Abstract vector spaces

Linear combinations

A linear combination looks the same as before, $c_1v_1 + \cdots + c_nv_n$. We know that this holds, as the vector space is closed under addition and scalar multiplication.

The span of a set is the set of all linear combinations. If $v = span(s), v$ is a spanning set for V.

We can now ask some fun questions:
is $3 - 2x^2$ in the span of $p(x) = 1 - x + x^2$ and $q(x) = 2 + x - 3x^2$
We can write out the polynomials as vectors and check if there exists c_1, c_2 such that

$$c_1 \begin{bmatrix} 1x^2 \\ -x \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3x^2 \\ x \\ 2 \end{bmatrix} = \begin{bmatrix} -2x^2 \\ 0 \\ 3 \end{bmatrix}$$

We can rewrite this more neatly

$$\begin{bmatrix} c_1 - 3c_2 = -2 \\ -c_1 + c_2 = 0 \\ c_1 + 2c_2 = 3 \end{bmatrix}$$

aka c1 c2 value we want

$$\begin{bmatrix} 1 & -3 & | & -2 \\ -1 & 1 & | & 0 \\ 1 & 2 & | & 3 \end{bmatrix}$$

Can we find a solution? We row reduce and get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

We see that $c_1 = 1, c_2 = 1$. So $3 - 2x^2 = 1(p(x)) + 1(q(x))$

Also, we can see that we don't span all of P_2, as we only have 2 equations and need to represent 3 variables c, x, x^2.

Subspaces:

Let v be a vector space an W be a nonempty subset of V. W is a #subspace if and only if :

- it contains the zero vector: $\vec{0} \in W$ where $\vec{0}$ is the zero vector of V
- closed under addition: if $\vec{u}, \vec{v} \in W$, then $\vec{u} + \vec{v} \in W$
- closed under scalar multiplication: if $\vec{u} \in W$ and C is scalar, then $c\vec{u} \in W$

Lets do some practice:

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad = 0,\ a, b, c, d \in \text{Re} \right\}$$

Is this a subspace of $M_{2,2}$? (answer - no)
For 2×2 matrices, we add and multiply component wise.

1:

This is a non empty set, as $\downarrow$

2:

we have the zero vector in the set $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

3:

we are closed under scalar multiplication since

$$s\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}, ab = 0, (sa)(sb) = 0, s(ab) = 0, s(0) = 0$$

so we are closed under multiplication.

4:

BUT! not closed under addition.

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

$(a_1, d_1) = 0 \text{ and } a_2 b_2 = 0$ does not imply
$(a_1 + a_2)(d_1 + d_2) = 0$
Because this term expands to

$$(a_1 d_1) + (a_1 d_2) + (a_2 d_1) + (a_2 d_2)$$
$$0 + a_1 d_2 + a_2 d_1 + 0 \text{ is not always } 0$$

Conclusion:

We didn't check all the boxes. Therefore, this is a subset of $M_{2,2}$ but not a subspace of $M_{2,2}$.