Change of basis

Exercise

Find a basis for P_1 the set of *all* polynomials
Every polynomial is a linear combination of monomials,
$a_0 + a_1x + a_2x^2 + a_3x^3 \dots a_nx^n$
So $B = \{1, x, x^2, x^3, \dots, x^n\}$ is a spanning set for P. It is linearly independent and also infinitely long ... cool.

Lets write a proof by contradiction that the set of al monomials is linearly independent:
Suppose there is some finite set of B with n vectors that is linearly *dependent* $\{x^{P_1}, x^{P_2}, \dots, x^{Pn}\}$ where $P_1 < p_2 < p_3 \dots < p_n$. Then there are some scalars (not all zero), $c_1, c_2, \dots, c_n$ so that
$c_1x^{P_1} + c_2x^{P_2} + \dots + x^{P_n} = 0$
This polynomial is 0 for all values of x. *BUT* the fundamental theorem of algebra says that a nonzero polynomial can have at most n roots. Thus, the polynomial would have to be the zero polynomial with $c_1 = c_2 = \dots = c_n = 0$. This is a contradiction, $\implies$ any finite set of B is linearly independent $\implies$ B is linearly independent.

Dimension

Def:

1. A vector is #finite_dimensional if it has a basis consisting of finitely many vectors
2. A vector space is #inifinite_dimensional if it has no finite basis
3. The vector space $\vec{0}$ is #zero_dimensional

Coordinates

Lets move to something that is more concrete. Lets say we have a finite basis $B = \left\{\vec{b}_1, \dots, \vec{b}_n\right\}$ for the vector space V.then any $\vec{v} \in V$ can be written uniquely as a linear combination of $\vec{v} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$

The unique coefficient $c_1, \dots, c_n$ are called the #coordinates of $\vec{v}$ relative to the basis β, (or #b-coordinates). The entries in the coordinate vector are the coordinates of this combination. We can see this easily with the x y plane - to specify a point $(2\vec{x}, 3\vec{y})$ we just write out $(2, 3)$. If instead we had a rotated representation, we would still be specifying points with the coefficients scaling each vector.

Take the vector space represented by $\begin{bmatrix} a \\ b \\ b \\ c \end{bmatrix}$

We have a basis

$$\beta = \left\{\left[\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right]\right\}$$

We can specify coordinates of a point $\begin{bmatrix} 2 \\ -7 \\ -7 \\ 8 \end{bmatrix}$ with $\begin{bmatrix} 2 \\ -7 \\ 8 \end{bmatrix}$, because those coordinates multiplied by the basis makes the point. We can make any basis and choose any coordinates to form that coordinate as long as β is a basis for V.

What are the coordinates of

$$\begin{bmatrix} 2 & -7 \\ -7 & 8 \end{bmatrix}$$

in the basis

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

The same, $\begin{bmatrix} 2 \\ -7 \\ 8 \end{bmatrix}$!

Some facts about coordinates

if B is a basis, then
$\left[\alpha_1\vec{u}_1 + \alpha_2\vec{u}_2 + \dots \alpha_n\vec{u}_n\right] = \alpha_1[\vec{u}_1] + \dots + \alpha_k[\vec{u}_k]$
This follows from the fact that
$[\vec{u} + \vec{v}]_\beta = [\vec{u}] + [\vec{v}]$, and $[\alpha\vec{u}]_\beta = \alpha[\vec{u}]_\beta$

We can use coordinate vectors to check the linear independence of vector spaces that are not $\mathbb{R}^n$.

$$\begin{aligned} &\left\{\vec{u}_1, \dots, \vec{u}_n\right\} \text{ is linearly independent} \\ \Leftrightarrow &\left\{[\vec{u}_1]_b, \dots, [\vec{u}_n]_b\right\} \text{ is linearly independent in } \mathbb{R}^n \end{aligned}$$

(make a careful note of the difference between k and n in the above).

Changing the basis

The choice of basis is important. How do we switch between bases?

In $\mathbb{R}^2$:
$B = \left\{ \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \right\}$
$\vec{u}, \vec{v}$

We can represent the point $[5, -1]_b$ as coordinates $\begin{bmatrix} 4 \\ -1 \end{bmatrix}$ in the standard basis.

What if we want to change to a different basis?

$$C = \left\{ \left[\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right] \right\}$$

We can represent x now as the solution to

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} = c_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

We can write this out as

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2c_2 \\ c_1 + 3c_2 \end{bmatrix}$$

Or, in a slightly nicer and simultaneously grosser way,

$$\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

We can see that $c_2 = 2$, meaning $c_1 = -7$.

However, this was a long process (find coordinates in the standard basis and then in the new). A far better way is to use a "change of basis" matrix.
$[\vec{x}]_c = \left([\vec{u}_1]_c [\vec{u}_2]_c \right) [\vec{x}]_b$
We take the basis vectors in c and multiply by the representation of $\vec{x}$ in b in order to find the representation of $\vec{x}$ in c.

We take in some vectors from basis B, compute the representations of those in c, and then we use that to represent the transformation from B to C. We represent this change of basis matrix as $P_{c \leftarrow b}$. Because matrix multiplication is from right to left, we take a matrix represented in B from the right and multiply it to get a matrix in C.

Suppose $B = \{\vec{u}_1, \ldots, \vec{u}_i\}$ and $C = \{v_1, \ldots, v_n\}$ then

The i^{th} column of $P_{c \leftarrow b}$ is $[\vec{u}_i]_\epsilon = \begin{bmatrix} p_{1,i} \\ p_{2,i} \\ \vdots \\ p_{n,i} \end{bmatrix}$ and $\vec{u}_i = p_{1,i}\vec{v}_i + \cdots + p_{n,i}\vec{v}_i$

If ϵ is any basis for V, then $[u_i]_\epsilon =$

$$\left([\vec{v}_1]_\epsilon, \ldots, \vec{v}_n]_\epsilon\right) \begin{bmatrix} p_{1,i} \\ p_{2,i} \\ \vdots \\ p_{n,i} \end{bmatrix}$$

We can solve this with gaussian elimination

$$[\ [\vec{v}_1]_\epsilon \quad [\vec{v}_2]_\epsilon \quad | \quad [\vec{u}_1]_\epsilon \quad [\vec{u}_2]_\epsilon\ ]$$

Lets do a concrete example.
Take

$$B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, C = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$$

We can compute the change of basis from $B \rightarrow C$

$$P_{c \leftarrow b} = \left(\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]_c, \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \right]_c \right)$$

We can row reduce

$$\begin{bmatrix} 0 & 2 & | & 1 & 1 \\ 1 & 3 & | & 0 & 1 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 1 & 0 & | & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & | & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

The right hand side is our change of basis matrix $P_{c \leftarrow b}$