Change of basis

Exercise

Find a basis for P_1 the set of *all* polynomials
Every polynomial is a linear combination of monomials,
$a_0 + a_1x + a_2x^2 + a_3x^3 \dots a_nx^n$
So $B = \{1, x, x^2, x^3, \dots, x^n\}$ is a spanning set for P. It is linearly independent and also infinitely long ... cool.

Lets write a proof by contradiction that the set of al monomials is linearly independent:
Suppose there is some finite set of B with n vectors that is linearly *dependent* $\{x^{P_1}, x^{P_2}, \dots, x^{Pn}\}$ where $P_1 < p_2 < p_3 \dots < p_n$. Then there are some scalars (not all zero), $c_1, c_2, \dots, c_n$ so that
$c_1x^{P_1} + c_2x^{P_2} + \dots + x^{P_n} = 0$
This polynomial is 0 for all values of x. *BUT* the fundamental theorem of algebra says that a nonzero polynomial can have at most n roots. Thus, the polynomial would have to be the zero polynomial with $c_1 = c_2 = \dots = c_n = 0$. This is a contradiction, $\implies$ any finite set of B is linearly independent $\implies$ B is linearly independent.

Dimension

Def:

1. A vector is #finite_dimensional if it has a basis consisting of finitely many vectors
2. A vector space is #inifinite_dimensional if it has no finite basis
3. The vector space $\vec{0}$ is #zero_dimensional

Coordinates

Lets move to something that is more concrete. Lets say we have a finite basis $B = \left\{\vec{b}_1, \ldots, \vec{b}_n\right\}$ for the vector space V.then any $\vec{v} \in V$ can be written uniquely as a linear combination of $\vec{v} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$

The unique coefficient $c_1, \ldots, c_n$ are called the #coordinates of $\vec{v}$ relative to the basis β, (or #b-coordinates). The entries in the coordinate vector are the coordinates of this combination. We can see this easily with the x y plane - to specify a point $(2\vec{x}, 3\vec{y})$ we just write out $(2, 3)$. If instead we had a rotated representation, we would still be specifying points with the coefficients scaling each vector.

Take the vectorspace represented by $\begin{bmatrix} a \\ b \\ b \\ c \end{bmatrix}$

We have a basis

$$\beta = \left\{\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]\right\}$$

We can specify coordinates of a point $\begin{bmatrix} 2 \\ -7 \\ -7 \\ 8 \end{bmatrix}$ with $\begin{bmatrix} 2 \\ -7 \\ 8 \end{bmatrix}$, because those coordinates multiplied by the basis makes the point. We can make any basis and choose any coordinates to form that coordinate as long as β is a basis for V.

What are the coordinates of

$\begin{bmatrix}2 & -7\\ -7 & 8\end{bmatrix} \$\$ inthebasis$

\beta=\left{ \begin{bmatrix}
1 & 0 \
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 1 \
1 & 0 \
\end{bmatrix},
\begin{bmatrix}
0 & 0 \
0 & 1
\end{bmatrix}
\right}

The same, $\begin{bmatrix}2\\-7\\8\end{bmatrix}$! ### Some facts about co

\begin{align}
\left{ \vec{u}{1},\dots,\vec{u}{n} \right}\text{ is linearly independent } \
\Leftrightarrow\left{ [\vec{u}_{1}]{b},\dots,[\vec{u}{n}]_{b} \right} \text{ is linearly independent in }\mathbb{R}^{n}
\end{align}

(make a careful note of the difference between k and n in the above). ##

C=\left{ \begin{bmatrix}
\begin{pmatrix}
0\1
\end{pmatrix}, & \begin{pmatrix}
2 \ 3

\end{pmatrix}
\end{bmatrix} \right}

$$Wecanrepresentxnowasthesolutionto$$

\begin{bmatrix}
4 \ -1
\end{bmatrix} = c*{1}\begin{pmatrix}*
0\1
\end{pmatrix}+c{2}\begin{pmatrix}
2\3
\end{pmatrix}

$$Wecanwritethisoutas$$

\begin{bmatrix}
4\-1
\end{bmatrix}=\begin{bmatrix}
2c*{2}\c*{1}+3c_{2}
\end{bmatrix}

$$Or,inaslightlynicerandsimultaneouslygrosserway,$$

\begin{bmatrix}
4\-1
\end{bmatrix}=\begin{bmatrix}
0 & 2 \
1 & 3
\end{bmatrix}\begin{bmatrix}
c*{1} \ c*{2}
\end{bmatrix}

$Wecanseethat\$c_2 = 2\$, meaning\$c_1 = -7\$. However, thiswasalongpro$