The Geometric and Sub-Geometric Completions of Doctrines

Joshua Wrigley
Università degli Studi dell'Insubria

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What is the idea behind doctrines?

In [1], Lawvere gave an elegant extension of Lindenbaum-Tarski algebras to the first-order setting.

Definition

A *doctrine* is any functor

$$P\colon \mathcal{C}^{op} \to \mathbf{PreOrd}.$$

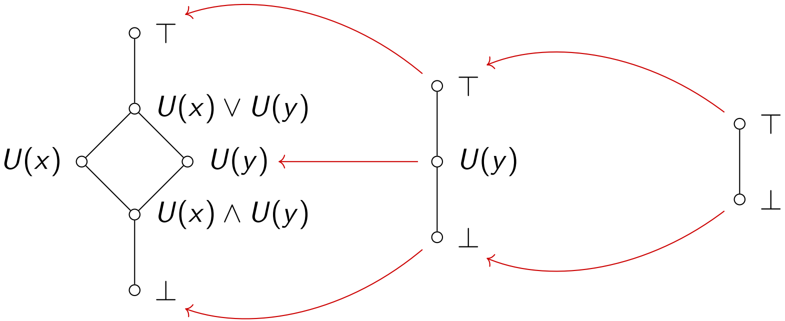
The category $\mathcal{C}$ should be interpreted as a category of *contexts* and *relabellings*.

Each fibre $P(c)$ should be interpreted as the algebra of *propositions* in context c.

An example of a doctrine

Doctrines capture the algebraic aspects of logical theories. E.g. adjoints to the substitution maps capture quantification.

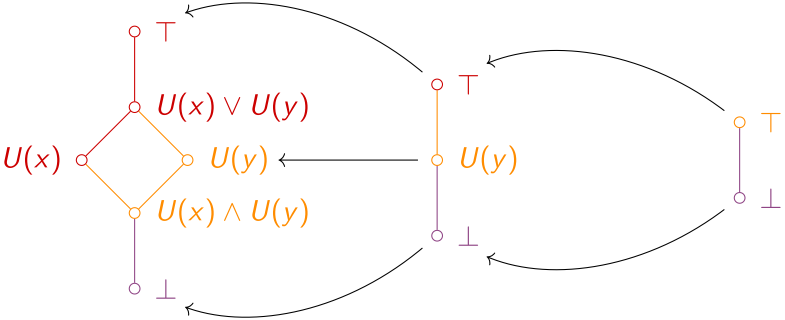
For example, the doctrine below represents the theory with one unary relation symbol.



An example of a doctrine

Doctrines capture the algebraic aspects of logical theories. E.g. adjoints to the substitution maps capture quantification.

For example, the doctrine below represents the theory with one unary relation symbol and the axiom $\top \vdash_{\emptyset} \exists x\, U(x)$.



Doctrines

Definition

The 2-category **Doc** has:

(i) as objects doctrines

$$P\colon \mathcal{C}^{op} \to \mathbf{PreOrd},$$

Doctrines

Definition

The 2-category **Doc** has:

(ii) as 1-cells pairs

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[dd, "F"'] \arrow[drr, "P"] \arrow[drr, phantom, shift right=6, "\Downarrow a"] & & \\ & & \mathbf{PreOrd}, \\ \mathcal{D}^{op} \arrow[urr, "Q"'] & & \end{tikzcd}$$

Doctrines

Definition

The 2-category **Doc** has:

(iii) and as 2-cells natural transformations

$$
\begin{tikzcd}
\mathcal{C}^{op} \arrow[dd, bend right, "F"'] \arrow[dd, bend left, "F'"] \arrow[dd, phantom, "\xRightarrow{\alpha}"] \arrow[dr, "P"] & \\
 & \mathbf{PreOrd}. \arrow[l, phantom, "a \Downarrow \; \leqslant \; \Downarrow a'"] \\
\mathcal{D}^{op} \arrow[ur, "Q"'] &
\end{tikzcd}
$$

Particular classes of doctrines

Certain 2-subcategories of **Doc** are the natural setting for the doctrinal approach to logics of various syntaxes.

Examples

(i) *Primary doctrines* interpret $\{\top, \wedge\}$. By **PrimDoc** we mean the 2-full 2-subcategory of **Doc**

(a) whose objects factor as $P\colon \mathcal{C}^{op} \to \mathbf{MSLat} \subseteq \mathbf{PreOrd}$, and $\mathcal{C}$ is *cartesian*,

(b) and whose 1-cells are the pairs where both $F\colon \mathcal{C} \to \mathcal{D}$ and $a_c\colon P(c) \to Q(F(c))$ are cartesian.

