Math 321: Intro to Advanced Mathematics
November 1, 2022

Writing assignment 12

Relations and equivalence relations. Write up careful solutions to all three problems.

Question 1. Define a relation on $\mathbb{N}$ by $\mathcal{W} = \{(m,n) \in \mathbb{N}^2 : 2m < n < 3m + 6\}$.

(a) Which of the following pairs are in $\mathcal{W}$ and which are not? Justify your answers (briefly):

$$(2,6) \qquad (5,21) \qquad (7,31) \qquad (3,5) \qquad (4,16)$$

(b) Give two more ordered pairs of natural numbers that are in $\mathcal{W}$ and two that are not. Be sure to justify your answers (briefly).

(c) Explain why $\mathcal{W}$ is **not** an equivalence relation.

Question 2. Try all three questions below, but just write up a careful solution for one of them to turn in.

(a) Let $\mathcal{R}$ be the empty relation on the integers. That is, $\mathcal{R} = \varnothing$ and for all $a, b \in \mathbb{Z}$, we have $(a,b) \notin \mathcal{R}$. Prove or disprove: $\mathcal{R}$ is an equivalence relation.

(b) Define a relation on $\mathbb{R}$ by $\mathcal{Q} = \{(x,y) \in \mathbb{R}^2 : (x-y) \in \mathbb{Q}\}$. Prove or disprove: $\mathcal{Q}$ is an equivalence relation.

(c) Define a relation on $\mathbb{Z}$ by $\mathcal{P} = \{(m,n) \in \mathbb{Z}^2 : m = 2^k n \text{ for some } k \in \mathbb{Z}\}$. Prove or disprove: $\mathcal{P}$ is an equivalence relation.

Question 3. Suppose that $\sim$ is an equivalence relation on a set S, and let $a, b \in S$. Let $[a]$ represent the equivalence class of a, the set of elements of S that are equivalent to a. In set notation,

$$[a] = \{x \in S : x \sim a\}.$$

Prove the following:

(a) If $a \sim b$ then $[a] = [b]$.
(b) If $a \nsim b$ then $[a] \neq [b]$.

Optional section: modular arithmetic proofs. Choose up to two problems from this section to prove using techniques from modular arithmetic.

Question 4. Prove that a number is divisible by 3 if the sum of its digits is divisible by 3.

Question 5. Prove that a number is divisible by 11 if the alternating sum of its digits is divisible by 11. (Here "alternating sum" means subtract then add then subtract then add, and so on. The alternating sum of the digits of 4563 is $4 - 5 + 6 - 3 = 2$.)

Question 6. Let $n \in \mathbb{Z}$. Prove that the units digit of n^4 is in the set $\{0, 1, 5, 6\}$.

Question 7. Show that if $n \in \mathbb{Z}$ is odd, then $16 \mid (n^4 + 4n^2 + 11)$.