Motzkin's Method and the Randomized Kaczmarz Method

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Joint work with Jesus De Loera and Deanna Needell

Optimization

I think about problems of the sort:

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Today we'll consider a specific form of optimization problem...

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$$\text{s.t. } Ax \leq b$$

$A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and we are optimizing over $x \in \mathbb{R}^n$.

Linear Feasibility Problem

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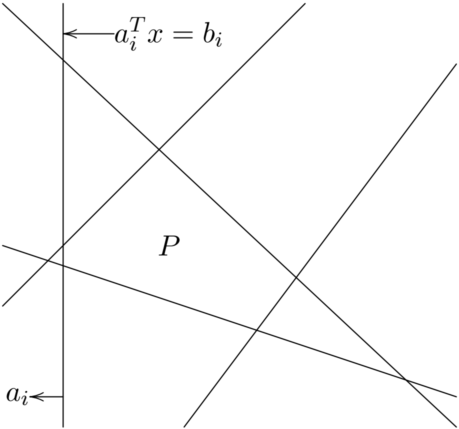
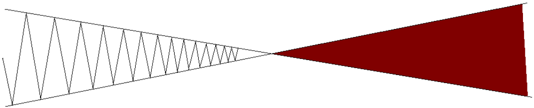
It can be shown that (LP) and (LF) are equivalent.

Linear Feasibility Problem

LF can be interpreted as seeking a point within a (possibly nonempty) polyhedron $P = \{x | Ax \leq b\}$:



Projection Methods



Motzkin

Kaczmarz

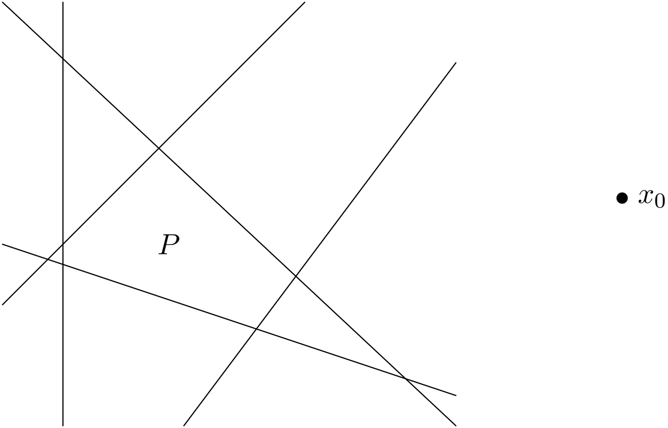
Motzkin's Relaxation Method(s)

Method

Suppose $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ *and* $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ *is nonempty. Fix* $0 < \lambda \leq 2$. *Given* $x_0 \in \mathbb{R}^n$, *iteratively construct approximations to* P *in the following way:*

1. *If* x_k *is feasible, stop.*
2. *Choose* $i_k \in [m]$ *as* $i_k := \underset{i\in[m]}{\operatorname{argmax}}\, a_i^T x_{k-1} - b_i$.
3. *Define* $x_k := x_{k-1} - \lambda \frac{a_{i_k}^T x_{k-1} - b_{i_k}}{||a_{i_k}||^2} a_{i_k}$.
4. *Repeat.*

Motzkin's Method



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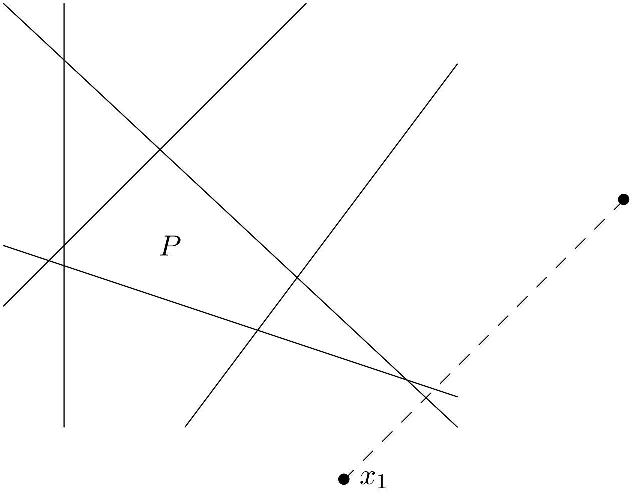

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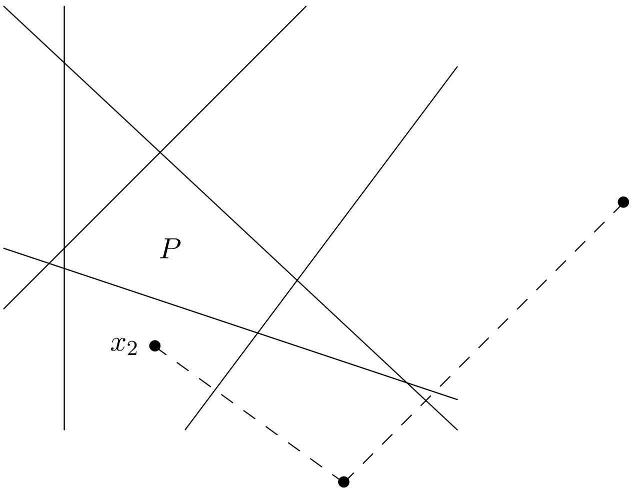

