Iterative Projection Methods

for noisy and corrupted systems of linear equations

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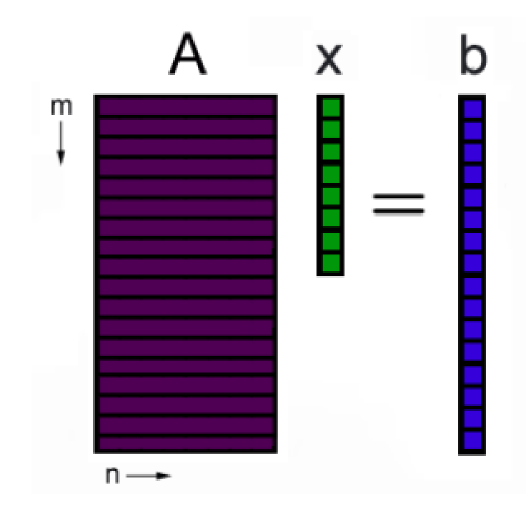
joint with Jesús De Loera and Deanna Needell
https://arxiv.org/abs/1605.01418 (SISC 2017) and forthcoming articles

Setup

We are interested in solving highly overdetermined systems of equations, $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$ and $m >> n$. Rows are denoted $\mathbf{a}_i^T$.



Projection Methods

If $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\}$ is nonempty, these methods construct an approximation to an element:

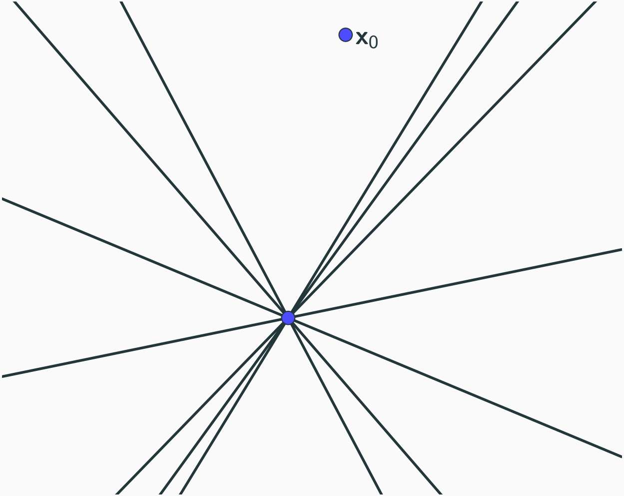
1. Randomized Kaczmarz Method
2. Motzkin's Method(s)
3. Sampling Kaczmarz-Motzkin Methods (SKM)

Randomized Kaczmarz Method

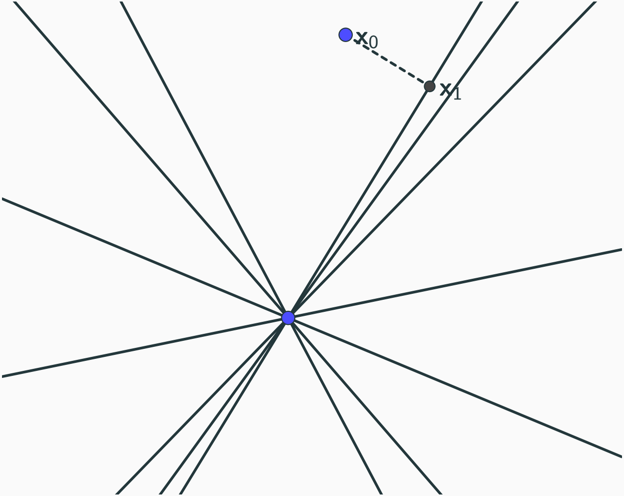
Given $\mathbf{x}_0 \in \mathbb{R}^n$:

1. Choose $i_k \in [m]$ with probability $\frac{\|\mathbf{a}_{i_k}\|^2}{\|A\|_F^2}$.
2. Define $\mathbf{x}_k := \mathbf{x}_{k-1} + \frac{b_{i_k} - \mathbf{a}_{i_k}^T \mathbf{x}_{k-1}}{||\mathbf{a}_{i_k}||^2} \mathbf{a}_{i_k}$.
3. Repeat.

Kaczmarz Method



Kaczmarz Method



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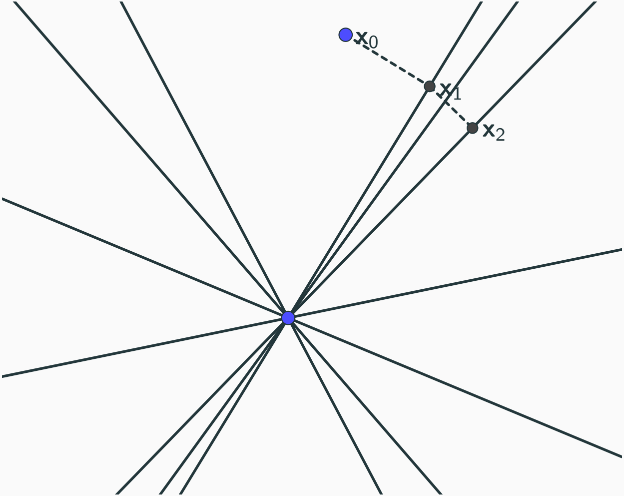

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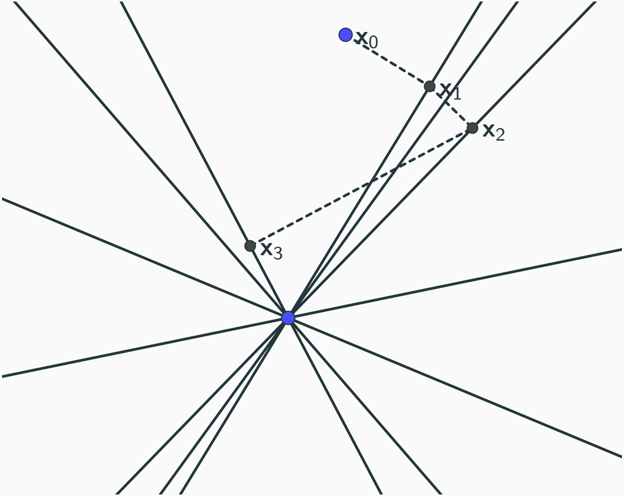


Convergence Rate

Theorem (Strohmer - Vershynin 2009)
Let $\mathbf{x}$ *be the solution to the consistent system of linear equations* $A\mathbf{x} = \mathbf{b}$*. Then the Random Kaczmarz method converges to* $\mathbf{x}$ *linearly in expectation:*

