Complex Eigenvalues

Complex Eigenvalues:

Recall that we can find eigenspaces. IE:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

with $\lambda = 3$.
Steps:

$$\begin{aligned} (A - \lambda I | 0) \\ \begin{bmatrix} -1 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \\ \implies x_1 = 3x_2 \\ \text{so} \\ \left\{ t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \forall t \in \mathbb{R} \right\} \\ = span(\begin{bmatrix} 3 \\ 1 \end{bmatrix}) \end{aligned}$$

are both ways of representing the eigenspace.

How to actually find eigenvalues?

$$\begin{aligned} A\vec{v} = \lambda \vec{v} \\ A\vec{v} = \lambda I \vec{v} \\ (A - \lambda I)\vec{v} = \vec{0} \end{aligned}$$

This implies $A - \lambda I$ is not invertible. We are looking for values that makes this not invertible. This happens if and only if $\det(A - \lambda I) = 0$.

λ is an eigenvalue of $A \Leftrightarrow \lambda$ satisfies the characteristic equation $\det(A - \lambda I) = 0$. Also remember, the eigenvalues are the diagonal values of any triangle matrix.

We can see something interesting:

For example: Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and find the basis for the eigenspace corresponding to each eigenvalue.

we need to find λ so that

$$\begin{aligned} \det(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}) = 0 \\ \implies \lambda^2 + 1 = 0 \\ \lambda = \pm i \end{aligned}$$

Complex eigenvalues still have eigenspaces:

$$\begin{aligned} \lambda = i \\ \left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \xrightarrow{iR_1 \to R_1} \\ \left[\begin{array}{cc|c} 1 & -i & 0 \\ 1 & -i & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \vec{x} = s\begin{pmatrix} i \\ 1 \end{pmatrix} \implies \left\{\begin{pmatrix} i \\ 1 \end{pmatrix}\right\} \text{is a basis for } E_i \end{aligned}$$

$$\begin{aligned} \lambda = -i : \\ \left[\begin{array}{cc|c} i & -1 & 0 \\ 1 & i & 0 \end{array}\right] \to \\ \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \vec{x} = s\begin{pmatrix} -i \\ 1 \end{pmatrix} \implies \left\{\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\} \text{ is a basis for} E_{-i} \end{aligned}$$

We find that doing this rotation matrix preserves lines in the complex space, which we have to interpret with the scalar multiplication.

Remember: The determinant of any triangle matrix is the product of the values along the diagonal

For example: find the eigenvalues of

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

As in the 2×2 case, eigenvalues of A are zeros of the characteristic polynomial,

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & 6 & 8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\ &= (3-\lambda)(-\lambda)(2-\lambda) \end{aligned}$$

For each of the three solutions, $\lambda = 3, \lambda = 0, \lambda = 2$
We can find a basis by plugging in that lambda and then reducing the matrix. We get:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} \right\}$$

Eigenvalues and invertibility

Theorem: Suppose A has a zero eigenvalue:

$$\begin{aligned} &\implies A\vec{x} = \vec{0} \text{ for some nontrivial } \vec{x} \\ &\implies \text{ A is not invertible} \end{aligned}$$

A is invertible $\Leftrightarrow$ 0 is **not** an eigenvalue of A.