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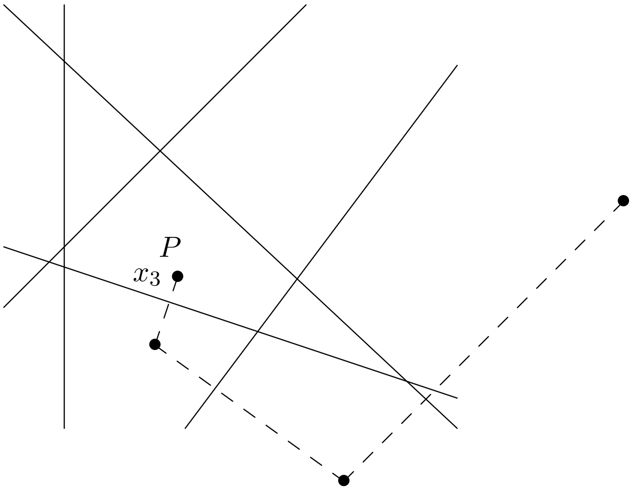


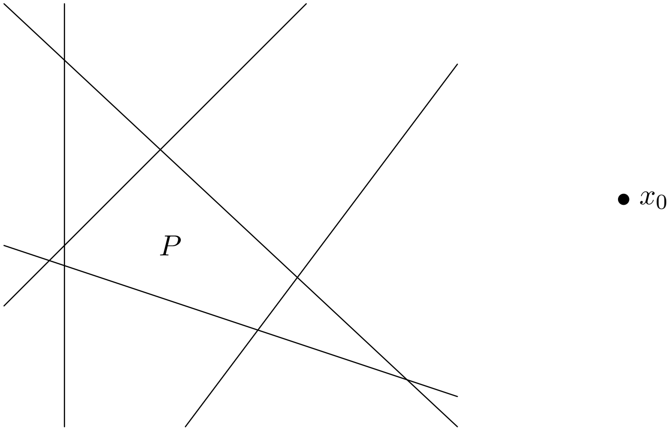
Randomized Kaczmarz Method

Method

Suppose $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ *and* $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ *is nonempty. Given* $x_0 \in \mathbb{R}^n$, *iteratively construct approximations to* P *in the following way:*

1. *If* x_k *is feasible, stop.*
2. *Choose* $i_k \in [m]$ *with probability* $\frac{||a_{i_k}||^2}{||A||_F^2}$.
3. *Define* $x_k := x_{k-1} - \frac{(a_{i_k}^T x_{k-1} - b_{i_k})^+}{||a_{i_k}||^2} a_{i_k}$.
4. *Repeat.*

Kaczmarz Method



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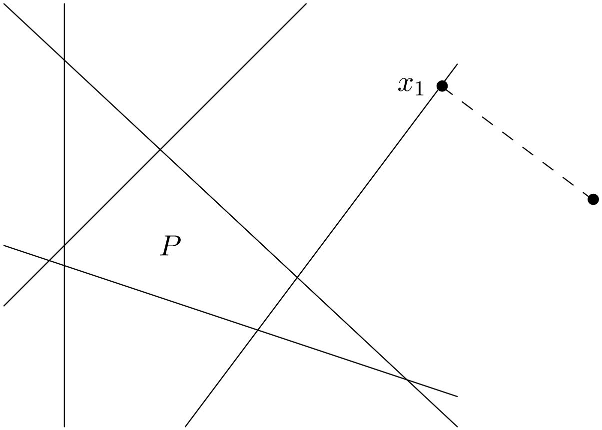

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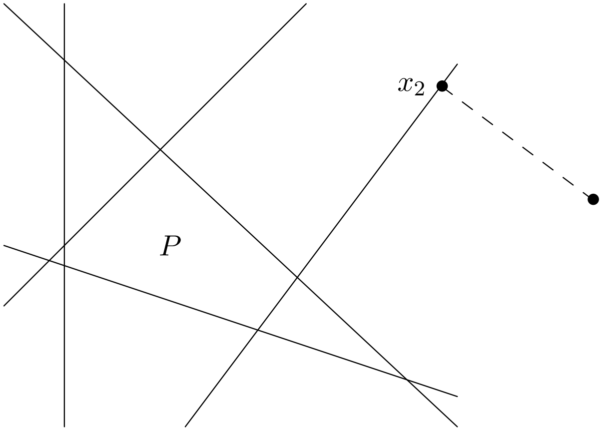

