In [1]:
```python
import numpy as np
from sympy import Matrix
```

1.Find the Jordan normal form of teh following matrices.

$(a)\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

The eigencvalues are along the diagonal.

$$P_A(x) = (x-1)^3$$

Now we find...

$$dimKer(A-I)^3 = 3$$
$$dimKer(A-I)^2 = 3$$
$$dimKer(A-I) = 2$$

Showing my work...

$$dimKer(A-I)^3 = 2$$
$$dimKer(\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})$$

In [4]:
```python
a = np.array([[ 0,  1, 1],
              [ 0,  0, 0],
              [ 0,  0, 0]])

np.matmul(a,a)
```

Out[4]:
```
array([[0, 0, 0],
       [0, 0, 0],
       [0, 0, 0]])
```

So this matrix contains a single length two chain and one length one chain. The minimal polynomial is $(x-1)^2$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Checking the jordan form using phyon.

In [11]:
```python
a = np.array([[ 1,  1, 1],
              [ 0,  1, 0],
              [ 0,  0, 1]])
m = Matrix(a)
```

```
p, j = m.jordan_form()
j
```

Out[11]:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(b) \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

Again, the eigenvalues are along the diagonal. The characteristic equation is...

$$P_A(x) = (x+1)^3$$

Now we find...

$$dimKer(A+I)^3 = 3$$
$$dimKer(A+I)^2 = 2$$
$$dimKer(A+I) = 1$$

In [33]:
```
a = np.array([[ 0, -1,  0],
              [ 0, 0,  -2],
              [ 0,  0,  0]])

np.matmul(a, a)
```

Out[33]:
```
array([[0, 0, 2],
       [0, 0, 0],
       [0, 0, 0]])
```

In [34]:
```
b = np.matmul(a, a)
np.matmul(b, a)
```

Out[34]:
```
array([[0, 0, 0],
       [0, 0, 0],
       [0, 0, 0]])
```

So this matrix contains a single three chain. The minimal polynomial is $(x+1)^3$ and the jordan form is....

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

checking my answer with python....

In [37]:
```
a = np.array([[-1, -1,  0],
              [ 0, -1, -2],
              [ 0,  0, -1]])
m = Matrix(a)
p, j = m.jordan_form()
j
```

Out[37]:

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$$(c) \begin{pmatrix} 0 & 1 & 0 & 0 \\ 11 & 6 & -4 & -4 \\ 22 & 15 & -8 & -9 \\ -3 & -2 & 1 & 2 \end{pmatrix}$$

$$det(\begin{pmatrix} \lambda & -1 & 0 & 0 \\ -11 & \lambda-6 & 4 & 4 \\ -22 & -15 & \lambda+8 & 9 \\ 3 & 2 & -1 & \lambda-2 \end{pmatrix})$$

Here we will do my favorate. Co-factor expansion to find $P_A(x)$

$$-1 * det(\begin{pmatrix} -11 & 4 & 4 \\ -22 & \lambda+8 & 9 \\ 3 & -1 & \lambda-2 \end{pmatrix})$$

$$\lambda * det(\begin{pmatrix} \lambda-6 & 4 & 4 \\ -15 & \lambda+8 & 9 \\ 2 & -1 & \lambda-2 \end{pmatrix})$$

This is painful. I will just use python.

In [45]:

```
a = np.array([[  0,  1,  0,  0],
              [ 11,  6, -4, -4],
              [ 22, 15, -8, -9],
              [ -3, -2,  1,  2]])
m = Matrix(a)
p, j = m.jordan_form()
j
```

Out[45]:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. For each of the matrices in the previous problem, find a formula for the matrix you get by raising it to the n^{th} power.