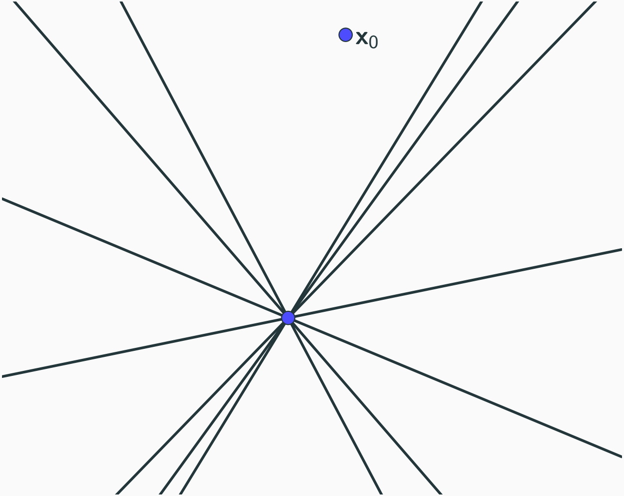
$$\mathbb{E}||\mathbf{x}_k - \mathbf{x}||_2^2 \leq \left(1 - \frac{1}{||A||_F^2 ||A^{-1}||_2^2}\right)^k ||\mathbf{x}_0 - \mathbf{x}||_2^2.$$

Motzkin's Relaxation Method(s)

Given $\mathbf{x}_0 \in \mathbb{R}^n$:

1. If $\mathbf{x}_k$ is feasible, stop.
2. Choose $i_k \in [m]$ as $i_k := \operatorname*{argmax}_{i \in [m]} |\mathbf{a}_i^T \mathbf{x}_{k-1} - b_i|$.
3. Define $\mathbf{x}_k := \mathbf{x}_{k-1} + \frac{b_{i_k} - \mathbf{a}_{i_k}^T \mathbf{x}_{k-1}}{||\mathbf{a}_{i_k}||^2} \mathbf{a}_{i_k}$.
4. Repeat.

Motzkin's Method



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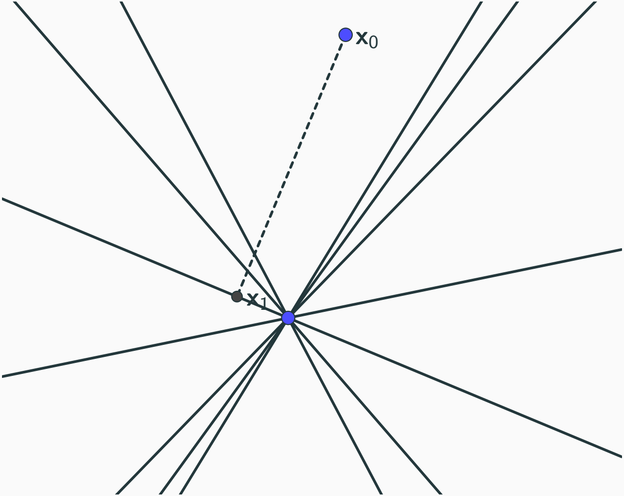

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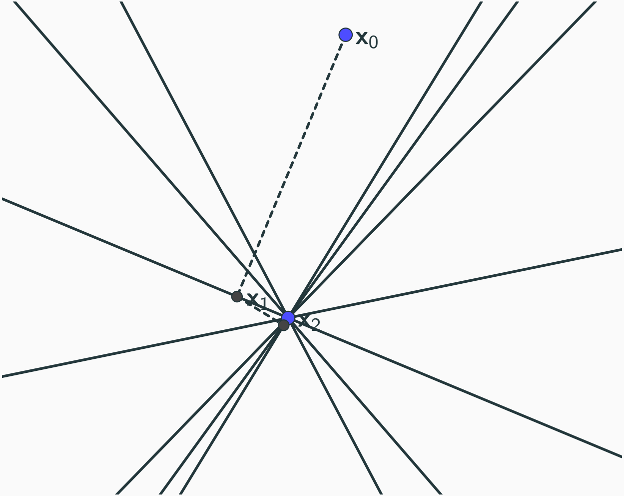


Convergence Rate

Theorem (Agmon 1954)

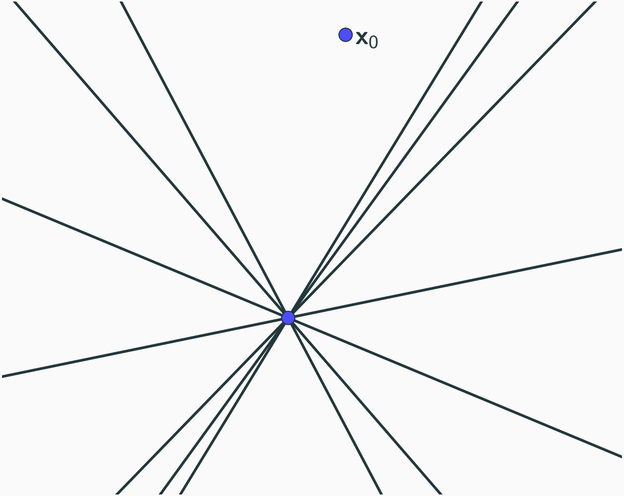
For a consistent, normalized system, $\|\mathbf{a}_i\| = 1$ for all $i = 1, ..., m$, Motzkin's method converges linearly to the solution $\mathbf{x}$:

$$\|\mathbf{x}_k - \mathbf{x}\|^2 \leq \left(1 - \frac{1}{m\|A^{-1}\|^2}\right)^k \|\mathbf{x}_0 - \mathbf{x}\|^2.$$

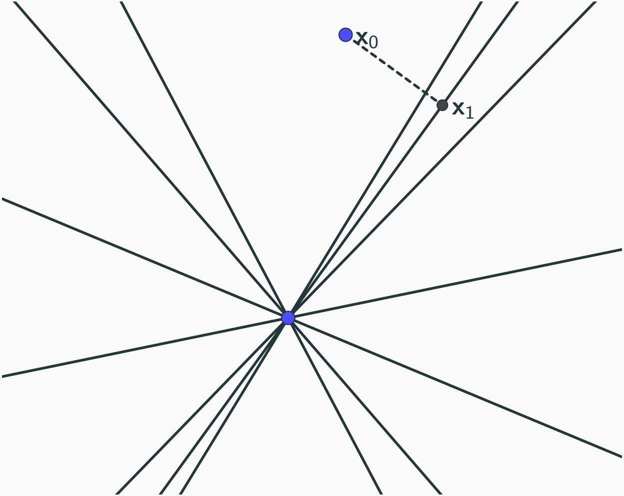
Our Hybrid Method (SKM)

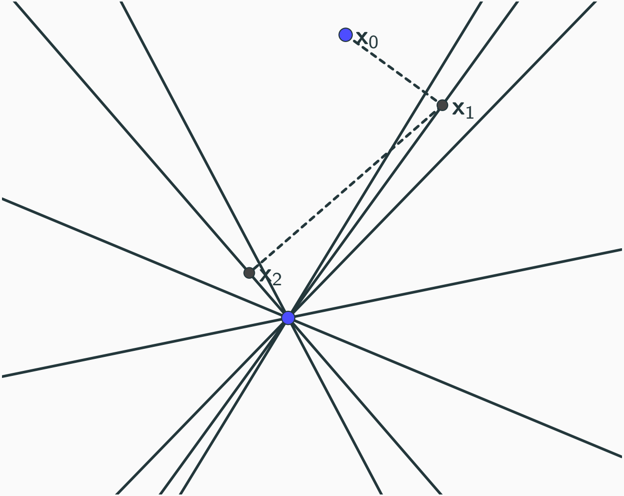
Given $\mathbf{x}_0 \in \mathbb{R}^n$:

1. Choose $\tau_k \subset [m]$ to be a sample of size β constraints chosen uniformly at random from among the rows of A.
2. From among these β rows, choose $i_k := \underset{i \in \tau_k}{\operatorname{argmax}} |\mathbf{a}_i^T \mathbf{x}_{k-1} - b_i|$.
3. Define $\mathbf{x}_k := \mathbf{x}_{k-1} + \frac{b_{i_k} - \mathbf{a}_{i_k}^T \mathbf{x}_{k-1}}{||\mathbf{a}_{i_k}||^2} \mathbf{a}_{i_k}$.
4. Repeat.


$\mathbf{x}_0$

SKM




$\mathbf{x}_0$
$\mathbf{x}_1$
$\mathbf{x}_2$

SKM Method Convergence Rate

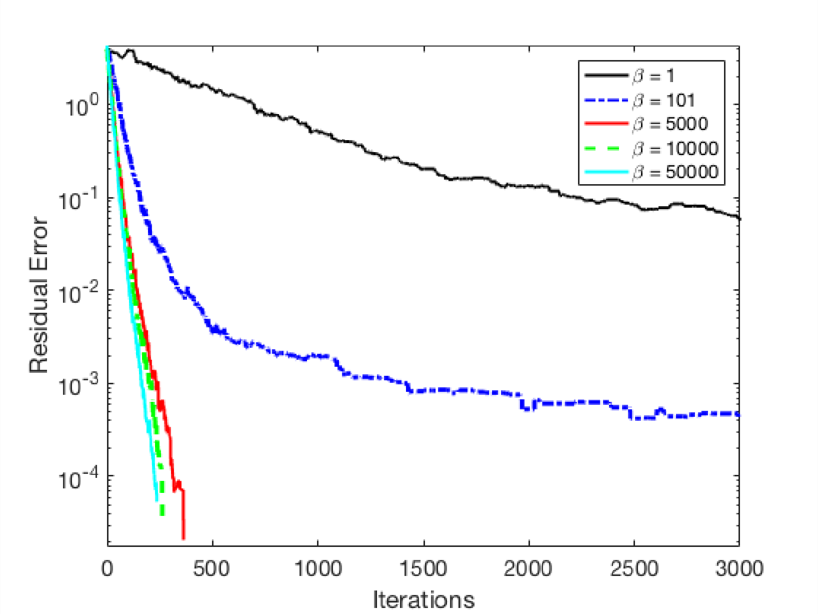
Theorem (De Loera - H. - Needell 2017)

For a consistent, normalized system the SKM method with samples of size β converges to the solution x at least linearly in expectation: If s_{k-1} is the number of constraints satisfied by x_{k-1} and $V_{k-1} = \max\{m - s_{k-1}, m - \beta + 1\}$ then

$$\mathbb{E}\|\mathbf{x}_k - \mathbf{x}\|^2 \leq \left(1 - \frac{1}{V_{k-1}\|A^{-1}\|^2}\right)\|\mathbf{x}_0 - \mathbf{x}\|^2$$

$$\leq \left(1 - \frac{1}{m\|A^{-1}\|^2}\right)^k \|\mathbf{x}_0 - \mathbf{x}\|^2.$$

Convergence



Convergence Rates

- ▷ RK: $\mathbb{E}||\mathbf{x}_k - \mathbf{x}||_2^2 \leq \left(1 - \frac{1}{||A||_F^2||A^{-1}||_2^2}\right)^k ||\mathbf{x}_0 - \mathbf{x}||_2^2.$

Convergence Rates

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- ▷ MM: $\|\mathbf{x}_k - \mathbf{x}\|^2 \leq \left(1 - \frac{1}{m\|A^{-1}\|^2}\right)^k \|\mathbf{x}_0 - \mathbf{x}\|^2.$

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- Why are these all the same?

An Accelerated Convergence Rate

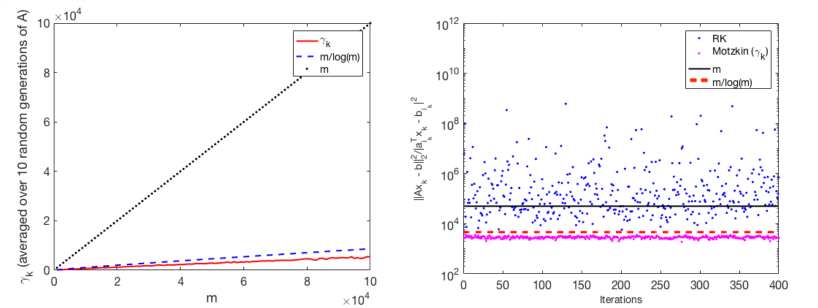
Theorem (H. - Needell 2018+)
Let $\mathbf{x}$ *denote the solution of the consistent, normalized system* $A\mathbf{x} = \mathbf{b}$*. Motzkin's method exhibits the (possibly highly accelerated) convergence rate:*

$$\|\mathbf{x}_T - \mathbf{x}\|^2 \leq \prod_{k=0}^{T-1} \left(1 - \frac{1}{4\gamma_k \|A^{-1}\|^2}\right) \cdot \|\mathbf{x}_0 - \mathbf{x}\|^2$$

Here γ_k *bounds the dynamic range of the* k*th residual,* $\gamma_k := \frac{\|A\mathbf{x}_k - A\mathbf{x}\|^2}{\|A\mathbf{x}_k - A\mathbf{x}\|_\infty^2}$.

