Spanning Sets and Linear Independence

It is at this time that i decided to make my notes into a comprehensive textbook, so things before this are incomplete. My apologies to the reader.

Homogenous Systems

#homogenous_systems A system of linear equations is homogenous if the constant term in each equation is zero. If a system is not homogenous, we say its #inhomogenous . Homogenous systems always pass through the origin)
For example,

$$\begin{aligned} x_1 - 7x_2 + 2x_3 = 0 \\ -4x_1 + x_2 - x_3 = 0 \end{aligned}$$

is a homogenous system. These equations form two planes that pass through the origin.

Homogenous systems always have at least one solution (the trivial solution, ie the $\vec{0}$).
From the rank theorem: If a homogenous system has m equations and n variables, and $m < n$, then the system has ∞ many solutions.

Linear combinations

given two vectors $\vec{a_1}, \vec{a_2}$ how do we find if $\vec{b}$ is a linear combination of a_1 or a_2?

We can write a system of equations to solve, letting x_1 and x_2 be real numbers.

$$x_1 a_1 + x_2 a_2 \stackrel{?}{=} b$$

Geometrically, this corresponds to adding together some number of each vector to define a point. If the 2d vectors are on the same line they can't define $\mathbb{R}^2$, but if they don't align then they can.

#span
the span of a set of vectors produces a subset of $\mathbb{R}^n$ and is denoted span$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}k\}$. if the vectors can produce $\mathbb{R}^n$ then they are called a #spanning_set for $\mathbb{R}^n$
If a vector is a linear combination of other vectors, it won't increase the span.

To calculate the span of a set, ie

$$a = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

We need to find $S_1 \text{and} S_2$, solutions to find any x y z.

$$s_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + s_2 \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

We write the system as an augmented matrix

$$\begin{bmatrix} 1 & 2 & | & x \\ 2 & 1 & | & y \\ 0 & -1 & | & z \end{bmatrix} = \begin{bmatrix} 1 & 1 & | & x \\ 0 & 1 & | & 2x - y \\ 0 & 0 & | & -2x + y - z \end{bmatrix}$$

So

$$\text{Span}\left\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}1\\1\\01\end{pmatrix}\right\}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}:-2x+y-z=0\right\}$$

This gives us the entire plane that the two vectors can define. the 0 0 row is a constraint as it can't be in $\mathbb{R}^3$. We only have two vectors, so there can only be 2 leading variables.
$-2x+y-z=0$ is the normal vector of the plane

Some more examples:

$$\begin{pmatrix}1 & -2 & | & x\\-1 & 2 & | & y\\2 & -4 & | & z\end{pmatrix}\to\begin{pmatrix}1 & -2 & | & x\\0 & 0 & | & x+y\\0 & 0 & | & z-2x\end{pmatrix}$$

We get a line as the intersection of two planes. The span of this is a single line that follows the constraints. The two vectors are a multiple of each other (-2 in this case)

If we add a new point that already satisfies the constraints given, it will not grow the span at all
IE if a new vector satisfies $z-2x=0$ and $0=x+y$ then it maintains the current span. Otherwise, if it breaks the constraints it is giving us more information, and allows us one more dimension to span.

Linear independence

A set of vectors is #linearly_dependent if there are scalars $c_1,c_2,\dots,c_k$ that are not all zero such that

$$c_1\vec{v}_1+c_2\vec{v}_2+\cdots+c_k\vec{v}_k=0$$

A set of vectors that is not linearly independent, it is #linearly_indepent
This is a property of the entire set of vectors.
We write the set of vectors in an augmented matrix with zero as last row.

Theorem: The set v1-vm is linearly dependent if and only if the homogenous linear system with the augmented matrix of each vector | 0 has a nontrivial solution.

see next: Matrix operations