

Relaxation dynamics of conserved quantities in a weakly non-integrable one-dimensional Bose gas

Giuseppe Brandino,¹ Jean-Sébastien Caux,¹ and Robert Konik²

¹*Institute for Theoretical Physics, Science Park 904,
University of Amsterdam, 1090 GL Amsterdam, The Netherlands*

²*CMPMS Dept., Brookhaven National Laboratory, Upton, NY 11973-5000, USA*

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Abstract

In this work we report preliminary results on the relaxational dynamics of one dimensional Bose gases, as described by the Lieb-Liniger model, upon release from a parabolic trap. We explore the effects of integrability and integrability breaking upon these dynamics by placing the gas post-release in an integrability breaking one-body cosine potential of variable amplitude. By studying the post-quench evolution of the conserved charges that would exist in the purely integrable limit, we begin to quantify the effects of the weak breaking of integrability on the long time thermalization of the gas.

I. INTRODUCTION

In their seminal experiment on one dimensional interacting cold atomic gases, Kinoshita et al. [1] argued for the possibility that the relaxational dynamics of such gases possessed memory of the gases' initial condition. Specifically, they observed that the momentum distribution of the gas did not rapidly evolve to a thermal equilibrium state, despite the presence of interactions between the gas' atoms. To explain this behavior, they conjectured that the gas possessed non-trivial conserved integrals of motion (beyond the energy of the gas), and that these integrals of motion were controlling the long time dynamics of the gas. These non-trivial integrals of motion should exist, they argued, because the underlying theoretical description of the gas, the Lieb-Liniger model, is known to be exactly solvable and has an infinite set of such integrals [2, 3]. In subsequent work Rigol et al. [4] sharpened this conjecture by arguing that while the gas did relax to a state governed by a thermodynamic ensemble, this ensemble was not the canonical (or microcanonical), but an ensemble aware of these additional integrals of motion, an ensemble they dubbed the generalized Gibbs ensemble.

In subsequent work this ensemble has been shown to govern the dynamics of a number of systems characterized by sets of non-trivial conserved quantities, both non-interacting [5–9] and interacting [10, 11]. However less studied has been the question of thermalization when the system has a set of weakly broken integrals of motion [12–14]. Does the weak breaking of the integrals of motion always lead to eventual thermalization of the gas as governed by the canonical ensemble? Is there a time scale of integrability breaking, τ_{IB} , for which for times $t < \tau_{IB}$ the dynamics appear integrable while for times $t > \tau_{IB}$, the dynamics are governed by the standard thermodynamic ensembles? Or is there a smooth crossover in thermalization as suggested in [12–14], where physical quantities interpolate between their values under the generalized Gibbs ensemble and their values in the canonical ensemble?

It is with these questions in mind that we study the following quantum quench problem. We begin by considering a one dimensional atomic Bose gas of N particles in a system of length L in the presence of a one-body parabolic trap, $V_{\text{para}}(x) = m\omega^2 x^2/2$. We describe the gas with the Hamiltonian,

$$H = H_{LL} + \sum_{i=1}^N V_{\text{para}}(x_i);$$

$$H_{LL} = -\frac{\hbar^2}{2m} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2c \sum_{\langle i,j \rangle} \delta(x_i - x_j), \quad (1)$$

where we work in units of $2m = \hbar = 1$. H_{LL} is the well known Lieb-Liniger model [2] with interaction strength c . We prepare the system in the ground state of this Hamiltonian, H . At $t = 0$ we remove the parabolic trap. In the absence of the parabolic trap, the Hamiltonian (now just the Lieb-Liniger model itself) is integrable, and we expect the subsequent dynamics of the gas to be governed by the non-trivial conserved charges in the system. If instead of simply removing the trap, we replace it with a different one-body potential, $V_{\cos}(x) = A \cos(\omega x)$, we break integrability, so changing the nature of the post-quench dynamics. By varying the amplitude A of this potential, we control the amount of integrability breaking in the system and its concomitant effects on the dynamics. This quantum quench is illustrated in Fig. 1.

To study these dynamics, we employ a combination of Bethe ansatz solvability and a numerical renormalization group. The Lieb-Liniger model is a model that can be solved with the Bethe ansatz, both determining its spectra [2] and its matrix elements [15]. Because we can compute matrix elements, we can compute correlations functions of this model using their Lehmann representations [16, 17]. This however is computationally intensive. In order to accomplish this task we employ an optimized set of routines known as ABACUS [16].

