Homework 3

1. For each of the following, give an example or explain why no such example exists.

(a) A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ that is onto.
(b) A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^3$ that is one to one.
(c) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ that is onto.
(d) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ that is one to one.
(e) For every $i = 0, 1, 2, 3, 4$, a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ whose rank is i.

2. Every year,1% in town A move to town B, and 2% of the people in town B move to town A. In 2015, there are 1000 people living in each town. Find a matrix M such that $M^n \begin{pmatrix} 1000 \\ 1000 \end{pmatrix}$ gives the number of people living in each town after n years.

3. Suppose $J \in M_{n,n}(\mathbb{R})$ is a matrix satisfying $J_{i,j} = 1$ if $j - i = 1$, and 0 otherwise. Calculate J^k for every $k \in \mathbb{N}$.

4. Consider the following method of encrypting information: Take a text, and break it up into chunks of four characters. So "linear algebra" turns into "line" "ar a" "lgeb" "ra ". Convert each chunk into numbers by taking numbers 1-26 for the letters a-z, numbers 27-36 for the digits 0-9 and number 0 for the space character. Think of these numbers as elements of F_{37}, and turn them into vectors. So "abcd" would turn into $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$. Multiply the vectors by the matrix $\begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 3 & 4 \\ 5 & 4 & 3 & 2 \end{pmatrix}$ (remember, the multiplication happens in the field F_{37}), and then convert the numbers back to letters. Decode the following message: "3sx ol0o". How would you decode a message in general?

5. Let $T : V \to W$, $S : W \to U$ be linear transformations. Show that $\operatorname{rank}(ST) \leq \operatorname{rank} S$ and $\operatorname{rank} ST \leq \operatorname{rank} T$. Now suppose that $U = V = W$ and that B is invertible. Show that $\operatorname{rank}(AB) = \operatorname{rank} A$.

6. In the city of Oddsville there were n citizens. The citizens were obsessed with participating in various clubs. The mayor of Oddsville felt that the citizen's passion for forming clubs was getting out of hand. As a result, she instituted two strange rules to curb the number of clubs. The first rule is that each club must have an odd number of members. The second rule is that every two different clubs must have an even number of members in common. Let m be the number of clubs in Oddsville. The mayor knew that with these rules, there couldn't be more clubs than there were people, that is: $m \leq n$, so these rules would prevent an excessive amount of clubs from forming. In this problem you'll figure out why.

(a) Name the citizens $1, 2, 3, \ldots, n$ and the clubs $C_1, \ldots, C_m$. Let A be the $m \times n$ matrix with coefficients in F_2, the field with 2 elements defined by: $A_{i,j} = 1$ if citizen j is in the club C_i and 0 otherwise. Make a sample town with five citizens and three clubs. Assign each citizen to as many clubs as you want and make the matrix A corresponding to your example. Make an assignment of citizens to clubs that satisfies the mayor's rules, and one that doesn't.

(b) Recall that A^T denotes the transpose matrix of A. So A^T is the $n \times m$ matrix with coefficients in F_2 such that $A_{i,j} = 1$ if citizen i is in club C_j and 0 otherwise. Write down the matrix A^T for the example you gave in the previous part.

(c) Let $B = AA^T$. This is an $m \times m$ matrix whose (i, j) coordinate is $\sum\limits_{k=1^n} A_{i,k}A_{kj,}$. Go over the formula for matrix multiplication and make sure that you understand how $B_{i,j}$ was calculated.

(d) Explain why the mayor's rules imply that $B = I_m$.

(e) Explain why this means that $\mathrm{Rank}(B) = m$.

(f) Show that $\mathrm{Rank}(A) \geq m$ (Hint: use problem 5).

(g) Explain why $\mathrm{Rank}(A) \leq n$ (Hint: it has n columns).

(h) Deduce that $m \leq n$, as required.