Inner product spaces

Inner products

Def: $\vec{u}, \vec{v}, \vec{w}$ are vectors in a vector space V and c is a real scalar. an #inner_product is an operation that assigns a real number $\langle \vec{u}, \vec{v} \rangle$ to each pair of vectors $\vec{u}, \vec{v}$ and is

- Symmetric:
  - $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
  - $\langle \vec{u}, \vec{v} + \vec{w} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{u}, \vec{w} \rangle$
- Linear:
  - $\langle c\vec{u}, \vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$
- Positive semi-definite:
  - $\langle \vec{u}, \vec{v} \rangle \geq 0$ with $\langle \vec{u}, \vec{u} \rangle = 0$ if and only if $\vec{u} = \vec{0}$.

We can define length, distance, and orthogonality for any inner product.

Def: let $\langle \vec{u}, \vec{v} \rangle$ be an inner product on vector space v. The length or norm of $\vec{v} \in V$ is $\left|\vec{v}\right| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$. A unit vector satisfies $\left|\vec{v}\right| = 1$. The distance between $\vec{u}$ and $\vec{v}$ is $\left|\vec{u} - \vec{v}\right|$. We say $\vec{u}$ and $\vec{v}$ are orthogonal if $\langle \vec{u}, \vec{v} \rangle = 0$.

IE, the distance between $\sin(x) \text{ and } \cos(x) \in [0, 2\pi]$ is

$$
\begin{aligned}
\langle \sin(x), \cos(x) \rangle &= \int_0^{2\pi} \sin(x)\cos(x)\,dx \\
&= \frac{1}{2}\int_0^{2\pi} 2\sin(x)\cos(x)\,d \\
&= \frac{1}{2}\int_0^{2\pi} \sin(2x)\,d \\
&= 0
\end{aligned}
$$

so $\sin(x)$ and $\cos(x)$ are orthogonal in $c[0, 2\pi]$ with respect to this inner product.

We can also generalize:

- Pythagorean Theorem: If $\vec{u}, \vec{v}$ are orthogonal, $\left|\vec{u}\right|^2 + \left|\vec{v}\right|^2 = |\vec{u} + \vec{v}|^2$
- Cauchy - Schwarz inequality: $\left|\langle \vec{u}, \vec{v} \rangle\right| \leq |\vec{u}||\vec{v}|$
- Triangle inequality: $\left|\vec{u} + \vec{v}\right| \leq |\vec{u}| + |\vec{v}|$

Gram-Schmidt

We can even generalize the Gram-Schmidt process to general vector spaces with an inner product!

For example: Find an orthogonal basis for $W = span\left\{2, x+1, x^2 - 1\right\}$
We take $v_1 = y_1 = 2$

$$
\begin{aligned}
v_2 &= y_2 - \frac{\left\langle \vec{y}_2, \vec{v}_1 \right\rangle}{\left\langle \vec{v}_1, \vec{v}_1 \right\rangle} \vec{v}_1 \\
&= c + 2 - \frac{\left(\int_0^1 2(x+1)\, dx \right)}{\int_0^1 2 * 2\, dx} 2 \\
&= x + 1 - \frac{3}{2} = x - \frac{1}{2} \\
v_3 &= y_3 - \frac{\langle v_3, v_1 \rangle v_1}{\langle v_1, v_1 \rangle} - \frac{\langle y_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2
\end{aligned}
$$

$$
= x^2 - 1 - \frac{\left(\int_0^1 2(x^2 - 1)\, dx \right)}{\int_0^1 4\, dx} 2 - \frac{\left(\int_0^1 \left(x - \frac{1}{2} \right) (x^2 - 1)\, dx \right)}{\left(\int_0^1 \left(x - \frac{1}{2} \right)^2 dx \right)} \left(x - \frac{1}{2} \right)
$$

$$
= x^2 - x + \frac{1}{6}
$$