

Categorical Quantum Dynamics

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Quantum Dynamics as Algebra

Dynamics	=	Time-translation symmetry
discrete periodic	→	π_T
discrete	→	π
continuous periodic	→	$i\pi/L$
continuous	→	iR

time energy

π_T	π_T
π	$iR/\pi L$
$i\pi/L$	π
iR	R

Quantising Groups

Work with the Hopf algebra

time-translation group

$\mathbb{C}[G]$

call this T , for time

$$\begin{array}{c} \text{Diagram: } \text{Two nodes connected by a vertical line, each with a loop on the left.} \\ |t\rangle \otimes |t\rangle \\ \uparrow \\ |\overline{t}\rangle \end{array}$$

(special commutative
t-Frobenius algebra)

$$\begin{array}{c} \text{Diagram: } \text{A single node with a loop.} \\ \frac{1}{\overline{1}} \\ \uparrow \\ |t\rangle \end{array}$$

$$\begin{array}{c} \text{Diagram: } \text{A node with a loop, labeled } N_0. \\ \text{inversible} \\ \uparrow \\ \text{Diagram: } \text{A node with a loop, labeled } N_0. \end{array}$$

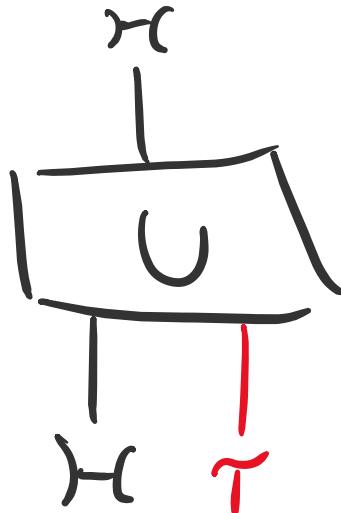
$$\begin{array}{c} \text{Diagram: } \text{A single node with a loop.} \\ |t+s\rangle \\ \uparrow \\ |t\rangle \otimes |s\rangle \end{array}$$

$$\begin{array}{c} \text{Diagram: } \text{A single node with a loop.} \\ |0\rangle \\ \uparrow \\ \frac{\overline{1}}{1} \end{array}$$

(quasi-special symmetric
t-Frobenius algebra)

$$\begin{array}{c} \text{Diagram: } \text{A node with a loop, labeled } \square = (\alpha_0) = r(\alpha_0) = \\ \text{antipode} \\ \uparrow \\ \text{Diagram: } \text{A node with a loop, labeled } \square = (\alpha_0) = r(\alpha_0) = \\ = l(\alpha_0) = l(\alpha_0) \end{array}$$

Quantising Algebras



s.t.

A diagram showing the time evolution of a unitary operator. On the left, there is a rectangle labeled U with inputs H and T , and output H . To its right is an equals sign. To the right of the equals sign is another rectangle labeled $U(t)$ with inputs H and T , and output H . A red arrow points from the bottom of the first U to the top of the second U , with a red circle around the arrowhead.

A diagram showing the composition of two unitary operators. On the left, there is a rectangle labeled U with inputs H and T , and output H . To its right is an equals sign. To the right of the equals sign is another rectangle labeled U with inputs H and T , and output H . A red curved arrow connects the bottom of the first U to the top of the second U , with a red circle around the arrowhead.

$$U(t+s) = U(s)U(t)$$

Algebra \rightarrow

A diagram showing the identity operator I represented by a rectangle with inputs H and T and output H . A red curved arrow connects the bottom of the rectangle to the top, with a red circle around the arrowhead.

$$U(0) = I$$

A diagram showing the adjoint operator U^+ represented by a rectangle with inputs H and T and output H . A red curved arrow connects the top of the rectangle to the bottom, with a red circle around the arrowhead.

$$U(-t) = U(t)^+$$

Unitary \nearrow

Quantum Dynamical Systems

No noise:

$$(\mu = 1 \text{ } \textcolor{red}{\bullet} \quad \eta = 1 \text{ } \textcolor{red}{\bullet})$$

$$\begin{array}{ccc} \kappa & \xrightarrow{\quad} & \kappa \otimes \tau \\ \boxed{F} & & \boxed{F} \otimes \tau \\ \downarrow & & \downarrow \end{array}$$

"add time on the side"

Objects:

$$\boxed{U}$$

quantum dynamical systems

↓ EN algebras (unitary)

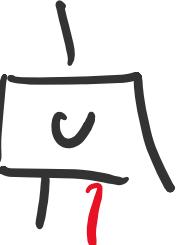
Morphisms:

$$U \xrightarrow{\quad} V$$
$$\boxed{F_U} = \boxed{F_V}$$

time-equivariant transformations

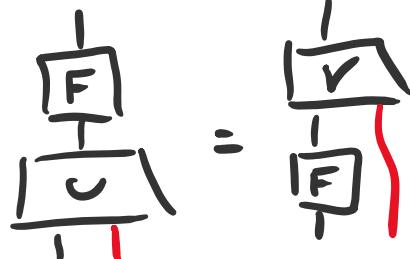
Quantum Dynamical Systems

Objects:



