Math 321: Intro to Advanced Mathematics
November 10, 2022

Assessment Carnival: Try one! Have some!

You may do up to 2 problems from each section.

Direct Proof.

Question 1. Let x be a positive real number. (That is, $x \in \mathbb{R}^+$. Prove that

$$x - \frac{2}{x} > 1 \quad \Longrightarrow \quad x > 2.$$

Question 2. A set $S \subseteq \mathbb{R}$ is called *convex* if the following holds for all $t \in [0,1]$:

$$(x \in S) \wedge (y \in S) \implies (xt + y(1-t)) \in S.$$

Let a and b be arbitrary real numbers. Prove that $L = \{x \in \mathbb{R} \mid ax \leq b\}$ is a convex set.

Logical equivalence / Contrapositive Proof.

Question 3. Which among the following expressions are logically equivalent to each other? (Prove what you say.)

(a) $Pand(P \Rightarrow Q)$

(b) Q

(c) $(P \wedge Q) \Leftrightarrow P$

(d) $P \Rightarrow Q$

(e) $P \wedge Q$

Question 4. Prove: For all non-negative real numbers a and b, if $\frac{a+b}{2} \neq \sqrt{ab}$, then $a \neq b$.

Proof by Contradiction / Quantifiers and Negation.

Question 5. Prove that there is no integer n such that $n \equiv 5 \pmod{14}$ and $n \equiv 3 \pmod{21}$.

Question 6. Prove that there do not exist three distinct positive integers a, b, and c such that each integer divides the difference of the other two.

Proof by Cases.

Question 7. Prove that for any integer n, at least one of the integers

$$n, \quad n+2, \quad n+4$$

is divisible by 3.

Question 8. Prove that for all $x \in \mathbb{R}$,

$$-5 \leq |x+2| - |x-3| \leq 5.$$

Mathematical Induction.

Question 9. Use mathematical induction to prove that $2^n > n^2$ for all integers $n \geq 5$.

Question 10. Given a finite collection of n points in the plane where $n \geq 3$ and not all of the points are colinear, prove that there exists a triangle in the plane having three of the n points as its vertices and none of the n points in its interior.

Quantifier Proof / Disproof.

Question 11. Prove or disprove: For any positive integer m there is a positive integer n such that $mn+1$ is a perfect square.

Question 12. Prove or disprove: There exists a positive real number x for which $x^2 < \sqrt{x}$.

Basic Set Theory.

Question 13. Let $U = \mathbb{Z}$ be our universal set, and define the following subsets:

$$\begin{aligned} A &= \{0,1,2\} \\ B &= \{-1,0,1\} \\ C &= \{4\}. \end{aligned}$$

List the elements of the set

$$\left(\overline{(A \setminus B)} \cap C\right) \times \mathcal{P}(C).$$

Question 14. Let A and B be sets in some universe U, so $A \times B$ is a set in the universe $U \times U$. Remember that for a set X, we write the complement of X as $\overline{X} = U \setminus X$. Prove that $\overline{A} \times \overline{B} \subseteq \overline{(A \times B)}$. Prove that the reverse inclusion is false (find a specific example where $\overline{(A \times B)} \nsubseteq \overline{A} \times \overline{B}$).

Equivalence classes / equivalence relations.

Question 15. Let $A = \{\text{cat}, \text{dog}, \text{monkey}\}$. Find **all** equivalence relations on A. Justify that you have them all.

Question 16. Suppose R and S are both equivalence relations on a set A. Is $R \cup S$ an equivalence relation on A? If so, prove it. Otherwise, provide a counterexample.

Modular arithmetic.

Question 17. Prove: For every $n \in \mathbb{N}$

$$9 \mid \left(4^{3^n} - 1\right).$$

Question 18. In class, we proved the following two statements for all $a, b, c \in \mathbb{Z}$ and all $m \in \mathbb{N}$.

(a) If $a \equiv b \pmod{m}$ then $a + c \equiv b + c \pmod{m}$.

(b) If $a \equiv b \pmod{m}$ then $ac \equiv bc \pmod{m}$.

For each statement given above:

- Write the **converse** of the statement.
- Decide if the converse is true or false, and prove you're right.