Homeswork 8

This is as far as I got by the end of teh day friday, but I plan to do more over the weekend. Maybe I'll do the thing where I email you (the professor) what I've done more to show I've put in the effort.

1.**Let J be a matrix which is in Jordan normal form. Suppose that J has the following blocks on its diagonal: $J_{5,\sqrt{2}}$ (a 5 x 5 block with 2 on the diagonal), 3 copies of $J_{4,7}$, 2 copies of $J_{2,7}$, 4 copies of $J_{1,7}$, two copies of $J_{3,\pi}$, and one copy of $J_{2,\pi}$.**
(a) What is the characteristic polynomial of J?
(b) What is the minimal polynomial of J?
(c) For each eigenvalue λ of J, what is the dimension of the generalized eigenspace corresponding to λ?
(d) For each eigenvalue λ of J, find the dimension of $Ker(J-\lambda I)^k$ for every k.

$$P_J(x) = (x-\sqrt{2})^5(x-7)^{20}(x-\pi)^8$$
$$m_J(x) = (x-\sqrt{2})^5(x-7)^4(x-\pi)^3$$

dimension of general eigenspace of $\lambda=\sqrt{2}$ is 5.
dimension of general eigenspace of $\lambda=7$ is 20.
dimension of general eigenspace of $\lambda=\pi$ is 8.

$Ker(J-\sqrt{2}I)^5 = 5$
$Ker(J-\sqrt{2}I)^4 = 4$
$Ker(J-\sqrt{2}I)^3 = 3$
$Ker(J-\sqrt{2}I)^2 = 2$
$Ker(J-\sqrt{2}I)^1 = 1$

$Ker(J-7I)^{20} = 20$
$Ker(J-7I)^4 = 20$
$Ker(J-7I)^3 = 17$
$Ker(J-7I)^2 = 14$
$Ker(J-7I)^1 = 9$

$Ker(J-\pi I)^8 = 8$
$Ker(J-\pi I)^3 = 8$
$Ker(J-\pi I)^2 = 6$
$Ker(J-\pi I)^1 = 3$

2.**Let W be the space of all continuous functions $R \to R$, and let $V \subset W$ be the subspace spanned by the functions x, x^2, x^3, e^x, e^{-x}. Let $T: V \to V$ be linear**

transformation that sends a function to its derivative (so $T(f) = f'$). Find the Jordan normal form of T.

Well, I really wish I did this question before the test today. Goofed that one up. Turning to inspirtion from the internet.

So

$$T(x) = 1$$
$$T(x^2) = 2x$$
$$T(x^3) = 3$$
$$T(e^x) = e^x$$
$$T(e^{-x}) = -e^{-x}$$

This is what our coordinat maps will look like.

$$\begin{pmatrix} \frac{d}{dx}x \\ \frac{d}{dx}x^2 \\ \frac{d}{dx}x^3 \\ \frac{d}{dx}e^x \\ \frac{d}{dx}e^{-x} \end{pmatrix}$$

$$T(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, T(x^2) = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, T(x^3) = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}, T(e^x) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, T(e^{-x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

Put those bad boys together... This is close I think but maybe not quit there. I don't fully understand it.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

3.**Let V be the space of all continuous functions $f : [0, 1] \to R$. Define the following inner product on $V :< f, g >= \int_0^1 f(t)g(t)dt$. Let $f(x) = 0$ be the constant function that is equal to 0, let $g(x) = x^2$, $h(x) = x^3 - 1$. Find the lengths of the sides and the angles of the triangle formed by f, g, h.**

I'm not able to draw the triangle in LateX but I'm treating $f(x) = 0$ as the origin, and $g(x), h(x), g(x) - h(x)$ as the vectors that tform the sides. To the length of the sides will be

$||g(x)||, ||h(x)||, ||g(x) - h(x)||$.

$$||g|| = \int_0^1 g(x)g(x)dx$$

Throw in $g(x) = x^2$, and integrate.

