A Sampling Kaczmarz-Motzkin Algorithm for Linear Feasibility

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These problems arise in machine learning classification, *support-vector machines* (Boser, Guyon, Vapnik 1992), (Cortes, Vapnik 1995).

Projection Methods

If $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ is nonempty, these methods construct an approximation to an element of P:

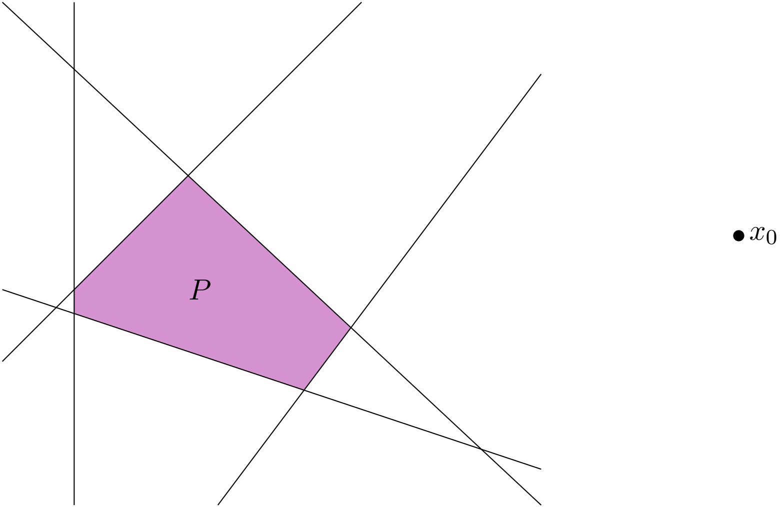
1. Motzkin's Relaxation Method(s)
2. Randomized Kaczmarz Method
3. Sampling Kaczmarz-Motzkin Method (SKM)

Motzkin's Relaxation Method(s)

Given $x_0 \in \mathbb{R}^n$, fix $0 < \lambda \leq 2$ and iteratively construct approximations to P:

1. If x_k is feasible, stop.
2. Choose $i_k \in [m]$ as $i_k := \underset{i \in [m]}{\operatorname{argmax}}\, a_i^T x_{k-1} - b_i$.
3. Define $x_k := x_{k-1} - \lambda \frac{a_{i_k}^T x_{k-1} - b_{i_k}}{||a_{i_k}||^2} a_{i_k}$.
4. Repeat.

Motzkin's Method



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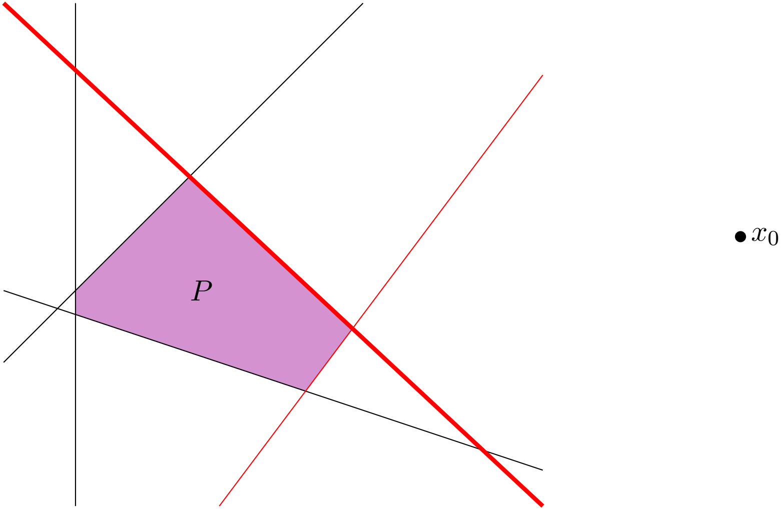