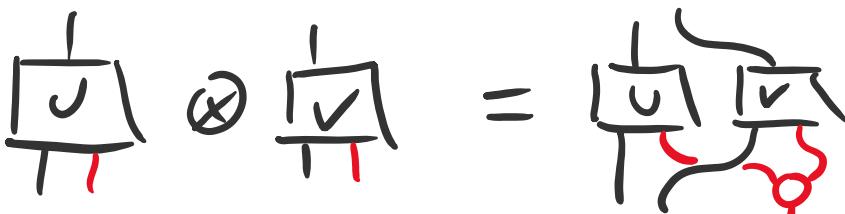
quantum dynamical
systems

Morphisms:

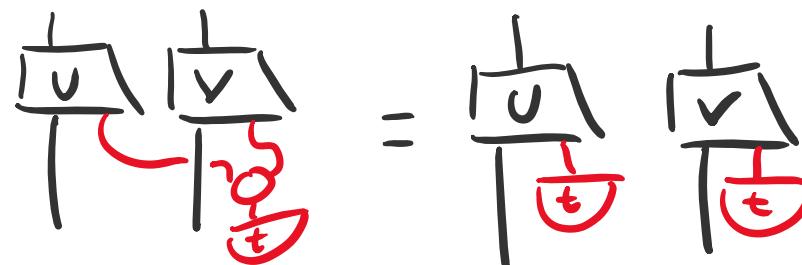

$$U \xrightarrow{F} V$$

time-equivariant transformations

Composition:

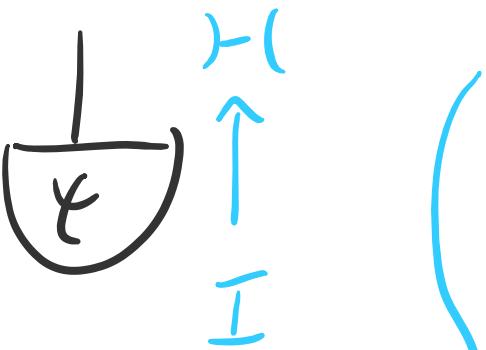


Synchronised
quantum dynamical \Rightarrow
systems



States and Histories

States:

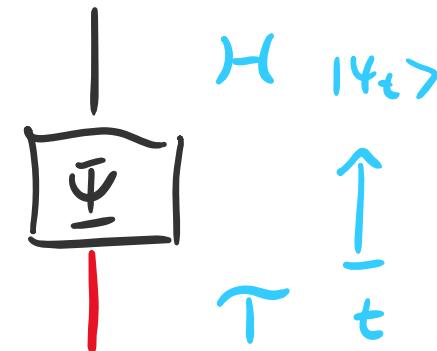


$$|\Psi\rangle = |\psi\rangle$$

$$|\Psi_t\rangle = U_t |\psi_0\rangle$$

this is the same as

Histories:



s.t.

$$|\Psi\rangle = |\psi\rangle$$

$$|\Psi_{t+s}\rangle = U_s |\Psi_t\rangle$$

$$(H, |\psi\rangle)$$

$$(T, \psi)$$

Hamiltonian as a Coalgebra

Time-Energy Duality:

$$\boxed{\frac{1}{P_E}} := \boxed{\frac{1}{U^+}}$$

Projector over energy E eigenstate.

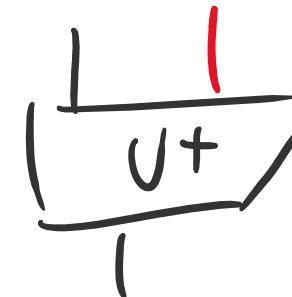
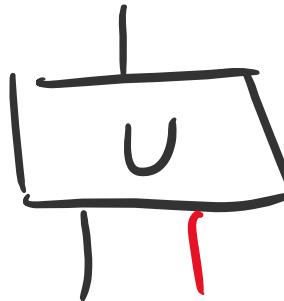
$$\boxed{P_{E'}} = \boxed{\frac{1}{U^+} \frac{1}{U^+}} = \boxed{\frac{1}{U^+}} = \boxed{\frac{1}{U^+} \delta_{EE'}} = \boxed{\frac{1}{P_E}} \delta_{EE'}$$

$$\begin{array}{ccc} \text{time} & & \text{energy} \\ \downarrow & & \downarrow \\ \mathbb{C}[G] \cong \mathbb{C}[G^\wedge] & & \\ \begin{array}{c} |t\rangle \\ \text{---} \\ t \quad t+s \\ \text{---} \\ |s\rangle \end{array} & & \begin{array}{c} |E\rangle \\ \text{---} \\ E \quad E \quad E+E' \\ \text{---} \\ |E'\rangle \end{array} \\ \text{time-translation} & & \text{energy-shift} \\ |E\rangle := \int e^{i\frac{Et}{\hbar}} |t\rangle dt & & \end{array}$$

e.g. Orthocompactness
 $P_E P_E = P_E \delta_{EE'}$

Hamiltonian as a Coalgebra

Dynamics
= algebra

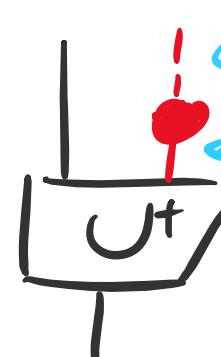
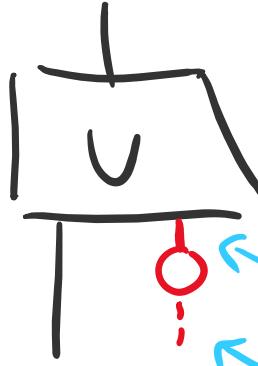


