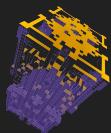


What is n-ary associativity?

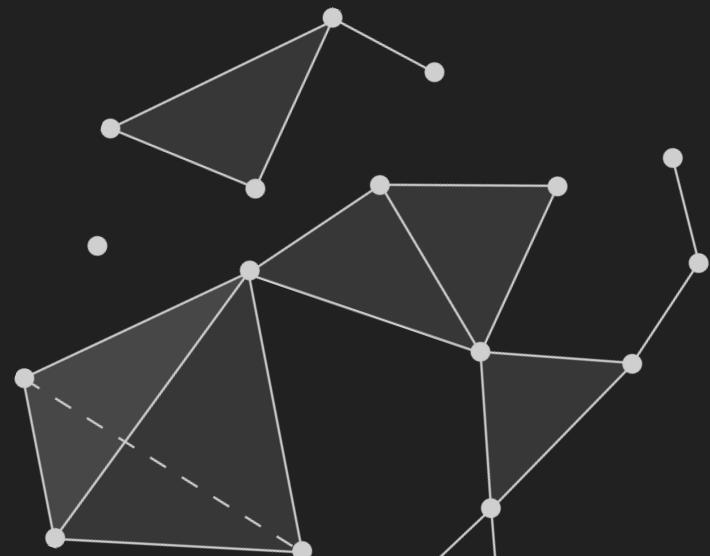
Carlos Zapata Carratalá



SEMF
Society for Multidisciplinary
and Fundamental Research



**Institute for the
Foundations of Computation**



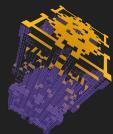
What is n-ary associativity?

*...or the odd tale of ternary mathematics and
the quest for the elusive notion of higher-arity category*

Carlos Zapata Carratalá



SEMF
Society for Multidisciplinary
and Fundamental Research



Institute for the
Foundations of Computation



Ternary Algebras

Ternary Algebras

3-Lie Algebras

$$(t, [., .]) \quad [-, -, -] \quad \wedge^3 t \rightarrow t$$

$$[x, y, [a, b, c]] = [[x, y, a], b, c] + [a, [x, y, b], c] + [a, b, [x, y, c]]$$

3-Lie Functor?

3-Lie Functor?

Lie functor : $(2\text{-})\text{LieAlg} \rightarrow \text{LieGrp}$
 $(\mathfrak{g}, [\cdot, \cdot]) \mapsto (G, *) : T_e G \cong \mathfrak{g}$

3-Lie Functor?

Lie functor : $(2\text{-})\text{LieAlg} \rightarrow \text{LieGrp}$
 $(\mathfrak{g}, [\cdot, \cdot]) \mapsto (G, *) : T_e G \cong \mathfrak{g}$

3-LieAlg \rightarrow ?
•

3-Lie Functor?

Lie functor : $(2\text{-})\text{LieAlg} \rightarrow \text{LieGrp}$

$(\mathfrak{g}, [\cdot, \cdot]) \mapsto (G, *) : T_e G \cong \mathfrak{g}$

$3\text{-}\text{LieAlg} \rightarrow ?$

OPEN
PROBLEM !!

Cubic Matrix Algebra

$$m_{ijk} \in \mathbb{F}^{N \times M \times L}$$

Cubic Matrix Algebra

$$m_{ijk} \in \mathbb{F}^{N \times M \times L} \quad (abc)_{\text{fish}} = \sum_{pqr} a_{ijp} \cdot b_{qrp} \cdot c_{qrk}$$