- ▷ improvement over previous result when $4\gamma_k < m$

γ_k: Gaussian systems



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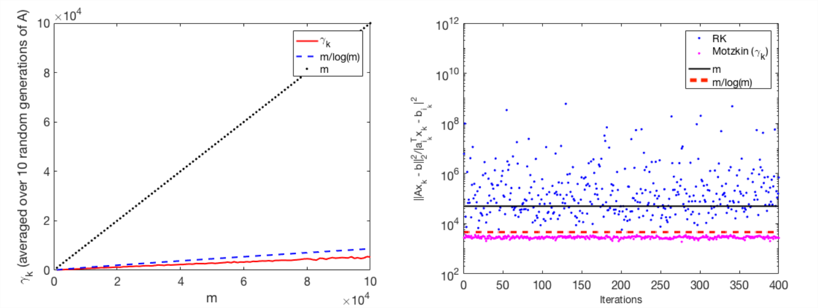
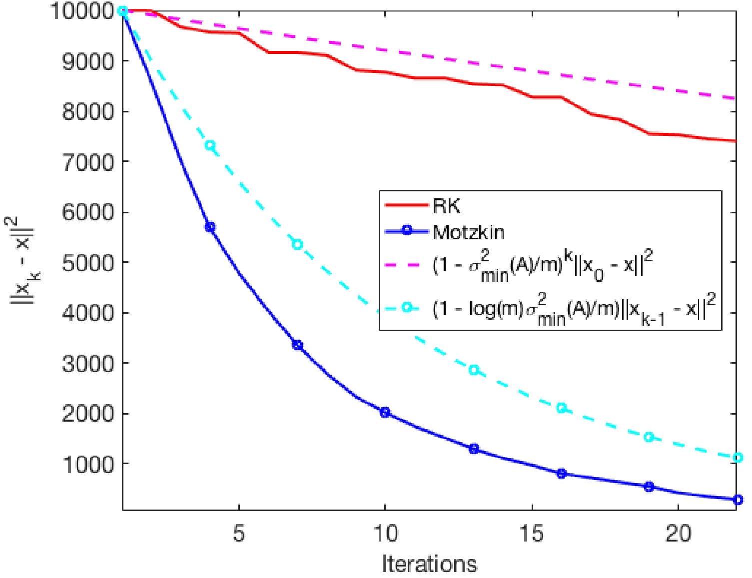
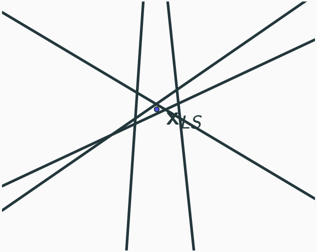


$$\gamma_k \lesssim \frac{m}{\log m}$$

Gaussian Convergence



Is this the right problem?



- ▷ noisy

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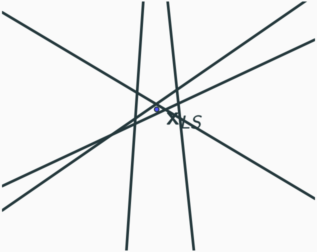

▷ noisy

▷ corrupted

Noisy Convergence Results

Theorem (Needell 2010)

Let A have full column rank, denote the desired solution to the system $A\mathbf{x} = \mathbf{b}$ by $\mathbf{x}$, and define the error term $\mathbf{e} = A\mathbf{x} - \mathbf{b}$. Then RK iterates satisfy

$$\mathbb{E}\|\mathbf{x}_k - \mathbf{x}\|^2 \leq \left(1 - \frac{1}{\|A\|_F^2 \|A^{-1}\|^2}\right)^k \|\mathbf{x}_0 - \mathbf{x}\|^2 + \|A\|_F^2 \|A^{-1}\|^2 \|\mathbf{e}\|_\infty^2$$

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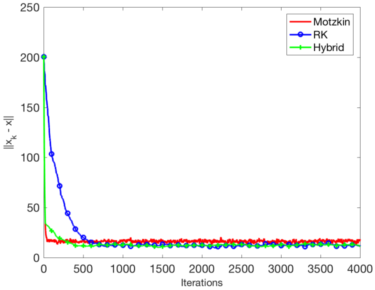
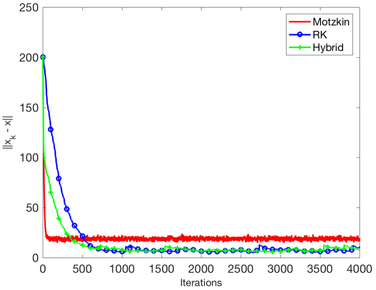
Theorem (H. - Needell 2018+)

Let $\mathbf{x}$ denote the desired solution of the system $A\mathbf{x} = \mathbf{b}$ and define the error term $\mathbf{e} = \mathbf{b} - A\mathbf{x}$. If Motzkin's method is run with stopping criterion $\|A\mathbf{x}_k - \mathbf{b}\|_\infty \leq 4\|\mathbf{e}\|_\infty$, then the iterates satisfy

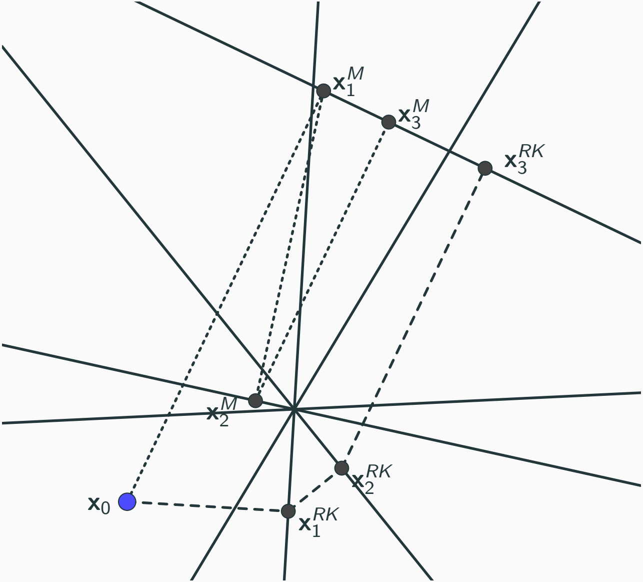
$$\|\mathbf{x}_T - \mathbf{x}\|^2 \leq \prod_{k=0}^{T-1} \left(1 - \frac{1}{4\gamma_k \|A^{-1}\|^2}\right) \cdot \|\mathbf{x}_0 - \mathbf{x}\|^2 + 2m\|A^{-1}\|^2 \|\mathbf{e}\|_\infty^2$$

Noisy Convergence





What about corruption?



Problem

	Problem:	$A\mathbf{x} = \mathbf{b} + \mathbf{e}$
(Corrupted)	**Error (e):**	sparse, arbitrarily large entries
	Solution ($\mathbf{x}^*$):	$\mathbf{x}^* \in \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$

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Applications: logic programming, error correction in telecommunications

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Applications: logic programming, error correction in telecommunications

	Problem:	$A\mathbf{x} = \mathbf{b} + \mathbf{e}$
~~**(Noisy)**~~	**Error (e):**	small, evenly distributed entries
	Solution ($\mathbf{x}_{\mathsf{LS}}$):	$\mathbf{x}_{\mathsf{LS}} \in \operatorname{argmin}\|A\mathbf{x} - \mathbf{b} - \mathbf{e}\|^2$

Why not least-squares?

MAX-FS: Given $A\mathbf{x} = \mathbf{b}$, determine the largest feasible subsystem.

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- MAX-FS is NP-hard even when restricted to homogenous systems with coefficients in $\{-1, 0, 1\}$ (Amaldi - Kann 1995)
- no PTAS unless $P = NP$

Proposed Method

Goal: Use RK to detect the corrupted equations with high probability.