$$||g(x)|| = \int_0^1 x^4 dx$$
$$||g(x)|| = \sqrt{1/5}$$

Do the same for $h(x)$, and $h(x) - g(x)$.

$$||h(x)|| = \int_0^1 (x^3 - 1)(x^3 - 1)dx$$
$$||h(x)|| = \sqrt{\frac{9}{14}}$$

Now for $||g(x) - h(x)||$...

$$||g(x) - h(x)|| = \sqrt{\int_0^1 x^2 - (x^3 - 1)}$$
$$||g(x) - h(x)|| = \sqrt{\frac{247}{210}}$$

Now finding the angles betweeen the vectors by defining the cosin function as...

$$cos\theta_1 = \frac{< g(x), h(x) >}{||g(x)|| * ||h(x)||}$$

$$cos\theta_1 = \frac{\int_0^1 x^2(x^3 - 1)dx}{x^2 * (x^3 - 1)}$$
$$cos\theta_1 = 0.464$$
$$\theta_1 = 62.35^\circ$$

or 1.08 rads

Now for the angle between g(x) and h(x)-g(x).

$$cos\theta_2 = \frac{< g(x), h(x) - g(x) >}{||g(x)|| * ||h(x) - g(x)||}$$

$$cos\theta_2 = \frac{\int_0^1 x^2(x^2 - x^3 + 1))dx}{\sqrt{\frac{1}{5}} * \sqrt{\frac{247}{210}}} = 0.755$$
$$\theta_2 = 40.97^\circ$$

or 0.715 rads.

Now for the angle between h(x) and h(x)-g(x).

$$cos\theta_3 = \frac{< h(x), h(x) - g(x) >}{||h(x)|| * ||h(x) - g(x)||}$$

$$cos\theta_3 = \frac{\int_0^1 (x^3 - 1)(x^2 - x^3 + 1))dx}{\sqrt{\frac{9}{14}} * \sqrt{\frac{247}{210}}}$$
$$cos\theta_3 = 0.9309$$
$$\theta_3 = 21.42^\circ$$

or 0.373 rads.

4.**Let W be the space of all continuous functions $R \to V \subset W$ be the subspace spanned by the functions x, x^2, x^3, e^x, e^{-x}. Let $U \subset W$ be the subspace spanned by x, x^2, x^3. Find the orthogonal projection of e^x to U.**

We willuse gram schmitt algorithm to express the supspace U as orthonormal basis. Then we will project the vector e^x onto the subspace U.

Step one is to set the length of the first vector, v_1 to 1, using $e_1 = \frac{v_1}{||v_1||}$.

Step two is to set the preceding vectors orthogonal to one another. Using...

$$e_j = \frac{v_j - < v_j, e_1 > e_1 - \ldots - < v_j, e_{j-1} >}{||v_j - < v_j, e_1 > e_1 - \ldots - < v_j, e_{j-1} > ||}$$

Step three is to project $e^x \in V$ onto U. This will be done by summing each component of the projection of e^x onto basis of U. This is represented by...

$$proj_U(v) = < v, e_1 > e_1 + < v, e_2 > e_2 + \ldots + < v, e_j > e_j$$

where $\{e_1, e_2, \ldots, e_j\}$ are the j basis vecotrs of U, that have been orthogonalized and brought to length one using the gram shmit process. Here $\{e_1, e_2, \ldots, e_j\}$ are derived from $\{x, x^2, x^3\}$ and are the basis vectors of U. The vector v is e^x and is a vector V.

