Homework 9

1. Let C_n be the graph consisting of n vertices all arranged in an n-gon. So C_3 is a triangle, C_4 is a square, C_5 is a pentagon, etc.

(a) For $n = 3, 4, 5$ find the eigenvalues of the adjacency matrix of C_n and their multiplicities.
(b) For which values of n will -2 be an eigenvalue of the adjacency matrix of C_n? Explain your answer.
(c) For $n = 4$, how many paths of length 100 are there that start and end at vertex 1?
(d) For $n = 4$, what is the probability that a random path of length 100 that starts at vertex 1 will end at vertex 3? If the length of the path gets very big, what number do you expect this probability to approach? Explain.

2. Consider the following two graphs: G_1 is the graph consisting of four vertices with edges forming a square (so, so far we have four vertices and four edges), together with one more vertex in the middle of the square that is connected to nothing. The graph G_2 is on the same vertices, and it has 4 edges: one from the middle vertex to each of the other vertices. So - it has one central vertex connected to four other vertices, and no further edges. This is called a star graph. These two graphs are very different. Show that their adjacency matrices have the same eigenvalues with the same multiplicities.

3. Let $N \in M_4(\mathbb{R})$ be the matrix whose entries are all 1's. Let $A = N - I$.

(a) The matrix A is an adjacency matrix of a graph. Which graph is it?
(b) Show that $A^2 - 2A - 3I = 0$.
(c) Show that 3 is an eigenvalue of A of multiplicity 1.
(d) Are there any other eigenvalues? If so, what are their multiplicities?
(e) What is the determinant of A?

4. Let $A \in M_3(\mathbb{R})$ be an orthogonal 3×3 matrix.

(a) Show that $p_A(x)$ has a real root.
(b) Show that any real eigenvalues of A have to be either 1 or -1.
(c) Show that if $\det A = 1$, then 1 must be an eigenvalue of A.
(d) Show that if $\det A = 1$ then A is a rotation about some axis.
(e) Show that the product of two rotations about the origin in $\mathbb{R}^3$ is again a rotation.

5. Explain why the adjacency matrix of a k-regular graph has $-k$ as an eigenvalue if and only if the graph has a bi-partite component.

6. Give examples of 4-regular graphs where the eigenvalues of the adjacency matrix have the following properties.

(a) 4 is an eigenvalue of multiplicity 1, and -4 is not an eigenvalue.
(b) 4 is an eigenvalue of multiplicity 3 and -4 is an eigenvalue of multiplicity 2.