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Lemma

Let $\epsilon^* = \min_{i \in supp(\mathbf{e})} |A\mathbf{x}^* - \mathbf{b}|_i = |e_i|$ *and suppose* $|supp(\mathbf{e})| = s$*. If* $||\mathbf{a}_i|| = 1$ *for* $i \in [m]$ *and* $||\mathbf{x} - \mathbf{x}^*|| < \frac{1}{2}\epsilon^*$ *we have that the* $d \leq s$ *indices of largest magnitude residual entries are contained in* $supp(\mathbf{e})$*. That is, we have* $D \subset supp(\mathbf{e})$*, where*

$$D = \underset{D \subset [A], |D| = d}{argmax} \sum_{i \in D} |A\mathbf{x} - \mathbf{b}|_i.$$

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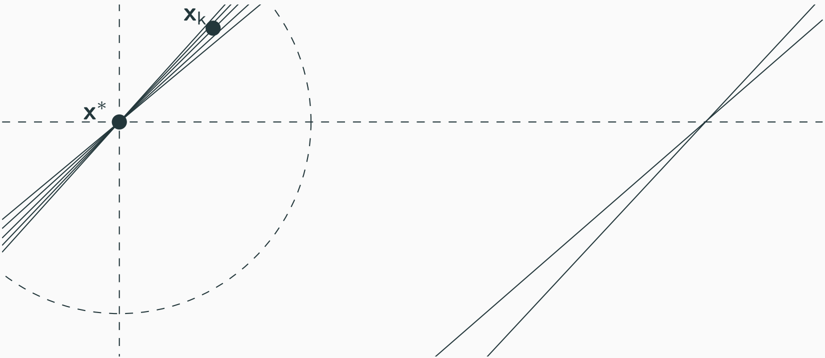

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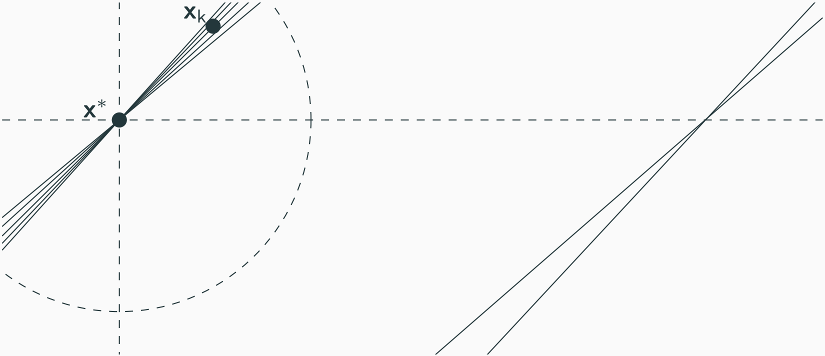


We call $\epsilon^*/2$ the *detection horizon*.

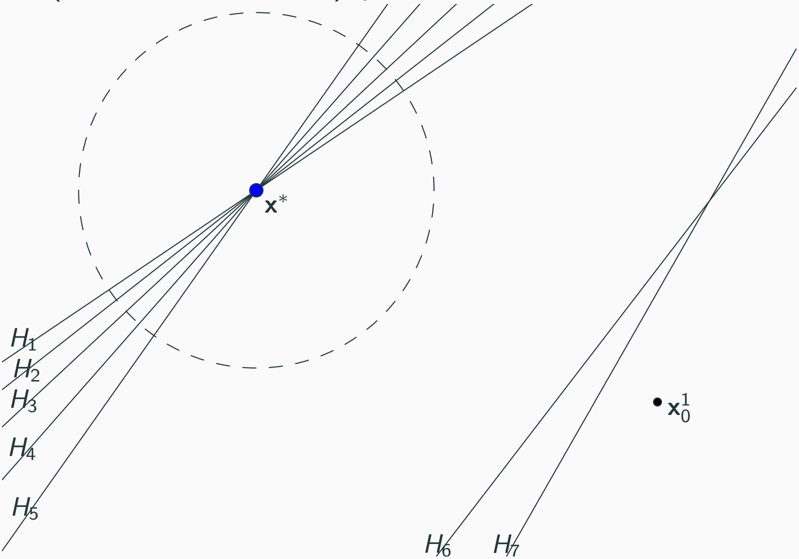
Proposed Method

Method 1 Windowed Kaczmarz

procedure $\mathrm{WK}(A, \mathbf{b}, k, W, d)$
    $S = \emptyset$
    for $i = 1, 2, \ldots W$ **do**
        $\mathbf{x}_k^i = k$th iterate produced by RK with $\mathbf{x}_0 = \mathbf{0}$, A, $\mathbf{b}$.
        $D = d$ indices of the largest entries of the residual, $|A\mathbf{x}_k^i - \mathbf{b}|$.
        $S = S \cup D$
    return $\mathbf{x}$, where $A_{S^C}\mathbf{x} = \mathbf{b}_{S^C}$

Example

WK(A**,**$\mathbf{b}$**,**$k = 2$**,**$W = 3$**,**$d = 1$**):** $j = 1,\ i = 1,\ S = \emptyset$



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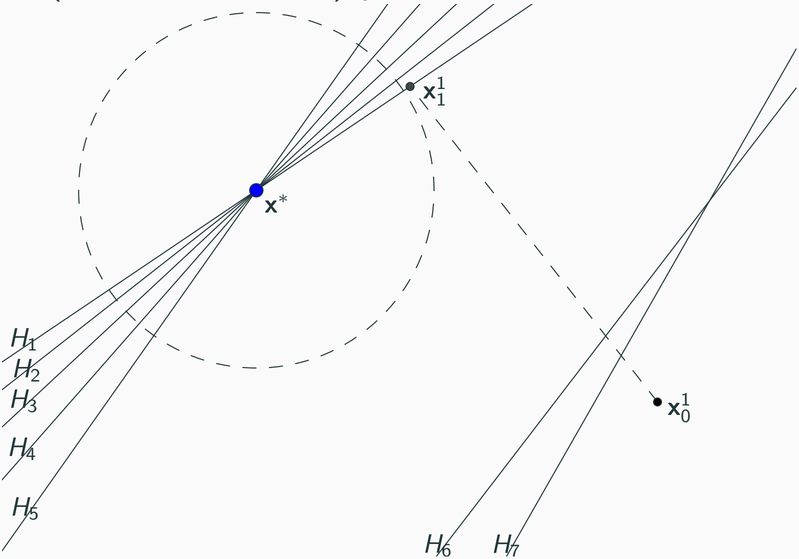
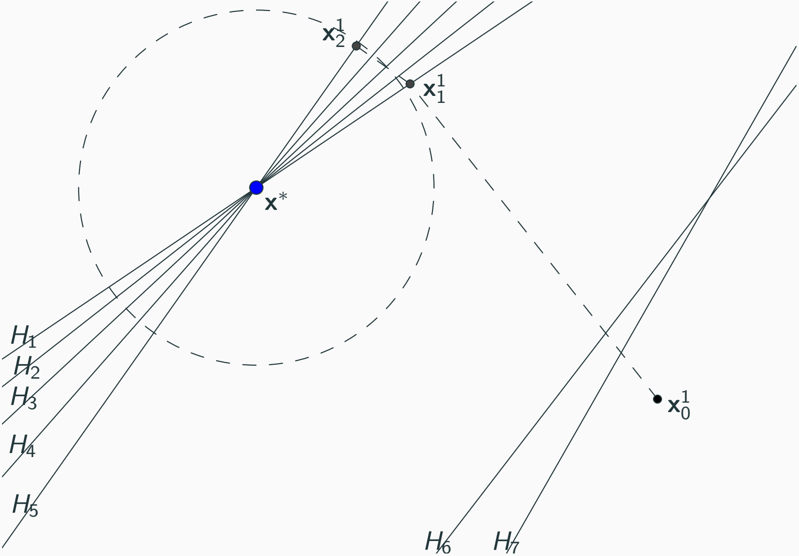


