For $E = (A \vee BD^*C)^*$, note that ~~any path to~~ for any two ~~$v_1, v$~~ $v, v' \in V_1$, if $\exists$ a ~~path to~~ $v - v'$ path we may partition into subpaths $P_1, \ldots, P_q$ s.t. no interior vertices are in $V_1$. Consider any such path $P_i$ ... it must ~~be~~ either be a single edge (i.e. in the support of matrix $A$) or ~~be~~ consist of ~~a $V_1 - V_2$ path (possibly empty) followed by a~~ an edge from $V_1$ to $V_2$, followed by a (possibly empty) $\underline{V_2 - V_2 \text{ path}}$ using only $V_2$ vertices!, followed by a $V_2 - V_1$ edge (this latter option is summarized in the matrix $BD^*C$.) Putting it all together, we have $A^* = (A \vee BD^*C)^*$. Now, noting the matrix

$F$ records all pairs $(v_2, v_1) \in V_2 \times V_1$ s.t. $\exists$ a $V_2 - V_1$ path. Such a path is precisely determined by the following:

$D^* \to$ (1) a (possibly empty) $V_2 - V_2$ path using only $V_2$ vertices

$B \to$ (2) an edge from $V_2 - V_1$

$E \to$ (3) an $\underline{\text{arbitrary}}$ $V_1 - V_1$ path.

Hence $F = EBD^*$. Similar argumentation (noting the role of

E] allow us to verify the remaining equations.

part ii ~~Now~~ w, our matrix E should give for each $v, v' \in V_1$ the length of a shortest $v_0 - v_0'$ path. Such a path may be decomposed as before — since subpaths of shortest paths, we want each $P_i$ to have min-weight. This is only an issue if we have to compute A w/ $BD^*C$, in which case we're good since "V" is now a coord-wise min. Moreover,

part ii ~~∅~~ ∅ since $u(AB)$ now simply adds $u(A)$ & $u(B)$, the resulting entry for $A[v, v']$ is the length of the shortest path.

part iii Divide & conquer: to get $w^*$, do (w/ $|V_1| = \lfloor \frac{|V|}{2} \rfloor$, say)

~~① compute $A^*$ & $D^*$ → 2 APSP(n/2) work~~

~~② compute~~

~~① compute $E = (A \vee BD^*C)^*$ →~~

① compute $D^*$ → APSP(n/2) work.

② compute E → $n^2$ for the sum $BD^*C$ ~~+ APSP(n/2) + B~~
+ MP(n/2) for $A \vee BD^*C$
+ APSP for final "*"

③ compute F, G, H → APSP(n/2) + 5 MSP(n/2).

Unrolling the recursion yields

$$APSP(n) = O\left( \sum_{k=1}^{\lg n} 6\,MSP(n/2^k) + O\left(\left(\tfrac{n}{2^k}\right)^2\right)\right)$$

$$= O\left(\lg n\,(MSP(n) + n^2)\right)$$

$$= \tilde{O}\left(MSP(n) + n^2\right)$$

where we allow $\tilde{O}$ since $MSP(n) >> \lg^k n$ $\forall k$ (any MSP-computer must read a $n^2$-size input).