Properties of symmetric matrices

"Spectral" $\implies$ eigen stuff

Symmetric matrices have nice spectral properties. For instance:

If we have a real and symmetric matrix, the eigenvalues are real.

Before: For any matrix, eigenvectors associated to distinct eigenvalues are linearly independent.
Now: For a symmetric matrix, eigenvectors associated to distinct eigenvalues are orthogonal.

Orthogonal Diagonalization

A matrix is #orthogonally_diagonalizable if there is an orthogonal matrix Q and a diagonal matrix D such that $Q^TAQ = D$.

The spectral theorem: Suppose A is a real, $n \times n$ matrix. Then A is symmetric if and only A is orthogonally diagonalizable. We have the nice property that $Q^{-1} = Q^T$.

Proof: Suppose $Q^TAQ = D$ Since Q is orthogonal, we have $QDQ^T = QQ^TAQQ^T = A$. Now,
$A^T = (QDQ^T)^T = QD^TQ^T = QDQ^T = A$ Thus $A^T = A$ so A is symmetric.

A is symmetric implies that A is orthogonally diagonalizable. Thus, we have the set $\text{symmetric} \subseteq \text{orthogonally diagonalizable}$.
We also have orthogonally diagonalizable matrix is symmetric, so we have $\text{OD} \subseteq \text{Sym}$
Thus, $\text{Orthogonally diagonalizable matrices} \Leftrightarrow \text{symmetric}$

The power of diagonalization is that we can easily find the eigendata or determinants etc.

For example:
take the symmetric matrix of all real components

$$A = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$$

The characteristic polynomial is $(\lambda - 7)^2(\lambda + 2)$ so the eigenvalues are $7, 7, 2$. The eigenspaces are

$$E_{-2} = span\left\{ \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix} \right\} E_7 = span\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

We can apply Gram Schmidt to get the orthogonal basis for E_7 (they aren't yet orthogonal).
We then normalize and get

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} & -\frac{2}{3} \\ 0 & \frac{2\sqrt{2}}{3} & -\frac{1}{3} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} & \frac{2}{3} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The judicious reader can check that $Q^T A Q = D$.