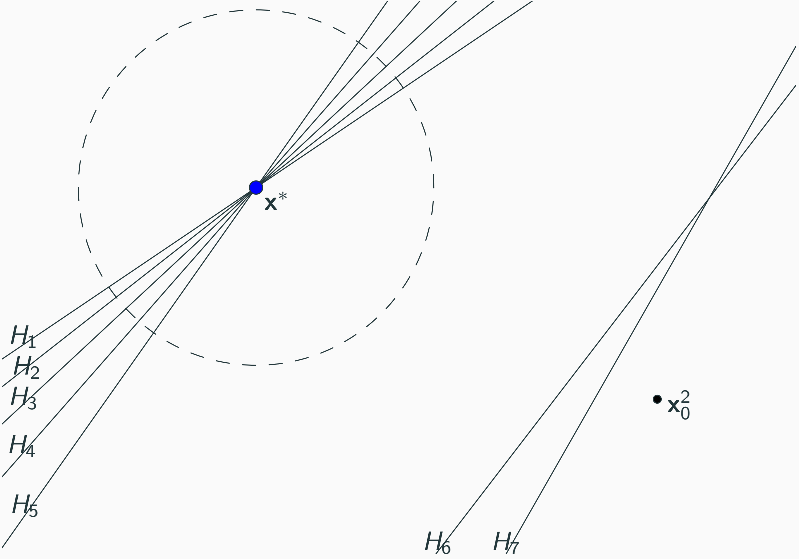
Example

WK(A**,b,**$k = 2$**,**$W = 3$**,**$d = 1$**):** $j = 2,\ i = 1,\ S = \{7\}$



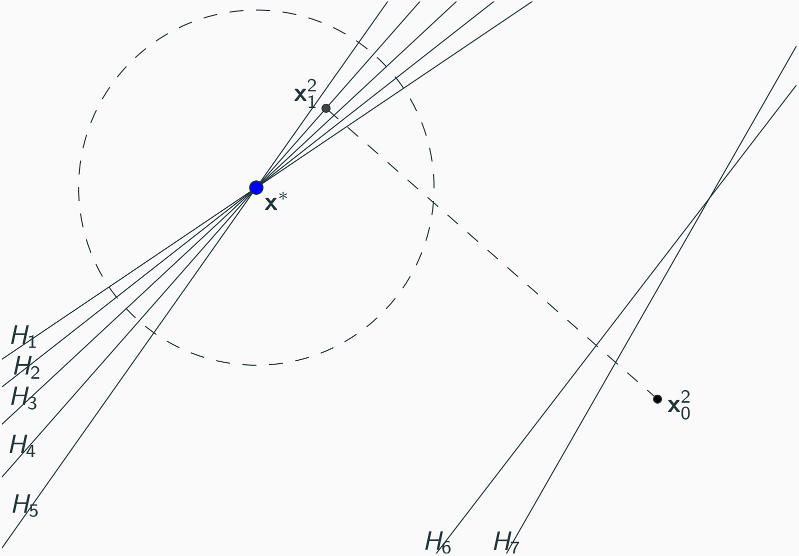
Example

WK(A**,b,**$k = 2$**,**$W = 3$**,**$d = 1$**):** $j = 1,\ i = 2,\ S = \{7\}$



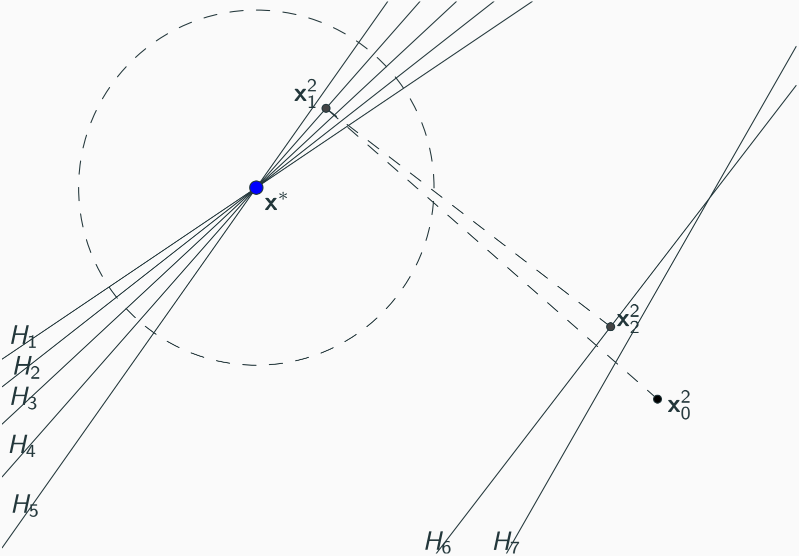
Example

WK(A,b,$k = 2$,$W = 3$,$d = 1$): $j = 1,\ i = 2,\ S = \{7\}$



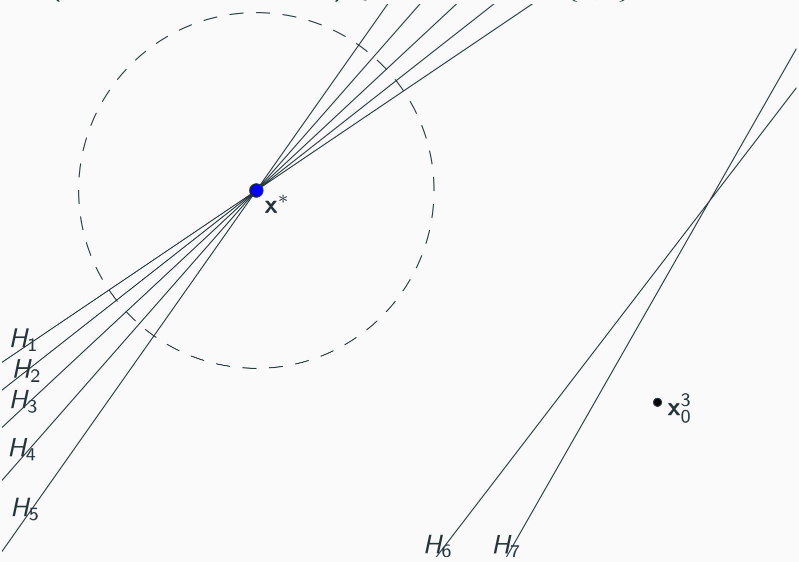
Example

WK(A,b,$k=2$,$W=3$,$d=1$): $j=2,\ i=2,\ S=\{7,5\}$



Example

WK(A**,b,**$k = 2$**,**$W = 3$**,**$d = 1$**):** $j = 1,\ i = 3,\ S = \{7, 5\}$



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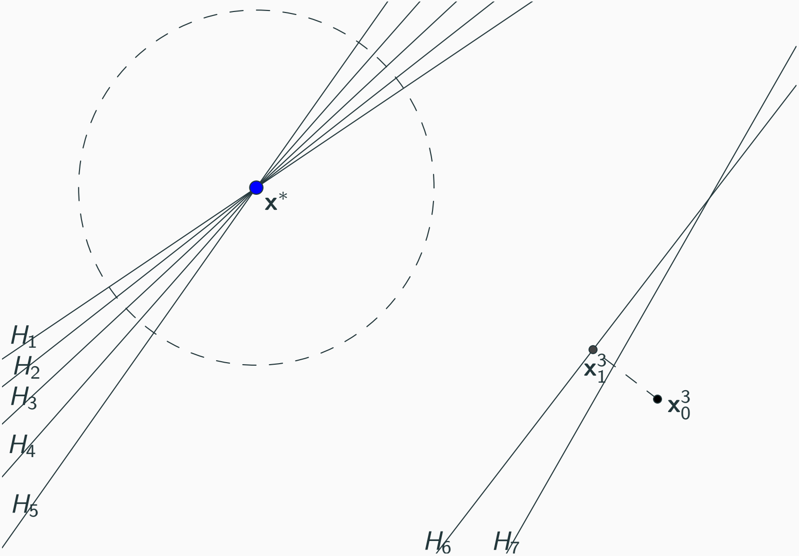
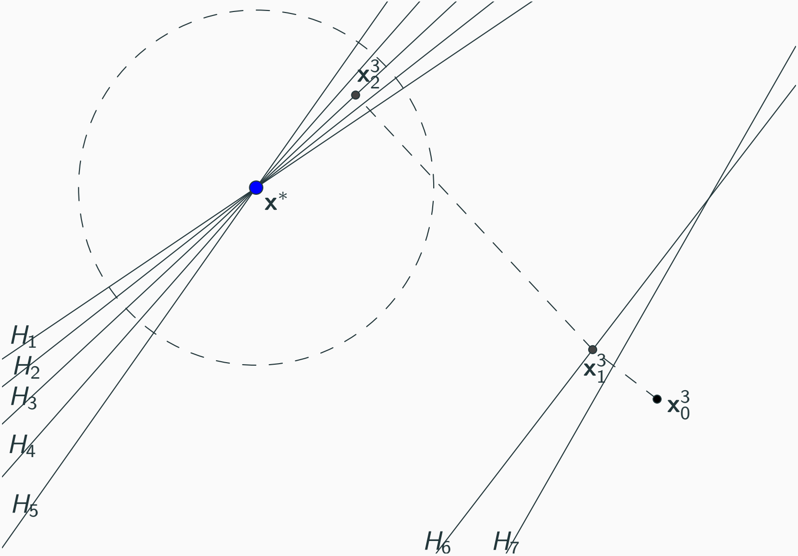


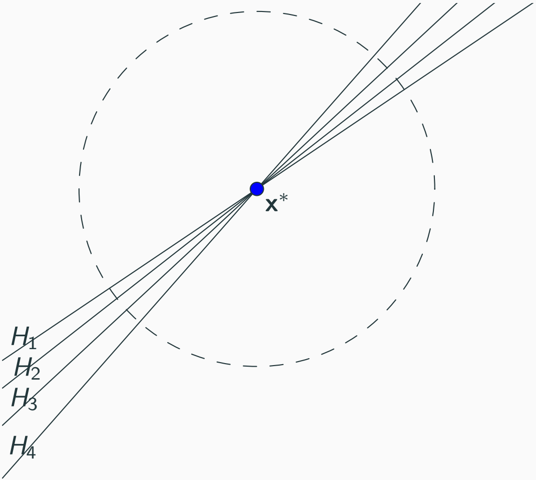
Example

WK(A,b,$k=2$,$W=3$,$d=1$): $j=2,\ i=3,\ S=\{7,5,6\}$



Example

