t_i are all distinct, the **polynomial interpolation** problem is to find a polynomial p of degree less than n such that $p(t_i) = y_i$ for all i.

1.1 Interpolation as a linear system

The *polynomial interpolation* problem in the definition above seeks a polynomial of the form

$$p(t) = c_1 + c_2 t + c_3 t^2 + \cdots + c_n t^{n-1}$$

such that $y_i = p(t_i)$ for all i. We can rewrite this as

$$\begin{array}{cccccc}
c_1 + & c_2 t_1 + & \cdots + & c_{n-1} t_1^{n-2} + & c_n t_1^{n-1} = & y_1 \\
c_1 + & c_2 t_2 + & \cdots + & c_{n-1} t_2^{n-2} + & c_n t_2^{n-1} = & y_2 \\
c_1 + & c_2 t_3 + & \cdots + & c_{n-1} t_3^{n-2} + & c_n t_3^{n-1} = & y_3 \\
 & & & & \vdots & \\
c_1 + & c_2 t_n + & \cdots + & c_{n-1} t_n^{n-2} + & c_n t_n^{n-1} = & y_n.
\end{array}$$

These equations can be written succinctly in our usual linear system form

$$\begin{bmatrix} 1 & t_1 & \cdots & t_1^{n-2} & t_1^{n-1} \\ 1 & t_2 & \cdots & t_2^{n-2} & t_2^{n-1} \\ 1 & t_3 & \cdots & t_3^{n-2} & t_3^{n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & t_n & \cdots & t_n^{n-2} & t_n^{n-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix},$$

which we denote $\mathbf{V}\mathbf{c} = \mathbf{y}$.

This is special type of matrix!

Definition: Vandermonde matrix

Given distinct values $t_1, \cdots, t_n$, a **Vandermonde matrix** for these values is the $n \times n$ matrix appearing above.