$$(abc)_{\text{Hilfsmittel}} = \sum_{pqk} a_{ipq} \cdot b_{pjk} \cdot c_{qrk}$$

$$(abc)_{\text{B.M.}} = \sum_p a_{ijp} \cdot b_{ipk} \cdot c_{pjk}$$

Cubic Matrix Algebra

$$m_{ijk} \in \mathbb{F}^{N \times M \times L}$$
$$(abc)_{\text{fish}} = \sum_{pqr} a_{ijp} \cdot b_{qrp} \cdot c_{qrk}$$
$$(abc)_{\text{Hilbert}} = \sum_{pqk} a_{ipq} \cdot b_{pjk} \cdot c_{qrk}$$
$$(abc)_{\text{B.M.}} = \sum_p a_{ijp} \cdot b_{ipk} \cdot c_{pjk}$$

$$(ab(cde)) = (a(dcb)e) = ((abc)d)e$$

Cubic Matrix Algebra

$$m_{ijk} \in \mathbb{F}^{N \times M \times L}$$

$$(abc)_{\text{fish}} = \sum_{pqr} a_{ijp} \cdot b_{qrp} \cdot c_{qrk}$$

$$(abc)_{\text{Hilbert}} = \sum_{pqr} a_{ipq} \cdot b_{pjr} \cdot c_{qrk}$$

$$(abc)_{\text{B.M.}} = \sum_p a_{ijp} \cdot b_{ipk} \cdot c_{pjk}$$

$$(ab(cde)) = (a(dcb)e) = ((abc)de)$$

no other known
forms of associativity !!

(Hyper)Graph-Theoretic Approach

(Hyper)Graph-Theoretic Approach

$$\sum_{\mathcal{P}} \alpha_{ip} \cdot \alpha_{pj} \sim$$



(Hyper)Graph-Theoretic Approach

$$\sum_p a_{ip} \cdot a_{pj} \sim$$

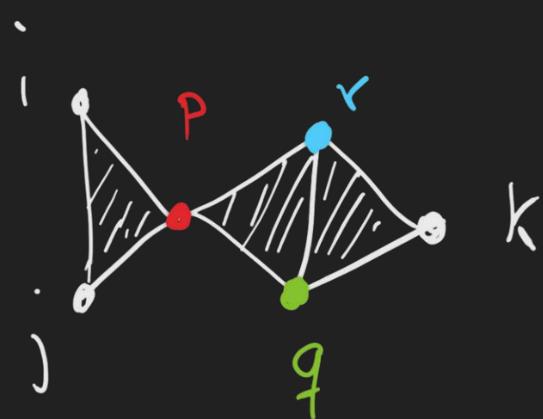


associativity \sim path transitivity

(Hyper)Graph-Theoretic Approach

(Hyper)Graph-Theoretic Approach

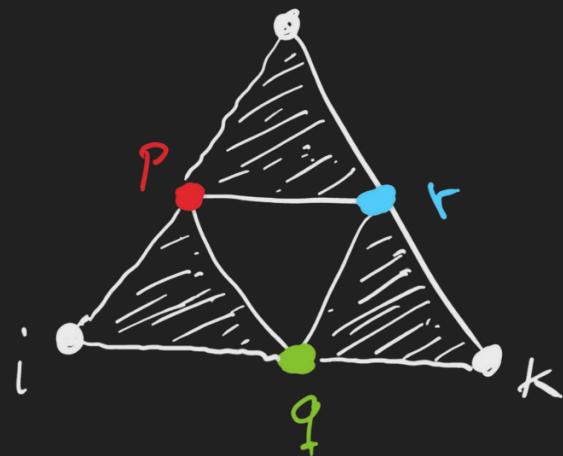
$$\sum_{pqk} \alpha_{ijp} \cdot \alpha_{qrp} \cdot \alpha_{qrk}$$



