Homework 11

2022-11-04

Question 1: For all $n \in \mathbb{N}$, show that...

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Proof: We proceed by induction.

(Base case) For the smallest values of $n \in \mathbb{N}$ we begin with 1, and 2...

$$\frac{1}{1 \times 2} = \frac{1}{1+1}$$
$$\frac{1}{2} = \frac{1}{2}$$

Looks good. Now for n = 2

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} = \frac{2}{2+1}$$
$$\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$
$$\frac{3}{6} + \frac{1}{6} = \frac{2}{3}$$
$$\frac{4}{6} = \frac{2}{3}$$
$$\frac{2}{3} = \frac{2}{3}$$

Solid. Moving on.

(Inductive step) Suppose $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$ for all $k \in \mathbb{N}$ and let $k > 2$. So then...

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1}$$

We add $\frac{1}{(k+1)(k+2)}$ to both sides.

$$\frac{1}{(k+1)(k+2)} + \sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

The sum symbol eats the $\frac{1}{(k+1)(k+2)}$

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

find a common denominator for the right side.

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

combine fractions.

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k(k+2)+1}{(k+1)(k+2)}$$

simplify...

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{(k+2)}$$

show right side as $k+1$..

$$\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{((k+1)+1)}$$

Therefor $k \implies k+1$ summarized as...

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{k}{k+1} \implies \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \frac{k+1}{((k+1)+1)}$$

So then through the inductive process we have shown that for all $n \in \mathbb{N}$

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

Question 3: In a chess tournament of n players, each pair of players plays exactly one match against each other. There are no draws. Prove that the players can be arranged in an order $p_1, p_2 ... p_n$ such that p_i defeats $p_i + 1$ for all $i = 1, 2, ...n - 1$

Proof: We proceed with strong induction.

(Base Cases) For n total players 0 to 1 the case is null. For 2 players arrange them as p_1, p_2 where p_1 defeated p_2 in a chess match.

(Inductive step) Suppose for the sake of contradiction that there is some smallest tournament of size k where we can not arrange the players in an order $p_1, p_2 ... p_k$ such that p_i defeats $p_i + 1$ for all $i = 1, 2, ...k - 1$

Remove one player from this player pool, while keeping all win/loss results of the other players the same. Let this new pool be player pool ℓ. Pool ℓ is now $k - 1$ players large, and $k - 1 < k$. Since k is the smallest player pool size that can't be arranged in such a way as described above, and ℓ is smaller than k, then ℓ can be arranged in an order $p_1, p_2 ... p_\ell$ such that p_i defeats $p_i + 1$ for all $i = 1, 2, ...\ell - 1$.

Arrange ℓ in this a way, then add a new player to the pool. Let this new player be player x. Allow player x to have the same win/loss results as the player originaly removed from k. Player pool ℓ is now equal to player pool k. Then sequentially check if this player has defeated each player in ℓ, right to left, from player p_ℓ to player p_1 in the $p_1, p_2 ... p_\ell$ standing.

For each check of player x to against player p_i across players in ℓ, there are two possible outcomes...

Case 1: Player x lost to player p_i. Place player x to the right of player p_i. The standing will maintains it's p_i defeated $p_i + 1$ order.

Case 2: Player x won against player p_i. Now repeat this process for player p_{i-1}. Move to the left in the $p_1, p_2 ... p_\ell$ standing, and repeat.

Now again, there are two possible outcomes to this process...

Case 1: Player x lost to someone, and will be placed in the standing as described. The standing will maintain it's p_i defeats $p_i + 1$ order.

Case 2: Player x has not lost to anyone (all wins).

If case 2 occurs, then we can place player x at the front of the $p_1, p_2 ... p_\ell$ standing, so that the p_i defeated $p_i + 1$ order will be maintained.

Now, in all possible cases of adding the player x back to pool ℓ, so that $\ell = k$, we have arranged k in the $p_1, p_2 ... p_\ell$ order in which player p_i has defeated player $p_i + 1$. But k is the smallest sized player pool that cannot be arranged in such a way. This is a contradiction. Therefor there is no smallest player pool k that can't be arranged in such a way. It must be that we can arrange any player pool of size $n \in \mathbb{N}$ in an order $p_1, p_2 ... p_n$ such that p_i defeats $p_i + 1$ for all $i = 1, 2, ...n - 1$

Question 4. Rewriting question. for any $n \in \mathbb{N}$ we can write write it as $n = 2^{x_1} + 2^{x_2} + ...2^{x_i}$ where x_0 to x_i are elements of the set $x \in P(\mathbb{N})$.

(Base case) Showing 1 to 7 can be writen as $2^{x_1} + 2^{x_2} + ...2^{x_i}$.

```
raise = function(x){
  sum(2^x)
}

# first few natural numbers
n = 0:2
# produce the power set of n
sets = powerSet(n, rev=TRUE)
# raise by 2 and sum each subset of n (in the powerset)
u = lapply(sets, raise)
# Just changing the data class to vector (from a list), and sriting them
u = unlist(u)
sort(u)
```

Let X be the set $\{x \in P(\mathbb{N}); 2^{x_1} + 2^{x_2} + ...2^{x_i}\}$. Suppose for the sake of argument that there is some $n \in \mathbb{N}$ where $n \notin X$.

Let k be this number, and t be the greatest factor of two that is less than k, and ℓ be the smallest factor of two greater than k. Such that $t < k < \ell$ and $2t = \ell$.

Then $k - t = m$ and $k = m + t$ where $m \in \mathbb{N}$, and $m < t < k$.
Since $m < t < k$ and k is the smallest number not in X, it follows that $t, m \in X$.

Similarly, because $t > m$, then $t + m \in X$. Note, if it where the case where $m > t$ then ℓ would be the greatest factor of two that is less than k, not t.
So $t + m = k$ and $t + m \in X$ but $k \notin X$. This is a contradiction.
Therefore, $\forall n \in X$.