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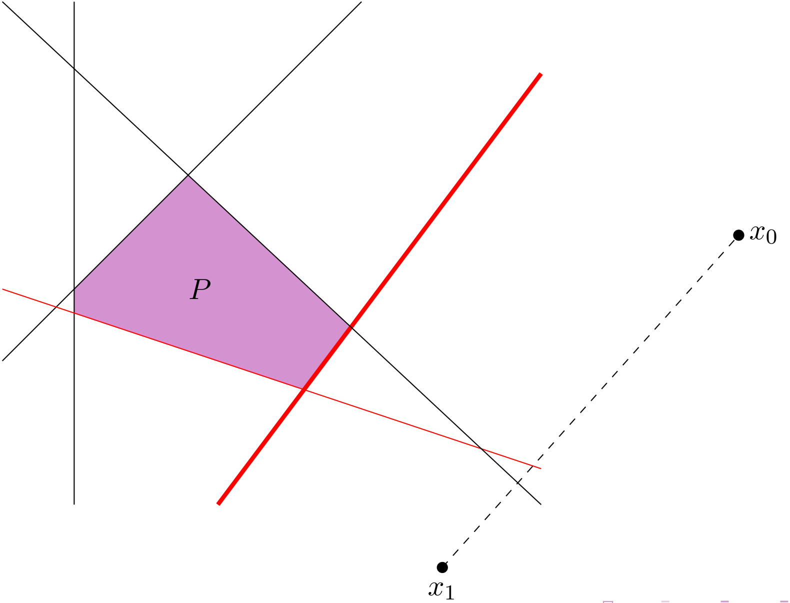

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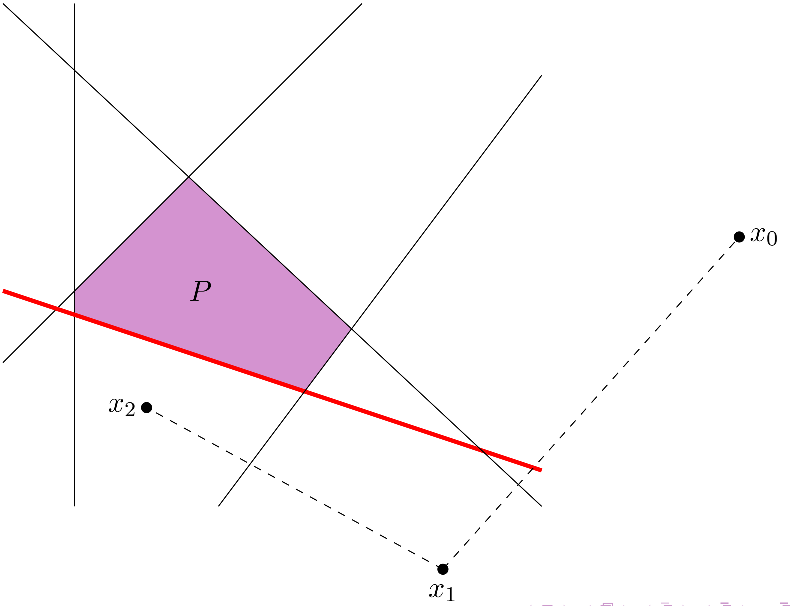

