Zernike Conventions

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Contents

1. Old Wavefront Model

The basic model for light propagation is Fresnel Diffraction:

$$\Psi(x,y,z,t) = \exp\left(\frac{\imath 2\pi}{\lambda}(z-t)\right) u_z(x,y) \ , \tag{1}$$

where we take the initial wavefront $u_{z=0}$ to be

$$u_0(x,y) \propto \exp\left(P(x,y)\right) \exp\left(\imath\Phi(x,y)\right)\right) \ , \tag{2}$$

where r is the two-dimensional radial coordinate, P is the pupil function, and Φ are the aberrations. u_0 is convolved with an atmospheric seeing kernel (Kolmogorov), then propagated to the focal plane. We measure $|u_z(x,y)|^2$. Aberrations in the wavefront ($\Phi \neq 0$) can introduce spurious shapes.

1.1. alt:

The basic model for light propagation is Fresnel Diffraction:

$$\Psi(x,y,z,t) = \exp\left(\frac{\imath 2\pi}{\lambda}(z-t)\right) u_z(x,y) \ , \tag{3}$$

where we take the initial wavefront $u_{z=0}$ to be

$$u_0(x,y) \propto \exp\left(P(x,y)\right) \exp\left(\imath\Phi(x,y)\right)\right) \ , \tag{4}$$

where r is the two-dimensional radial coordinate, P is the pupil function, and Φ are the aberrations.

u_0 is convolved with an atmospheric seeing kernel (Kolmogorov), then propagated to the focal plane:

$$u_z = \frac{\exp\left(\imath\pi r^2/\lambda z\right)}{\imath\lambda z} \mathrm{FT}\left[\exp\left(\frac{\imath\pi r^2}{\lambda z}\right) u_0(x,y)\right] \ . \tag{5}$$

What we measure is the irradiance:

$$I_z(x,y) = |u_z(x,y)|^2. \tag{6}$$

Aberrations in the wavefront ($\Phi \neq 0$) can introduce spurious shapes.

These aberrations vary across the focal plane; $d, a\ldots$ are really $d(x,y)$, functions of focal plane location. Call $d_0, a_0 \ldots$ the ‘normal’ optics contribution to wavefront aberrations. Defocus and misalignment of the focal plane can be characterized by linear shifts and tilts of the normal wavefront, so that for some exposure i the defocus d is:

$$d_i(x,y) = (1+\Delta_d+\theta_{y,d}x+\theta_{x,d}y)d_0(x,y)\ . \tag{7}$$

Measurements of these variations using the focus chips is how the active optics system works.

We would like to obtain this same information for the focal plane, not just the alignment chips. But because the focal plane is near focus, directly modeling Φ is difficult. Instead, parameterize the shape into a sum of moments:

$$M_{pq,z} = \frac{\int \mathrm{d}A\ w(x,y)I_z(x,y)(x-\bar{x})^p(y-\bar{y})^q}{\int \mathrm{d}A\ w(x,y)I_z(x,y)}\ , \tag{8}$$

where $\bar{x}$ and $\bar{y}$ are the centroids of the image and w is a weighting function.

We measure the moments here using the *hsm* weighting scheme, an iterative weighting optimized for measuring second moments used in galsim.

2. Zernikes – Noll, Complex Number

Want table like one from cpd2014MayCollaboration but also including azimuthal etc degree and form as a complex number z and $x+\imath y$

also astigmatism -x and y as well as trefoil need to be renamed

Type	Noll	$(n,\ m)$	Variable	Polar Polynomial
Piston	1	(0, 0)	0	1
Tilt-0	2	(0, 1)	0	$2r\cos\theta$
Tilt-90	3	(0, 1)	0	$2r\sin\theta$
Defocus	4	(2, 0)	d	$\sqrt{3}(2r^2-1)$
Astigmatism-45	5	(2, 2)	$\Im[a]$	$\sqrt{6}r^2\sin 2\theta$
Astigmatism-0	6	(2, 2)	$\Re[a]$	$\sqrt{6}r^2\cos 2\theta$
Coma-90	7	(3, 1)	$\Im[c]$	$\sqrt{8}(3r^3-2r)\sin\theta$
Coma-0	8	(3, 1)	$\Re[c]$	$\sqrt{8}(3r^3-2r)\cos\theta$
Trefoil-30	9	(3, 3)	$\Im[t]$	$\sqrt{8}r^3\sin 3\theta$
Trefoil-0	10	(3, 3)	$\Re[t]$	$\sqrt{8}r^3\cos 3\theta$
Spherical Defocus	11	(4, 0)	s	$\sqrt{5}(6r^4-6r^2+1)$

3. Moments and Adaptive Moments

both the conventional definition and talk about how I use Hirata and Seljak adaptive moments.

4. Whisker Conventions

Lift from whiskerconvention for w?

