Complex Eigenvalues

Complex Eigenvalues:

Recall that we can find eigenspaces. IE:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$$

with $\lambda = 3$.
Steps:

$$\begin{aligned} &(A - \lambda I | 0) \\ &\left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ \implies &x_1 = 3x_2 \\ &\text{so} \\ &\left\{ t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \forall t \in \mathbb{R} \right\} \\ &= span(\begin{bmatrix} 3 \\ 1 \end{bmatrix}) \end{aligned}$$

are both ways of representing the eigenspace.

How to actually find eigenvalues?

$$\begin{aligned} A\vec{v} &= \lambda \vec{v} \\ A\vec{v} &= \lambda I \vec{v} \\ (A - \lambda I)\vec{v} &= \vec{0} \end{aligned}$$

This implies $A - \lambda I$ is not invertible. We are looking for values that makes this not invertible. This happens if and only if $\det(A - \lambda I) = 0$.

λ is an eigenvalue of $A \Leftrightarrow \lambda$ satisfies the characteristic equation $\det(A - \lambda I) = 0$. Also remember, the eigenvalues are the diagonal values of any triangle matrix.

We can see something interesting:

For example: Find the eigenvalues of $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and find the basis for the eigenspace corresponding to each eigenvalue.

we need to find λ so that

$$\begin{aligned} \det(\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix}) &= 0 \\ \implies \lambda^2 + 1 &= 0 \\ \lambda &= \pm i \end{aligned}$$

Complex eigenvalues still have eigenspaces:

$$\lambda = i$$

$$\begin{bmatrix} -i & -1 & | & 0 \\ 1 & -i & | & 0 \end{bmatrix} \xrightarrow{iR_1 \to R_1}$$

$$\begin{bmatrix} 1 & -i & | & 0 \\ 1 & -i & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -i & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\vec{x} = s \begin{pmatrix} i \\ 1 \end{pmatrix} \implies \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix} \right\} \text{is a basis for } E_i$$

$$\lambda = -i:$$

$$\begin{bmatrix} i & -1 & | & 0 \\ 1 & i & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & i & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\vec{x} = s \begin{pmatrix} -i \\ 1 \end{pmatrix} \implies \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for} E_{-i}$$

We find that doing this rotation matrix preserves lines in the complex space, which we have to interpret with the scalar multiplication.

Remember: The determinant of any triangle matrix is the product of the values along the diagonal

For example: find the eigenvalues of $$
A = \begin{bmatrix}
3 & 6 & -8 \
0 & 0 & 6 \
0 & 0 & 2
\end{bmatrix}

$Asinthe\$2 \times 2\$case, eigenvaluesof\$A\$arezerosofthecharacteristicpol$

\begin{align}
\det(A-\lambda I)=\det \begin{bmatrix}
3-\lambda & 6 & 8 \
0 & -\lambda & 6 \
0 & 0 & 2-\lambda
\end{bmatrix} \
= (3-\lambda)(-\lambda)(2-\lambda)
\end{align}

$Foreachofthethreesolutions, \$\lambda = 3, \lambda = 0, \lambda = 2\$Wecanfindabasisby$

\left{
\begin{bmatrix}
1 \ 0 \ 0\end{bmatrix},
\begin{bmatrix}
-2 \ 1 \ 0
\end{bmatrix},\begin{bmatrix}
-10 \ 3 \ 1
\end{bmatrix}\right}

Eigenvalues and invertibility Theorem: Suppose A has a zero

\begin{align}
\implies &A\vec{x}=\vec{0} \text{ for some nontrivial }\vec{x} \
\implies &\text{ A is not invertible }
\end{align}

$$\$A\$isinvertible\$ \Leftrightarrow \$0is**not**aneigenvalueofA.$$