HW 14: Cardinality

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Question 1. Prove each of the following. I each case, you should create a bijection between the two sets.Briefly justify that your function are in fact bijections.

(a) $|\{\heartsuit, \clubsuit, \spadesuit\}| = |\{\circ, \Box, \triangle\}|$

Informal proof Let $A = \{\heartsuit, \clubsuit, \spadesuit\}$ and $B = \{\circ, \Box, \triangle\}$ and the funciton $f : A \to B$ be $f = \{(\heartsuit, \circ), (\clubsuit, \Box), (\spadesuit, \triangle)\}$. f is a bijection from A to B, so then $|\{\heartsuit, \clubsuit, \spadesuit\}| = |\{\circ, \Box, \triangle\}|$. f is injective because every element in A maps to a unique element in B. f is a surjection because every element in B is mapped to an element in A. f's surgective and injectivenes can be easily seen because A and B are finite and small. $\blacksquare$.

(b) $|\mathbb{N}| = |\{oddnaturalnumbers\}|$

Informal proof Let O be the set of odd numbers. Then $f : \mathbb{N} \to O$ as $f(n) = 2n + 1$, where $n \in \mathbb{N}$, is a bijection. f is a injection because every element in $\mathbb{N}$ maps to a unique element in O. That is all of $\mathbb{N}$ are in the domain of f, and f is increasing so order is preserved, and each output is unique. f is a surjection because every element in O maps to an element in $\mathbb{B}$, through the inverse function $n = (o - 1)/2$ where $o \in O$. This is further evidence that $f(n) = 2n + 1$ is a bijection, because it's inverse is also a function. $\blacksquare$

(c) $|A \times \{1\}| = |A|$, where A is any set.

Informal proof The set $A \times \{1\}$ looks like... $A \times \{1\} = \{(a_1, 1), (a_2, 1), (a_3, 1)...(a_i, 1)\}$ where a_i is the i'th element in $A \times \{1\}$. The projection of $A \times \{1\}$ onto it's first element is a bijeciton from $A \times \{1\} \to A$. Writen as $f(a, 1) = a$. f is an injection because f accepts the set $A \times \{1\}$, and outputs the set A. We know the elements output by f are unique because all elements in A are unique (no repeating elements in a set). It is a surjection because all elements of A are mapped to an element of $A \times \{1\}$. $\blacksquare$

Question 2. Let $\mathcal{F}$ denote the set of all functions from $\mathbb{N}$ to $\{0, 1\}$.

(a) Describe at least three functions in the set $\mathcal{F}$.

$f : \mathbb{N} \to \{0, 1\}$ where $f(x) = x * 0$, or $f(x) = x^0$, or $f(x) = \{$if x is even assign a 1, if x is odd assign a 0$\}$.

(b) Prove that $|\mathcal{F}| = |P(\mathbb{N})|$.

Proof We proceed by proof of equivalent cases (not sure if that's a thing). First I will show that $|\mathcal{F}| = 2^{|\mathbb{N}|}$. Then I will show that $|P(\mathbb{N})| = 2^{|\mathbb{N}|}$

case 1. By the definition of equal functions, for each element $n \in \mathbb{N}$ there are only two possible functions that can send $n \to \{0, 1\}$. So then given the multiplcatory rule of combinitorics (pg. 69 of textbook), there are $2^{|\mathbb{N}|}$ functions that can send $\mathbb{N} \to \{0, 1\}$. Thus $|\mathcal{F}| = 2^{|\mathbb{N}|}$.

case 2. In class we showed that $\mid P(\mathbb{N}) \mid = 2^{|\mathbb{N}|}$.

Therefore $\mid P(\mathbb{N}) \mid = 2^{|\mathbb{N}|} = \mid F \mid$, meaning $\mid P(\mathbb{N}) \mid = \mid F \mid$. ■

Question 3. Let X be a set. Prove that "it has the same cardinality as" is an equivalence relation on $P(X)$.

We approach this by addressing three claims. We show that the "it has the same cardinality as" relation, is a bijection that is transitive, reflexive and symmetric.

