1. We first show that $E = (A \vee BD^*C)^*$.

Let m be the number of rows of A and let V_1 be the vertices corresponding to the first m rows.

Let $V_2 = V \setminus V_1$.

Define $G_1 = (V_1, E_1)$ where $e = uv \in E_1$ if $(u, v \in V_1)$ either uv is an edge in G or there exists a uv-path in G where all internal vertices are in V_2.

Then clearly v is reachable from u in G iff v is reachable from u in G_1.

So it suffices to show that the adjacency matrix W_1 of G_1 is equal to $A \vee BD^*C$.

Indeed, A corresponds to the edges in G and, for $u, v \in V_1$, there is a uv-path in G with all internal vertices in V_2 iff

$$e_u^T BD^*Ce_v = 1 \quad \text{, where } e_v \text{ is } (0, 0, \cdots, 0, \underbrace{1}_{v\text{-coordinate}}, 0, \cdots)$$

since $e_u^T B$, Ce_v have 1's only on edges on u, v with neighbors in V_2, and so $(e_u^T B) D^* (Ce_v)$ indicates the reachability in V_2 of the neighbors of u, v in V_2.

Therefore $E = (A \vee BD^*C)^*$.

Similarly, for $u \in V_2$ and $v \in V_1$, there is a uv-path in G iff for some edge $e = ab \in G$, $a \in V_2$, $b \in V_1$, there is a V_2-path from u to a and a V_1-path from b to v. Therefore $G = D^*CE$.

Same argument with V_1, V_2 switched gives

$$F = EBD^*.$$

To show $H = D^* \vee GBD^*$, note that for $u, v \in V_2$, there is a uv-path in G iff either there is a uv-path in V_2 (corresponding to D^*) or there exists an edge $e = ab$ such that $a \in V_1$, $b \in V_2$, and:

- There is a ua-path in G (corresponds to the matrix G)
- ab is an edge in G (" " " B)
- There is a bv-path in V_2 (" " " D^*)

Therefore $H = D^* \vee GBD^*$,

and this completes the proof of this part.

2. $E = (A \vee BD^*C)^*$: For $u, v \in V_1$, every uv-path is a concatenation of paths with internal edges in V_2 and edges in V_1. Concatenation of paths is denoted by products of matrices, which sums the weights. The $\vee$ operation takes the minimum of the two kinds of paths, so $E = (A \vee BD^*C)^*$.

For $u \in V_2$, $v \in V_1$, every uv-path can be decomposed as before, with D^* giving shortest paths within V_2 and E giving shortest paths between two vertices of V_1 in G. The product D^*CE finds all valid concatenations and gives the minimum. Thus $G = D^*CE$.

Same argument gives $F = EBD^*$

H gives shortest paths between two vertices of V_2 in G, which is the min. of paths contained in V_2 and paths that use V_1, so $H = D^* \vee GBD^*$.

3. Given a graph G on n vertices, we can compute APSP on G by ~~computing~~ dividing the vertices into two (near) equal sized parts, then applying the previous ideas to compute W^*.

This requires computing:

- D^* — takes $APSP(n/2)$ time
- $(A \vee BD^*C)^*$ — $2 MSP(n/2) + O(n^2) + APSP(n/2)$
- D^*CE — $2 MSP(n/2)$
- EBD^* — $2 MSP(n/2)$
- $D^* \vee GBD^*$ — $2 MSP(n/2) + O(n^2)$.

Note that (BD^*) ~~used~~ occurs 3 times, so we can skip two $MSP(n/2)$ operations.

This gives a total of $2 \cdot APSP(n/2) + 6 MSP(n/2) + O(n^2)$ running time.

Computing APSP recursively in this manner, noting that APSP(1) can be computed in constant time, we get:

$$APSP(n) = 2\,APSP(n/2) + 6\,MSP(n/2) + O(n^2)$$

$$= 4\,APSP(n/4) + 6\,MSP(n/2) + 12\,MSP(n/4) + O(n^2) + 2O((\tfrac{n}{2})^2)$$

$$\vdots$$

$$= \sum_{k=1}^{\log n} 2^k O(1) + 6 \cdot 2^{k-1} MSP(n/2^k) + 2^{k-1} O((n/2^{k-1})^2)$$

$$= O(n) + 6 \sum_{k=1}^{\log n} 2^{k-1} MSP(n/2^k) + O(n^2) \underbrace{\sum_{k=1}^{\log n} O(1/2^{k-1})}_{\leq O(1)}$$

$$= 6 \sum_{k=1}^{\log n} 2^{k-1} MSP(n/2^k) + O(n^2)$$

Since MSP is superlinear, it follows that

$$APSP(n) = O(MSP(n) + n^2)$$