Because we are interested in perturbing the gas by introducing integrability breaking one-body potentials, both pre- and post-quench, the solvability of the Lieb-Liniger model is insufficient for the task at hand. Instead to study such deformations, we employ a numerical renormalization group (NRG) [11, 18–21] able to study perturbations of integrable and conformal continuum field theories. This approach, as it is an extension of a methodology known as the truncated conformal spectrum approach [22, 23], has been primarily used to study perturbations of relativistic field theories [18–21], but has recently been applied to the Lieb-Liniger model perturbed by a one-body potential [11], the problem at hand. The NRG uses the eigenstates of the Lieb-Liniger model as a computational basis. Because this basis accounts for the interactions of the Bose gas particles with one another, this numerical method builds in the strong correlations present in the problem right at the start.

The paper is organized as follows. In Section 2 we demonstrate that we can compute the equilibrium properties of the gas in the one-body potentials. While we have shown in Ref. [11] that we can accurately describe the ground state and first few excited states in

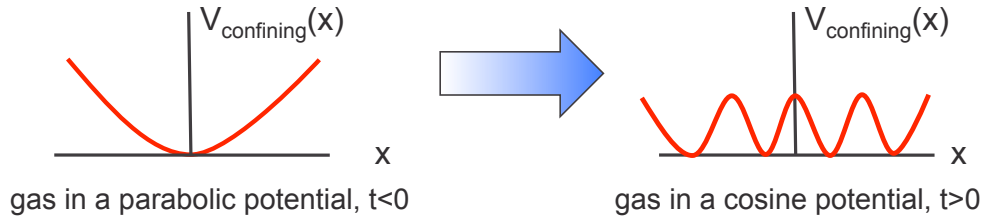


FIG. 1: Sketch of the quench protocol. For $t < 0$, we prepare the gas in its ground state in a parabolic potential. At $t = 0$, we remove the parabolic potential and replace it with a small cosine potential.

such a potential, here we show that we can obtain with reasonably accuracy a wide range of the spectrum. This will be important for the determination of the post-quench time evolution of the Lieb-Liniger conserved charges. In Section 3, we consider the evolution of the expectation values of these charges. This will allow us to begin to understand the consequences of integrability breaking. In Section 4, we discuss these results briefly and examine possible further directions for this work.

II. EQUILIBRIUM PROPERTIES OF THE GAS IN THE ONE-BODY POTENTIALS

In this section we demonstrate that we can compute the equilibrium properties of the gas in the one-body potentials necessary to describe the dynamics of the system post-quench. We first show that we can prepare the initial state, the ground state of the gas in the presence of the parabolic potential, V_{para} , accurately. To this end we show in the central panel of Fig. 2 the density profile, $\rho(x)$, of the gas in this trap. We see that we get good agreement between the NRG numerics for $N = L = 14, c = 7200$ and an analytical computation of the density profile of the gas in its Tonks-Girardeau limit ($c = \infty$). Such analytics are possible because the gas, for certain quantities such as $\rho(x)$, can be treated as equivalent to free