To find a formula for each matrix above raised to the nth power I would find their jordan normal form (which I have), the rais that to the nth power. So...

$$J = X^{-1}JX$$
$$J^2 = X^{-1}J^nXX^{-1}J^nX$$
$$J^2 = X^{-1}J^2X$$
$$\dots$$
$$J^n = X^{-1}J^nX$$

A the very least the characteristic polynomial is easy to find. The eigenvectors stay the same, but the eigenvalues are raised to the nth power.

$$(a)P_{A^n}(x) = (x - 1^n)^2$$

$$(b)P_{A^n}(x) = (x + 1^n)^3$$

Did some googleing and looked back at the notes. The general form of a Jordan block isas follows...

$$J_k(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \dots & \binom{n}{k-1}\lambda^{n-k+1} \\ & \lambda^n & \binom{n}{1}\lambda^n & \dots & \binom{n}{k-2}\lambda^{n-k+2} \\ & & \dots & \dots & \dots \\ & & & \lambda^n & \binom{n}{1}\lambda^{n-1} \\ & & & & \lambda^n \end{pmatrix}$$

For (a) we have two blocks. 1 and (1,1,0,1). The single 1 block is easy. That's 1. For the 2x2 block we get...

$$J_k(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} \\ & \lambda^n \end{pmatrix} = \begin{pmatrix} 1 & \binom{n}{1}1^{n-1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1+n-1 \\ 0 & 1 \end{pmatrix}$$

The complete jordan form matrix is...

$$J(\lambda) = \begin{pmatrix} 1 & 1+n-1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For (b) we have a single three by three block with only the -1 eigenvalue.

Using the jordan block general form we get the equation for A^n.

$$J(\lambda)^n = \begin{pmatrix} \lambda^n & \binom{n}{1}\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ & \lambda^n & \binom{n}{1}\lambda^n \\ & & \lambda^n \end{pmatrix} = \begin{pmatrix} -1^n & \binom{n}{1}(-1)^{n-1} & \binom{n}{2}(-1)^{n-2} \\ & -1^n & \binom{n}{1}(-1)^n \\ & & (-1)^n \end{pmatrix}$$

For (c) we have two 2x2 blocks. One for -1 eigenvalue and one for the 1 eigenvalue.

$$J_k(1)^n = \begin{pmatrix} 1 & \binom{n}{1}1^{n-1} \\ & 1 \end{pmatrix}$$

$$J_k(-1)^n = \begin{pmatrix} (-1)^n & \binom{n}{1}(-1)^{n-1} \\ & (-1)^n \end{pmatrix}$$

The complete jordan amtrix is...

$$J(\lambda)^n = \begin{pmatrix} (-1)^n & \binom{n}{1}(-1)^{n-1} & & \\ & (-1)^n & & \\ & & 1 & \binom{n}{1}1^{n-1} \\ & & & 1 \end{pmatrix}$$

3. Let A be a matrix such that $pA(x) = x^4(x-1)^2(x-3)^6$.
Suppose that...
$\dim\mathrm{Ker}(A^4) = 5$ **I'm changing this to 4 not 5.**
$\dim\mathrm{Ker}(A^3) = 3$
$\dim\mathrm{Ker}(A^2) = 2$
$\dim\mathrm{Ker}(A) = 1$
$\dim\mathrm{Ker}(A-I)^2 = 2$
$\dim\mathrm{Ker}(A-I) = 2$
$\dim\mathrm{Ker}(A-3I)^6 = 6$
$\dim\mathrm{Ker}(A-3I)^5 = 6$
$\dim\mathrm{Ker}(A-3I)^4 = 6$
$\dim\mathrm{Ker}(A-3I)^3 = 6$
$\dim\mathrm{Ker}(A-3I)^2 = 5$
$\dim\mathrm{Ker}(A-3I) = 3$
Find the Jordan normal form of A.

So we have a 12 x 12 matrix. Fill in the eigenvalues...

$$J = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&3&0&0&0&0&0 \\ 0&0&0&0&0&0&0&3&0&0&0&0 \\ 0&0&0&0&0&0&0&0&3&0&0&0 \\ 0&0&0&0&0&0&0&0&0&3&0&0 \\ 0&0&0&0&0&0&0&0&0&0&3&0 \\ 0&0&0&0&0&0&0&0&0&0&0&3 \end{pmatrix}$$

now fill in the chains using 1's. There's one length 4 chain for \lambda = 0. There's two length 1 chain for \lambda = 1. There's three chains for $\lambda = 3$, one of length 1, one of length 2, and one of thength 3.

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

4. An $n \times n$ matrix A is said to have finite order if there exists $k > 0$ such that $A^k = In$. Show that if we are working over the field C, every finite order matrix is diagonalizable.

To show A is invertible we will to show that it has n independent eigenvectors and eigenvalues. A matrix is independent iff it has linearly independent eigenvectors, and all unique eigenvalues, with no eigenvalues being 0.