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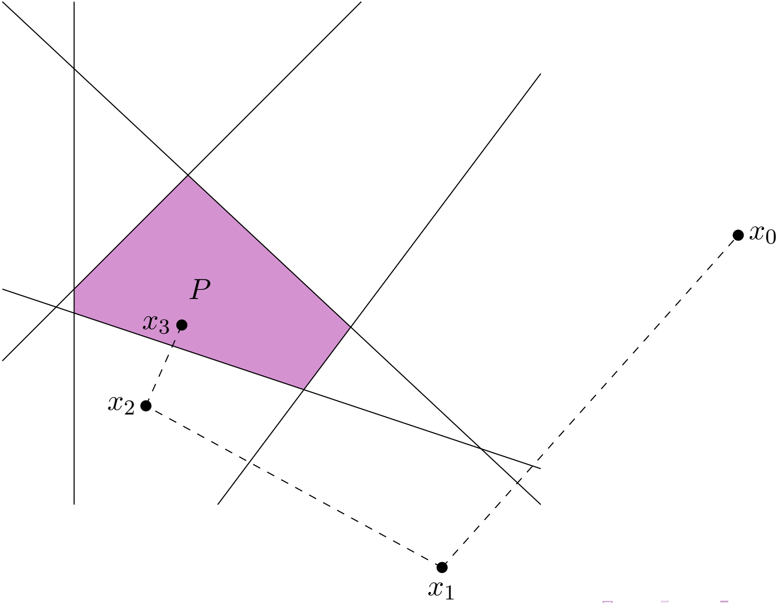
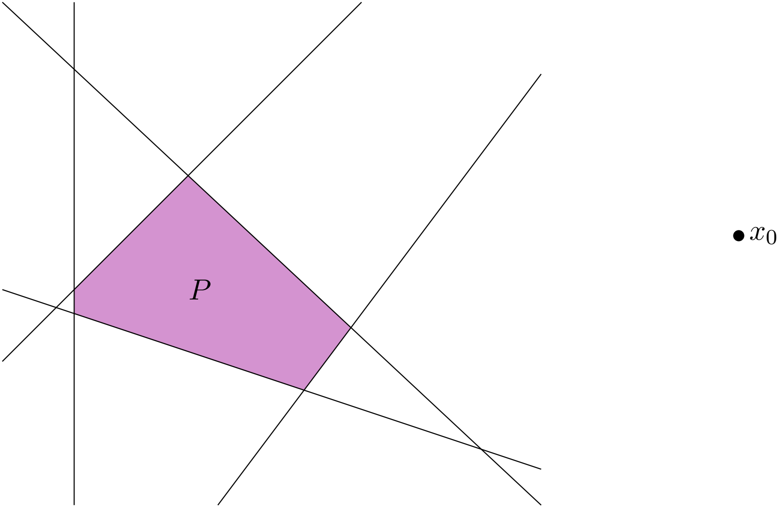


Randomized Kaczmarz Method

Given $x_0 \in \mathbb{R}^n$, iteratively construct approximations to P:

1. If x_k is feasible, stop.
2. Choose $i_k \in [m]$ with probability $\frac{||a_{i_k}||^2}{||A||_F^2}$.
3. Define $x_k := x_{k-1} - \frac{(a_{i_k}^T x_{k-1} - b_{i_k})^+}{||a_{i_k}||^2} a_{i_k}$.
4. Repeat.

Kaczmarz Method



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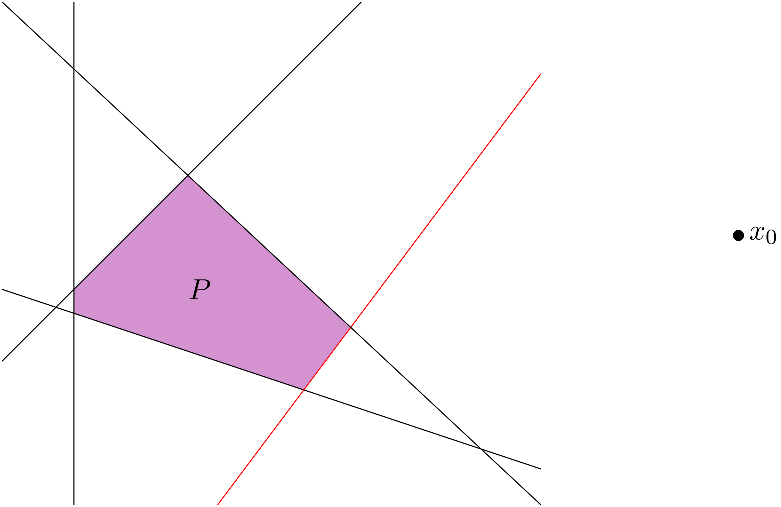

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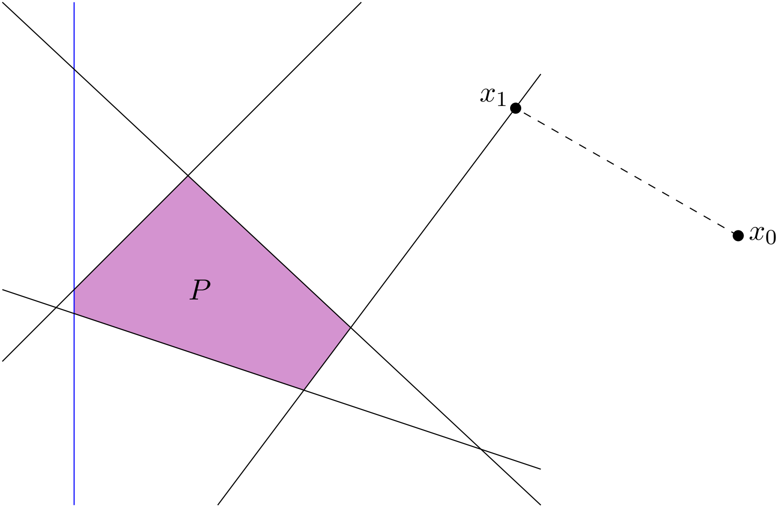