fish

(Hyper)Graph-Theoretic Approach

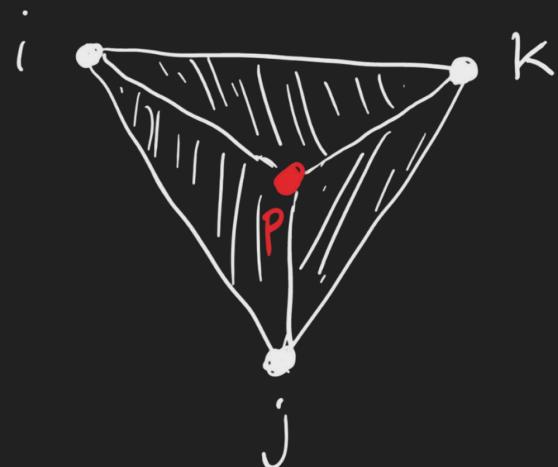
$$\sum_{PQR} \alpha_{ipq} \cdot \alpha_{pr} \cdot \alpha_{qrk}$$



triforce

(Hyper)Graph-Theoretic Approach

$$\sum_{\mathcal{P}} \alpha_{ij\mathcal{P}} \cdot \alpha_{i\mathcal{P}k} \cdot \alpha_{\mathcal{P}jk}$$



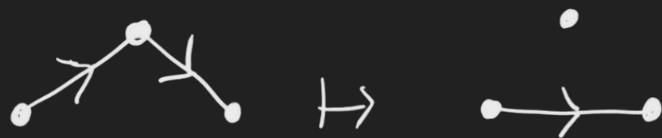
Bhattacharya - Mesner

n-ary Categories?

n-ary Categories?



n-ary Categories?



n-ary Categories?

semicategory



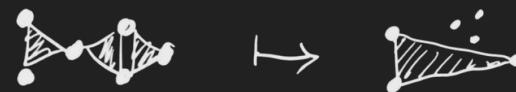
univ



dagger



Semireapoid



operad



how do we implement axioms?

what is n-ary associativity?

The Principle of Diagrammatic Simplicity

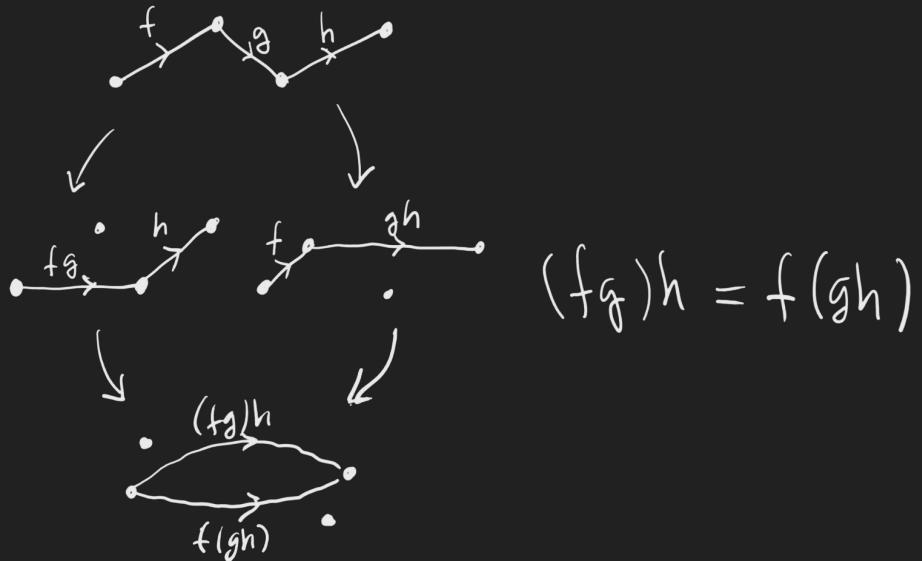
The Principle of Diagrammatic Simplicity

