Homework 4

1. Calculate the determinants of the following matrices.

(a) $\begin{pmatrix} -1 & 6 & -2 \\ 3 & 4 & 5 \\ 5 & 2 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 4 & 1 & -3 \\ 2 & 10 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & -2 & 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{pmatrix}$

2. Use the determinant formula given in class to calculate the inverse of $\begin{pmatrix} -1 & 6 & -2 \\ 3 & 4 & 5 \\ 5 & 2 & 1 \end{pmatrix}$.)

3. Show that if (a, b), (c, d) are points in the plane, then the unique line passing through them is given by the equation

$$\det \begin{pmatrix} 1 & 1 & 1 \\ x & a & c \\ y & b & d \end{pmatrix} = 0$$

Show that if (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are three points in $\mathbb{R}^3$ that do not lie on a straight line, then the unique plane passing through them has equation:

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ x & x_1 & x_2 & x_3 \\ y & y_1 & y_2 & y_3 \\ z & z_1 & z_2 & z_3 \end{pmatrix} = 0$$

4. Given that $1898, 3471, 7215, 8164$ are all divisible by 13, show that

$$\det \begin{pmatrix} 1 & 8 & 9 & 8 \\ 3 & 4 & 7 & 1 \\ 7 & 2 & 1 & 5 \\ 8 & 1 & 6 & 4 \end{pmatrix}$$

is divisible by 13, without actually calculating it.

5. Let V be the vector space of all sequences. A vector in V is of the form $(a_0, a_1, a_2, a_3, \ldots)$. Consider the following two linear transformation $R, L : V \to V$. $R(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$ and $L(a_0, a_1, a_2, \ldots) = (a_1, a_2, a_3, \ldots)$.

(a) Show that $LR = I$, but $RL \neq I$.

(b) If W is a finite dimensional vector space, and $S, T : W \to W$, is it possible that $ST = I$, but $TS \neq I$? Explain your answer.

6. You know that it's possible for two $n \times n$ matrices not to commute. In other words, it's possible to find matrices A, B such that $AB \neq BA$. If A, B are invertible, we can multiply both sides by $A^{-1}B^{-1}$, and we can write this inequality as $ABA^{-1}B^{-1} \neq I_n$. Is it possible to find A, B invertible that satisfy no non-obvious relations at all? (So: $AB^3A^2B^{-1} \neq I_n$ and $A^{-1}BA^3 \neq I_n$, and all other possible relations, except for the obvious ones that have to hold true like $AA^{-1} = I_n$ and $B^{-1}B = I_n$ etc.).= If we can find such A, B we say that they *generate a free group*. The technical tool for showing that two matrices generate a free group is called the ping pong lemma.

(a) Let A, B be two invertible $n \times n$ matrices. Suppose that $\exists X, Y \subset \mathbb{R}^n$ such that $X \not\subset Y$ and $Y \not\subset X$ and such that for any $i \in \mathbb{Z} \setminus \{0\}$ we have that $A^i(X) \subset Y$, $B^i(Y) \subset X$. Show that A and B generate a free group. Hint: start by showing that products of the form $B^{i_1}A^{i_2}B^{i_3} \ldots B^{i_{k-1}}A^{i_k}B^{i_{k+1}}$ cannot be equal to I_n.

(b) The result in the previous part is often called the ping pong lemma. Why do you think it's called that?

(c) Let $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Show that A and B are invertible, and find their inverses.

(d) Let $X = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 | |y| > |x|\}$ and let $Y = \{\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 | |y| < |x|\}$. Prove that for any $i \in \mathbb{Z} \setminus \{0\}$ we have that $A^i(X) \subset Y$, $B^i(Y) \subset X$. Deduce that A, B generate a free group.

(e) For any n, find two invertible $n \times n$ matrices that generate a free group.