Motzkin's Method and the Randomized Kaczmarz Method

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Joint work with Jesús De Loera and Deanna Needell

Optimization

I think about problems of the sort:

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Today we'll consider a specific form of optimization problem...

Linear Programs

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$$\begin{aligned} &\min c^T x \\ &\text{s.t. } Ax \leq b \end{aligned} \qquad (LP)$$

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$$\min c^T x \qquad (LP)$$
$$\text{s.t. } Ax \leq b$$

$A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ and we are optimizing over $x \in \mathbb{R}^n$.

Linear Feasibility Problem

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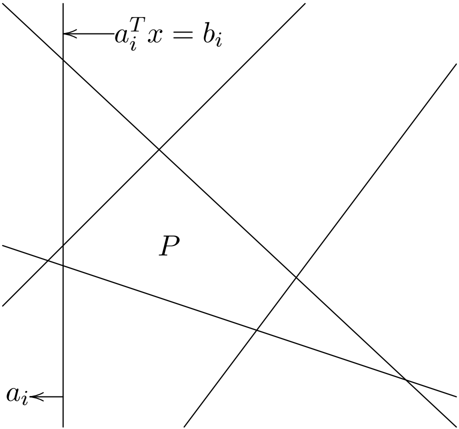
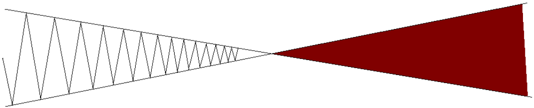
It can be shown that (LP) and (LF) are equivalent.

Linear Feasibility Problem

LF can be interpreted as seeking a point within a (possibly nonempty) polyhedron $P = \{x|Ax \leq b\}$:



Projection Methods



Motzkin

Kaczmarz

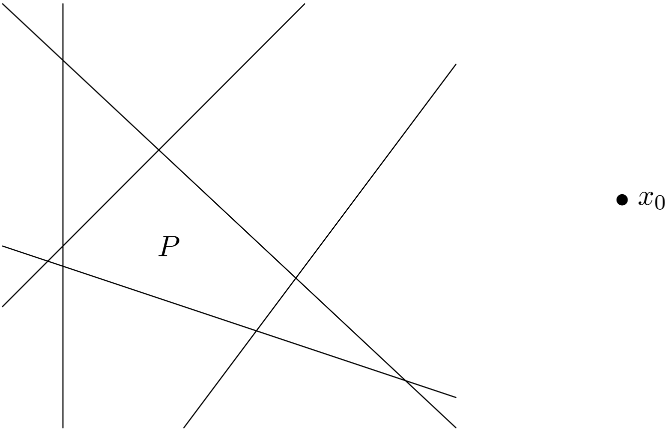
Motzkin's Relaxation Method(s)

Method

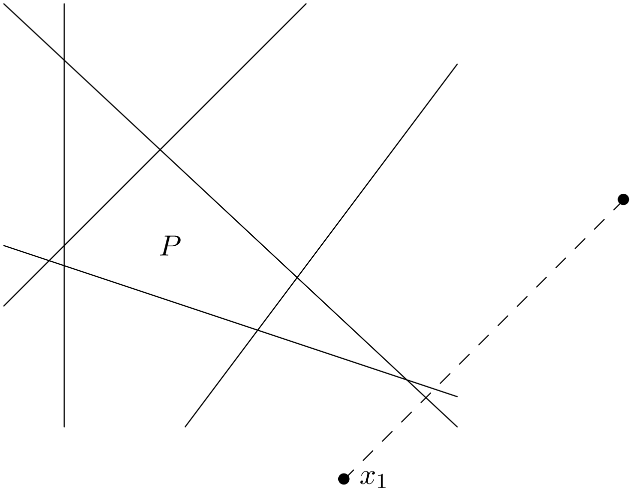
Suppose $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ *and* $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ *is nonempty. Fix* $0 < \lambda \leq 2$. *Given* $x_0 \in \mathbb{R}^n$, *iteratively construct approximations to* P *in the following way:*

1. *If* x_k *is feasible, stop.*
2. *Choose* $i_k \in [m]$ *as* $i_k := \underset{i \in [m]}{\text{argmax}}\, a_i^T x_{k-1} - b_i$.
3. *Define* $x_k := x_{k-1} - \lambda \frac{a_{i_k}^T x_{k-1} - b_{i_k}}{||a_{i_k}||^2} a_{i_k}$.
4. *Repeat.*