Particular classes of doctrines

Examples

(ii) By **ExDoc** $\subseteq$ **PrimDoc** we denote the 2-full 2-subcategory of *existential doctrines*:

(a) whose objects are primary doctrines $P\colon \mathcal{C}^{op} \to \mathbf{MSLat}$ where each $P(f)$ has a left adjoint $\exists_{P(f)}$ satisfying the *Frobenius* and *Beck-Chevalley* conditions,

(b) and whose 1-cells are those morphisms of primary doctrines for which

$$\begin{tikzcd} P(d) \arrow[r, "\exists_{P(f)}"] \arrow[d, "a_d"] & P(c) \arrow[d, "a_c"] \\ Q(F(d)) \arrow[r, "\exists_{Q(F(f))}"] & Q(F(c)) \end{tikzcd}$$

commutes.

Particular classes of doctrines

Examples

(iii) The 2-full 2-subcategory **CohDoc** $\subseteq$ **ExDoc** is the 2-category of *coherent doctrines* where

(a) objects are existential doctrines that factor as

$$P\colon \mathcal{C}^{op} \to \mathbf{DLat} \subseteq \mathbf{MSLat}.$$

(b) and for each 1-cell (F, a), a_c is a lattice homomorphism.

(iv) The 2-full 2-subcategory $\mathbf{GeomDoc}_{\text{cart}} \subseteq$ **CohDoc** of *geometric doctrines* (over a cartesian base) has

(a) as objects those coherent doctrines that factor as

$$P\colon \mathcal{C}^{op} \to \mathbf{Frm} \subseteq \mathbf{DLat},$$

(b) and for each 1-cell (F, a), a_c is a frame homomorphism.

Completing to richer syntax

So we have a hierarchy of syntaxes

$$\mathbf{GeomDoc}_{\text{cart}} \hookrightarrow \mathbf{CohDoc} \hookrightarrow \mathbf{ExDoc} \hookrightarrow \mathbf{PrimDoc}$$

Can we universally complete a doctrine to a richer syntax?

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Yes! This is the *existential completion* due to Trotta [2].

The existential completion

Definition (Trotta, [2])

Let $P \colon \mathcal{C}^{op} \to \mathbf{MSLat}$ be a *primary doctrine*. The *existential completion* of P is as follows.

(i) Take the pairs (f, x) where

$$d \xrightarrow{f} c \in \mathcal{C}, \quad x \in P(d),$$

ordered by $(g, y) \leqslant (f, x)$ if there exists $e \xrightarrow{h} d \in \mathcal{C}$ such that

$$\begin{tikzcd} e \arrow[d, "h"'] \arrow[dr, "g"] & \\ d \arrow[r, "f"'] & c \end{tikzcd}$$

and $y \leqslant P(h)(x)$. Let $P^{\exists}(c)$ be the posetal reflection.

The existential completion

Definition (Trotta, [2])

(ii) For each $e \xrightarrow{g} c$, $P^{\exists}(g)\colon P^{\exists}(c) \to P^{\exists}(e)$ sends (f, x) to $(k, P(h)(x))$, where

$$\begin{tikzcd} e \times_c d \arrow[r, "k"] \arrow[d, "h"] & e \arrow[d, "g"] \\ d \arrow[r, "f"] & c \end{tikzcd}$$

is a pullback in $\mathcal{C}$.

The free geometric completion

We can obtain a left 2-adjoint to $\mathbf{GeomDoc}_{\mathsf{cart}} \hookrightarrow \mathbf{PrimDoc}$ by first existentially completing a primary doctrine, and then freely adding joins.

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Definition

Explicitly, the *free geometric completion* $\mathfrak{Z}_{\mathrm{Fr}}(P) \colon \mathcal{C}^{op} \to \mathbf{Frm}_{\mathrm{open}}$ of P is the doctrine where

(i) the elements of $\mathfrak{Z}(P)(c)$ are up-sets of $P^{\exists}(c)$, i.e. sets S of pairs (f, x) with $d \xrightarrow{f} c \in \mathcal{C}$ and $x \in P(d)$ where, given $e \xrightarrow{h} d$ and $y \leqslant P(h)(x)$,

$$\text{if } (f, x) \in S \text{ then } (f \circ h, y) \in S,$$

(ii) meanwhile, $\mathfrak{Z}_{\mathrm{Fr}}(g) \colon \mathfrak{Z}_{\mathrm{Fr}}(P)(c) \to \mathfrak{Z}_{\mathrm{Fr}}(P)(d)$ sends S to

$$g^*(S) = \{ (h, y) \mid (f \circ h, y) \in S \}.$$

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This is not surprising. Taking the free group on a set of generators is not idempotent, i.e.

