

Flux-induced persistent motion of solitons in topological Josephson junctions

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We propose a Hamiltonian describing the collective motion of a phase soliton (Josephson vortex) within a Josephson junction separating two topological superconductors. Using the Hamiltonian we derive the equations of motion for the soliton trapped in an annular Josephson junction. We find a universal phase accumulated by the soliton when it encircles the junction, that depends on the parity of the number of vortices enclosed by the junction. We demonstrate that the presence of this phase can be measured through its effect on the junction's resistance.

Introduction.— Topological order, a remarkable type of quantum order that admits no local order parameter [1–5], was manifested recently in many forms in several laboratories worldwide [6–10]. Most recently, ground-breaking experiments in solid-state devices have potentially observed signatures of Majorana fermions [8–10], quasi-particles that are their own quasi-holes [11]. An unambiguous discovery of these highly-prized quasi-particles may carry far-reaching consequences for basic science and technology alike: they entail a novel type of non-commutative quantum statistics [12–17] that beckons a paradigm shift in quantum computation [18, 19].

Some of the most fascinating constructs that have emerged out of these studies are the topological Josephson junctions (TJJs) [20, 21], which differ from their non-topological counterparts by the presence of a pair of one-dimensional counter-propagating Majorana states nucleated at the junction. The resulting dynamics for the junction is consequently governed by a modified sine-Gordon Hamiltonian, where the regular bosonic degrees of freedom couple with the low-lying Majorana fermions [22]. In particular, properties of Josephson vortices through the junction are modified so that each vortex carries a Majorana zero mode [20, 22, 24, 26]. TJJs have been argued to be a promising venue for realizing itinerant non-Abelian anyons taking the form of Josephson vortices [22, 24, 25].

In this paper we derive an effective quantum Hamiltonian for a Josephson vortex in a TJJ (see Eq. (18)), unveiling a remarkable property of these vortices. It is shown that the presence of the low-lying Majorana edge states that interact with the vortex crucially affects its dynamics. As a consequence, for the case of a Josephson vortex going around an annular Josephson junction [27, 28] (see Fig. 1), the vortex accumulates a universal phase. This phase can be exploited to induce a persistent motion of the vortex around the junction, triggered by the nucleation of an additional vortex in the region enclosed by the junction (i.e. by changing the magnetic flux Φ through the central hole). The induced motion drives the Josephson junction into its finite resistance state [23], revealing the presence of the phase. For non-topological Josephson junctions, an externally induced charge Q can drive the vortex into a similar persistent motion [27] through the Aharonov-Casher effect [29–31];

The system acts as an elegant dual of an Aharonov-Bohm ring for electrons. In contrast, for TJJs the persistent motion of the Josephson vortex can be controlled through two knobs instead of one: i) Continuously, using the induced charge Q in the region enclosed by the junction; and (ii) Using the enclosed flux, whose value is quantized in units of $h/2e$. In units of the charge, the nucleation of an extra vortex within the central region is equivalent to an $e/4$ shift in the Aharonov-Casher charge Q (where e is the electronic charge).

We proceed to derive the equations governing the motion of a Josephson vortex. Our starting point is the Hamiltonian describing the TJJ [22], which we now briefly describe in terms of its bosonic and fermionic degrees of freedom.

Description of the junction.— The following Hamiltonian governs the dynamics of the relative phase degree of freedom across the junction [32]

$$\mathcal{H}_\varphi = \frac{\hbar c}{g^2} \left\{ \frac{1}{2c^2} (\partial_t \varphi)^2 + \frac{1}{2} (\partial_x \varphi)^2 + \frac{1}{\lambda^2} [1 - \cos \varphi(x)] \right\}, \quad (1)$$

where $\varphi(x)$ is the relative phase, x is the coordinate run-

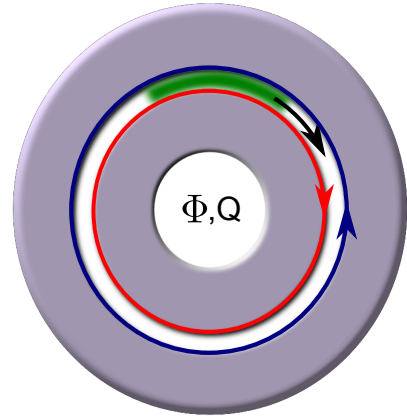


FIG. 1. (Color online) An annular topological Josephson junction trapping a single soliton (green). Counter-propagating Majorana edge states (red and blue lines) are nucleated at the junction. A charge Q and phase Φ are induced externally within the central region.

