Homework: Mathematical Induction

Math 321: Intro to Advanced Mathematics

Due - Thursday, November 3, 2022

Choose at least two problems from section 1, and at least one problem from section 2 and one from section 3 to write up a careful solution (so at least four total). Use some kind induction to prove each statement that you choose.

§1 Mathematical Induction

Problem 1. For all $n \in \mathbb{N}$, show that

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + ... + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

Problem 2. For all integers $n \in \mathbb{N}$, show that

$$\sum_{i=1}^{2^n} \frac{1}{i} = \frac{1}{1} + \frac{1}{2} + ... + \frac{1}{2^n} \geq 1 + \frac{n}{2}.$$

Problem 3. In a chess tournament of n players, each pair of players plays exactly one match against each other. There are no draws. Prove that the players can be arranged in an order $P_1, P_2, ..., P_n$ such that P_i defeats P_{i+1} for all $i = 1, 2, ..., n-1$.

§2 Strong Induction and Smallest Counterexample

Problem 4. In class we defined the Fibonacci sequence as $f_1 = 1, f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. Show that for all $n \in \mathbb{N}$, we have the following inequality

$$\left(\frac{3}{2}\right)^{n-2} \leq f_n.$$

Problem 5. Prove by smallest counterexample. Show that for all $n \in \mathbb{N}$, n can be written as a sum of distinct powers of two.

§3 Even More Induction

This section has problems where it might not be obvious straight away how to use induction.

Problem 6. Consider a grid of squares that is 2^n squares wide by 2^n squares long, where $n \in \mathbb{N}$. One of the squares has been cut out, but you do not know which one! You have a bunch of L-shapes made up of 3 squares, call them "triominos". Prove that you can perfectly cover this chessboard with triominos (with no overlap) for any $n \in \mathbb{N}$.

Problem 7. Let $n \in \mathbb{N}$. Suppose n chords are drawn in a circle so that each chord intersects every other, but no three intersect at one point. Prove that the chords cut the circle into $\frac{n^2+n+2}{2}$ regions.