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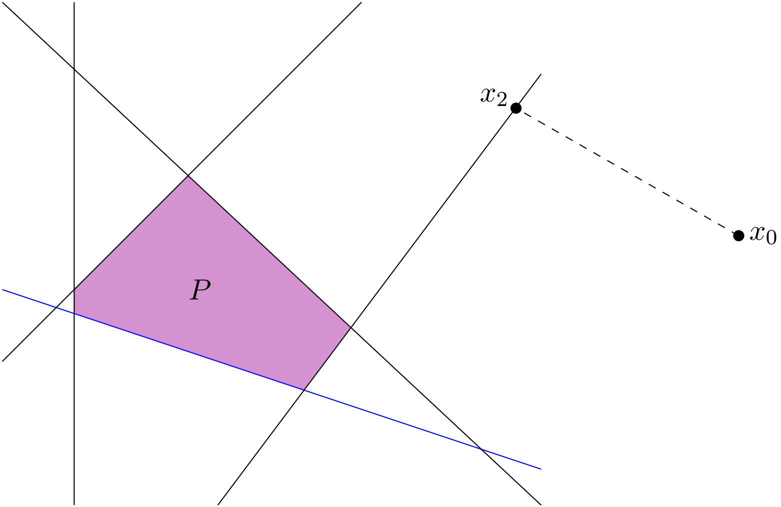

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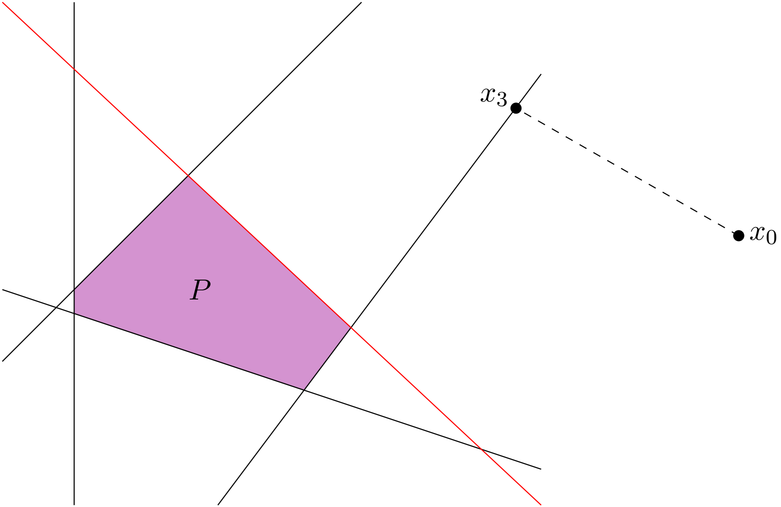

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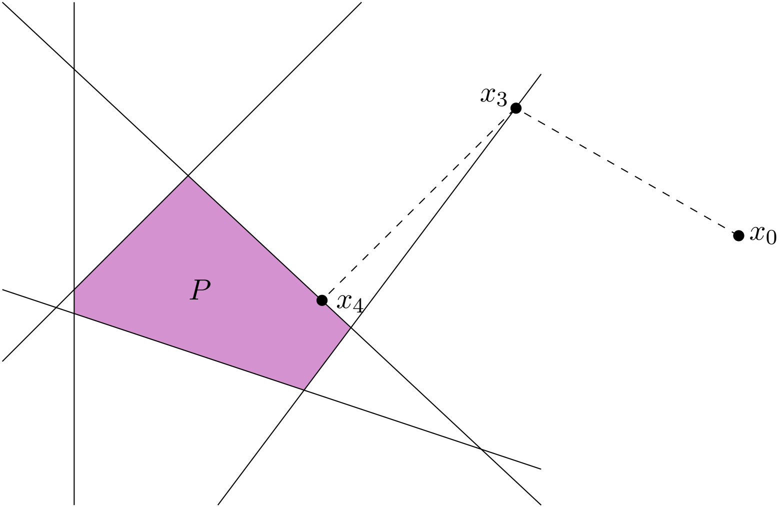

