Math 321: Intro to Advanced Mathematics
October 4, 2022

Tell me a (COMBINATORIAL) story...

For each question, try to tell a combinatorial story that shows why the formula is always true. You can start with some specific examples and see if you can generalize them. The back of this page has hints.

Question 1. $\sum_{i=0}^{n-1} i = \binom{n}{2}$.

Question 2. $\sum_{j=0}^{n-1} (j+1)(n-j) = \binom{n+2}{3}$.

Question 3. $\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}$.

Question 4. $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$.

Question 5. $\binom{n}{2}\binom{n-2}{k-2} = \binom{n}{k}\binom{k}{2}$.

Question 6. $\binom{2n}{2} = 2\binom{n}{2} + n^2$.

Question 7. $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

Question 8. $\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$.

Hint 1 This describes one of the patterns you noticed in class: the 3rd column in Pascal's triangle has terms that increase by 1 then by 2 then by 3 then by 4 then ...

$\binom{n}{2}$ means how many ways you can choose two things the set $\{1, 2, 3, \ldots n\}$. Think about if the bigger of the two elements is 1: how many ways? If the bigger of the two elements is 2: how many ways? If the bigger of the two elements is 3? etc.

Hint 2 Write out the sum

$$\sum_{j=0}^{n-1}(j+1)(n-j) = 1(n-0) + 2(n-1) + 3(n-2) + \cdots + (n-1)2 + n(1)$$

This feels similar to Question 1. Does thinking along the same lines as in Hint 1 help?

Hint 3 This is the "hockey stick" property that Prof. Manes showed on Pascal's triangle: the sum of the numbers on the shaft equal the number on the blade.

For $\binom{n+1}{m+1}$, you can think about forming a committee of $m+1$ people out of $n+1$ candidates. You can break this process into cases by forcing some people to be on the committee and some to be off the committee.

Hint 4 $\binom{2n}{n}$ is the number of ways to choose n objects out of $2n$ choices. Suppose those choices are split into two equal groups of size n.

Also remember that $\binom{n}{k} = \binom{n}{n-k}$. Rewriting the sum might make things more clear.

Hint 5 On the right side, I feel like I'm choosing a committee of k people out of n candidates, and then electing a president and a vice-president from the members of that committee. What about the right side?

Hint 6 Similar to Hint 4, $\binom{2n}{2}$ is the number of ways to choose 2 objects out of $2n$ choices. Suppose those choices are split into two equal groups of size n.

Hints 7 & 8 These feel similar to the result that the sum of the nth row of Pascal's triangle is 2^n. How did we prove that?