Let R be the bijective relation "it has the same cardinality as", and let A be any set.

claim 1. R is reflexive. The identity function i_A is a bijection on $A \to A$, so $|A| = |A|$.

claim 2. R is symetric. Suppose $|A| = |B|$, then there is a bijection $A \to B$, and it's inverse is $f^{-1} : B \to A$. Therefore $|B| = |A|$

claim 3. R is transitive. Suppose $|A| = |B|$ and $|B| = |C|$. Similar to the symmetric case, there are bijections $f : A \to B$ and $g : B \to C$. Their composite $g \circ f$, is a bijection on $A \to C$. Therefore $|A| = |C|$.

We've shown that the relation R, is reflexive, symetric, and transitive on any set A. Since $P(X)$ is a set, then this proof holds for $P(X)$. ■

Reference: This uses theorem 12.2, and a paragraph on page 273 of the Book of Proofs.

Question 4. Prove or disprove: The set $\{a_1, a_2, a_3 ... a_i : a \in \mathbb{Z}\}$ of infinite sequence of integers is countably infinite.

Proof We show there exists a bijection $f : \mathbb{N} \to \{a_1, a_2, a_3 ... a_i : a \in \mathbb{Z}\}$, so that $\{a_1, a_2, a_3 ... a_i : a \in \mathbb{Z}\}$ is countably infinite.

Let Y be the set $\{a_1, a_2, a_3 ... a_i : a \in \mathbb{Z}\}$. The following table describe the bijection $f : \mathbb{N} \to Y$. I also show the set of f in set notation.

Set...

$$f = \{(a_1, 1), (a_1, 2), (a_1, 3), (a_1, 4), (a_1, 5) ... (a_\infty, \infty)\}$$

Table...

```
n = 1:10
a = rep("a", length(n))
z = paste(a, as.character(n),sep = "")
data.frame(n, z)
```

```
##     n   z
## 1   1  a1
## 2   2  a2
## 3   3  a3
## 4   4  a4
## 5   5  a5
## 6   6  a6
## 7   7  a7
## 8   8  a8
## 9   9  a9
## 10 10 a10
```

f is both surgective and injective. Every integer appears once in the column "z" or in the first coordinate of f. So, given any integer $b \in \mathbb{Z}$, there is some natural number n such that $f(n) = b$, so f is surgective. f is injective because the way the data frame is constructed it forces $f(m) \neq f(n)$ when $m, n \in \mathbb{N}$ and $m \neq n$. From this bijection we find $\mid \mathbb{N} \mid = \mid Y \mid$, therefore Y is countably infinite. Y being the set $\{a_1, a_2, a_3 ... a_i : a \in \mathbb{Z}\}$. ■

Reference: This uses example 14.2 of the Book of Proofs

Question 5: Prove that A and B are finite sets with $\mid A \mid = \mid B \mid$, then any injection $f : A \to B$ is also a surjection. Show this is not necessarily true if A and B are not finite.

Direct proof!!! (p.s. I do not have all my direct proof check marks)...
Suppose that A and B are finite sets with $\mid A \mid = \mid B \mid$.
Then the number of injective functions that can be made from A to B $f : A \to B$, is described using the binomial function $\binom{|B|}{|A|}$. Similarly, the number of bijective functions from $A \to B$ is given by $\binom{|A|}{|B|}$.
Since A and B are of equal cardinality, then $\binom{|B|}{|A|} = \binom{|A|}{|B|}$.
Thus the number of injective relationships equals the number of bijective relationships.
This means all injective functions between A and B are surjective.
Therefore any injection, $f : A \to B$, is also a surjection. ■

*Note: When describing the number of bijective functions as $\binom{|A|}{|B|}$, if $\mid A \mid < \mid B \mid$ then $\binom{|A|}{|B|} = 0$, which agrees with the pigeon hole principle that if $\mid A \mid < \mid B \mid$ then there is no surjective function from A to B, and thus no bijective function from A to B.

This is not necessarily true if A and B are not finite.