MAT 258A: Numerical Optimization - Homework 1

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1. (Trefethen & Bau, Lecture 6) Let $P \in \mathbb{R}^{n \times n}$ be a nonzeros projector. Show that $||P||_2 \geq 1$, and that this holds with equality $\iff$ P is an orthoganal projector.

   Proof. First,
$$||P||_2 = ||P^2||_2 \leq ||P||_2^2 \quad \text{so} \quad 1 \leq ||P||_2.$$
   ($\Leftarrow$) If P is an orthoganal projector then $P^T = P$, so
$$||P||^2 = \sup_{||x||=1} ||Px||^2 = \sup_{||x||=1} (Px)^T Px = \sup_{||x||=1} x^T P^T P x = \sup_{||x||=1} x^T P^2 x = \sup_{||x||=1} x^T P x$$
$$\leq \sup_{||x||=1} ||Px|| ||x|| = \sup_{||x||=1} ||Px|| = ||P||.$$
   Thus, $||P|| \geq 1$, and with the above, we have equality. $\square$

2. Suppose that A is m-by-n, with $m > n$ and A full rank. Show that a minimizer x and corresponding residual r of the problem
$$\underset{x}{\text{minimize}} \frac{1}{2}||Ax - b||^2$$
   can be obtained by solving the augmented linear system
$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}.$$
   Now write down the augmented linear system that gives the solution and residual of
$$\underset{x}{\text{minimize}} \frac{1}{2}||Ax - b||^2 + \frac{1}{2}\delta^2 ||x||^2 + c^T x.$$

3. Set A by m-by-n, with the SVD $A = U\Sigma V^T$. Compute the SVDs of the following matrices in terms of the factors of A:
   (a) $(A^T A)^{-1}$
   (b) $(A^T A)^{-1} A^T$
   (c) $A(A^T A)^{-1}$
   (d) $A(A^T A)^{-1} A^T$

4. Suppose that the m-by-n matrix, with $m < n$, is full rank. Then the problem $\min ||Ax - b||$ is underdetermined. Show that the solution is an $(n - m)$-dimensional set. Show how to compute the unique minimum 2-norm solution using an appropriately modified (i) normal equations, (ii) QR decomposition, and (iii) SVD.