Solve $A_{S^c}\mathbf{x} = \mathbf{b}_{S^c}$.



Theoretical Guarantees

Lemma

Let $\epsilon^* = \min_{i \in supp(\mathbf{e})} |A\mathbf{x}^* - \mathbf{b}|_i = |e_i|$ *and suppose* $|supp(\mathbf{e})| = s$. *Assume that* $||\mathbf{a}_i|| = 1$ *for all* $i \in [m]$ *and let* $0 < \delta < 1$. *Define*

$$k^* = \left\lceil \frac{\log\left(\frac{\delta(\epsilon^*)^2}{4||\mathbf{x}^*||^2}\right)}{\log\left(1 - \frac{\sigma^2_{\min}(A_{supp(\mathbf{e})^C})}{m-s}\right)} \right\rceil.$$

Then in window i *of the Windowed Kaczmarz method, the iterate produced by the RK iterations,* $\mathbf{x}^i_{k^*}$ *satisfies*

$$\mathbb{P}\Big[||\mathbf{x}^i_{k^*} - \mathbf{x}^*|| \leq \frac{1}{2}\epsilon^*\Big] \geq p := (1-\delta)\Big(\frac{m-s}{m}\Big)^{k^*}.$$

Theoretical Guarantees

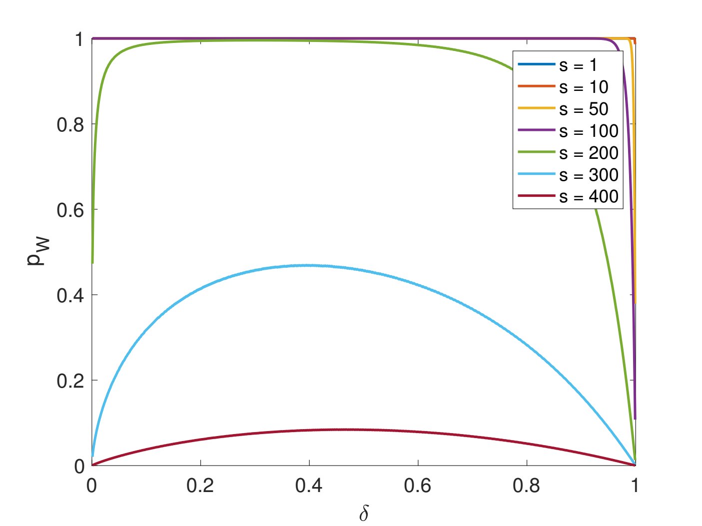
Theorem (H. - Needell 2018+)