Energy observable
= coalgebra

prepare
in O

measure
in \bullet

classical
time
evolution



classical
energy
measurement

G^*
 $C[G^*]$

Schrödinger's Equation

$$H |\psi_E\rangle = E |\psi_E\rangle$$

$$\Updownarrow U(t) |\psi_E\rangle = e^{-i\frac{Et}{\hbar}} |\psi_E\rangle$$

$$U(t) = \exp\left[-i\frac{Ht}{\hbar}\right]$$

$$U(t) |\psi_E\rangle = e^{-i\frac{Et}{\hbar}} |\psi_E\rangle$$

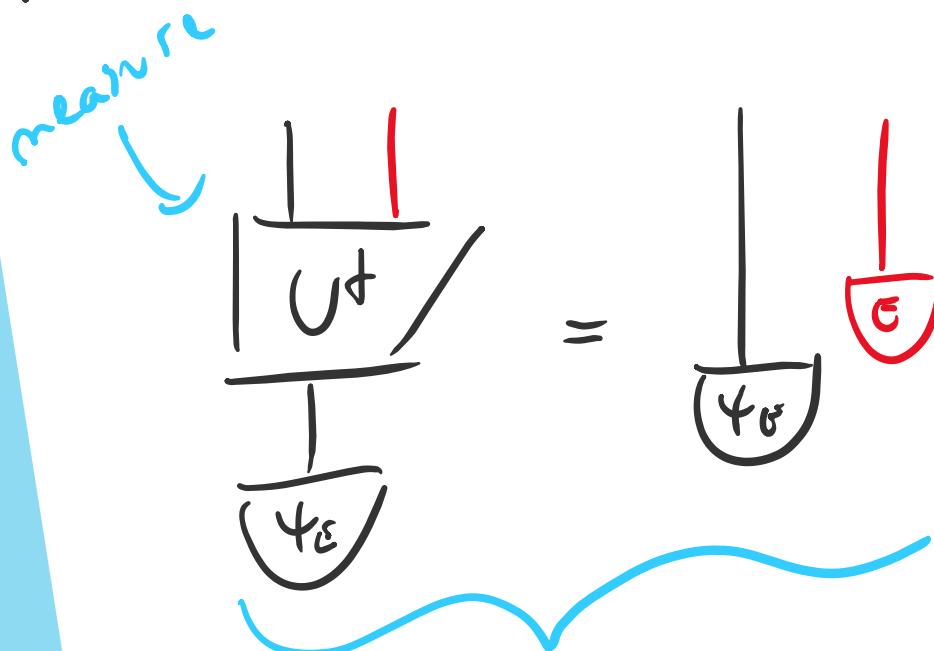
$$i\hbar \frac{d}{dt} |\psi_t\rangle = H |\psi_t\rangle$$

$$\Rightarrow |\psi_{t+s_t}\rangle = U(s_t) |\psi_t\rangle$$

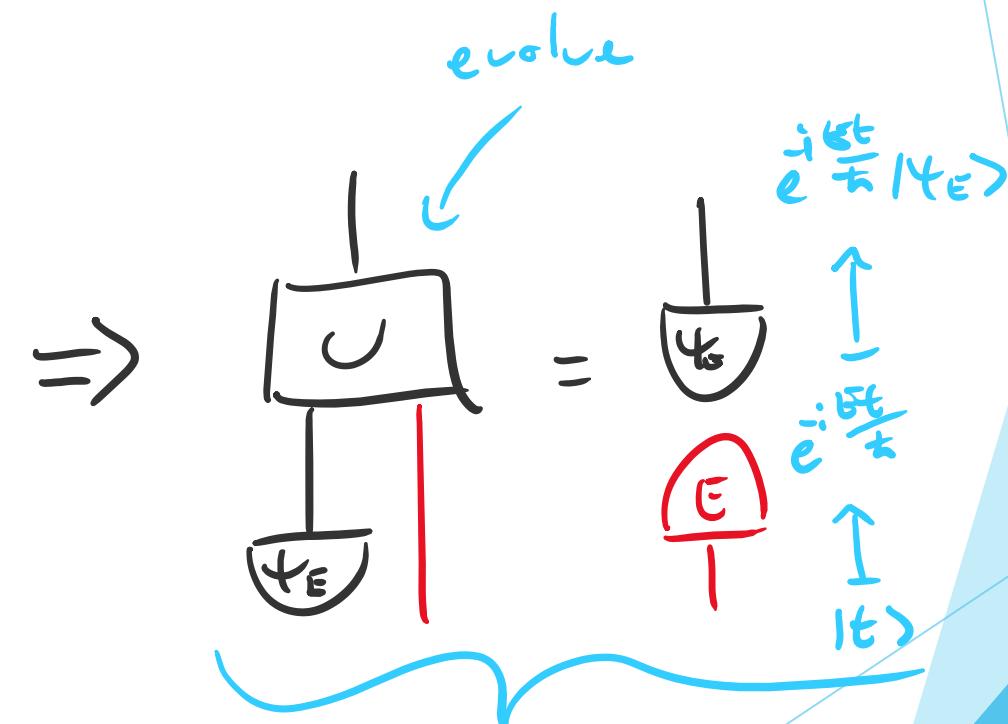
Schrödinger's Equation

$$U(t) |\psi_E\rangle = e^{-i\frac{E}{\hbar}t} |\psi_E\rangle$$

Proof:



Definition of energy eigenstate



Schrödinger's Equation

Stone's Theorem (with a really long name)

$$U(t) = \int e^{-i \frac{E t}{\hbar}} dP_{\sigma}$$

Proof:

$$\begin{aligned} U(t) &= U^+ + iU^- \\ &= U^+ - iU^- \\ &= U^+ - i \sum_{\sigma} E_{\sigma} P_{\sigma} \\ &= U^+ - i \sum_{\sigma} E_{\sigma} \delta(t - E_{\sigma}) \end{aligned}$$

von Neumann's Ergodic Theorem

$$\lim_{T \rightarrow \infty} \int_0^T e^{i \frac{Bt}{\hbar}} U(t) \frac{dt}{T} = P_E$$

Proof:

$$\sum_t \left(\begin{array}{c} + \\ G \\ + \end{array} \right) \left(\begin{array}{c} U \\ \hline T \\ + \end{array} \right) \frac{1}{N} = \left(\begin{array}{c} U \\ \hline T \\ \frac{1}{N} \end{array} \right) = \left(\begin{array}{c} \frac{1}{N} \\ \hline U \end{array} \right)$$

Weyl Canonical Commutation Relations

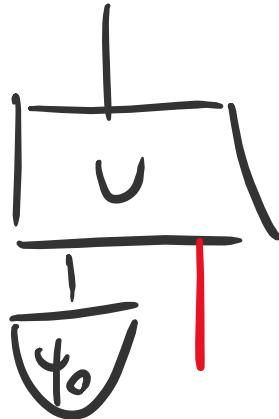
pref:

$$V_E U_t = e^{-\frac{iE t}{\hbar}} U_t V_E$$

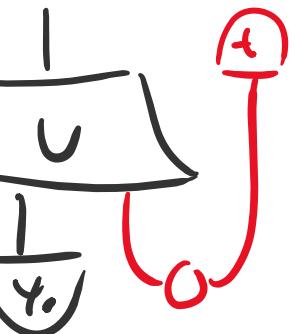
time-translation
energy-shift

= =

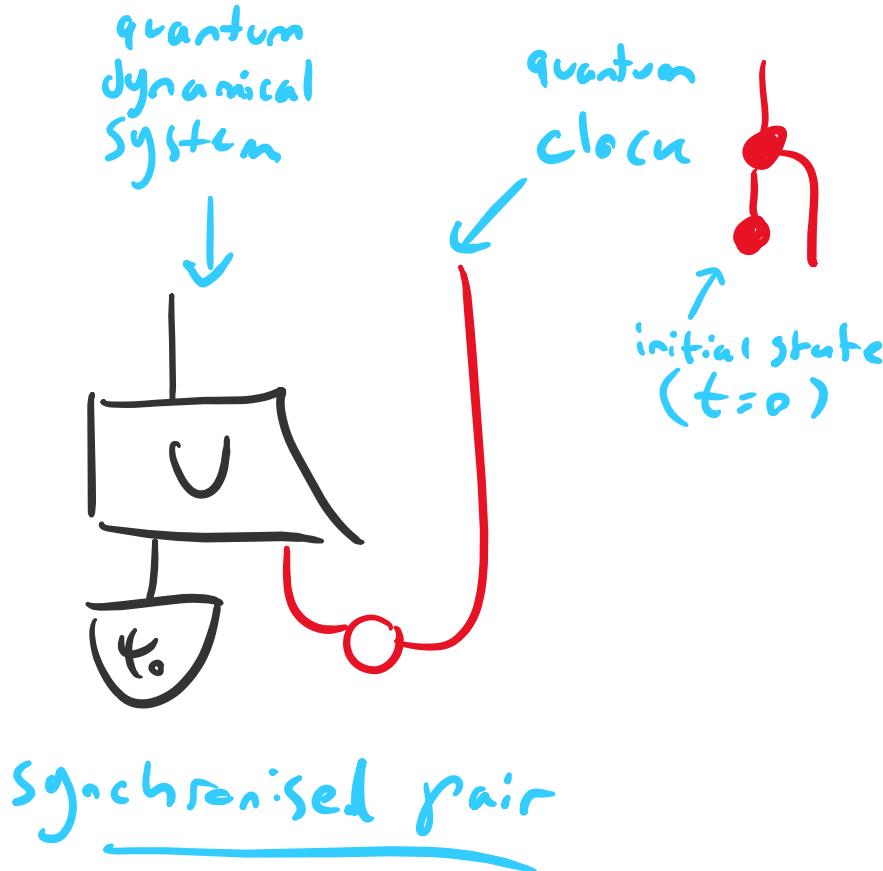
Quantising Time



history

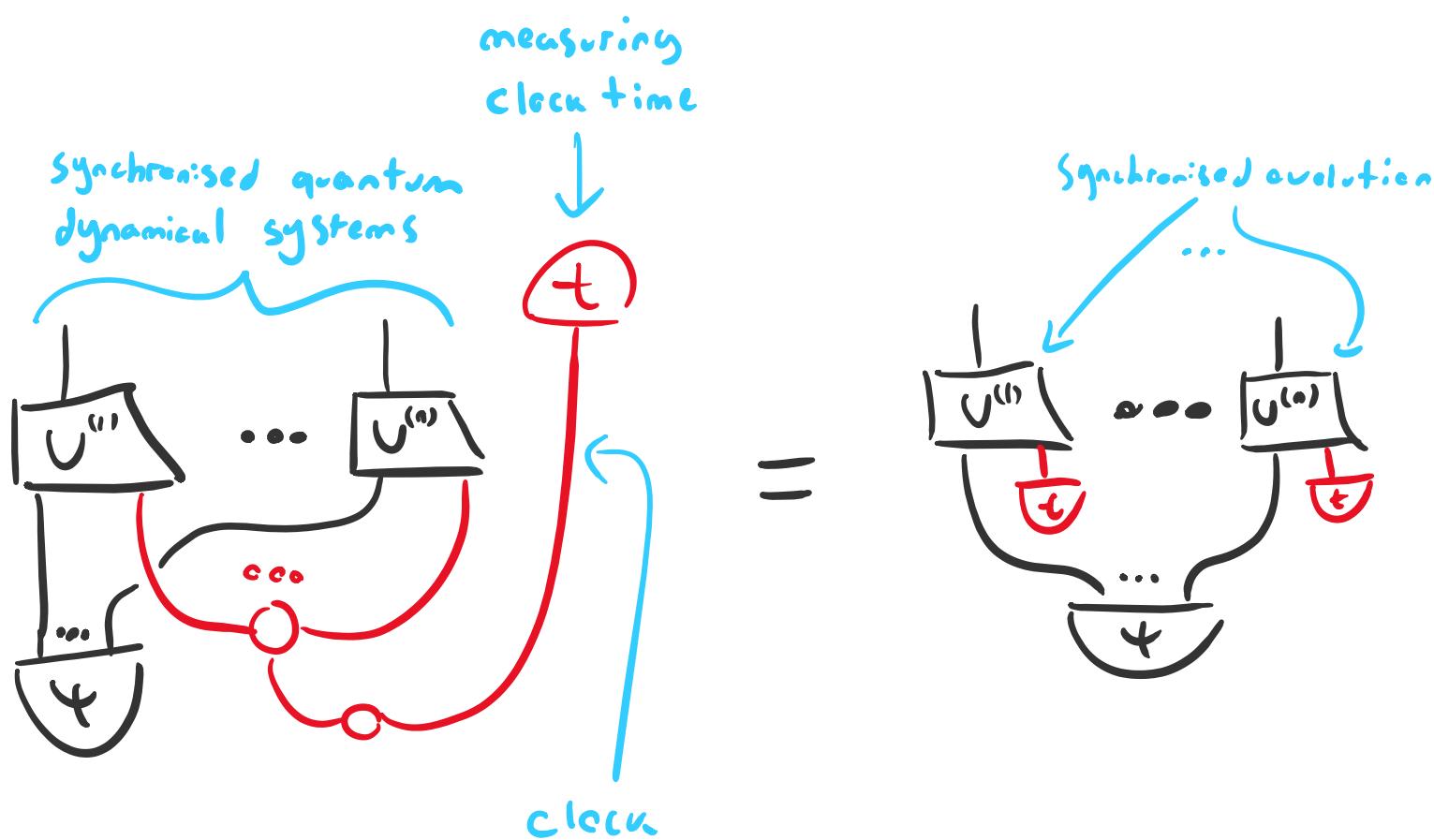


measuring
clock time :