5. Higher Order Moments

In analogy with gravitational shear and flexion, these moments can be decomposed into linear combinations with convenient rotational symmetries:

$$e_0 = M_{20} + M_{02} \tag{9}$$
$$e_1 = M_{20} - M_{02} \tag{10}$$
$$e_2 = 2M_{11} \tag{11}$$
$$\zeta_1 = M_{30} + M_{12} \tag{12}$$
$$\zeta_2 = M_{03} + M_{21} \tag{13}$$
$$\delta_1 = M_{30} - 3M_{12} \tag{14}$$
$$\delta_2 = -M_{03} + 3M_{21} \tag{15}$$
$$\tag{16}$$

$\epsilon = (e_1 + \imath e_2)/e_0$ is one of the common definitions for ellipticity, while ζ and δ correspond to unnormalized F-1 and F-3 flexion, respectively.

Is it ϵ or χ?

Wavefront aberrations lead to measurable moments. Plugging (5) into (8) and carrying out the math for the Zernike aberrations defocus, astigmatism, coma, trefoil, and spherical defocus

(d, a, c, t, s) we find analytically:

$$\begin{aligned}
e_0 = &\quad 24d^2 + 16\sqrt{15}ds + 120s^2 + 12|a|^2 + 56|c|^2 + 24|t|^2 \\
\mathbf{e} = e_1 + \imath e_2 = &\quad 8\sqrt{2}(3d + \sqrt{15}s)a + 32c^2 + 45\bar{c}t \\
\boldsymbol{\zeta} = \zeta_1 + \imath\zeta_2 = &\quad 64(3d + 2\sqrt{15}s)(2a\bar{c} + \bar{a}t) + \\
&\quad 16\sqrt{2}(18d^2 + 22\sqrt{15}ds + 120s^2 + 6|a|^2 + 25|c|^2 + 15|t|^2)c \\
&\quad + 48\sqrt{2}a^2\bar{t} + 240\sqrt{2}\bar{c}^2 t \\
\boldsymbol{\delta} = \delta_1 + \imath\delta_2 = &\quad 192(3d + 2\sqrt{15}s)ac \\
&\quad + 144\sqrt{2}(2d^2 + 2\sqrt{15}ds + 12s^2 + 5|c|^2)t \\
&\quad + 144\sqrt{2}a^2\bar{c} + 160\sqrt{2}c^3
\end{aligned}$$

6. Zernike v Seidel

maybe work out the field aberration stuff with seidel first and then with zernike (the maths certainly will be at least a /little/ simpler...)

7. Everything before here is kinda poopish.

8. Wavefront Model

Define our pupil function to be in the (u, v) pupil plane coordinates as:

$$\exp(\imath\Phi(u, v)) \tag{17}$$

This equation includes both the pupil function from the spider and any annulus, and the aberrations to the wavefront, which go as polynomial terms in Φ. Up to some set of constants, we can represent the Point Spread Function (PSF) as follows[1]:

$$\mathrm{PSF}(x, y) = \left| \int \mathrm{d}u\mathrm{d}v \exp(-\imath\Phi(u, v) - \imath xu - \imath yv) \right|^2 \tag{18}$$

[1] I will play extremely fast and loose with Fourier Transform conventions and will also assume everything is properly normalized.

A useful other term is the Optical Transfer Function (OTF), which is the Fourier Transform of the PSF and hence the autocorrelation of the pupil function.

One thing I have seen used is that we can represent the measured moments as integrals over the derivatives of the wavefront, ie:

$$\int \mathrm{d}x\mathrm{d}y x^p y^q \mathrm{PSF}(x,y) = \int \mathrm{d}u\mathrm{d}v (\partial_u \Phi)^p (\partial_v \Phi)^q \tag{19}$$

Or at least I think it is that, and not the following:

$$\int \mathrm{d}x\mathrm{d}y x^p y^q \mathrm{PSF}(x,y) = \int \mathrm{d}u\mathrm{d}v (\partial_u^p \Phi)(\partial_v^q \Phi) \tag{20}$$

Anyways let us proceed with the lefthand side of the expression. The PSF is the Inverse Fourier Transform of the OTF, which is a convolution of the pupil function with its complex conjugate (note that $\bar{\Phi} = \Phi$):

$$\int \mathrm{d}x\mathrm{d}y x^p y^q \mathrm{PSF}(x,y) = \int \mathrm{d}x\mathrm{d}y x^p y^q \int \mathrm{d}u\mathrm{d}v \exp(\imath x u + \imath y v) \int \mathrm{d}u'\mathrm{d}v' \exp\left(\imath\Phi(u',v') + -\imath\Phi(u'-u, v'-v)\right) \tag{21}$$

(Note: it may be $u+u'$ instead of $u-u'$. I keep alternating the answer each time I check it!)