ning along the junction, λ the Josephson penetration length, \bar{c} the renormalized velocity of light, and g a dimensionless constant which depends on the parameters of the junction. In addition, the junction supports two counter-propagating Majorana edge states with a Hamiltonian $H_\psi = H + \bar{H}$ (H describes the external edge and \bar{H} the internal)

$$\begin{aligned} H &= i\frac{v}{2} \int dx \psi(x) \partial_x \psi(x), \\ \bar{H} &= -i\frac{v}{2} \int dx \bar{\psi}(x) \partial_x \bar{\psi}(x). \end{aligned} \quad (2)$$

Here $x \in [0, L]$ runs around the circumference of the edge and v is the neutral edge velocity. In addition, there exists a tunneling term between the two edge states which couples the phase degree of freedom and the two Majorana degrees of freedom, taking the form

$$H_{tun} = i \int dx W(x) \psi(x) \bar{\psi}(x), \quad (3)$$

where $W(x) = m \cos[\varphi(x)/2]$ is the Majorana mass term [20, 22]. The full Hamiltonian for the junction, $H_\varphi + H_\psi + H_{tun}$ [22], is an extension of the supersymmetric Sine-Gordon model for general values of m [26].

We perform the following mode expansion,

$$\begin{aligned} \psi(x) &= \sqrt{\frac{1}{L}} \sum_n e^{-2\pi i n x/L} \psi_n, \\ \bar{\psi}(x) &= \sqrt{\frac{1}{L}} \sum_n e^{2\pi i n x/L} \bar{\psi}_n. \end{aligned} \quad (4)$$

The fields obey anti-commutation relations of the form

$$\{\psi(x), \psi(x')\} = \frac{1}{L} \sum_n e^{-2\pi i n (x-x')/L} = \delta(x-x'),$$

where the modes ψ_n satisfy $\{\psi_n, \psi_{n'}\} = \delta_{n+n',0}$ (and similarly for the opposite chirality). In addition, $\{\psi(x), \bar{\psi}(x')\} = 0$. Note that in particular this implies $2\psi_0^2 = 1$, or $\psi_0^2 = 1/2$ (for either chirality). Plugging this into the Hamiltonian we get,

$$\begin{aligned} H &= \frac{2\pi v}{L} \left[\frac{1}{2} \sum_n n \psi_{-n} \psi_n \right] \equiv \frac{2\pi v}{L} \mathcal{L}, \\ \bar{H} &= \frac{2\pi v}{L} \left[\frac{1}{2} \sum_n n \bar{\psi}_{-n} \bar{\psi}_n \right] \equiv \frac{2\pi v}{L} \bar{\mathcal{L}}. \end{aligned} \quad (5)$$

It is the quantity \mathcal{L} (and $\bar{\mathcal{L}}$), the dimensionless momentum operator, that we will pursue in the following. Using Eq. (4), periodic boundary conditions on the Majorana field imply $n \in \mathbb{Z}$, while anti-periodic boundary conditions imply $n \in \mathbb{Z} + \frac{1}{2}$.

A quantized shift of the edge states' total momentum.— To understand the change in the momentum when the

boundary conditions are exchanged between periodic and anti-periodic, we write \mathcal{L} as

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \sum_n n \psi_{-n} \psi_n = \sum_{n>0} n \psi_{-n} \psi_n - \frac{1}{2} \sum_{n>0} n \\ &\equiv \sum_{n>0} n \psi_{-n} \psi_n + \mathcal{L}_0, \end{aligned} \quad (6)$$

where \mathcal{L}_0 is a constant, which can be thought of as the ground state contribution; the other part consists of excitations above the ground state. If there is a vortex enclosed by the edge, $n \in \mathbb{Z}$, otherwise $n \in \mathbb{Z} + 1/2$.

