HW12 Jamie Ash

Question 1. Define a relation on $\mathbb{N}$ by $\mathbb{W} = \{(m, n) \in \mathbb{N} : 2m < n < 3m + 6\}$

(a) Which of the following pairs are in $\mathbb{W}$ and which are not? Justify your answers

$$\{(2, 6), (5, 21), (7, 31), (3, 5), (4, 16)\}$$

Answer: The pairs (2, 6), and (4, 16) are in $\mathbb{W}$. The pairs (5, 21), (7, 31), and (3, 5) are not in $\mathbb{W}$.

For (2, 6)...

$$\begin{aligned} 2m < n < 3m + 6 \\ 2(2) < 6 < 3(2) + 6 \\ 4 < 6 < 12 \\ TRUE \end{aligned}$$

For (4, 16)...

$$\begin{aligned} 2m < n < 3m + 6 \\ 2(4) < 16 < 3(4) + 6 \\ 8 < 16 < 18 \\ TRUE \end{aligned}$$

For (5, 21)...

$$\begin{aligned} 2m < n < 3m + 6 \\ 2(5) < 21 < 3(5) + 6 \\ 10 < 21 < 21 \\ FALSE \end{aligned}$$

For (7, 31)...

$$\begin{aligned} 2m < n < 3m + 6 \\ 2(7) < 31 < 3(7) + 6 \\ 14 < 31 < 27 \\ FALSE \end{aligned}$$

For (3, 5)...

$$\begin{aligned} 2m < n < 3m + 6 \\ 2(3) < 5 < 3(3) + 6 \\ 6 < 5 < 15 \\ FALSE \end{aligned}$$

(b) Give two more ordered pairs of natural numbers that are in $\mathbb{W}$ and two that are not. Be sure to justify your answers.

Answer: The pairs (2, 5) and (1, 3) are in $\mathbb{W}$. The pairs (1, 10) and (2, 3) are not in $\mathbb{W}$.

For (2, 5)...

$$\begin{aligned} 2m < n < 3m + 6 \\ 2(2) < 5 < 3(2) + 6 \\ 4 < 5 < 12 \\ TRUE \end{aligned}$$

For (1, 3)...

$$\begin{aligned} 2m < n &< 3m + 6 \\ 2(1) < 3 &< 3(1) + 6 \\ 2 < 3 &< 9 \\ &TRUE \end{aligned}$$

For (1, 10)...

$$\begin{aligned} 2m < n &< 3m + 6 \\ 2(1) < 10 &< 3(1) + 6 \\ 2 < 10 &< 9 \\ &FALSE \end{aligned}$$

For (2, 3)...

$$\begin{aligned} 2m < n &< 3m + 6 \\ 2(2) < 3 &< 3(2) + 6 \\ 4 < 3 &< 12 \\ &FALSE \end{aligned}$$

(c) Explain why $\mathbb{W}$ is not an equivalence relation.

Answer: $\mathbb{W}$ is not an equivalent relation because $\mathbb{W} = \{(m,n) \in \mathbb{N} : 2m < n < 3m+6\}$ is not reflexive. That is $m\mathbb{W}n$ does not imply $n\mathbb{W}m$. A relation must be reflexive, symmetric and transitive to be an equivalence relationship. Here's an example of an ordered pair in $\mathbb{W}$ that is not reflexive.

For (1, 3)...

$$\begin{aligned} 2m < n &< 3m + 6 \\ 2(1) < 3 &< 3(1) + 6 \\ 2 < 3 &< 9 \\ &TRUE \end{aligned}$$

For (3, 1)...

$$\begin{aligned} 2m < n &< 3m + 6 \\ 2(3) < 1 &< 3(3) + 6 \\ 6 < 1 &< 15 \\ &FALSE \end{aligned}$$

Question 2. Write up a careful solution:

(a) Let $\mathbb{R}$ be the empty set relation on the integers. That is, $R = \emptyset$ and for all $a, b \in \mathbb{Z}$ we have $(a, b) \in R$. Prove or disprove: R is an equivalence relation.

Answer: $\mathbb{R}$ is not an equivalent relation because it is not symmetric. A relation is equivalent if it is symmetric, transitive, and reflexive. The relation $\mathbb{R}$ happens to be transitive and symmetric because transitive and symmetric use conditional statements in their definition. That is, a relation is symmetric if, $\forall x, y \in \mathbb{Z}$, $x\mathbb{R}y \implies y\mathbb{R}x$. Similarly, a relation is transitive if $\forall x, y, z \in \mathbb{Z}$, $x\mathbb{R}y \cup x\mathbb{R}y \implies y\mathbb{R}x$. For the relation of the empty set, there exists no $x\mathbb{R}y$ so the statements $x\mathbb{R}y \cup x\mathbb{R}y \implies y\mathbb{R}x$ and $x\mathbb{R}y \implies y\mathbb{R}x$ are always true.

For the relation $\mathbb{R}$ to be reflexive it must satisfy $\forall x \in \mathbb{Z}$, $x\mathbb{R}x$. Because there does exists $x \in \mathbb{Z}$, but there does not exist $x \in \mathbb{R}$, then $x\mathbb{R}x$ does not hold. Therefore R is not a reflexive relation, and consequently not an equivalence relation.

Question 3. Suppose that $\mathbb{R}$ is an equivalence relation on a set S, and let $a, b \ \in \mathbb{S}$. Let $[a]$ represent the equivalence class of a, the set of elements of S that are equivalent to a. In set notation.

$$[a] = \{x \in S : x \sim a\}$$

Prove the following:

(a) If $a \sim b$ then $[a] = [b]$.

Proof: I'm changing the $\sim$ to $\mathbb{R}$.

Suppose $aRb\ldots$

Because R is an equivalence relationship on S, and $a, b \in S$, then there exists some equivalence class $[a]$ and $[b]$.

By definition the equivalence class $[b]$ is $\{x \in S : xRb\}$ and $[a]$ is $\{x \in S : xRa\}$. Because $[a] = [b]$ then $\{x \in S : xRb\} = \{x \in S : xRa\}$. But $a \in S$ so then aRb and $b \in S$ so bRa.

We will show $[a] \subset [b]$ and $[b] \subset [a]$. To show $[a] \subset [b]$ suppose $c \in [a]$. As $c \in [a] = \{x \in S : xRa\}$ we find cRa. We've shown cRa and aRb so by the transitive property of R we have cRb. This implies $c \in \{x \in S : xRb\} = [b]$. So $c \in [a] \implies c \in [b]$, so $[a] \subset [b]$.

We do nearly the same to show $[b] \subset [a]$.

So $[b] \subset [a]$ and $[a] \subset [b]$, therefore $[a] = [b]$.

(b) If $a \nsim b$ then $[a] \neq [b]$.

Proof: We proceed by the contrapositive case. That is, if $[a] = [b]$ then $a \sim b$.

Suppose $[a] = [b]\ldots$
Show $[a] = [b]$ implies symmetric
Show $[a] = [b]$ implies reflexive
Show $[a] = [b]$ implies transative.

Therefor $a \sim b$