Oof i was working to proive invertibility, not diagonalizability.
All invertible matrixes are diagonalizable? No. Eigenvalues of 0 can show up in diagonal matrix's. No, I'm just showing eigenvectors are independent, meaning they have unique eigenvalues, and the matrix A can be diagonalized.

Let $a_1 \ldots a_n$ be scalars where $a \in \mathbb{C}$, and $v_1 \ldots v_n$ be the eigenvectors of A.

We must show that when $A^k = I_n$ for $k > 0$ we have...
$a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ only when $a_1 = a_2 = \ldots = a_n = 0$
If we have unique eigenvectors, then we must have unique eigenvalues.
If we have unique eigenvalues then we have a diagonalizable matrix.

Suppose for contradicion that...

Multiplly both sides by A. Keep multiplying both sides by A.

Let's say it's invertable.

For contradiction lets supose $a_1 v_1 + a_2 v_2 + \ldots + a_n v_n = 0$ when $a_1 \neq a_2 \neq \ldots \neq a_n \neq 0$

Note. I'm thinking of doing a proof by contracdiction and using the knowledge that $A^k = I_n$

Starting over and taking a new approuch.

Lets put A^k into jordan form, call it $J(\lambda)^k$. $A^k = In$, means A is invertible and has no 0 eigenvalues. Then the Jordan blocks of A, call them $B(\lambda)$ are, $B_m(\lambda)^k = I_m$.
So then...

$$A^k = X^{-1}J^kX$$

and

$$A^k = I_n$$

which is already in jordan normal form, so...

$$J^k = I_n$$

We need to show that m = 1 for all jordan blocks in the jordan neomal form of A.

Raising J^k to the k^{th} power does not change the size of the blocks in J. So the block sizes in J^k are equal to the block sizes in J. The block sizes in J^k are all one, because J^k is the identity matrix. So the block sies in J are all one. Aditionaly, J is the jordan normal form of A.

$$A = X^{-1}JX$$

So then J can be represented as the diagonal matrix of A, call it Λ, because J is a diagonal matrix with the eigenvalues λ of A along it's diagonal. So then..

$$A = X^{-1}JX = X^{-1}\Lambda X$$

Therefore A, any finite dimensional matrix, is diagonalizable. ■

Recall that for any real number x, $e^x = \Sigma_{i=0}^{\inf} \frac{1}{i!} x^i$.

We can define a similar operation for matrices, which is useful in many areas of mathematics. If A is an nxn matrix, then $e^A == \Sigma_{i=0}^{\inf} \frac{1}{i!} A^i$.

(a) If A is a diagonal matrix with the numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$ on the diagonal, explain why e^A is a diagonal matrix with the numbers $e^{\lambda_1}, \ldots, e^{\lambda_k}$ on the diagonal.

$$e^A = \Sigma_{i=0}^{\inf} \frac{1}{i!} A^i$$

Lets open A and carry out some arithmatic...

$$\Sigma_{i=0}^{\text{inf}} \frac{1}{i!} \begin{pmatrix} \lambda_1 & \dots & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \lambda_k \end{pmatrix}^i$$

$$\Sigma_{i=0}^{\text{inf}} \begin{pmatrix} \frac{1}{i!}\lambda_1^i & \dots & \dots & 0 \\ 0 & \frac{1}{i!}\lambda_2^i & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \frac{1}{i!}\lambda_k^i \end{pmatrix}$$

Similarly the sum caries out during matrix adition...

$$\begin{pmatrix} \Sigma_{i=0}^{\text{inf}} \frac{1}{i!}\lambda_1^i & \dots & \dots & 0 \\ 0 & \Sigma_{i=0}^{\text{inf}} \frac{1}{i!}\lambda_2^i & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \Sigma_{i=0}^{\text{inf}} \frac{1}{i!}\lambda_k^i \end{pmatrix}$$

Each element along the diagonal now equals $e^{\lambda_1}, \dots, e^{\lambda_k}$, so... $$

e^A

$$\begin{pmatrix} e^{\lambda_1} & \dots & \dots & 0 \\ 0 & e^{\lambda_2} & \dots & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & e^{\lambda_k} \end{pmatrix}$$

\ \blacksquare{} $$

(b) If A is an $n \times n$ matrix with λ’s on the diagonal and 1’s on the off diagonal, explain how to calculate e^A.

$$e^A = \Sigma_{i=0}^{\text{inf}} \frac{1}{i!} A^i$$

$$A = \begin{pmatrix} \lambda_1 & 1 & \dots & 0 \\ 0 & \lambda_2 & 1 & 0 \\ 0 & \dots & \dots & 0 \\ 0 & \dots & \dots & \lambda_k \end{pmatrix}$$

Assuming 1's run the entire off diagonal, thenm A is a jordan matrix of one block.