semicategory

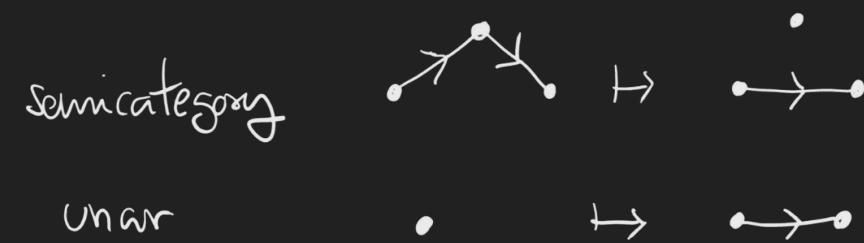


The Principle of Diagrammatic Simplicity

semicategory



The Principle of Diagrammatic Simplicity

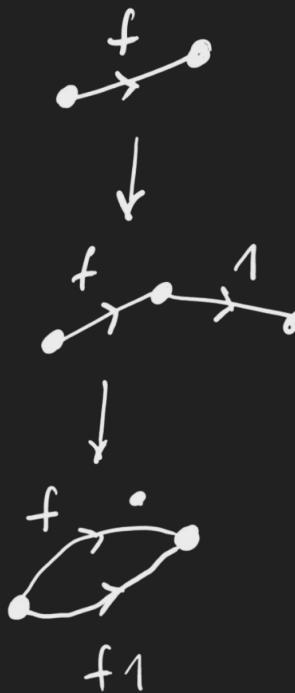


The Principle of Diagrammatic Simplicity

semicategory



unarr



$$f \circ 1 = f$$

The Principle of Diagrammatic Simplicity

dagger

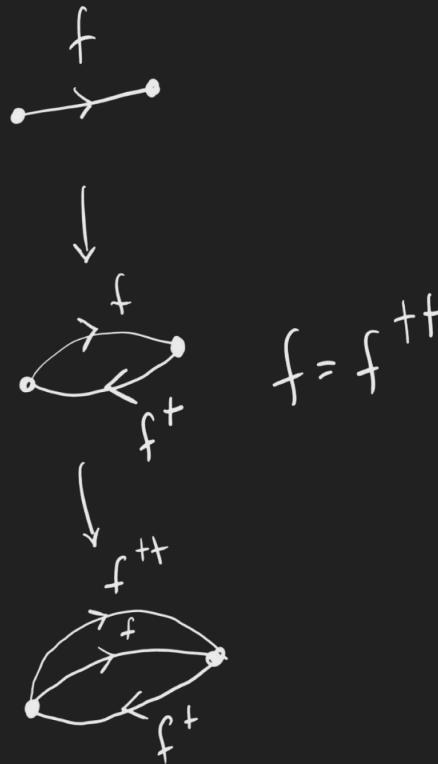


The Principle of Diagrammatic Simplicity

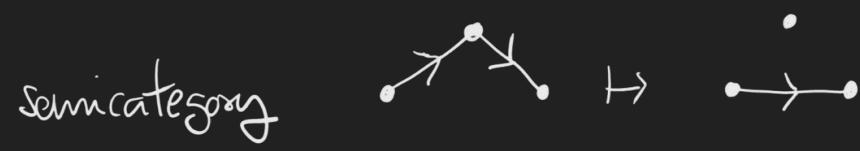
dagger



\mapsto



The Principle of Diagrammatic Simplicity

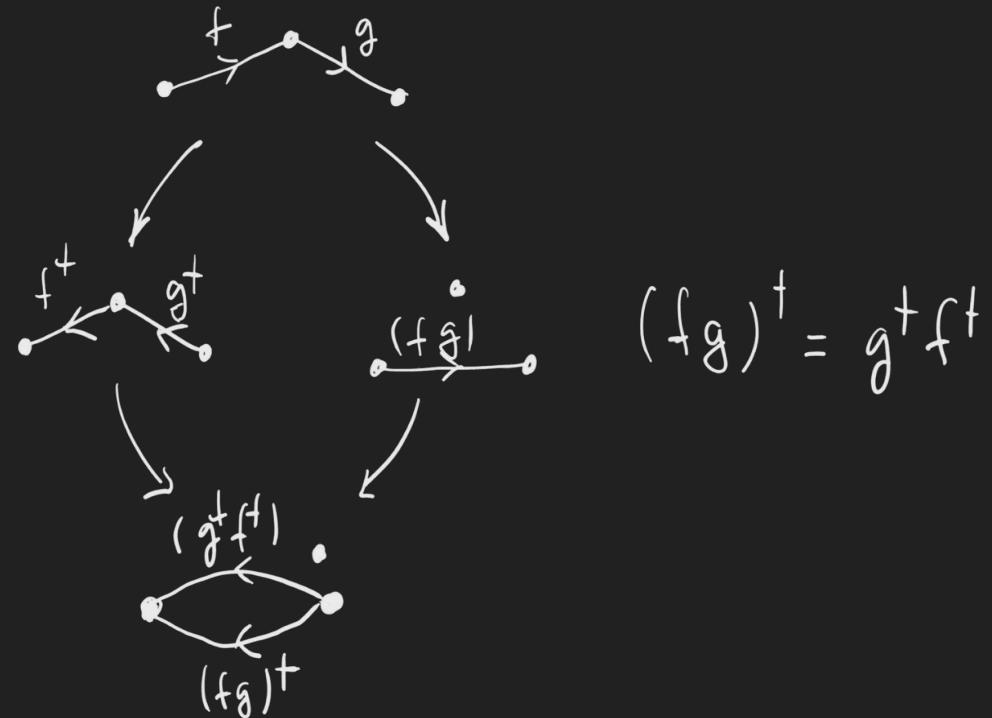


The Principle of Diagrammatic Simplicity

semicategory



