Inner product spaces

Inner products

Def: $\vec{u}, \vec{v}, \vec{w}$ are vectors in a vector space V and c is a real scalar. an #inner_product is an operation that assigns a real number $\langle \vec{u}, \vec{v} \rangle$ to each pair of vectors \vec{u}, \vec{v} and is

- Symmetric:
 - ullet $\langle ec{u},ec{v}
 angle = \langle ec{v},ec{u}
 angle$
 - $ullet \left\langle ec{u},ec{v}+ec{w}
 ight
 angle =\left\langle ec{u},ec{v}
 ight
 angle +\left\langle ec{u},ec{w}
 ight
 angle$
- Linear:
 - $ullet \left\langle cec{u},ec{v}
 ight
 angle =c\left\langle ec{u},ec{v}
 ight
 angle$
- Positive semi-definite:
 - $\langle ec{u}, ec{v}
 angle \geq 0$ with $\langle ec{u}, ec{u}
 angle = 0$ if and only if $ec{u} = ec{0}$.

We can define length, distance, and orthogonality for any inner product.

Def: let $\langle \vec{u}, \vec{v} \rangle$ be an inner product on vector space v. The length or norm of $\vec{v} \in V$ is $|\vec{v}| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. A unit vector satisfies $|\vec{v}| = 1$. The distance between \vec{u} and \vec{v} is $|\vec{u} - \vec{v}|$. We say \vec{u} and \vec{v} are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$

IE, the distance between $\sin(x) \ \mathrm{and} \ \cos(x) \in [0,2\pi]$ is

$$egin{align} \langle \sin(x),\cos(x)
angle &= \int_0^{2\pi}\sin(x)\cos(x)\,dx\ &= rac{1}{2}\int_0^{2\pi}2\sin(x)\cos(x)\,d\ &= rac{1}{2}\int_0^{2\pi}\sin(2x)\,d\ &= 0 \end{split}$$

so $\sin(x)$ and $\cos(x)$ are orthogonal in $c[0,2\pi]$ with respect to this inner product.

We can also generalize:

- Pythagorean Theorem: If $ec{u}, ec{v}$ are orthogonal, $\left|ec{u}\right|^2 + \left|ec{v}\right|^2 = \left|ec{u} + ec{v}\right|^2$
- Cauchy Schwarz inequality: $\left|\left\langle ec{u}, ec{v}
 ight
 angle \right| \leq |ec{u}| |ec{v}|$
- Triangle inequality: $\left| ec{u} + ec{v}
 ight| \leq \left| ec{u}
 ight| + \left| ec{v}
 ight|$

Gram-Schmidt

We can even generalize the Gram-Schmidt process to general vector spaces with an inner product!

For example: Find an orthogonal basis for $W=span\left\{2,x+1,x^2-1
ight\}$ We take $v_1=y_1=2$

$$egin{aligned} v_2 &= y_2 - rac{\left< ec{y}_2, ec{v}_1
ight>}{\left< ec{v}_1, ec{v}_1
ight>} ec{v}_1 \ &= c + 2 - rac{\left(\int_0^1 2(x+1) \, dx
ight)}{\int_0^1 2 * 2 \, dx} 2 \ &= x + 1 - rac{3}{2} = x - rac{1}{2} \ &v_3 &= y_3 - rac{\left< v_3, v_1
ight> v_1}{\left< v_1, v_1
ight>} - rac{\left< y_3, v_2
ight>}{\left< v_2, v_2
ight>} v_2 \ &= x^2 - 1 - rac{\left(\int_0^1 2(x^2-1) \, dx
ight)}{\int_0^1 4 \, dx} 2 - rac{\left(\int_0^1 \left(x - rac{1}{2}
ight) (x^2-1) \, dx
ight)}{\left(\int_0^1 \left(x - rac{1}{2}
ight)^2 \, dx
ight)} \left(x - rac{1}{2}
ight) \ &= x^2 - x + rac{1}{6} \end{aligned}$$