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But it *is* idempotent if we also allow for relations –

$$\langle \langle X \mid \rangle | R_{\langle X \rangle} \rangle \cong \langle X | \, \rangle.$$

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So what should relations for categorical logic look like?

For geometric logic at least, *Grothendieck topologies*.

The Grothendieck construction

Definition

Given a doctrine $P\colon \mathcal{C}^{op} \to \mathbf{PreOrd}$, we denote the *Grothendieck construction* by $\mathcal{C} \rtimes P$, the category

(i) whose objects are pairs (c, x), $x \in P(c)$,

(ii) and whose arrows

$$(d, y) \xrightarrow{f} (c, x)$$

are arrows $d \xrightarrow{f} c$ such that $y \leqslant P(f)(x)$.

The Grothendieck construction

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Given a doctrine $P \colon \mathcal{C}^{op} \to \mathbf{PreOrd}$, we denote the *Grothendieck construction* by $\mathcal{C} \rtimes P$, the category

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are arrows $d \xrightarrow{f} c$ such that $y \leqslant P(f)(x)$.

Given a morphism of doctrines $(F, a) \colon P \to Q$, we will write

$$F \rtimes a \colon \mathcal{C} \rtimes P \to \mathcal{D} \rtimes Q$$

for the functor that sends (c, x) to $(F(c), a_c(x))$.

Doctrinal sites

Definition

The 2-category **DocSites** of *doctrinal sites* has

(i) as objects pairs (P, J) where $P \colon \mathcal{C}^{op} \to \mathbf{PreOrd}$ is a doctrine and J is a Grothendieck topology on $\mathcal{C} \rtimes P$,

(ii) as 1-cells morphisms of doctrines $(F, a) \colon P \to Q$ such that

$$F \rtimes a \colon (\mathcal{C} \rtimes P, J) \to (\mathcal{D} \rtimes Q, K)$$

is *continuous* and *flat*, and $F \colon \mathcal{C} \to \mathcal{D}$ is *flat*.

(iii) and 2-cells are the same as in **Doc**.

Is this a sensible thing to do?

Examples

(i) **PrimDoc** is equivalent to the 1-full 2-subcategory of **DocSites** on objects (P, J_{triv}) where P is a primary doctrine.

(ii) **ExDoc** is the 1-full 2-subcategory of **DocSites** on objects (P, J_{Ex}) where P is existential and J_{Ex} is the topology generated by covers

$$(d, x) \xrightarrow{f} (c, \exists_f x).$$

(iii) **CohDoc** is the 1-full 2-subcategory of **DocSites** on objects (P, J_{Coh}) where P is coherent and J_{Coh} is the topology generated by covers

$$(d, x) \xrightarrow{f} (c, \exists_f x \vee \exists_g y) \xleftarrow{g} (e, y).$$

Geometric doctrines

Definition

By **GeomDoc** we denote the 1-full 2-subcategory of **DocSites** on objects of the form $(\mathbb{L}, K_{\mathbb{L}})$ where

(a) $\mathbb{L}$ is a functor taking values in $\mathbf{Frm}_{\text{open}}$,

(b) and $K_{\mathbb{L}}$ is the Grothendieck topology on $\mathcal{C} \rtimes \mathbb{L}$ where

$$\left\{ (d_i, x_i) \xrightarrow{f_i} (c, y) \,\middle|\, i \in I \right\} \in K_{\mathbb{L}}(c, y)$$

if and only if $y = \bigvee_{i \in I} \exists_{f_i} x_i$.

The 2-category $\mathbf{GeomDoc}_{\text{cart}}$ is a 1-full 2-subcategory of $\mathbf{GeomDoc}$, $\mathbf{DocSites}$.

The geometric completion

Can we build a left adjoint to **GeomDoc** $\hookrightarrow$ **DocSites**?