For a closed circular Josephson junction, in the absence of tunneling, we have the following

$$\mathcal{L} = \sum_{n>0} n \psi_{-n} \psi_n + \mathcal{L}_0(N_v \bmod 2), \quad (7)$$

$$\bar{\mathcal{L}} = \sum_{m>0} m \bar{\psi}_{-m} \bar{\psi}_m + \bar{\mathcal{L}}_0(\bar{N}_v \bmod 2), \quad (8)$$

where N_v (\bar{N}_v) denotes the number of vortices enclosed by the external (internal) edge. We now calculate the difference in the ground state contribution in the presence of a Josephson vortex within the junction, i.e. $N_v = 1$ and $\bar{N}_v = 0$. We employ a regularizing function $F(x)$ such that $F'(x) = \partial_x F(x)$ decays to zero faster than $1/x^2$ when $x \rightarrow \infty$, and $F'(0) = 1$. We calculate the regularized sum [33]

$$\begin{aligned} \Delta \mathcal{L}_0 &= \mathcal{L}_0(1) - \bar{\mathcal{L}}_0(0) \\ &= -\frac{1}{2} \sum_{n=1}^{\infty} \left[n F'(\alpha n) - (n - \frac{1}{2}) F'(\alpha(n - \frac{1}{2})) \right]. \end{aligned} \quad (9)$$

By taking the limit $\alpha \rightarrow 0$ we now get

$$\begin{aligned} \Delta \mathcal{L}_0 &= -\frac{1}{2} \partial_\alpha \sum_{n=1}^{\infty} \left[F(\alpha n) - F(\alpha(n - \frac{1}{2})) \right] \\ &= -\frac{1}{2} \partial_\alpha \sum_{n=1}^{\infty} \left[\frac{\alpha}{2} F'(\alpha n) - \left(\frac{\alpha}{2}\right)^2 \frac{1}{2} F''(\alpha n) \right] \\ &= -\frac{1}{2} \partial_\alpha \int_{\alpha/2}^{\infty} d(\alpha n) \left[\frac{1}{2} F'(\alpha n) - \frac{\alpha}{8} F''(\alpha n) \right] \\ &= \frac{1}{16} [F'(0) + F'(\infty)] = \frac{1}{16}. \end{aligned} \quad (10)$$

This result is well-known from conformal field theory (see e.g. [34]) where a zeta function regularization is typically used to obtain the same value. In the following we explore how this quantized shift can affect the dynamics of the soliton, which is governed by field theories that are explicitly non-conformal.

Effective Hamiltonian for a Josephson vortex.— We consider the solution for a classical soliton (in the non-relativistic limit and for $\lambda \ll L$)

$$\varphi_s(x, q(t)) = 4 \arctan \left[\exp \left(\frac{x - q(t)}{\lambda} \right) \right], \quad (11)$$

with a center of mass coordinate at $q(t)$. We plug the solution into the Lagrangian, using $\partial_t \varphi_s = \dot{q} \partial_x \varphi_s$, and the integrals $\int_{-\infty}^{\infty} dx (\partial_x \varphi_s)^2 = \frac{8}{\lambda}$ and $\int_{-\infty}^{\infty} dx (1 - \cos \varphi_s) = 4\lambda$ to get

$$\frac{1}{2} m_s \dot{q}^2 + E_0 \quad (12)$$

where we defined the soliton mass m_s and the soliton rest energy. We now proceed to the Majorana sector, with $\Psi = (\psi \ \bar{\psi})^T$

$$H_\psi = \int dx \Psi^T H_0 \Psi, \quad (13)$$

where

$$H_0 = \frac{1}{2} \begin{bmatrix} iv \partial_x & iW(x, q(t)) \\ -iW(x, q(t)) & -iv \partial_x \end{bmatrix} \quad (14)$$

and

$$\begin{aligned} W(x, q(t)) &= m \cos[\varphi_s(x, q(t))/2] \\ &= m \tanh[(x - q(t))/\lambda] \equiv W(x - q(t)). \end{aligned} \quad (15)$$

The equations simplify considerably by taking a boost to the moving frame

$$\begin{aligned} x' &= x - q(t), \\ t' &= t, \\ \partial_x &= \partial_{x'}, \\ \partial_t &= -\dot{q} \partial_{x'} + \partial_{t'}. \end{aligned} \quad (16)$$

