Math 321: Intro to Advanced Mathematics
October 13, 2022

Homework: Sets and congruences

Sets and power sets: You may want to read Sections 8.1–8.3 about sets. You can also review material from Chapter 1.

Question 1. Let $m, n \in \mathbb{Z}$. Consider three sets:

$$S_m = \{k \in \mathbb{Z} \colon m \mid k\}, \quad S_n = \{k \in \mathbb{Z} \colon n \mid k\}, \quad S_{mn} = \{k \in \mathbb{Z} \colon mn \mid k\}.$$

(a) Prove that for all $m, n \in \mathbb{Z}$,

$$S_m \cap S_n \neq \varnothing.$$

(b) Prove that for all $m, n \in \mathbb{Z}$ we have

$$S_{mn} \subseteq S_m \cap S_n.$$

(c) Can you find specific $m, n \in \mathbb{Z}$ where

$$S_{mn} = S_m \cap S_n?$$

(d) Can you find specific $m, n \in \mathbb{Z}$ where

$$S_{mn} \subsetneq S_m \cap S_n?$$

Question 2. Remember that for a set A, the power set of A is the set of all subsets of A. We write $\mathcal{P}(A)$.

(a) Prove that if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

(b) Prove that in general $\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$.

(c) Can you find sets A and B where it is true that $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$?

(d) Prove or disprove: $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$ for all sets A and B.

Computations: You can review modular arithmetic and congruences in Section 5.2.

Question 3. Find 3 different values for x. Then describe **all** integers that satisfy the congruence.

(a) $x \equiv 1 \pmod 2$
(b) $x \equiv 0 \pmod 5$
(c) $x \equiv 3 \pmod{10}$

Question 4. Find all values of n that make the congruence true.

(a) $5 \equiv 25 \pmod n$
(b) $20 \equiv 0 \pmod n$
(c) $37 \equiv 1 \pmod n$

Proofs: Prove the following using any technique at your disposal (direct proof, contrapositive, proof by contradiction, cases ...).

Question 5. If n is odd, then $8 \mid (n^2 - 1)$.

Question 6. If $a, b \in \mathbb{Z}$, then $(a+b)^3 \equiv a^3 + b^3 \pmod 3$.

Question 7. For any integer n, prove that $3 \mid (n^3 + 2n)$.