A Hybrid Method (SKM)

Given $x_0 \in \mathbb{R}^n$, fix $0 < \lambda \leq 2$ and iteratively construct approximations to P in the following way:

1. If x_k is feasible, stop.
2. Choose $\tau_k \subset [m]$ to be a sample of size β constraints chosen uniformly at random from among the rows of A.
3. From among these β rows, choose $i_k := \underset{i \in \tau_k}{\operatorname{argmax}}\, a_i^T x_{k-1} - b_i$.
4. Define $x_k := x_{k-1} - \lambda \frac{(a_{i_k}^T x_{k-1} - b_{i_k})^+}{||a_{i_k}||^2} a_{i_k}$.
5. Repeat.

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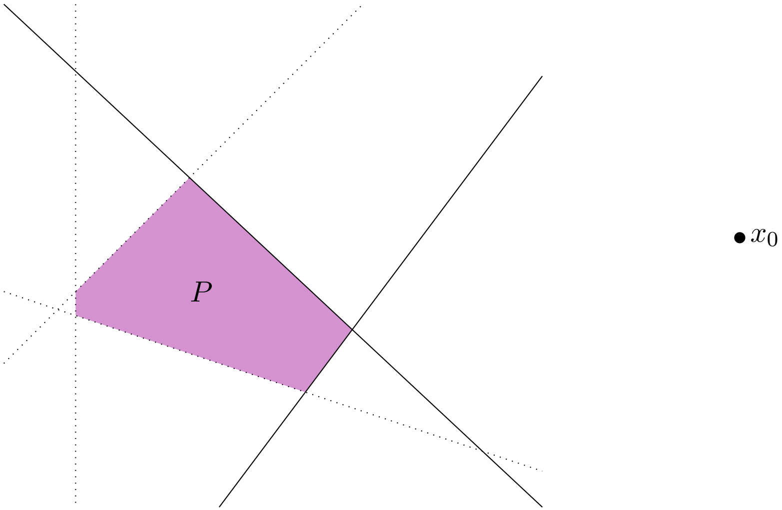

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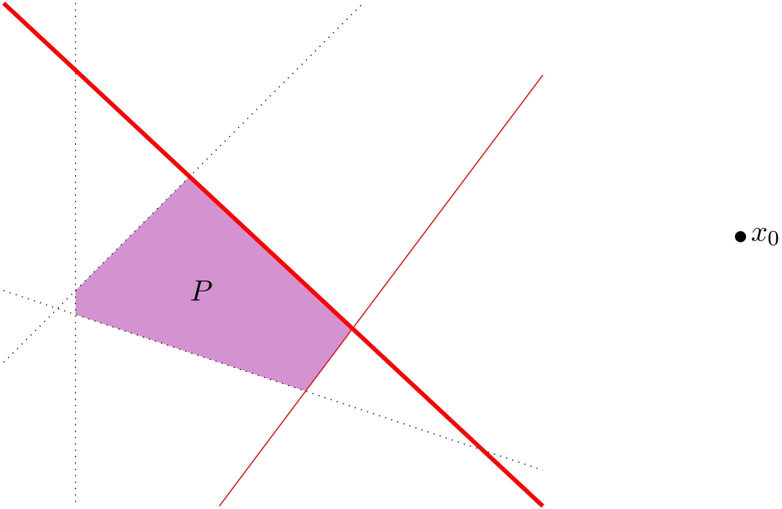