We see that the Majorana fields couple to the center of mass velocity of the soliton via a vector-potential like term that measures the total momentum carried by the two counter-propagating edge states, taking the form:

$$i\dot{q} \int dx (\psi \partial_x \psi + \bar{\psi} \partial_x \bar{\psi}) = \frac{2\pi}{L} \dot{q} (\mathcal{L} - \bar{\mathcal{L}}). \quad (17)$$

The full Hamiltonian $H_\varphi + H_\psi + H_{tun}$, written in the background of a single soliton, is given in terms of the soliton's center of mass momentum \hat{p} (which we now re-instate as a quantum operator) as

$$\begin{aligned} H_s &= E_0 + \frac{1}{2m_s} \left[\hat{p} - \frac{2\pi}{L} (\mathcal{L} - \bar{\mathcal{L}}) \right]^2 \\ &+ \frac{2\pi v}{L} (\mathcal{L} + \bar{\mathcal{L}}) + i \int dx W(x) \psi(x) \bar{\psi}(x). \end{aligned} \quad (18)$$

This Hamiltonian describes the dynamics of the Josephson vortex within the junction and is a central result of this paper. Not only it captures universal features as we now describe, it can also be used to extract potential deviations from universality.

The Hamiltonian admits a subtle symmetry, that the edge states' total momentum operator and tunneling operator, given respectively by

$$P = \frac{2\pi}{L} (\mathcal{L} - \bar{\mathcal{L}}), \quad (19)$$

$$T = i \int dx W(x) \psi(x) \bar{\psi}(x), \quad (20)$$

in fact commute. To show this, one can use the following argument. First, we calculate the commutator

$$[P, T] = \frac{2\pi}{L} \int dx \partial_x W(x) \psi(x) \bar{\psi}(x). \quad (21)$$

For our choice of φ_s , the term $\partial_x W(x)$ is localized around the center of the soliton. We can formally write the commutator in Eq. (21) as $[P, T] = \frac{2\pi i}{L} \lim_{x \rightarrow x_0} \partial_{x_0} T[x_0]$, where $T[x_0]$ is obtained from Eq. (20) by replacing $W(x)$ by $W(x - x_0)$. Since $T[x_0]$ does not depend on x_0 owing to translational symmetry, the derivative term is zero. Hence $[P, T] = 0$. The free spectrum of the operator P , given by

$$\frac{2\pi}{L} \left(\pm \frac{1}{16} + \frac{n}{2} \right), \quad n \in \mathbb{Z}, \quad (22)$$

remains unaffected by the tunneling, with \pm related to the parity of the number of vortices within the path. This demonstrates a certain robustness of the $1/16$ result even for this non-conformal case.

Proposed experiment for detecting the phase shift.— In the absence of a Josephson vortex within the junction, the tunneling results in the Majorana modes gapping out. When a Josephson vortex is then introduced, a zero energy bound state nucleates on the vortex while the rest of the modes remain gapped. There are two cases of interest: (i) If there is an even number of vortices in the central region of the sample, the total dimensionless momentum carried by the two counter-propagating edge states is $1/16$; and (ii) If there is an odd number of vortices the same momentum is $-1/16$. Hence, while the total energy for the two edges is the same for the two cases, a certain part of the momentum operator, $\mathcal{L}_0 - \bar{\mathcal{L}}_0$, changes sign.

The energy spectrum of the Josephson vortex can be derived from Eq. (18), and in the presence of an externally induced Aharonov-Casher charge Q within the central region, is given by

$$E_s = E_c \left[\frac{Q}{2e} + \left(\frac{n_f}{4} + \frac{n_v}{16} \right) - n_b \right]^2, \quad (23)$$

where E_c is the charging energy, $n_f = \pm$ is the fermion parity within the enclosed path of the Josephson vortex, $n_v = \pm$ the parity of the number of vortices within the same region, and $n_b \in \mathbb{Z}$ the relative number of Cooper pairs between the two superconducting plates. In the low energy sector there is an emergent dependence between n_f and n_v : If $n_v = 1$ then $n_f = 1$, but if $n_v = -1$ then n_f is free [22].

We start from the case $n_f = 1$ and $n_v = 1$, tuning the junction into its zero resistance state by shifting the induced Aharonov-Casher charge Q ; the Josephson vortex accordingly acquires a vanishing velocity. Next we add an extra vortex within the central region of the sample, shifting the value of n_v to -1 . The Josephson vortex then