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Definition (Caramello, [3])

The *geometric completion* of a doctrinal site (P, J) is the doctrine $\mathfrak{Z}(P, J) \colon \mathcal{C}^{op} \to \mathbf{Frm}_{\text{open}}$ where

(i) an element $S \in \mathfrak{Z}(P, J)(c)$ is a set of pairs (f, x), consisting of $d \xrightarrow{f} c \in \mathcal{C}$ and $x \in P(d)$, such that

  (a) if $(f, x) \in S$ then $(f \circ g, y) \in S$ for each $e \xrightarrow{g} d$ and $y \leqslant P(g)(x)$,

  (b) if $\{ (e_i, y_i) \xrightarrow{g_i} (d, x) \mid i \in I \}$ is J-covering and, for each $i \in I$, $(f \circ g_i, y_i) \in S$, then $(f, x) \in S$ too,

(ii) meanwhile $\mathfrak{Z}(P, J)(g) \colon \mathfrak{Z}(P, J)(c) \to \mathfrak{Z}(P, J)(e)$ sends S to

$$\mathfrak{Z}(P, J)(g)(S) = \{ (h, y) \mid (f \circ h, y) \in S \}.$$

The universal property of the geometric completion

This constitutes the action on objects of a strict left 2-adjoint to **GeomDoc** $\hookrightarrow$ **DocSites**.

Theorem

The geometric completion $\mathfrak{Z}(P, J)$ of a doctrinal site is:

(i) **Universal** - for every morphism of doctrinal sites $(F, a) \colon (P, J) \to (\mathbb{L}, K_{\mathbb{L}})$, there is a unique morphism of geometric doctrines $(F, \mathfrak{a})$ for which the triangle commutes

$$\begin{tikzcd} P \arrow[r, "\eta^{(P,J)}"] \arrow[dr, "a"'] & \mathfrak{Z}(P,J) \arrow[d, "\mathfrak{a}"] \\ & \mathbb{L} \circ F^{op}; \end{tikzcd}$$

(ii) **Idempotent** - $\mathfrak{Z}(P, J) \cong \mathfrak{Z}(\mathfrak{Z}(P, J), K_{\mathfrak{Z}(P,J)})$.

The relative Beck-Chevalley condition

Note that a geometric doctrine $\mathbb{L} \colon \mathcal{C}^{op} \to \mathbf{Frm}_{\text{open}}$ need not be fibred over a cartesian category.

Instead, a functor $\mathbb{L} \colon \mathcal{C}^{op} \to \mathbf{Frm}_{\text{open}}$ is a geometric doctrine if one of the following equivalent conditions is satisfied:

(i) the assignment

$$\left\{ (d_i, x_i) \xrightarrow{f_i} \left(c, \bigvee_{i \in I} \exists_{f_i} x_i\right) \;\middle|\; i \in I \right\} \in K_{\mathbb{L}} \left(c, \bigvee_{i \in I} \exists_{f_i} x_i\right)$$

defines a Grothendieck topology;

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Note that a geometric doctrine $\mathbb{L} \colon \mathcal{C}^{op} \to \mathbf{Frm}_{\text{open}}$ need not be fibred over a cartesian category.

Instead, a functor $\mathbb{L} \colon \mathcal{C}^{op} \to \mathbf{Frm}_{\text{open}}$ is a geometric doctrine if one of the following equivalent conditions is satisfied:

(ii) $\mathbb{L}$ is an *internal frame* of $\mathbf{Sets}^{\mathcal{C}^{op}}$;

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Instead, a functor $\mathbb{L} \colon \mathcal{C}^{op} \to \mathbf{Frm}_{\text{open}}$ is a geometric doctrine if one of the following equivalent conditions is satisfied:

(iii) the *relative Beck-Chevalley condition* is satisfied – given a set $S \in \mathfrak{Z}_{\mathrm{Fr}}(\mathbb{L})(c)$ and a map $e \xrightarrow{g} c \in \mathcal{C}$,

$$\mathbb{L}(g)\left(\bigvee_{(f,x)\in S} \exists_f x\right) = \bigvee_{(h,y)\in g^*(S)} \exists_h y.$$

Proposition

If $\mathcal{C}$ has pullbacks, then $\mathbb{L}$ satisfies the relative Beck-Chevalley condition if and only if $\mathbb{L}$ satisfies the Beck-Chevalley condition.

Completing to fragments of geometric logic

We saw that the free geometric completion is the existential completion followed by the point-wise join completion.

Equivalently,

$$\mathfrak{Z}_{\mathsf{Fr}}(P) \cong \mathfrak{Z}(P^{\exists}, J_{\mathsf{Ex}}).$$

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In other words, freely geometric completing is the same as completing P via the *existential monad* $T^{\exists}$, keeping track of this new information by a topology, and then geometrically completing.

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In other words, freely geometric completing is the same as completing P via the *existential monad* $T^{\exists}$, keeping track of this new information by a topology, and then geometrically completing.