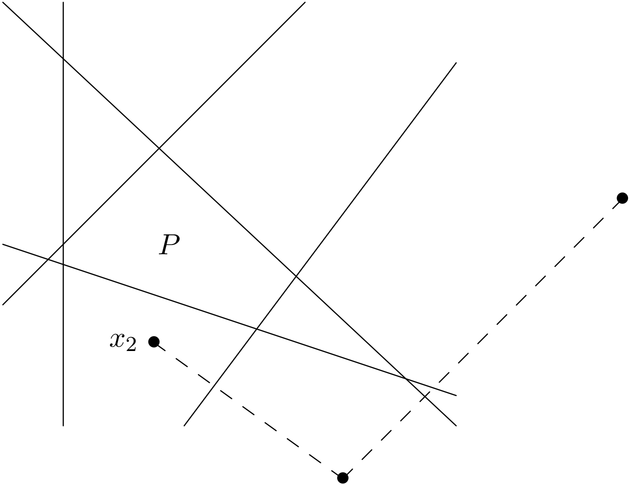
Motzkin's Method



Motzkin's Method



Motzkin's Method



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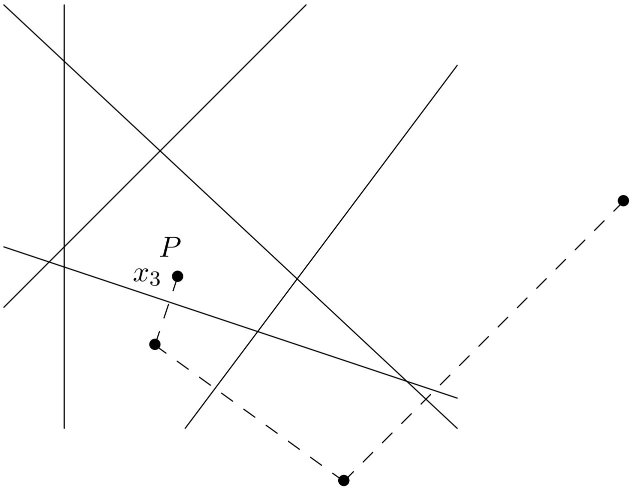


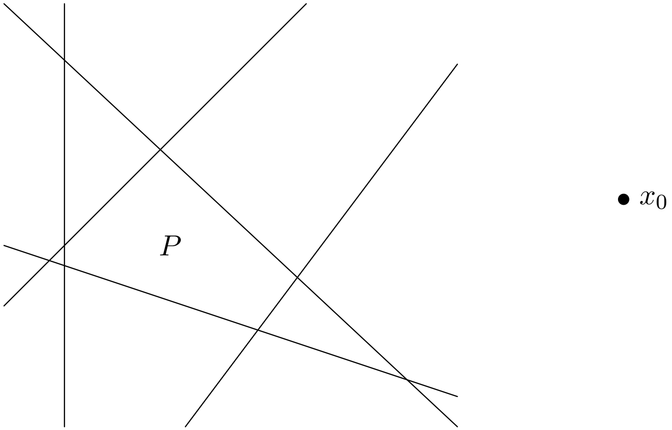
Randomized Kaczmarz Method

Method

Suppose $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ *and* $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ *is nonempty. Given* $x_0 \in \mathbb{R}^n$, *iteratively construct approximations to* P *in the following way:*

1. *If* x_k *is feasible, stop.*
2. *Choose* $i_k \in [m]$ *with probability* $\frac{||a_{i_k}||^2}{||A||_F^2}$.
3. *Define* $x_k := x_{k-1} - \frac{(a_{i_k}^T x_{k-1} - b_{i_k})^+}{||a_{i_k}||^2} a_{i_k}$.
4. *Repeat.*

Kaczmarz Method



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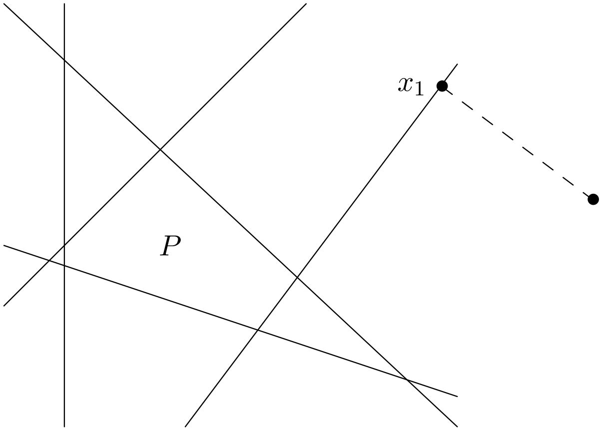