dgger



The Principle of Diagrammatic Simplicity

operad



The Principle of Diagrammatic Simplicity

operad



$$\begin{array}{c} \text{Diagrammatic expression} \\ \downarrow \\ \text{Simplification} \\ \downarrow \\ \text{Simpler diagram} \\ \downarrow \\ \text{Final simplified form} \end{array}$$

Diagrammatic expression:

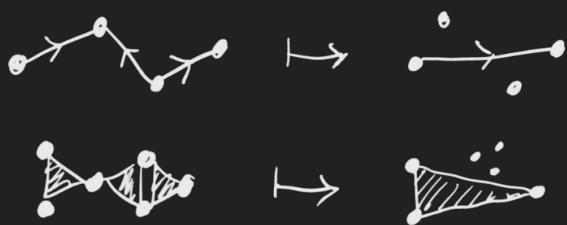
Simplification:

Final simplified form:

$$\left(\begin{matrix} \bar{a} \\ b \\ c \end{matrix} \right) = \left(\begin{matrix} a \\ \bar{b} \\ c \end{matrix} \right)$$

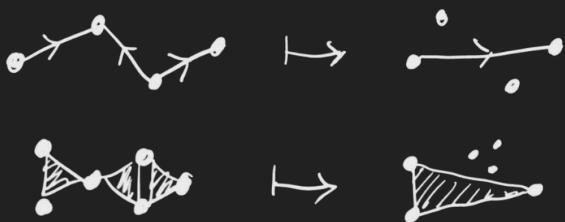
The Principle of Diagrammatic Simplicity