Assume that $\|\mathbf{a}_i\| = 1$ *for all* $i \in [m]$ *and let* $0 < \delta < 1$. *Suppose* $d \geq s = |supp(\mathbf{e})|$, $W \leq \lfloor \frac{m-n}{d} \rfloor$ *and* k^* *is as given in the previous lemma. Then the Windowed Kaczmarz method on* $A, \mathbf{b}$ *will detect the corrupted equations* ($supp(\mathbf{e}) \subset S$) *and the remaining equations given by* $A_{[m]-S}, \mathbf{b}_{[m]-S}$ *will have solution* $\mathbf{x}^*$ *with probability at least*

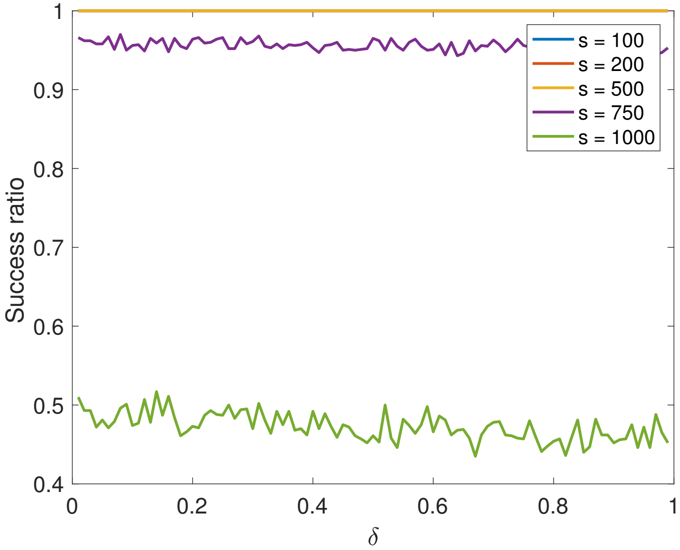
$$p_W := 1 - \left[1 - (1-\delta)\left(\frac{m-s}{m}\right)^{k^*}\right]^W.$$

Theoretical Guarantee Values (Gaussian $A \in \mathbb{R}^{50000\times 100}$)

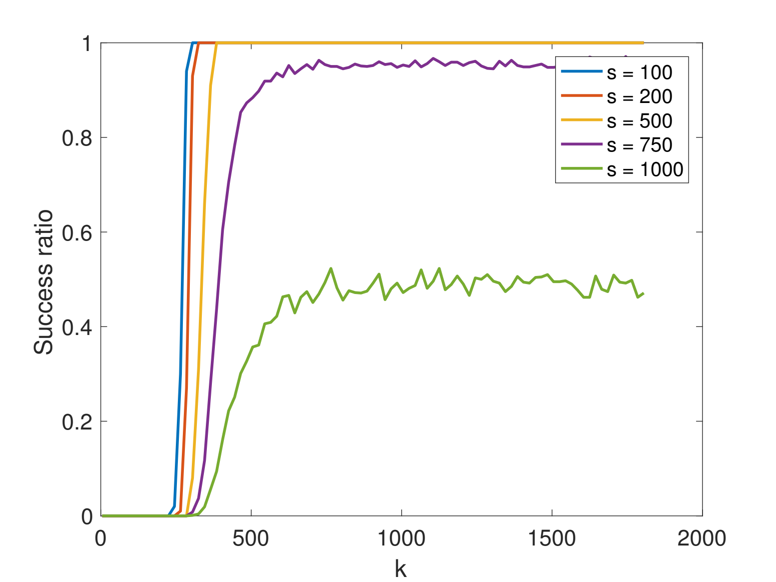
$$p_W := 1 - \left[1 - (1-\delta)\left(\frac{m-s}{m}\right)^{k^*}\right]^W$$



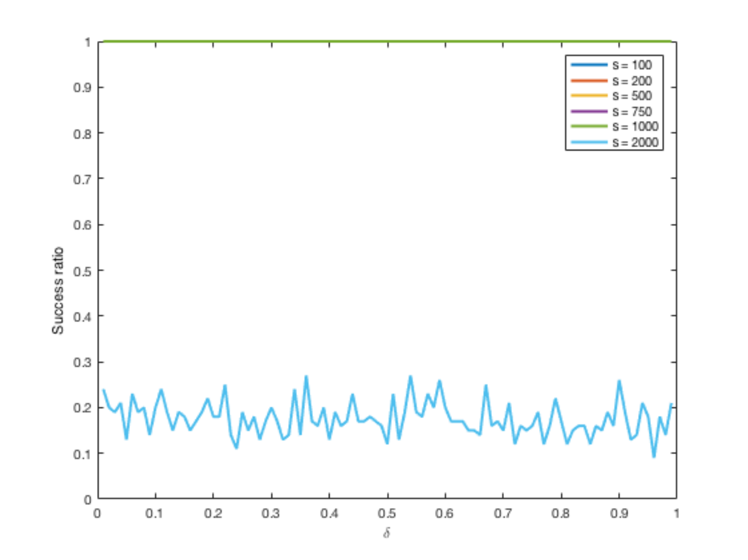
Experimental Values (Gaussian $A \in \mathbb{R}^{50000 \times 100}$)



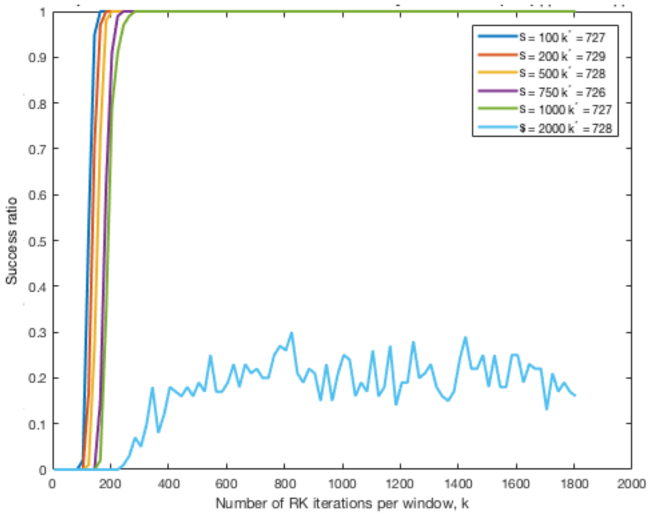
Experimental Values (Gaussian $A \in \mathbb{R}^{50000\times 100}$)



Experimental Values (Gaussian $A \in \mathbb{R}^{50000\times 100}$)



Experimental Values (Gaussian $A \in \mathbb{R}^{50000\times 100}$)



Conclusions and Future Work

- Motzkin's method is accelerated even in the presence of noise
- RK methods may be used to detect corruption

- identify useful bounds on γ_k for other useful systems
- reduce dependence on artificial parameters in corruption detection bounds