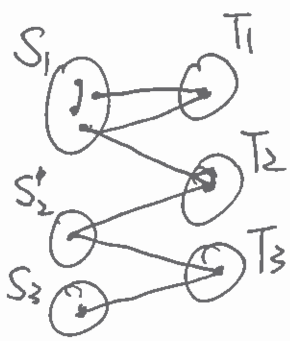
Problem 2: Obviously A & D are square matrices. (By checking multiplication)

①. Show $E = (A \vee BD^*C)^*$. For convenience, call $V = S \uplus T$ where S corresponds to the rows and columns in A; T corresponds to columns and rows in D. Thus E is the reachability matrix from S to S.

Two vertices $u, v \in S$ are connected if: i) exist a u–v path in S, or at least ii) exist a path from u to T to v.

A is the adjacency matrix for S. BD^*C represents the "reachbility" that via a connected component in T, $u \in S$ and $v \in S$ can reach that in one step. Then it's enough to build E. Within steps u could reach v, either in S or ~~usin~~ traversing some vertices in T (not necessarily in one connected component). like my drawing on the left.



$$E = (A \vee BD^*C)^*.$$

For vertices in each connected component fn S, reachability is 1; else if it can traverse T to another component in S, it also works. Thus take * to get E.

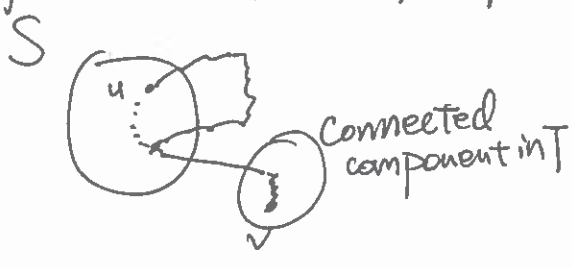
Note at this part of question ~~stage~~ addition is "or", multiplication is "and", hence for multiplication with a ~~reachability~~ matrix, resulting matrix only contains 0 or 1. Same for ②. ③.

② $F = EBD^* \;\&\; G = D^* CE$.

These two are similar, in fact $F = G^T$ since graph G is ~~undirect~~ undirected. So we only prove one.

F represents the reachability from S to T. (G: T to S).

EBD^* represents all possible ways a vertex $u \in S$ to reach a vertex $v \in T$. $EBD^*(u,v) = \sum_x E(u,x)(BD^*)(x,v)$ $= \sum_x E(u,x) \cdot \sum_y B(x,y) D^*(y,v)$. That is, for any "bridge"/connection between S and T, i.e. x, y here, as long as u could reach x (could be same) and y could reach v, with $x \sim y$, $u - v$ path exists. E and D^* represents reachability from S to S within G, and T to T (within T only)



③ $H = D^* V G B D^*$

This comes from the idea that a vertex $u \in T$ is connected to $v \in T$ if i) u, v in same connected component, which covered by D^*, or ii) u connects to x in S (by G); v connects to y (by D^*), and $x \sim y$ (by B). Note reachability is an equivalence relation, thus x and y can be chosen properly to satisfy the description, if $u-v$ path does exist through S.

For "V" operation it is obvious, similar to existence of path covered by either component, once we find shortest path, it comes from the min of the existings. And for matrix "multiplication" this can be seen as well: consider $MN(x,y)$
$MN(x,y) = \min_s \{M(x,s) + N(s,y)\}$ where s is the appropriate vertex in designated set (corresponding to column index for X, and row index for Y). Thus multiplication counts for min length of path through a designated vertex set.

To see $APSP(n) \leq 2ASAP(\frac{n}{2}) + 6MSP(\frac{n}{2}) + O(n^2)$, consider a bipartition of V with equal sizes (approximately) $V = V_1 \uplus V_2$. Running D^* ~~gives~~ consumes $APSP$ ~~ASAP~~ $(\frac{n}{2})$.

Then running BD^* consumes $MSP(\frac{n}{2})$; $E(BD^*)$ $MSP(\frac{n}{2})$; $(BD^*)C$ $MSP(\frac{n}{2})$; $G(BD^*)$ $MSP(\frac{n}{2})$; D^*C $MSP(\frac{n}{2})$; $(D^*C)E$ $MSP(\frac{n}{2})$. Taking "$\vee$" operation is element-wise comparison, of $c_1(n^2+c_2)$ time, $O(n^2)$. Thus $D^* \vee GBD^*$ and $A \vee BD^*C$ each takes $O(\frac{n^2}{4})$, in total $O(\frac{n^2}{2})$, or $O(n^2)$

Last $APSP$ ~~ASAP~~ $(\frac{n}{2})$ comes from $(A \vee BD^*C)^*$ operation.

Such divide-and-conquer takes $2APSP$ ~~ASAP~~ $(\frac{n}{2}) + 6MSP(\frac{n}{2}) + O(n^2)$ as worst scenario upper bound for $APSP(n)$.

$$APSP(n) \leq 2APSP(\tfrac{n}{2}) + 6MSP(\tfrac{n}{2}) + cn^2 \leq 4APSP(\tfrac{n}{4}) + 6(MSP(\tfrac{n}{2}) + 2MSP(\tfrac{n}{4})) + \tfrac{3}{2}cn^2$$

$$\cdots \leq 2^k \cdot APSP(\tfrac{n}{2^k}) + (2 - \tfrac{1}{2^k}) \cdot cn^2 + 6(MSP(\tfrac{n}{2}) + 2MSP(\tfrac{n}{4}) + \cdots + 2^{k-1} MSP(\tfrac{n}{2^k}))$$

$$\approx n \cdot C' + 2cn^2 + 6 \sum_{i=1}^{k-1} MSP(\tfrac{n}{2^i})$$

Since MSP superquardratic, $MSP(\frac{n}{2^i}) \geq 2^2 MSP(\frac{n}{2^{i+1}})$ $\sum_{i=1}^{k-1} 2^i MSP(\frac{n}{2^i}) \approx O(MSP(n))$

Therefore $APSP(n) \approx O(n^2 + MSP(n))$. $APSP(n) = \hat{O}(MSP(n) + n^2)$