(actually I just looked it up. It looks like I got the IFT right, but I flipped the signs so that in the barred part it is $u'-u$ instead of the other way. Also the pupil function is $exp\imath\Phi$ and not the inverse of that. It is worth pointing out that Φ (phase aberration = $\frac{2\pi}{\lambda}W$) is real-valued. This combined with the proper sign of $u'-u$ means that the phase cancels. Also, a property of the cross-correlation)

Let us flip the order of integration and integrate over x and y. Doing so yields (modulo some set of (-1)'s and $\imath$'s):

$$\int \mathrm{d}u\mathrm{d}v\mathrm{d}u'\mathrm{d}v' e^{\imath\Phi(u',v')} e^{-\imath\Phi(u'-u,v'-v)} \delta^{(p)}(u)\delta^{(q)}(v) \tag{22}$$

where $\delta^{(p)}(u)$ is the p-th derivative of the Dirac delta distribution function. We then proceed with the integration over the unprimed variables (again prefactors are ignored because I'm lazy):

$$\int \mathrm{d}u'\mathrm{d}v' e^{\imath\Phi(u',v')} \left[\left(\frac{\partial}{\partial u}\right)^p \left(\frac{\partial}{\partial v}\right)^q e^{-\imath\Phi(u'-u,v'-v)}\right]_{(u,v)=(0,0)} \tag{23}$$

Conveniently, after the derivatives are taken, the phase aberration cancels. Let's give a couple examples of what the terms are for each moment (constants in front arbitrary and complex):

$$sum(p,q) = 1 \propto \Phi_a \tag{24}$$

$$sum(p,q) = 2 \propto \alpha\Phi_{ab} + \beta\Phi_a\Phi_b \tag{25}$$

$$sum(p,q) = 3 \propto \alpha\Phi_{abc} + \beta(\Phi_{ab}\Phi_c + \Phi_{ac}\Phi_b + \Phi_{bc}\Phi_a) + \gamma\Phi_a\Phi_b\Phi_c \tag{26}$$

where subscripts refer to derivative with respect to arbitrary u or v. In other words, we need to make a call about the relative magnitude of multiplying first derivatives together compared with a second derivative in the wavefront.

Finally note that we measure moments over time (so there is an implicit integration over and dependency on time) and from a fixed center that is usually taken to be the centroid of the final (time-integrated) object. Shaking in the telescope can take a wavefront that is circular and produce an ellipticity. One can model this as a variation in the tilt Zernikes with time.

9. Zernikes

Type	Noll	$(n,\ m)$	Coef	Polar Polynomial	Complex Polynomial
Piston	1	(0, 0)	0	1	1
Tilt-0	2	(0, 1)	0	$2r\cos\theta$	$z+\bar{z}$
Tilt-90	3	(0, 1)	0	$2r\sin\theta$	$-\imath(z-\bar{z})$
Defocus	4	(2, 0)	d	$\sqrt{3}(2r^2-1)$	$\sqrt{3}(2z\bar{z}-1)$
Astigmatism-45	5	(2, 2)	$\Im[a]$	$\sqrt{6}\,r^2\sin 2\theta$	$-\frac{\imath\sqrt{6}}{2}(z^2-\bar{z}^2)$
Astigmatism-0	6	(2, 2)	$\Re[a]$	$\sqrt{6}\,r^2\cos 2\theta$	$\frac{\sqrt{6}}{2}(z^2+\bar{z}^2)$
Coma-90	7	(3, 1)	$\Im[c]$	$\sqrt{8}(3r^3-2r)\sin\theta$	$-\frac{\imath\sqrt{8}}{2}(3z\bar{z}-2)(z-\bar{z})$
Coma-0	8	(3, 1)	$\Re[c]$	$\sqrt{8}(3r^3-2r)\cos\theta$	$\frac{\sqrt{8}}{2}(3z\bar{z}-2)(z+\bar{z})$
Trefoil-30	9	(3, 3)	$\Im[t]$	$\sqrt{8}\,r^3\sin 3\theta$	$-\frac{\imath\sqrt{8}}{2}(z^3-\bar{z}^3)$
Trefoil-0	10	(3, 3)	$\Re[t]$	$\sqrt{8}\,r^3\cos 3\theta$	$\frac{\sqrt{8}}{2}(z^3+\bar{z}^3)$
Spherical Defocus	11	(4, 0)	s	$\sqrt{5}(6r^4-6r^2+1)$	$\frac{\sqrt{5}}{2}(3(2z\bar{z}-1)^2-1)$

Double check these rules to make sure that they are correct!

Once we have the format of the desired derivatives, the calculation of moments is then just a matter of integrating the polynomials over u and v using $z = u + \imath v$. It is convenient to use complex notation to describe the u and v coordinates. Hence, the derivatives are of the form $\left(\frac{\partial}{\partial u} - \imath\frac{\partial}{\partial v}\right) = \frac{\partial}{\partial \bar{z}}$. It is also very convenient to consider the polar integration and realize that any resultant terms of the form $z^\alpha \bar{z}^\beta$ will integrate to zero if $\alpha \neq \beta$ and if they are equal than the integral will be $\frac{2}{\alpha+2}$. (I might have also switched integration range from -1 to 1 to 0 to 1. That might have been a silly mistake.) Next also note that $\mathrm{d}u\mathrm{d}v = \frac{\imath}{2}\mathrm{d}z\mathrm{d}\bar{z}$ and that $\bar{z}^2 = \overline{z^2}$.