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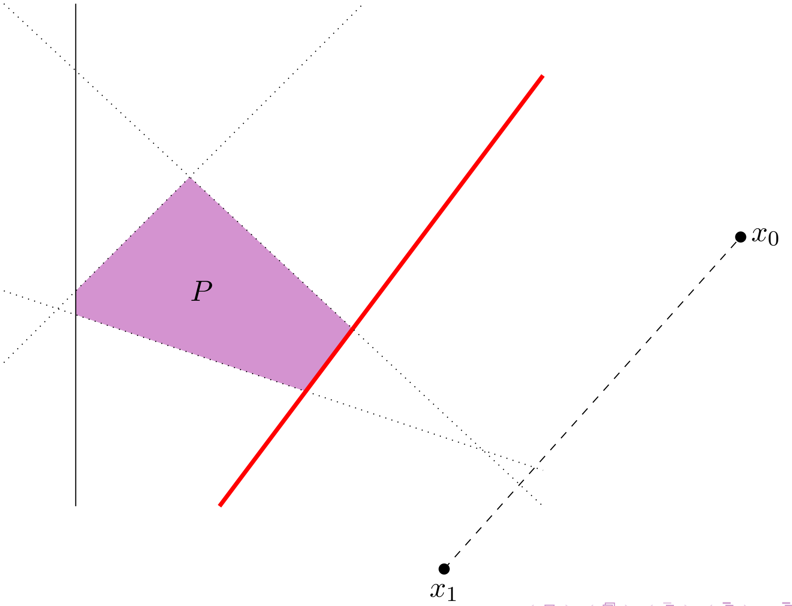

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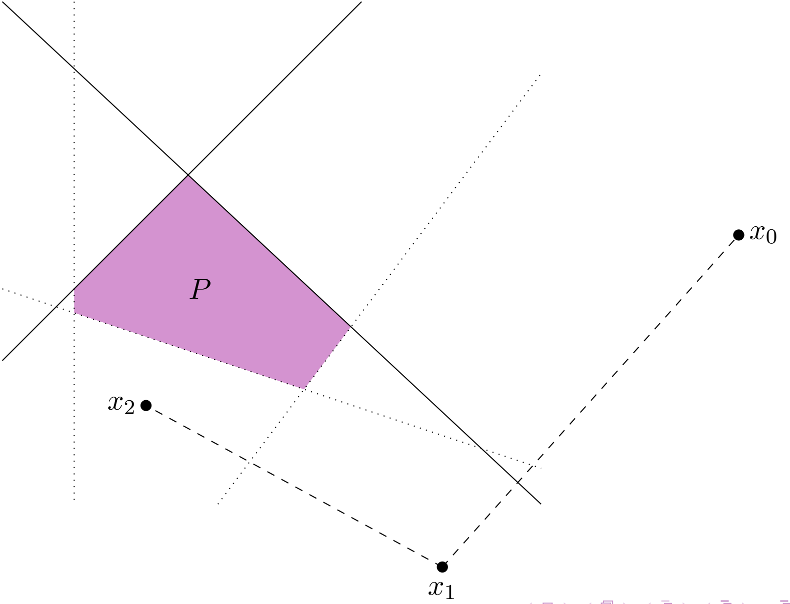

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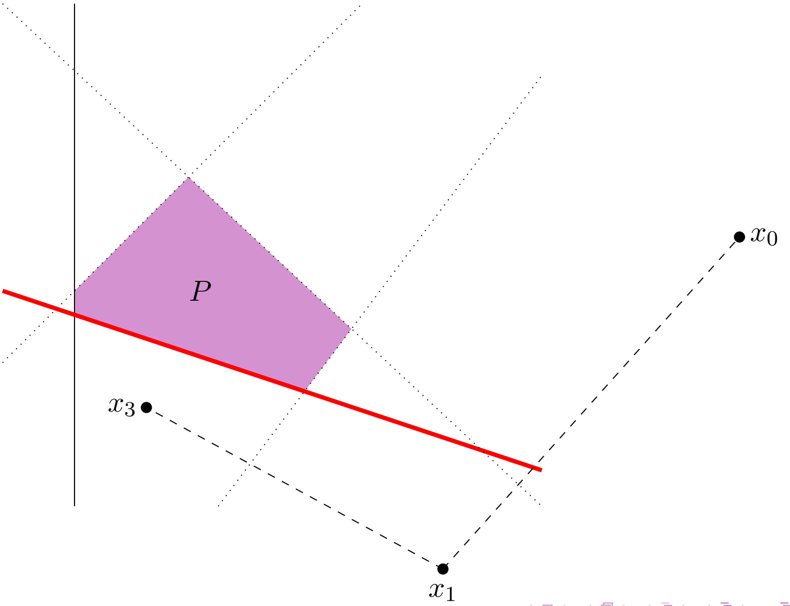

