Homework 5

1. Diagonalize the following matrices.

(a) $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$

(b) $\begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$

2. Diagonalize the linear transformation $S : M_n(F) \to M_n(F)$ given by $S(A) = A^T$.

3. Given $A \in M_n(F)$, the *trace* of A, or $\mathrm{tr}(A)$ is the number $A_{1,1} + \ldots + A_{n,n}$.

(a) Use the formula for matrix multiplication to explain why $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for any $A, B \in M_n(F)$.

(b) Show explicitly that for any 2 matrix A, $P_A(x) = x^2 - \mathrm{tr}(A)x + \det(A)$.

(c) Suppose that A, B are similar matrices (so $B = X^{-1}AX$ for some X). Explain why $p_A = p_B$.

(d) Suppose that A is a diagonalizable $n \times n$ matrix. Explain why $p_A(x) = x^n - \mathrm{tr}(A)x^{n-1} + a_{n-2}x^{n-1} + \ldots + a_x + (-1)^n \det(A)$.

(e) Express $\mathrm{tr}(A)$, $\mathrm{set}(A)$ and the other coefficients that appear in p_A in terms of the eigenvalues of A.

4. Let $f_0 = 1$, $f_1 = 1$, $f_{n+2} = f_{n+1} + f_n$. The numbers $f_0, f_1, f_2, \ldots$ are called the Fibonacci numbers.

(a) Let $F = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Explain why for every n:

$$F^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}$$

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(b) Diagonalize F.

(c) Suppose that $A = X^{-1}DX$ for a diagonal matrix D. Show that for any n, $A^n = X^{-1}D^nX$.

(d) Find an explicit formula for the number f_n.

5. Recall the game we played on the first week with a 2×2 grid with black and white squares. To show that one can pass from any configuration to any other configuration, you showed that a certain 4×4 matrix with entries in F_2 is invertible. Write down this matrix, but now consider its entries as elements of $\mathbb{Z}$ (so use 0 instead of [0], and 1 instead of [1]). Calculate the determinant of this matrix. Now find all primes p such that if the game is played on a 2×2 grid with p colors instead of 2, then there are some configurations that can't be reached from other configurations. If the game is played on a $n \times n$ grid, show that there is a always some p where some configurations can't be reached from other configurations. (Challenge: for every n, show that there are only finitely many such p).