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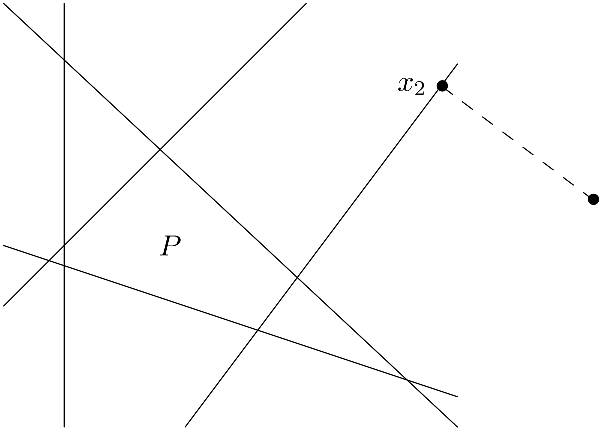

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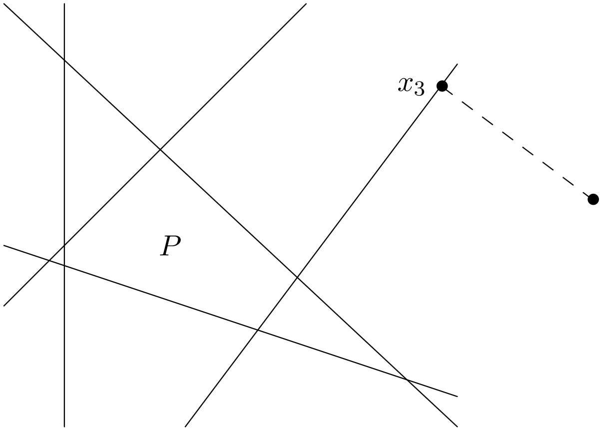
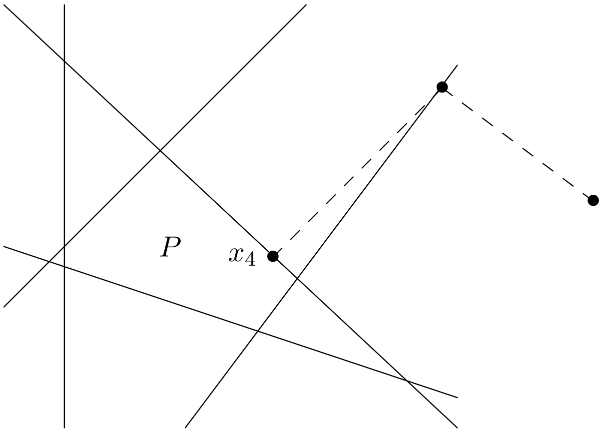
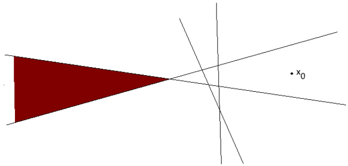
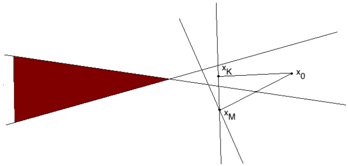


Kaczmarz Method



Comparison of the Methods





Motivation

Motzkin's Method
Pro: convergence produces monotone decreasing distance sequence
Con: computationally expensive for large systems

Kaczmarz Method
Pro: computationally inexpensive, able to analyze the expected convergence rate
Con: slow convergence near the polyhedral solution set

A Hybrid Method

Method (SKMM)

Suppose $A \in \mathbb{R}^{m\times n}$, $b \in \mathbb{R}^m$ *and* $P := \{x \in \mathbb{R}^n : Ax \leq b\}$ *is nonempty. Fix* $0 < \lambda \leq 2$. *Given* $x_0 \in \mathbb{R}^n$, *iteratively construct approximations to* P *in the following way:*

1. *If* x_k *is feasible, stop.*
2. *Choose* $\tau_k \subset [m]$ *to be a sample of size* β *constraints chosen uniformly at random from among the rows of* A.
3. *From among these* β *rows, choose* $i_k := \operatorname*{argmax}_{i\in\tau_k} a_i^T x_{k-1} - b_i$.
4. *Define* $x_k := x_{k-1} - \lambda \frac{(a_{i_k}^T x_{k-1} - b_{i_k})^+}{||a_{i_k}||^2} a_{i_k}$.
5. *Repeat.*

Generalized Method

Note that both previous methods are captured by the class of SKMM methods:

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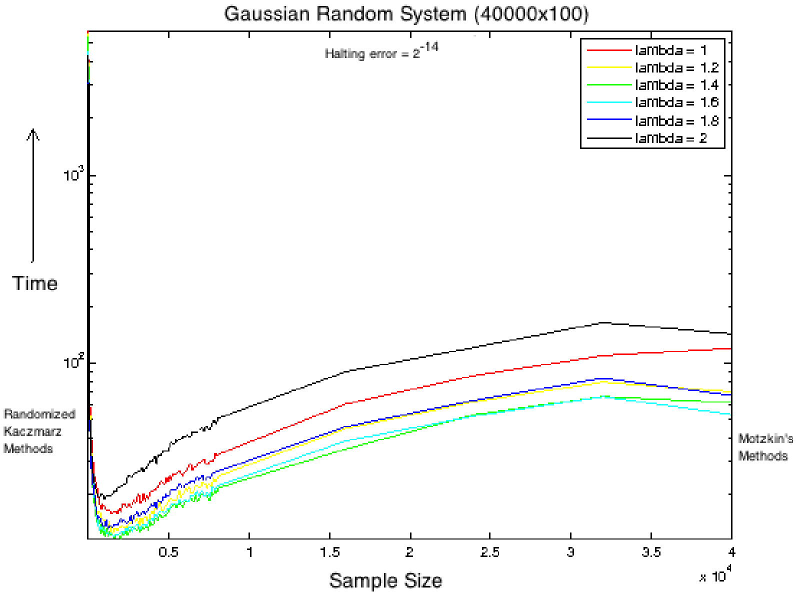
Generalized Method

Note that both previous methods are captured by the class of SKMM methods:

1. The Kaczmarz method is SKMM where the sample size, $\beta = 1$ and the relaxation parameter, $\lambda = 1$.
2. Motzkin's Relaxation methods are SKMM where the sample size, $\beta = m$.

Experimental Results



Motzkin's Method Convergence Rate

Theorem (Agmon)

For a normalized system, $||a_i|| = 1$ for all $i = 1, ..., m$, if the feasible region, $P := \{x | Ax \leq b\}$, is nonempty then the relaxation methods converges linearly:

$$d(x_k, P)^2 \leq \left(1 - \frac{2\lambda - \lambda^2}{mL_2^2}\right)^k d(x_0, P)^2.$$

Random Kaczmarz Method Convergence Rate

Theorem (Lewis, Leventhal)

If the feasible region, $P := \{x | Ax \leq b\}$, is nonempty then the Randomized Kaczmarz method with relaxation parameter λ converges linearly in expectation:

$$\mathbb{E}[d(x_k, P)^2] \leq \left(1 - \frac{2\lambda - \lambda^2}{||A||_F^2 L_2^2}\right)^k d(x_0, P)^2.$$

SKM Method Convergence Rate

Theorem (De Loera, H., Needell)

If the feasible region (for normalized A) is nonempty, then the SKM methods with samples of size β converges at least linearly in expectation: In each iteration,

$$\mathbb{E}[d(x_k, P)^2] \leq \left(1 - \frac{2\lambda - \lambda^2}{S_{k-1} L_2^2}\right) d(x_{k-1}, P)^2$$

where $S_{k-1} = \max\{m - s_{k-1}, m - \beta + 1\}$ and s_{k-1} is the number of constraints satsifed by x_{k-1}. Clearly then,

$$\mathbb{E}[d(x_k, P)^2] \leq \left(1 - \frac{2\lambda - \lambda^2}{m L_2^2}\right)^k d(x_0, P)^2.$$

Improving the rate:

Lemma

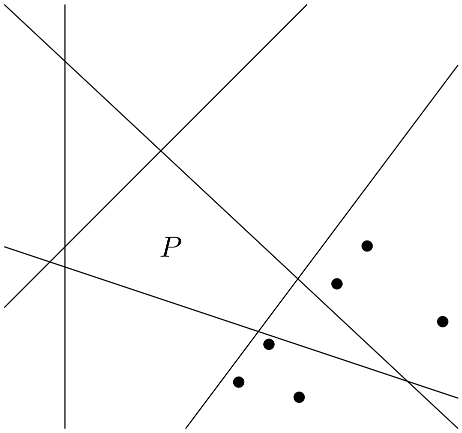
If the sequence generated by an SKM method, $\{x_k\}$, converges to P (without terminating) then $\{x_k\}$ converges to a unique, inclusion-minimal F_{τ^*}, some face of P.

Improving the rate:

Lemma

There exists K after which the only constraints violated by x_k for $k \geq K$ are those which define or include F_{τ^*}, the unique, inclusion-minimal face to which the iterates are converging, i.e. if $a_j^T x_k > b_j$ for $k \geq K$ then $j \in \tau^*$ and $F_{\tau^*} \subset \{x | a_j^T x = b_j\}$.