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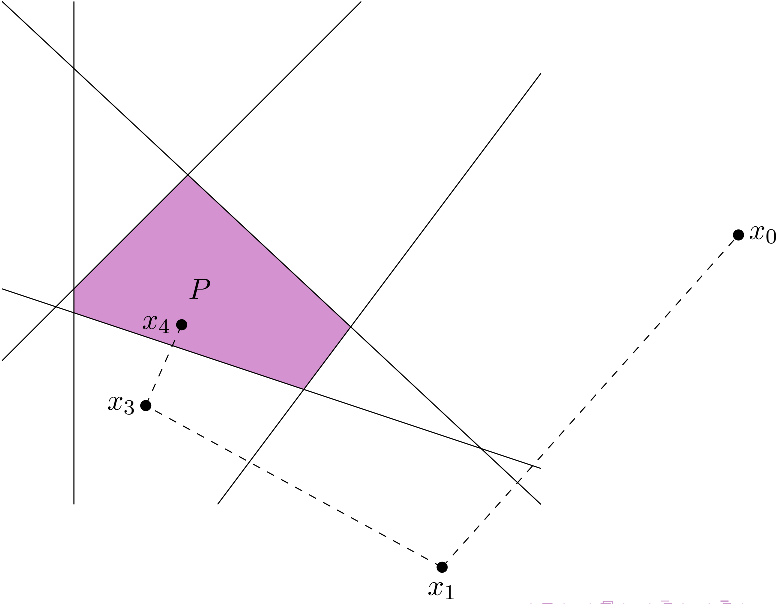
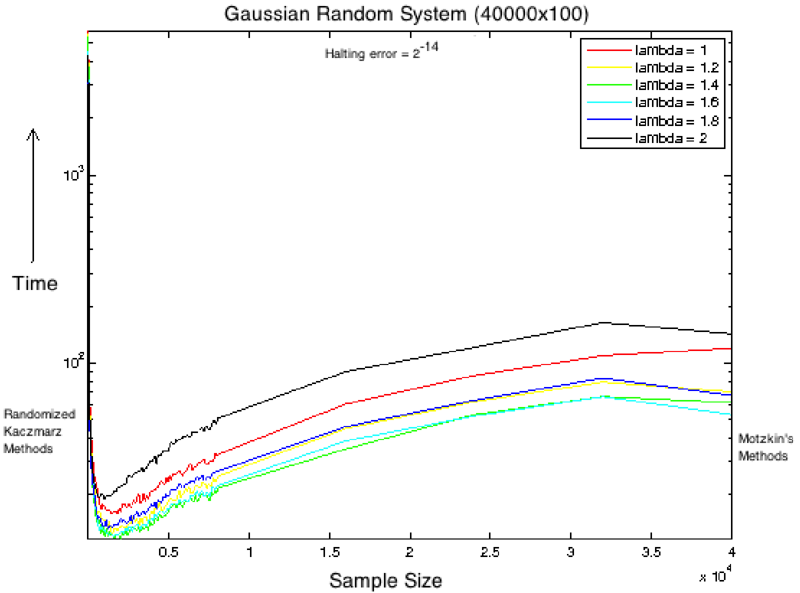


Experimental Results



Previously Known Convergence Rates

Theorem (Relaxation Method, Agmon 1954)

For a normalized system, $||a_i|| = 1$ for all $i = 1, ..., m$, where $P := \{x | Ax \leq b\}$ is nonempty, the relaxation methods converge linearly:

$$d(x_k, P)^2 \leq \left(1 - \frac{2\lambda - \lambda^2}{mL_2^2}\right)^k d(x_0, P)^2.$$

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The *Hoffman constant*, L_2 is an error bound defined as the minimum constant that satisfies

$$d(x, P) \leq L_2 ||(Ax - b)^+||_2.$$

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Theorem (Random Kaczmarz Method, Lewis, Leventhal 2008)