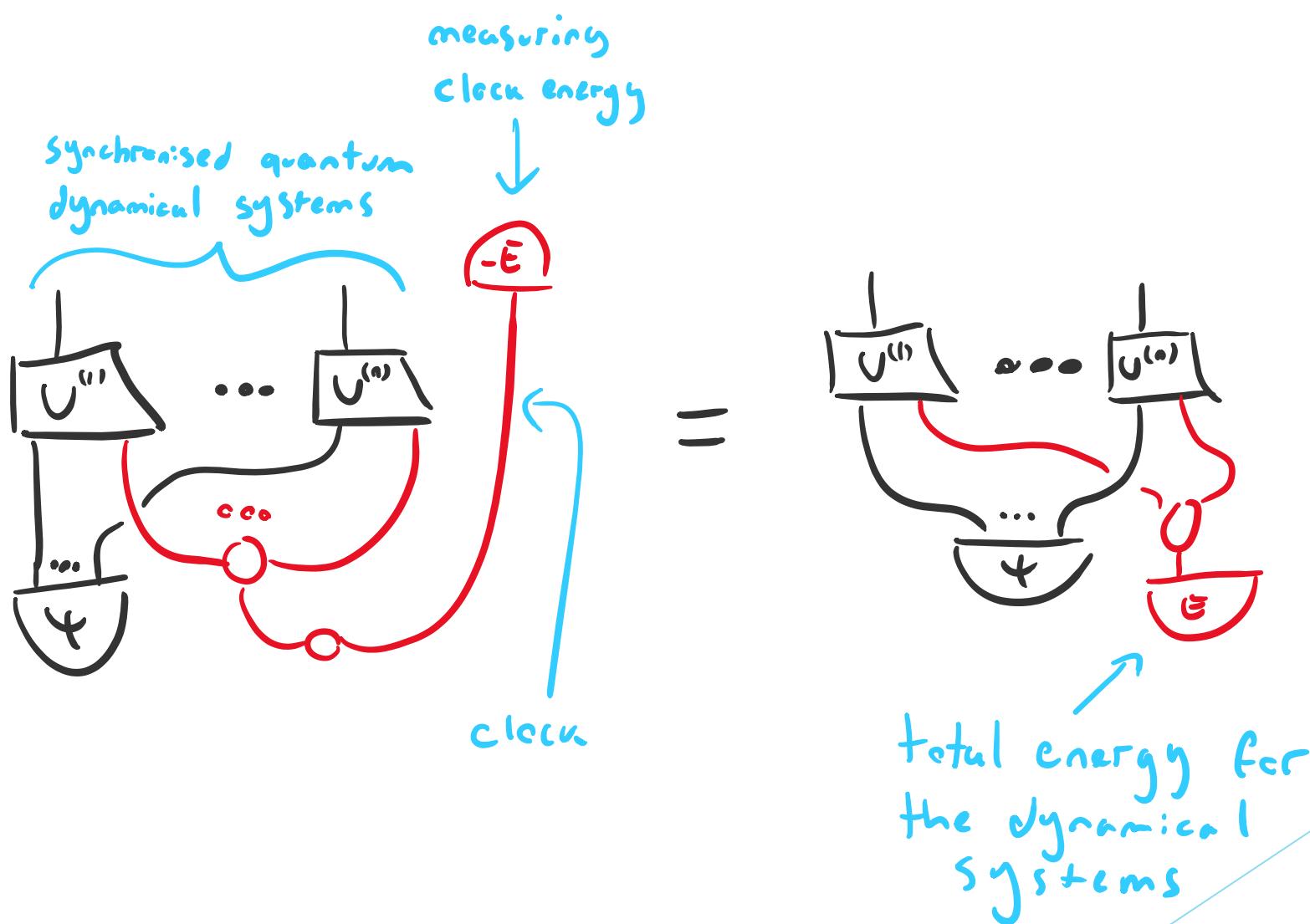


$$\text{measuring clock time :} = \text{quantum dynamical system} = \text{quantum clock}$$

