Inverse of a Matrix

Theorem: if A is an invertible matrix, its inverse is unique.
Proof: By way of contradiction, suppose A has two inverses: $A' \text{ and } A''$
This means that $A'A = I = AA'$ and $A''A = I = AA''$.
Well now show that $A' = A''$.

$$A' = A'I = A'(AA'') = (A'A)A'' = IA'' = A''$$

Thus we have shown that there can only be 1 inverse. I still feel iffy about this proof that there can only exist one inverse of a function, but sure.

How can we compute the inverse?

For a 2×2 matrix, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and A is invertible, then
$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
General case intuition: If A is invertible, and B is the inverse of A, then
$AB = I$
We can think of this as a set of linear systems. We can use the matrix-column representation of systems to decouple this system.

$$A \left[\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n\right] = I = [\vec{e}_1, \vec{e}_2, \ldots, \vec{e}]$$

$\vec{e}_1 \ldots \vec{e}_n$ represents the identity matrix.

$$A\vec{b}_1 \text{ can be solved with } [A|\vec{e}_1]$$
$$A\vec{b}_2 \text{ can be solved with } [A|\vec{e}_2]$$
$$A\vec{b}_n \text{ can be solved with } [A|\vec{e}_n]$$

note: Solving the system $[A|\vec{b}]$ uses the same sequence of ERO's for any $\vec{b}$! The form of A determines what sequence of EROs are necessary!

for each $A\vec{b}_n$ we turn the matrix into $\left[RREF(A)|\vec{b}_n\right]$
Since there is only one inverse matrix, this will be $\left[I|\vec{b}_1\right]$
We can stack together all of these vectors back, which is performing $[A|I] \rightarrow [I|B]$. To summarize: in order to compute the inverse, we use the eros defined by a and act them on b and end up with a matrix B that will make the identity.

Rigorously, what is going on here?
Each ERO is multiplying the matrix A and multiplying it by a particular matrix. #ERO **is really multiplication by elementary matrices** #elementary_matrices: Any matrix that can be obtained by performing an ERO on an identity matrix.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{swaps rows}$$
$$\begin{bmatrix} 1 & 0 \\ 0 & -3 \end{bmatrix} \text{scales by -3}$$
$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{does } r_1 \rightarrow r_1 + r_2$$

We can see this is due to swapping the identity, ie

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

What does this look like for a 3x3 matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

We can undo ERO's by using their inverse matrixes.

For row swapping EROs, the inverse is itself

$$E = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, E^{-1=} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -7 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{7} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

see previous: Matrix Algebra and Inverse