Math 321: Intro to Advanced Mathematics
December 1, 2022

Classwork: I put my thang down, flip it, and INVERT it

Question 1. Let $f : X \to Y$ and $g : Y \to X$ both be functions. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $g \circ f$ is injective, then f is injective.

(b) If $g \circ f$ is injective, then g is injective.

(c) If $g \circ f$ is surjective, then f is surjective.

(d) If $g \circ f$ is surjective, then g is surjective.

Question 2. Define $f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x^2$. List the elements in each set:

(a) $f(\{0, 1, 2\})$.

(b) $f^{-1}(\{0, 1, 4\})$.

(c) $f^{-1}(\{-2, 2\})$.

Question 3. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(z) = 3z^2 - 4$. Find each of the following sets.

(a) $g(\{-1, 1\})$.

(b) $g([-1, 1])$. (Remember $[a, b]$ means the closed interval from a to b in $\mathbb{R}$.)

(c) $g([-2, 4])$.

(d) $g^{-1}((-10, 1))$. (Remember (a, b) means the openar interval from a to b in $\mathbb{R}$.)

(e) $g(\varnothing)$.

(f) $g^{-1}(\varnothing)$.

(g) $g(\mathbb{R})$.

(h) $g^{-1}(\mathbb{R})$.

Question 4. Let $h : \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = x^2$.

(a) Find two nonempty subsets $A, B \subseteq \mathbb{R}$ such that $A \cap B = \varnothing$ but $h^{-1}(A) = h^{-1}(B)$.

(b) Find two nonempty subsets $A, B \subseteq \mathbb{R}$ such that $A \cap B = \varnothing$ but $h(A) = h(B)$.

Question 5. Suppose $f : X \to Y$ is an injection and A and B are disjoint subsets of X. Are $f(A)$ and $f(B)$ necessarily disjoint subsets of Y? If so, prove it. Otherwise, provide a counterexample.

Question 6. Let $f : X \to Y$ be a function, and let $A, B \subseteq X$ and $C, D \subseteq Y$. Determine if each statement is true or false. If it's true, prove it. If it's false, provide a counterexample.

(a) If $A \subseteq B$ then $f(A) \subseteq f(B)$.

(b) If $C \subseteq D$ then $f^{-1}(C) \subseteq f^{-1}(D)$.

(c) $f(A \cup B) \subseteq f(A) \cup f(B)$.

(d) $f(A \cup B) \supseteq f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(f) $f(A \cap B) \supseteq f(A) \cap f(B)$.

(g) $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

(h) $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.

(i) $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

(j) $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.

(k) $A \subseteq f^{-1}(f(A))$.

(l) $A \supseteq f^{-1}(f(A))$.

(m) $C \subseteq f\left(f^{-1}(C)\right)$.

(n) $C \supseteq f\left(f^{-1}(C)\right)$.

Question 7. Let $f : \mathbb{R} \to \mathbb{R}$ be an **additive** funciton, meaning that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Prove that $f(0) = 0$.

(b) Prove that $f(-x) = -f(x)$.

(c) Prove that f is injective if and only if $f^{-1}(\{0\}) = \{0\}$.