Semileapoid



The Principle of Diagrammatic Simplicity

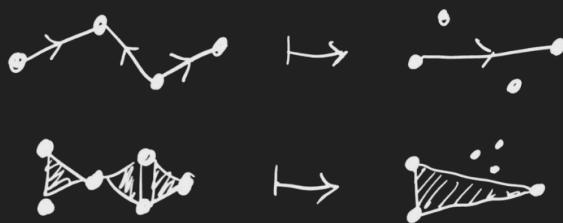
Semileapoid



$$(abc)de = a(dcb)e = ab(cde)$$

The Principle of Diagrammatic Simplicity

Semihopoid



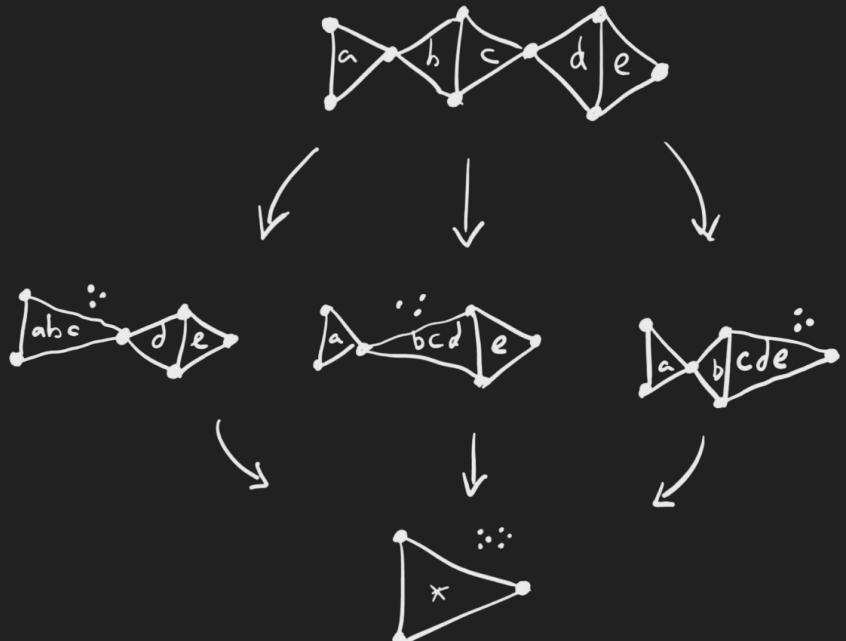
arXiv > math > arXiv:2205.05456

Mathematics > Rings and Algebras

[Submitted on 1 May 2022]

Heaps of Fish: arrays, generalized associativity and heapoids

Carlos Zapata-Carratalá, Xerxes D. Arsiwalla, Taliesin Beynon



$$(abc)de = a(dcb)e = ab(cde)$$

Chemoids - a higher-arity generalization of categories

Chemoids - a higher-arity generalization of categories

- atoms : set of primitive elements (e.g. objects)

Chemoids - a higher-arity generalization of categories

- atoms : set of primitive elements (e.g. objects)
- bonds : sets of data structures constructed from atoms (e.g. morphisms)

Chemoids - a higher-arity generalization of categories

- atoms : set of primitive elements (e.g. objects)
- bonds : sets of data structures constructed from atoms (e.g. morphisms)
- molecules : hypergraphs constructed from bonds (e.g. diagrams)

Chemoids - a higher-arity generalization of categories

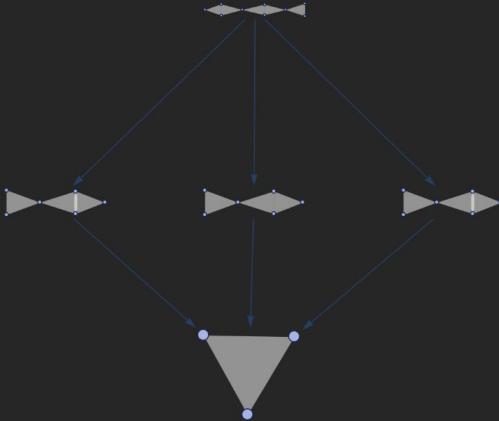
- atoms : set of primitive elements (e.g. objects)
- bonds : sets of data structures constructed from atoms (e.g. morphisms)
- molecules : hypergraphs constructed from bonds (e.g. diagrams)
- reactions : hypergraph rewrite rules (e.g. diagrammatic composition)