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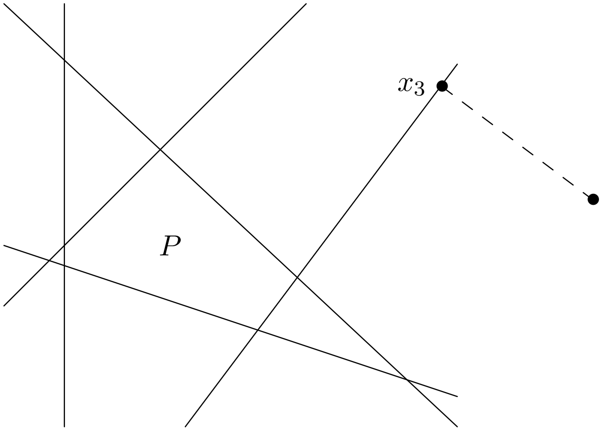

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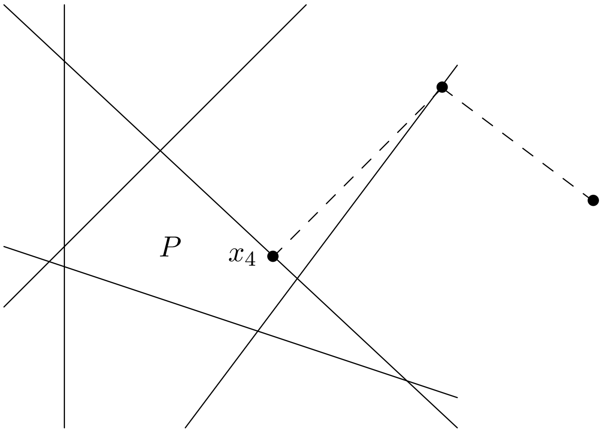

Motivation

Motzkin's Method
Pro: convergence produces monotone decreasing distance sequence
Con: computationally expensive for large systems

Kaczmarz Method
Pro: computationally inexpensive, able to analyze the expected convergence rate
Con: slow convergence near the polyhedral solution set

A Hybrid Method

Method (SKMM)

Suppose $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ *and* $P := \{x \in \mathbb{R}^n : Ax \le b\}$ *is nonempty. Fix* $0 < \lambda \le 2$. *Given* $x_0 \in \mathbb{R}^n$, *iteratively construct approximations to* P *in the following way:*

1. *If* x_k *is feasible, stop.*
2. *Choose* $\tau_k \subset [m]$ *to be a sample of size* β *constraints chosen uniformly at random from among the rows of* A.
3. *From among these* β *rows, choose* $i_k := \underset{i\in\tau_k}{\text{argmax}}\, a_i^T x_{k-1} - b_i$.
4. *Define* $x_k := x_{k-1} - \lambda \frac{(a_{i_k}^T x_{k-1} - b_{i_k})^+}{||a_{i_k}||^2} a_{i_k}$.
5. *Repeat.*

Generalized Method

Note that both previous methods are captured by the class of SKMM methods:

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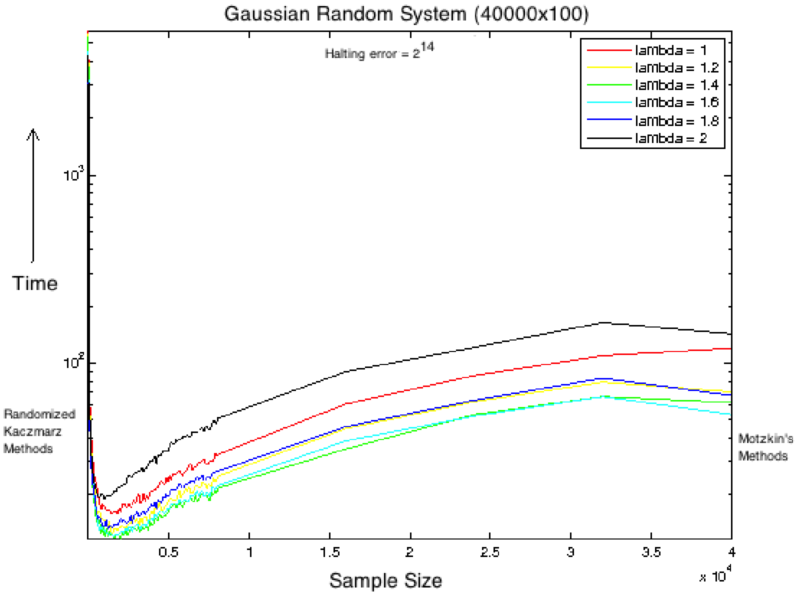
Note that both previous methods are captured by the class of SKMM methods:

1. The Kaczmarz method is SKMM where the sample size, $\beta = 1$ and the relaxation parameter, $\lambda = 1$.
2. Motzkin's Relaxation methods are SKMM where the sample size, $\beta = m$.

Experimental Results



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2. It is well known that the Random Kaczmarz update step is equivalent to the coordinate descent update step applied to the dual problem. I will explore connections of SKMM to variants of randomized coordinate descent in the dual variable space.
3. Provide theoretical guidance for selection of the optimal sample size, β, and optimal overshooting parameter, λ.

Acknowledgements

Thanks to you for attending!

Are there any questions?

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