Row, Col, Null

def: Suppose A is a $n \times m$ matrix. The row space of A is the subspace $row(A)$ or $\mathbb{R}^m$ spanned by the rows

The column space of A is the subspace $row(A)$ or $\mathbb{R}^m$ spanned by the columns.

If we take a matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

row(A)

$$= \begin{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} & \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \end{bmatrix}$$

and col(a) =

$$\begin{bmatrix} \begin{pmatrix} 1 \\ 4 \end{pmatrix} & \begin{pmatrix} 2 \\ 5 \end{pmatrix} & \begin{pmatrix} 3 \\ 6 \end{pmatrix} \end{bmatrix}$$

We can see that the dimension of the Row space is 2, so the column-space also has dimension 2.

A #consistent system has either 1 or infinitely many solutions. ∞ is if there is a free variable, or #inconsistent if there is an impossible row, ie $\begin{bmatrix} 0 & 0 & 0 & | & nonzero \end{bmatrix}$

col(A) tells us what $\vec{b}$ vectors yield consistent linear solutions. These are the possible outputs $A\vec{x}$.

row(A) tells us the possible results of doing EROs on A.

Null(A) = $\{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$.
Ex:

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$$

Null(A):

$$A\vec{u} = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$.
proof: Note $A\vec{0} = \vec{0}$, so $\vec{0} \in null(A)$.
Suppose $\vec{v}, \vec{u} \in null(A)$. Note that
$A(c\vec{u} + d\vec{v}) = cA\vec{u} + dA\vec{v} = c_0 0 + d_0 0 = \vec{0}$, so $c\vec{u} + d\vec{v} \in \text{ null } (A)$

The idea behind the null space of a matrix is that it is precisely those vectors in the domain being sent to the $\vec{0}$ vector in the codomain. We have to find the solution set of $Ax = 0$.

We do this by finding the null space of a reduced row echelon form of A, which has the same null space as A. That is, if B is the reduced row echelon form for A , $Ax = 0$ if and only if $Bx = 0$ So, $N(B) = N(A)$

For example, take

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

We reduce to RREF, then find the free variables and write the result in a nice form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} num \\ num \\ num \\ num \\ num \end{bmatrix} + x_2 \begin{bmatrix} num \\ num \\ num \\ num \\ num \end{bmatrix} \ldots$$

whatever, to use the free variables to define every variable. The vectors attached with these free variables is the Null space of A.
in this example, we find the null space as

$$\begin{bmatrix} \begin{pmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} & \begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \end{bmatrix}$$

so dim(null(A)) = 2.

Any two bases of a subspace contain the same number of vectors
Def: The *dimension* of a nonzero subspace S of $\mathbb{R}^n$ is the number of nonzero vectors in a basis for S. We can write this as $dim(S)$. Special case: $dim(\vec{0}) = 0$.

Fact: Any set of n *linearly independent* vectors forms a basis for $\mathbb{R}^n$.

Rank nullity theorem:

#rank : dimension of the row or column space =
$\#$ of rows or columns $=$ # leading variables.
$rank = dim(row(A)) = dim(col(A))$

$\text{nullity} = \text{number of free variables}$
$\text{nullspace} = \text{linearly dependent columns}$

$\text{num vectors} = \text{ rank} + \text{ nulity}$
↑ this is the Rank Nullity Theorem. Number of free leading variables + number of free variables = total number of variables. This fact seems obvious, but it is a very useful fact to have stated.
I f there are no free variables then null(A) = 0 and rank(A) = n.

Take an $n \times n$ matrix where $Rank(A) = n$

$$\begin{aligned}\Leftrightarrow \text{ nullity } (A) = 0 \Leftrightarrow \text{ null A } = \{\vec{0}\} \\ \Leftrightarrow \text{ The only solutino to } A\vec{x} = \vec{0} \text{ is } \vec{x} = \vec{0} \\ \Leftrightarrow A \text{ is invertible}\end{aligned}$$