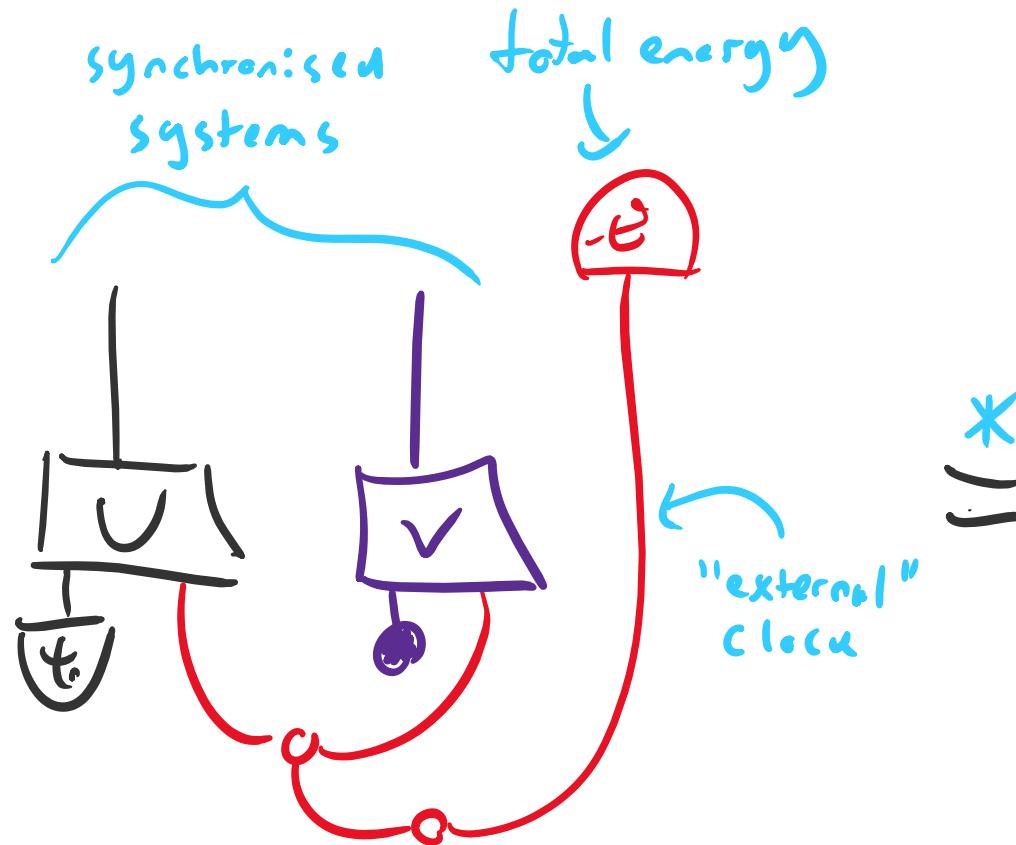
Quantising Time



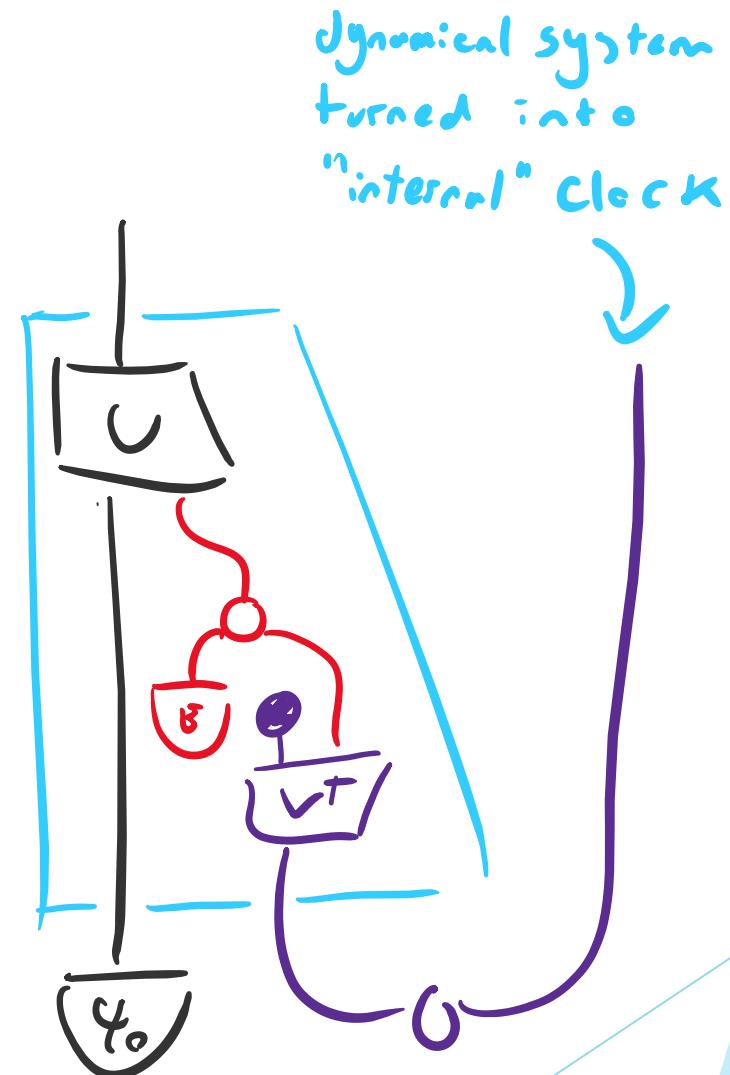
Quantising Time



Quantising Time

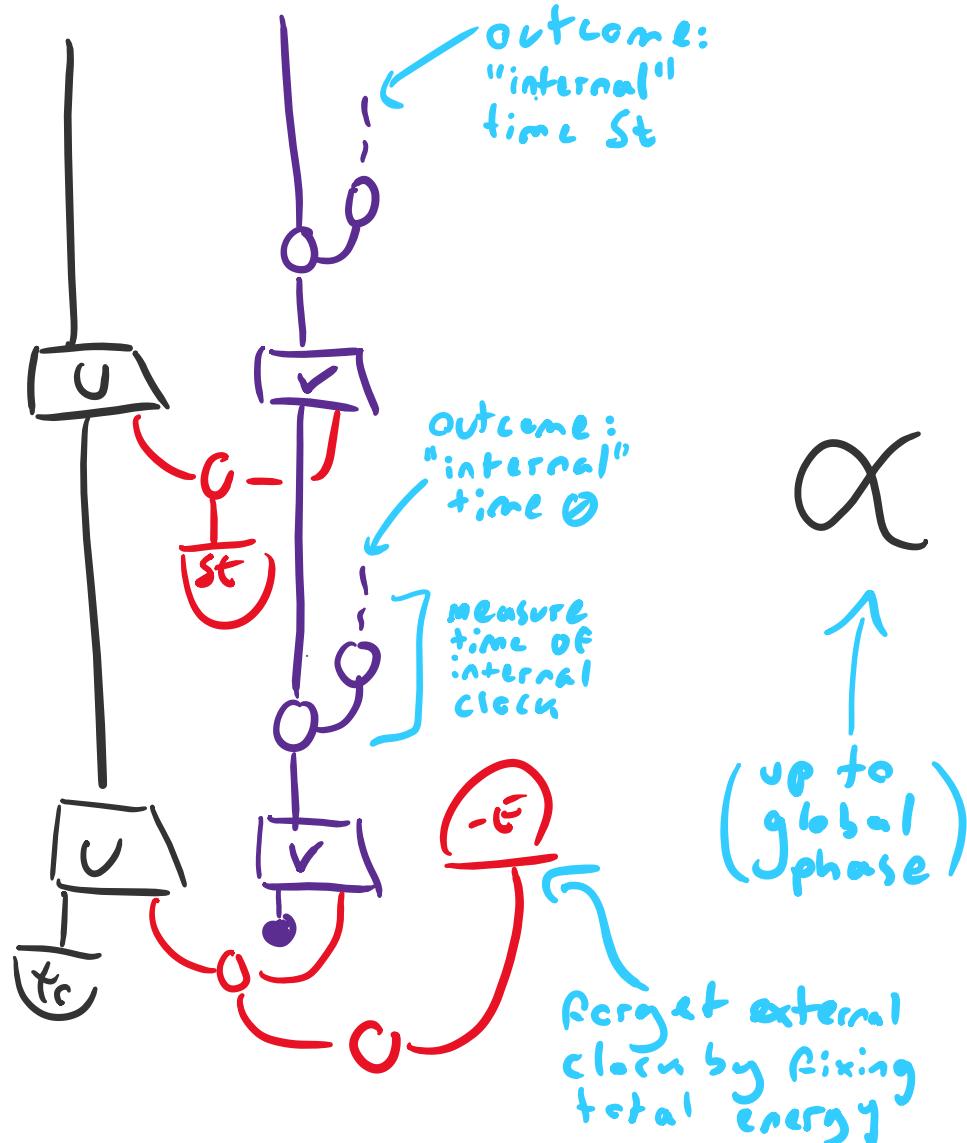