If $P := \{x|Ax \leq b\}$ is nonempty then the Randomized Kaczmarz method with relaxation parameter λ converges linearly in expectation:

$$\mathbb{E}[d(x_k, P)^2] \leq \left(1 - \frac{2\lambda - \lambda^2}{||A||_F^2 L_2^2}\right)^k d(x_0, P)^2.$$

SKM Method Convergence Rate

Theorem (De Loera, H., Needell)

If the feasible region (for normalized A) is nonempty, then the SKM methods with samples of size β converge at least linearly in expectation: In each iteration,

$$\mathbb{E}[d(x_k, P)^2] \leq \left(1 - \frac{2\lambda - \lambda^2}{S_{k-1} L_2^2}\right) d(x_{k-1}, P)^2$$

where $S_{k-1} = \max\{m - s_{k-1}, m - \beta + 1\}$ and s_{k-1} is the number of constraints satisfed by x_{k-1}. Then,

$$\mathbb{E}[d(x_k, P)^2] \leq \left(1 - \frac{2\lambda - \lambda^2}{m L_2^2}\right)^k d(x_0, P)^2.$$

Improved Rate

Theorem (De Loera, H., Needell)

If the feasible region, $P = \{x|Ax \leq b\}$ is nondegenerate (generic) and nonempty (for normalized A), then an SKM method with samples of size $\beta \leq m - n$ is guaranteed an increased convergence rate after some K:

$$\mathbb{E}[d(x_k, P)^2] \leq \left(1 - \frac{2\lambda - \lambda^2}{mL_2^2}\right)^K \left(1 - \frac{2\lambda - \lambda^2}{(m - \beta + 1)L_2^2}\right)^{k-K} d(x_0, P)^2.$$

Finiteness of Motzkin's Method

Theorem (Goffin 1980, Telgen 1982)

Either the relaxation method detects feasibility of the rational, normalized system, $Ax \leq b$ (A, b have binary encoding size Σ), within $k = \left\lceil \frac{2^{4\Sigma}}{n\lambda(2-\lambda)} \right\rceil$ iterations or the system is infeasible.*

*with $x_0 = 0$

Expected Finiteness of SKM methods

Theorem (De Loera, H., Needell)

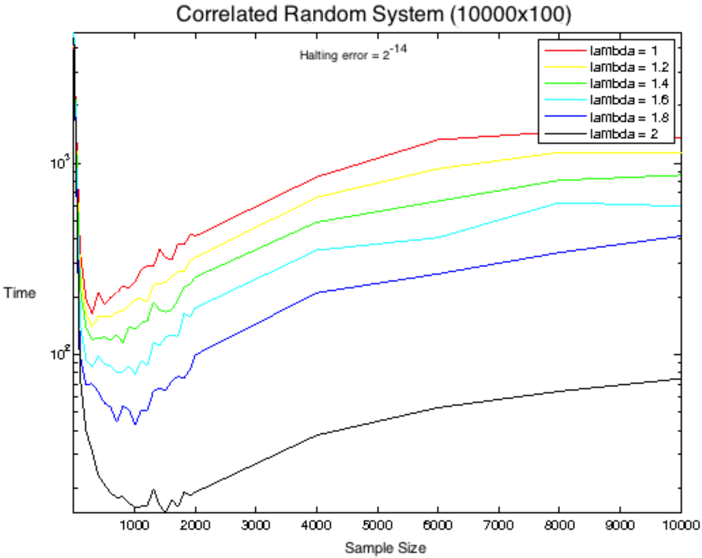
If the rational, normalized system $Ax \leq b$ is feasible (A, b with binary encoding size Σ), then the SKM methods can detect feasibility.

The expected number of steps required for the SKM methods with projection parameter $0 < \lambda < 2$ to detect feasibility is no more than*

$$\left\lceil \frac{4\Sigma - 4 - \log n}{\log\left(\frac{m^2 L_2^2}{m^2 L_2^2 - (2\lambda - \lambda^2)}\right)} \right\rceil.$$

*with $x_0 = 0$

Conclusions



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2. Describe the K after which the convergence rate is guaranteed to be improved.
3. Explore connections of SKM to variants of randomized coordinate descent in the dual variable space.

Acknowledgements

Thanks to you for attending!

Are there any questions?

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