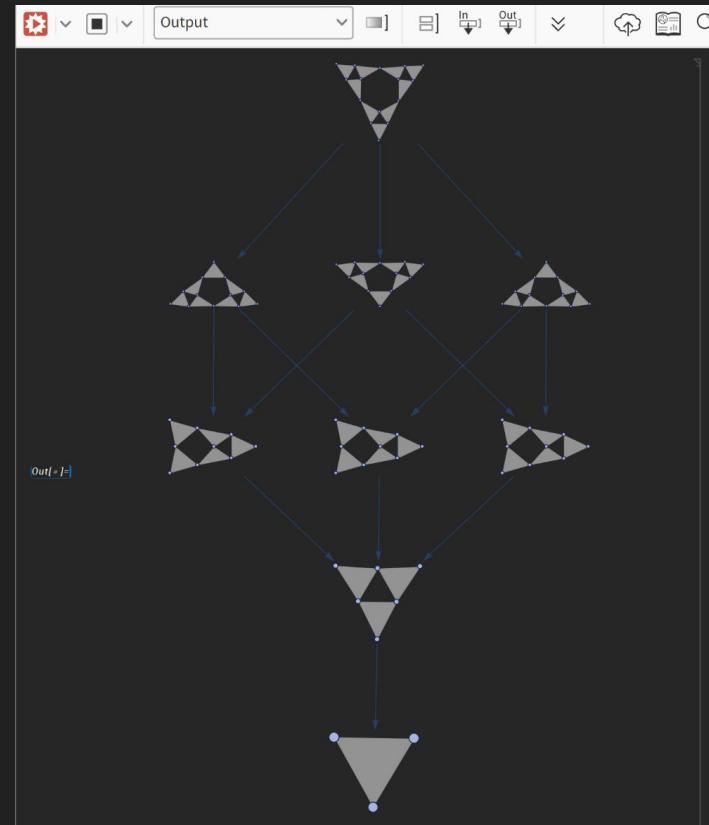
Chemoids - a higher-arity generalization of categories

- atoms : set of primitive elements (e.g. objects)
- bonds : sets of data structures constructed from atoms (e.g. morphisms)
- molecules : hypergraphs constructed from bonds (e.g. diagrams)
- reactions : hypergraph rewrite rules (e.g. diagrammatic composition)
 - ⊕ enforce the Principle of Diagrammatic Simplicity

Chemoids - a higher-arity generalization of categories

In[+]:= ResourceFunction["NestGraphTagged"] [
First@ResourceFunction["MultiReplace"] [#,
{{OrderlessPatternSequence[x_, y_, p_]}, {OrderlessPatternSequence[q_, r_, p_]},
{OrderlessPatternSequence[q_, r_, z_]}] → {Nothing, Nothing, {x, y, z}},
"Mode" → "Subsets"] &,
{DisFish},
10,
"StateLabeling" → True,
VertexShapeFunction →
→ Function[{pos, hg, size},
Inset@ResourceFunction["WolframModelPlot"] [hg, ImageSize → Tiny,
Background → Transparent, "ArrowheadLength" → 0,
EdgeStyle → <| {_, _, _} → Transparent |>,
"EdgePolygonStyle" → Directive[White, Opacity[.5]]], pos, size]],
"RuleStyling" → None
]


Out[+]=



<https://arity.science/>

Arity Science

compositional structure of higher-order systems

\neq

Arity Science is a multi-project research programme that aims to develop novel mathematical and computational frameworks, generalizing existing formalisms such as category theory, relational algebra and network theory, that can accommodate the complexity of higher-order systems while remaining rigorous and computable. Arity Science is a transdisciplinary effort that focuses on the formal investigation of mathematical objects such as n-ary algebras or hypergraphs, and their subsequent application for the modelling of higher-order phenomena across scientific disciplines, including brain connectomics, rewriting and automata, genome topology, art history or molecular computing.

Get in touch!



c.zapata.carratala@gmail.com

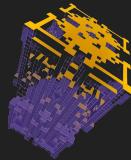


@ZapataCarratala



SEMF

Society for Multidisciplinary
and Fundamental Research



**Institute for the
Foundations of Computation**



SEMF

Society for Multidisciplinary
and Fundamental Research



<https://www.youtube.com/@SEMF>



@semf_nexus

<https://semf.org.es>