Lastly, and this is not a step, but needs to be done first, we need to make V an inner product space. This means we need to define a bilinear map that meets the requirments of an inner product on the vectors in V. From chapter six of the "lin alg. done right" textbook, we are given an inner product for the real continous function space. Typicaly we would need to prove that it meets the requirments of being an inner product on the real continous functions, but I will not do that here. Anyhow... $$

<f(t), g(t)> = \int_{-1}^1 f(t) g(t) dt $$

So, I did this by hand and it was a doosy. I do not want to typ up every grusome detail. I'll but what I found the orthonormal basis of U to be.

$$e_1 = \frac{x}{\sqrt{2/3}}, e_2 = \frac{x^2}{\sqrt{2/5}}, e_2 = \frac{175x^3 + 105x}{8}$$

From here I use decimals to represent each value because it was a real pain....

$$e_1 = 1.22x, e_2 = 1.58x^2, e_3 = 21.87x^3 + 13.12x$$

So I don't complete the computation because it was so taxing. but from here I would do...

$$proj_U(e^x) =< e^x, 1.22x > 1.22x + < e^x, 1.58x^2 > 1.58x^2 + < e^x, 21.87x^3 + 13.12x > 2$$

Ask the prof if this is correct. I'm not sure about the formula for the projection onto multiple basis vectors. Also, can you project two basis vectors onto a subspace?

5.**Let $V = M_3(R)$ be the space of 3×3 matrices. Define an inner product on V by setting $hA, Bi = trace(ATB)$. Let $U \subset V$ be the subspace of antisymmetric matrices (a matrix A is antisymmetric if $A^T = -A$. Find:**
(a) The closest point in U to the identity matrix.
(b) The distance between the identity matrix and the space U.

All 3×3 asymetric matrices have the form

$$\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

So this is a three dimensional space, given by a, b, and c. All asymetric matrixes have 0's along the idagonal. That is because 0 is the only real number, given by $k \in \mathbb{R}$, where $-k = k$. The space is spanned by the standard beasis...

$$span(U) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

We can now project I onto U.

The closest point of I to U is given by the projection, which is the sum of the projection of I onto each basis for U, shown below.

$$proj_u(I) = \frac{< I, u_i > u_1}{||u_1||} + \frac{< I, u_2 > u_2}{||u_2||} + \frac{< I, u_i > u_3}{||u_3||}$$

To begin, let's find the norm of the standard basis for U. Lets start calling those standard basis $\{u_1, u_2, u_3\}$.

$$\text{norm } u_1 = \sqrt{trace(\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}^T \times \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix})} = \sqrt{2}$$

Good, an irrational number. Turns out the norm of all three basis are equal.

$$||u_1|| = ||u_2|| = ||u_3|| = \sqrt{2}$$

Now lets find the inner products of the basis vectors...

$$< u_1, I >= trace(u_1^T I) = trace(-u_1 I) = 0$$

So I is orthogonal to u_1. This is interesting. Because I is symetric and diagonmal, it has only 0's for the non-diagonal components. This means the trace of I times and asymetric matrix (0's along the dioagonal) will be 0. So I is orthogonal to the basis of U. $$

<u_2, I> = trace(u_2^T I) = trace(-u_2 I) = 0 $$

$$< u_3, I >= trace(u_3^T I) = trace(-u_3 I) = 0$$

This makes our projection pretty easy.

$$proj_u(I) = \frac{0 * u_1}{\sqrt{2}} + \frac{0 * u_2}{\sqrt{2}} + \frac{0 * u_3}{\sqrt{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So the closest point is the zero matrix. I have a though on this. If we can diagonalize any symetric matrix (spectral theorum), then are all symetric matrixes orthogonal to the space of asymetric matrixes? Since any diagonal matrix times an asymetric matrix (zero's along te diagonal) will be the zero matrix.

Now to find the distance between I and the zero matrix. I bet I'll need to do distances for the final test.

We have $dist(U, I) = dist(0_3, I) = ||I - 0_3||$. This is...

$$dist(0_3, I) = \sqrt{trace(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})^T \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix})} = \sqrt{3}$$

So ya, the angle of I and the space or asymetric matixes, (U) is orthogonal, the closest point in U to I is the zero matrix, and the distance from the space U to I is $\sqrt{3}$.