Improving the rate:



Improved Rate

Theorem (De Loera, H., Needell)

If the feasible region, $P = \{x|Ax \leq b\}$ is generic and nonempty (for normalized A), then an SKM method with samples of size $\beta \leq m - n$ is guaranteed an increased convergence rate after some K:

$$\mathbb{E}[d(x_k, P)^2] \leq \left(1 - \frac{2\lambda - \lambda^2}{mL_2^2}\right)^K \left(1 - \frac{2\lambda - \lambda^2}{(m - \beta + 1)L_2^2}\right)^{k-K} d(x_0, P)^2.$$

Finiteness of Motzkin's Method

Theorem (Telgen)

Either the relaxation method detects feasibility of the system, $Ax \leq b$ (with A normalized), within $k = \left\lceil \frac{2^{4L}}{n\lambda(2-\lambda)} \right\rceil$ iterations or the system is infeasible.*

*with $x_0 = 0$

Expected Finiteness of SKM methods

Theorem (De Loera, H., Needell)

If the system, $Ax \leq b$ is feasible, then with high probability the Sampling Kaczmarz-Motzkin method with relaxation parameter $0 < \lambda < 2$ will detect feasibility within a given number of steps.*

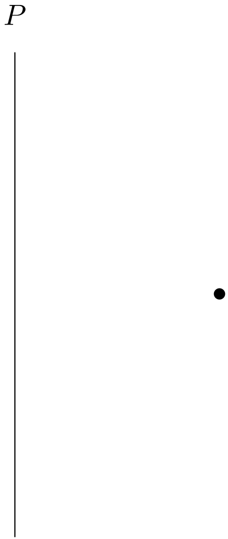
*with $x_0 = 0$

Termination of Motzkin's Reflection Method

Theorem (Motzkin, Schoenberg)

When $P = \{x | Ax \leq b\}$ is full dimensional, then Motzkin's method with $\lambda = 2$ will terminate with a solution.

Termination:



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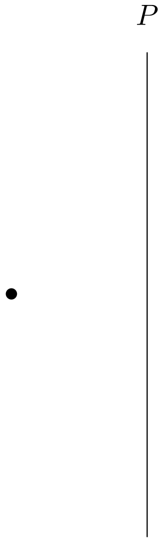

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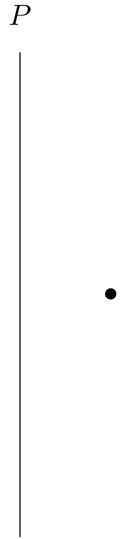

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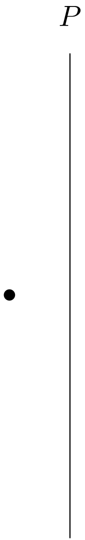
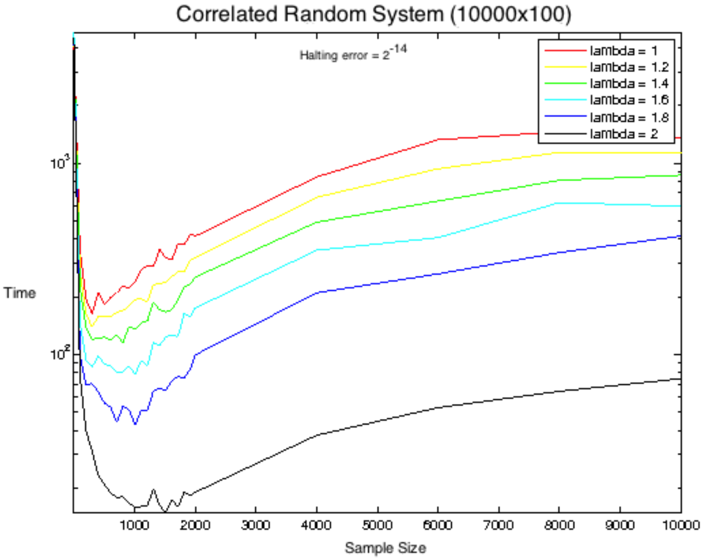


Termination of SKM reflection method

Theorem (De Loera, H., Needell)

When $P = \{x | Ax \leq b\}$ is full dimensional, then the SKM methods with $\lambda = 2$ will terminate with a solution.

Conclusions



Future work:

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1. Provide a smoothed analysis for the SKM methods.
2. It is well known that the Random Kaczmarz update step is equivalent to the coordinate descent update step applied to the dual problem. I will explore connections of SKMM to variants of randomized coordinate descent in the dual variable space.
3. Provide theoretical guidance for selection of the optimal sample size, β, and optimal overshooting parameter, λ.

Acknowledgements

Thanks to you for attending!

Are there any questions?

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