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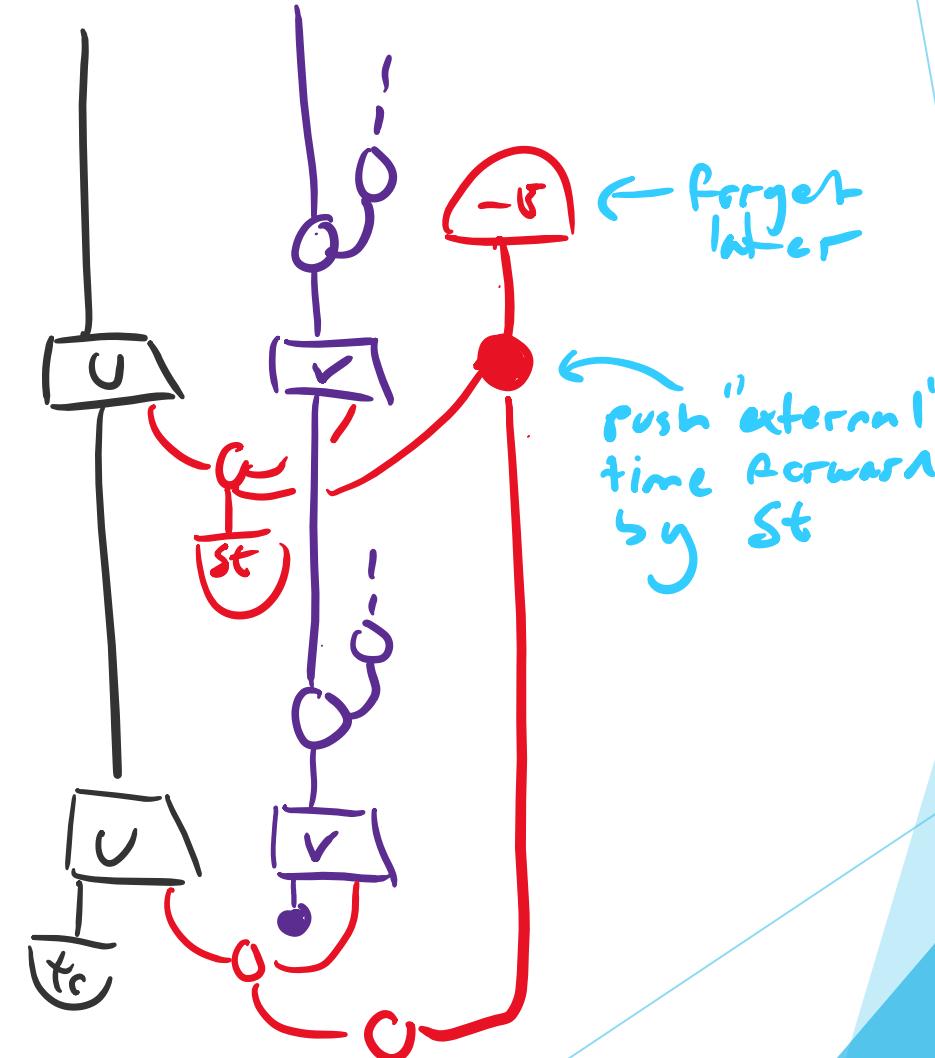


* Terms and conditions apply.

As time goes by...



α
(up to global phase)



THANK YOU!

Any Questions?