This behaviour is not exclusive to the existential completion.

Flat morphisms of doctrines

Notation

We write $\mathbf{Doc}_{\text{flat}}$ for the 1-full 2-subcategory of **DocSites** on objects of the form (P, J_{triv}).

Equivalently, $\mathbf{Doc}_{\text{flat}}$ is the 2-full 2-subcategory of **Doc** whose 1-cells are doctrine morphisms such that

$$F \rtimes a \colon \mathcal{C} \rtimes P \to \mathcal{D} \rtimes Q$$

is *flat*.

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is *flat*.

The free geometric completion extends to give a 2-functor

$$\mathbf{PrimDoc} \subseteq \mathbf{Doc}_{\mathsf{flat}} \xrightarrow{\mathfrak{Z}_{\mathsf{Fr}}} \mathbf{GeomDoc}.$$

Notation

Given a 2-subcategory A-**Doc** $\subseteq$ **Doc**, we write A-$\mathbf{Doc}_{\mathsf{flat}}$ for the 2-full 2-subcategory of A-**Doc** whose 1-cells are flat.

Sub-geometric completions

Definition

Let A-**Doc** be 2-full 2-subcategory of **PrimDoc** that contains the image of

$$A\text{-}\mathbf{Doc}_{\mathrm{flat}} \hookrightarrow \mathbf{Doc}_{\mathrm{flat}} \xrightarrow{\mathfrak{Z}_{\mathrm{Fr}}} \mathbf{GeomDoc},$$

and, for each A-doctrine P, the morphism $\eta^{(P,J_{\mathrm{triv}})} \colon P \to \mathfrak{Z}_{\mathrm{Fr}}(P)$.

A 2-monad (T, ε, ν) on A-**Doc** is *sub-geometric* if, for each A-doctrine P,

(i) there is a morphism

$$\xi_P \colon T\mathfrak{Z}_{\mathrm{Fr}}(P) \to \mathfrak{Z}_{\mathrm{Fr}}(P)$$

for which $(\mathfrak{Z}_{\mathrm{Fr}}(P), \xi_P)$ is a (strict) T-algebra,

Sub-geometric completions

Definition

(ii) and there is a topology J_P^T on $\mathcal{D} \rtimes TP$ such that

$$\varepsilon^P \colon (P, J_{\mathrm{triv}}) \to (TP, J_P^T),$$

$$\xi^P \colon (T\mathfrak{Z}_{\mathrm{Fr}}(P), J^T_{\mathfrak{Z}_{\mathrm{Fr}}(P)}) \to (\mathfrak{Z}_{\mathrm{Fr}}(P), K_{\mathfrak{Z}_{\mathrm{Fr}}(P)}),$$

and $T\theta \colon (TP, J_P^T) \to (TQ, J_Q^T)$, for all $P \xrightarrow{\theta} Q \in A\text{-}\mathbf{Doc}$,

are all morphisms of doctrinal sites.

Sub-geometric completions

Theorem

For each sub-geometric completion (T, ε, ν), the square

$$\begin{tikzcd} A\text{-}\mathbf{Doc}_{\mathsf{flat}} \arrow[r, hook] \arrow[d, "J^T"] & \mathbf{Doc}_{\mathsf{flat}} \arrow[d, "\mathfrak{Z}_{\mathsf{Fr}}"] \\ \mathbf{DocSites} \arrow[r, "\mathfrak{Z}"] & \mathbf{GeomDoc}, \end{tikzcd}$$

commutes up to iso., where J^T is the 2-functor $P \mapsto (TP, J^T_P)$.

In particular, there is an isomorphism $\mathfrak{Z}_{\mathsf{Fr}}(P) \cong \mathfrak{Z}(TP, J^T_P)$.

Corollary

Completing a doctrine with respect to any subset of the logical symbols $\{\top, \wedge, \vee, \bigvee, \exists\}$ is sub-geometric.

Thank you for your attention

Doctrine theory:

[1] F.W. Lawvere, "Adjointness in foundations", *Dialectica*, vol. 23, no. 3/4, pp. 281-296, 1969.

[2] D. Trotta, "The existential completion", *Theory and Applications of Categories*, vol. 35, pp. 1576-1607, 2020.

The geometric completion:

[3] O. Caramello, "Fibred sites and existential toposes", arXiv: 2212.11693 [math.AG], 2022

[4] J.W., "The geometric completion of a doctrine", arXiv: 2304.07539 [math.CT], 2023.