So raising it to a power should be easy. We us the structure...

$$J_k^i = \begin{pmatrix} \lambda^1 & \binom{i}{1}\lambda^{i-1} & \binom{i}{2}\lambda^{i-2} & \dots & \binom{i}{n-1}\lambda^{i-n+1} \\ & \lambda^2 & \binom{i}{1}\lambda^{i} & \dots & \binom{i}{n-2}\lambda^{i-n+2} \\ & & \dots & \dots & \dots \\ & & & \lambda^{k-1} & \binom{i}{1}\lambda^{i-1} \\ & & & & \lambda^k \end{pmatrix}$$

furthermore, A is already in jordan form so we don't need to use the X^{-1} and X matrixes.

We jump straight to....

$$e^A = \begin{pmatrix} e^{\lambda_1} & \binom{i}{1}\lambda^{i-1} & \binom{i}{2}\lambda^{i-2} & \dots & \binom{i}{n-1}\lambda^{i-n+1} \\ & e^{\lambda_1} & \binom{i}{1}\lambda^{i} & \dots & \binom{i}{n-2}\lambda^{i-n+2} \\ & & \dots & \dots & \dots \\ & & & e^{\lambda_1} & \binom{i}{1}\lambda^{i-1} \\ & & & & e^{\lambda_1} \end{pmatrix}$$

I'm not sure how to simplify what's going on above the diagonal. I have not carried out the multiplication, and I want to represent it in the form of e^{λ}

(c) if $B = X^{-1}AX$, explain why $e^B = X^{-1}e^AX$.

$$e^B = \Sigma_{i=0}^{\inf} \frac{1}{i!} B^i$$

substitute $X^{-1}AX$ for B, and expand...

$$e^B = \Sigma_{i=0}^{\inf} \frac{1}{i!} (X^{-1}AX)^i$$
$$= \Sigma_{i=0}^{\inf} \frac{1}{i!} X^{-1}A^iX$$

now the expansion of the sum...

$$= \frac{1}{0!} X^{-1}A^0X + \dots + \frac{1}{\inf!} X^{-1}A^{\inf}X$$

pull X^{-1} out of the sum...

$$= X^{-1}(\frac{1}{0!} A^0X + \dots + \frac{1}{\inf!} A^{\inf}X)$$

bring the sum back down to earth...

$$= X^{-1}\Sigma_{i=0}^{\inf} \frac{1}{i!} A^iX$$

that middle part is e^A

$$= X^{-1}e^A X$$

$$\blacksquare$$

(d) sing Jordan normal form, explain how to calculate e^A for any A.

Any matrix A can be written as $A = X^{-1}JX$ where J is the jordan normal from of A, in some basis.

So takig a similar aprouch as the last question, we get...

$$A^B = \Sigma_{i=0}^{\text{inf}} \frac{1}{i!}(X^{-1}JX)^i$$

$$= \Sigma_{i=0}^{\text{inf}} \frac{1}{i!} X^{-1}J^i X$$

pulling the matrix out of the sum as I did before.

$$= X^{-1}\Sigma_{i=0}^{\text{inf}} \frac{1}{i!} J^i X$$

$$e^A = X^{-1}e^J X$$

So that's how we do it.

(e) Let $A = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$. Calculate e^A

I never actually do this but will need to for a test I bet.
It's a diagonalizable matrix, so that simpliies things, making X easier to find.

6. Let $A = J_{m,\lambda}$ be the $m \times m$ matrix defined in class, with with λ's on the diagonal and 1's on the off diagonal. Find examples of the following.

(a) m, λ where A 2 has a different Jordan normal form than A.

Any matrix with eigenvalues not all equal to 1 or 0.
This will $make J^2 \neq J$ because the eigenvalues will not be eaqual.

(b) m, λ where A 2 has the same Jordan normal form as A.

observe the identity matrix, I.