Linear Transformations

Def: A #linear_transformation from a vector space V to a vector space W is a mapping $T : V \to W$ such that

- $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$
- $T(c\vec{u}) = cT(\vec{u})$
  for $\vec{u}, \vec{v} \in V$ and c scalar
  These requirements mean that T preserves linear combinations.

Properties:

- $T(\vec{0}_v) = \vec{0}_w$
- $T(-\vec{v}) = -T(\vec{v})$
- $T(\vec{u} - \vec{v}) = T(\vec{u}) - T(\vec{v})$
  for all $\vec{u}, \vec{v} \in V$.

Composition

We can say the same for compositions of linear combinations.

$$(S \circ F)v = S(F(v))$$
$$\text{we can prove two things:}$$
$$\begin{aligned} &(S \circ F)(\vec{u} + \vec{t}) \\ &= S(F(\vec{u})) + S(F(\vec{v})) \\ &= (S \circ F)\vec{u} + (s \circ F)\vec{v} \end{aligned}$$

$$\begin{aligned} \text{and } &(S \circ F)(c\vec{u}) \\ &= S(F(c\vec{u})) \\ &= c(S(F(\vec{u}))) \\ &= c(S \circ F)(\vec{u}) \end{aligned}$$

kernel and range

The kernel of T is the set of all vectors that are mapped to zero in the output space W, ie $ker(t) = \left\{\vec{v} \in V : T(\vec{v}) = \vec{0}_w\right\}$
The range is all possible outputs, ie $T = \mathscr{P}_3 \to \mathscr{P}_2$
$T(\vec{u}) = \frac{d\vec{u}}{dx}$
$range(t) = \mathscr{P}_2$
$ker(T) = \mathscr{P}_0$

dim(ker(T))+dim(range(T))= dim(input space)

one to one and onto

One to one: every input has a unique output vector.
Onto: for a transformation $T : V \to W$, every vector in the set W has a corresponding vector in V that transforms onto it. To prove one to one, we have to look at the kernel and check that it is $\left\{\vec{0}\right\}$.

A linear transformation is called an #isomorphism if it is one to one and onto

a transformation is invertible if there is an inverse
Theorem 1: T is invertible $\Leftrightarrow$ T is one to one and onto.

Theorem 2: A linear transformation $T : V \to W$ is one to one $\Rightarrow$ $ker(T) = \left\{\vec{0}\right\}$

Proof for 2 (In the forward direction):
Assume T is one to one. By definition of kernel, $T(\vec{v}) = \vec{0}$ if $V \in ker(T)$. We know $T(\vec{0}) = \vec{0}$ so $\vec{0} \in ker(T)$. But T is one to one so if $T(\vec{v}) = \vec{0} = T(\vec{0})$ then $\vec{v} = \vec{0}$. Thus, $ker(T) = \left\{\vec{0}\right\}$.

It is an exercise to the reader to prove the backwards direction. for the second theorem.

Theorem 3: If T is one-to-one, $\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n$ is linearly independent. This implies that $T(\vec{v}_1), T(\vec{v}_2), \ldots, T(\vec{v}_n)$ is linearly independent.

Theorem 4: Let $T : V \to W$ be a linear transformation with $dim(V) = dim(W)$, then T $\text{is one to one } \Leftrightarrow \text{ T is onto .}$

is $\mathscr{P}_2 \text{ isomorphic to } \mathbb{R}^3$?

- they are the same dimension
- Can we provide a transformation tat is 1-1 and onto?

We can take the constants for each monomial in $\mathscr{P}_2$ and write it as the coordinate vector in $\mathbb{R}^3$. We get that it is onto and one to one because they are both 3 dimensional, so we can use theorem 4. We can easily show either one-to-one or onto and prove that it is an isomorphism.

Two finite dimensional vector spaces are isomorphic $\Leftrightarrow$ their dimensions are the same.

Matrix of linear transformation

If $T : V \to W$ is a linear transformation, and V, W have bases B, C, then there exists $A = \left[T(\vec{v}_1)_c, \ldots, T(\vec{v}_n)_c\right]$ that satisfies

$$A[\vec{v}]_b = T(\vec{v})_c$$

side note: from this arises change of basis if the transformation is the identity.

Example: let $D : \mathscr{P}_3 \to \mathscr{P}_2$ be the differential operator $D(p(x)) = p'(x)$

Find the matrix A of D with respect to bases $B=\{1, x, x^2, x^3\}$ for $\mathscr{P}_3$ and $C=\{1, x, x^2\}$ for $\mathscr{P}_2$.

A takes in vectors in $\mathbb{R}^4$ and spits out vectors in $\mathbb{R}^3$, so A is a 3×4 matrix.
We apply D to every element of B.
$D(1)=0, D(x)=1, D(x^2)=2x, D(x^3)=3x^2$
Now we find the coordinate vectors of each of these with respect to c.
We get

$$A=\left[\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\2\\0\end{pmatrix},\begin{pmatrix}0\\0\\3\end{pmatrix}\right]$$

$$A=\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}$$

We can now take derivatives of anything in $\mathscr{P}^3$
Take, for example, $D(7-2x+9x^2-4x^3)=-2+18x-12x^2$.
We can compute this with matrices by taking the coordinates from the input space and multiplying by A.
EX:

$$\begin{bmatrix}0&1&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}\begin{pmatrix}7\\-2\\9\\-4\end{pmatrix}=\begin{pmatrix}-2\\18\\-12\end{pmatrix}$$

A question: Is this invertible?

Theorem: T is invertible $\Leftrightarrow$ the matrix A is invertible.

Note that

$$T = I \circ T = T \circ I \text{ so}$$
$$[T]_{b \leftarrow c} = [I]_{b \leftarrow c}[T]_{c \leftarrow c}$$
$$[T]_{b \leftarrow c} = [T]_{b \leftarrow b}[I]_{b \leftarrow c}$$

We have a somewhat familiar - (similar!) - equation that arises

$$[T]_{c \leftarrow c} = P_{b \leftarrow c}^{-1} T_{b \leftarrow b} P_{b \leftarrow c}$$

We have finished the fundamental theorem of invertible matrices:

A is invertible

Transformations:
T is invertible
T is one to one
T is onto
$\ker(T) = \vec{0}$
range(T)=W

Solutions + Matrix forms:
$a\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$
$A\vec{x} = \vec{0}$ has only the trivial solution
The RREF of A is I_n
A is a product of elementary matrices

Columns:
The column vectors of A are linearly independent
The column vectors of A span $\mathbb{R}^n$
The column vectors of A for a basis for $\mathbb{R}^n$

Subspaces:
Rank(A) $= n$
Nullity(A) $= 0$

Rows:

The row vectors of A are linearly indepenent

The row vectors of A span $\mathbb{R}^n$

The row vectors of A form a basis for $\mathbb{R}^n$