Homework 7

1. Find the Jordan normal form of the following matrices.

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 11 & 6 & -4 & -4 \\ 22 & 15 & -8 & -9 \\ -3 & -2 & 1 & 2 \end{pmatrix}$

2. For each of the matrices in the previous problem, find a formula for the matrix you get by raising it to the n^{th} power.

3. Let A be a matrix such that $p_A(x) = x^4(x-1)^2(x-3)^6$. Suppose that $\dim\mathrm{Ker}(A^4) = 5$, $\dim\mathrm{Ker}(A^3) = 3$, $\dim\mathrm{Ker}(A^2) = 2$, $\dim\mathrm{Ker}(A) = 1$, $\dim\mathrm{Ker}(A-I)^2 = 2$, $\dim\mathrm{Ker}(A-I) = 2$, $\dim\mathrm{Ker}(A-3I)^6 = 6$, $\dim\mathrm{Ker}(A-3I)^5 = 6$, $\dim\mathrm{Ker}(A-3I)^4 = 6$, $\dim\mathrm{Ker}(A-3I)^3 = 6$, $\dim\mathrm{Ker}(A-3I)^2 = 5$, $\dim\mathrm{Ker}(A-3I) = 3$. Find the Jordan normal form of A.

4. An $n \times n$ matrix A is said to have *finite order* if there exists $k > 0$ such that $A^k = I_n$. Show that if we are working over the field $\mathbb{C}$, every finite order matrix is diagonalizable.

5. Recall that for any real number x, $e^x = \sum\limits_{i=0}^{\infty} \frac{1}{i!}x^i$. We can define a similar operation for matrices, which is useful in many areas of mathematics. If A is an $n \times n$ matrix, then $e^A = \sum\limits_{i=0}^{\infty} \frac{1}{i!}A^i$.

(a) If A is a diagonal matrix with the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ on the diagonal, explain why e^A is a diagonal matrix with the numbers $e^{\lambda_1}, \ldots, e^{\lambda_k}$ on the diagonal.

(b) If A is an $n \times n$ matrix with λ's on the diagonal and 1's on the off diagonal, explain how to calculate e^A.

(c) if $B = X^{-1}AX$, explain why $e^B = X^{-1}e^AX$.

(d) Using Jordan normal form, explain how to calculate e^A for any A.

(e) Let $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. Calculate e^A.

6. Let $A = J_{m,\lambda}$ be the $m \times m$ matrix defined in class, with with λ's on the diagonal and 1's on the off diagonal. Find examples of the following.

(a) m, λ where A^2 has a different Jordan normal form than A.
(b) m, λ where A^2 has the same Jordan normal form as A.