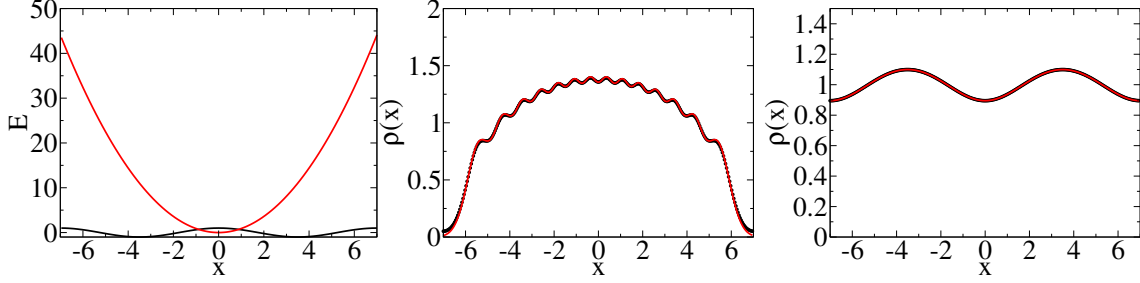


FIG. 2: Right: The parabolic one-body potential, $V_{\text{para}}(x) = 0.9x^2$ (red), used to prepare the initial state and the post-quench one-body cosine potential, $V_{\text{cos}}(x) = \cos(4\pi x/L)$ with amplitude $A = 1$ (black). Center: The density, $\rho(x)$, of the gas ($N = L = 14, c = 7200$) in its ground state in the presence of the parabolic potential: red (analytics), black (NRG). Right: The density, $\rho(x)$, of the gas ($N = L = 14, c = 7200$) in its ground state in the presence of the cosine potential with amplitude $A = 2$: red (analytics), black (NRG).

fermions. In Fig. 2 (right panel) we also demonstrate that we can accurately compute the density profile of the gas in a cosine potential.

However in order to compute post-quench dynamics, we need to be able to describe not only the ground state in the cosine potential, but some large number of excited states. In our quench protocol, we take as our initial $t = 0$ state the ground state of the gas in the parabolic potential, $|\psi_{GS,\text{para}}\rangle$. If we can compute a wide range of eigenstates in the cosine potential, both ground and excited states, $|\psi_{\alpha,\text{cos}}\rangle$, we can expand this initial state in terms of the post-quench basis:

$$|\psi_{GS,\text{para}}\rangle = \sum_{\alpha} c_{\alpha} |\psi_{\alpha,\text{cos}}\rangle. \quad (2)$$

Of course for this expansion to be exact, we would need to know *all* of the eigenstates of the gas in the parabolic potential. We will instead settle for a determination of the post-quench eigenbasis that allows us to include enough states so that $\sum_{\alpha} |c_{\alpha}|^2 > 0.99$.

Because we use the eigenstates of the Lieb-Liniger model absent a one-body potential, $|\psi_{\alpha,LL}\rangle$, as the computational basis of the NRG, the NRG gives any eigenstate in a one-body potential as a linear combination of such states:

$$|\psi_{\text{one-body}}\rangle = \sum_{\alpha} b_{\alpha} |\psi_{\alpha,LL}\rangle. \quad (3)$$

Each Lieb-Liniger state $|\psi\rangle_{\alpha,LL}$ is characterized by N -rapidities, λ_i , $i = 1, \dots, N$. These rapidities govern the energy, E_{α} , and momentum, P_{α} , of the state (relative to the Hamiltonian,

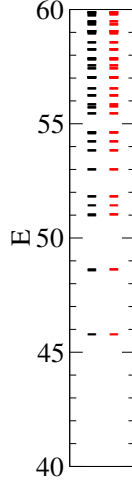


FIG. 3: A plot of the energies of the gas ($N = L = 14, c = 7200$) in the cosine potential as determined by the NRG (black) and by analytics (red) in the hardcore limit ($c = \infty$).

H_{LL}):

$$E_\alpha = \sum_{i=1}^N \lambda_{\alpha,i}^2; \quad P_\alpha = \sum_{i=1}^N \lambda_{\alpha,i}. \quad (4)$$

These rapidities are found as solutions of the Bethe equations:

$$e^{i\lambda_n L} = \prod_{m \neq n}^N \frac{\lambda_n - \lambda_m + ic}{\lambda_n - \lambda_m - ic}, \quad n = 1, \dots, N. \quad (5)$$

In the limit of $c = 0$, we see that the Bethe equations collapse to the momentum quantization condition for a particle in a periodic system of length L .

In computing the spectrum of states in the cosine potential, we employ the variant of the NRG discussed in Ref. [20]. The NRG in its plain vanilla formulation [18] can compute the spectrum of the low lying states of the gas in the one-body potential [11]. But to capture accurately an appreciable fraction of the spectrum, we need to employ a sweeping routine [20] analogous to that used in the finite volume routine of the density matrix renormalization group [24].

In Fig. 3 we present the results for the spectrum of the gas in the $V_{\cos}(x) = \cos(4\pi x/L)$ ($A = 1$) as computed with the NRG and with analytics in the $c = \infty$ limit. We see we are able to describe accurately a wide range of the spectrum. For higher energy states, there are some slight differences between analytics and the NRG which we believe can be ascribed to $1/c$ corrections, which while small are still present.

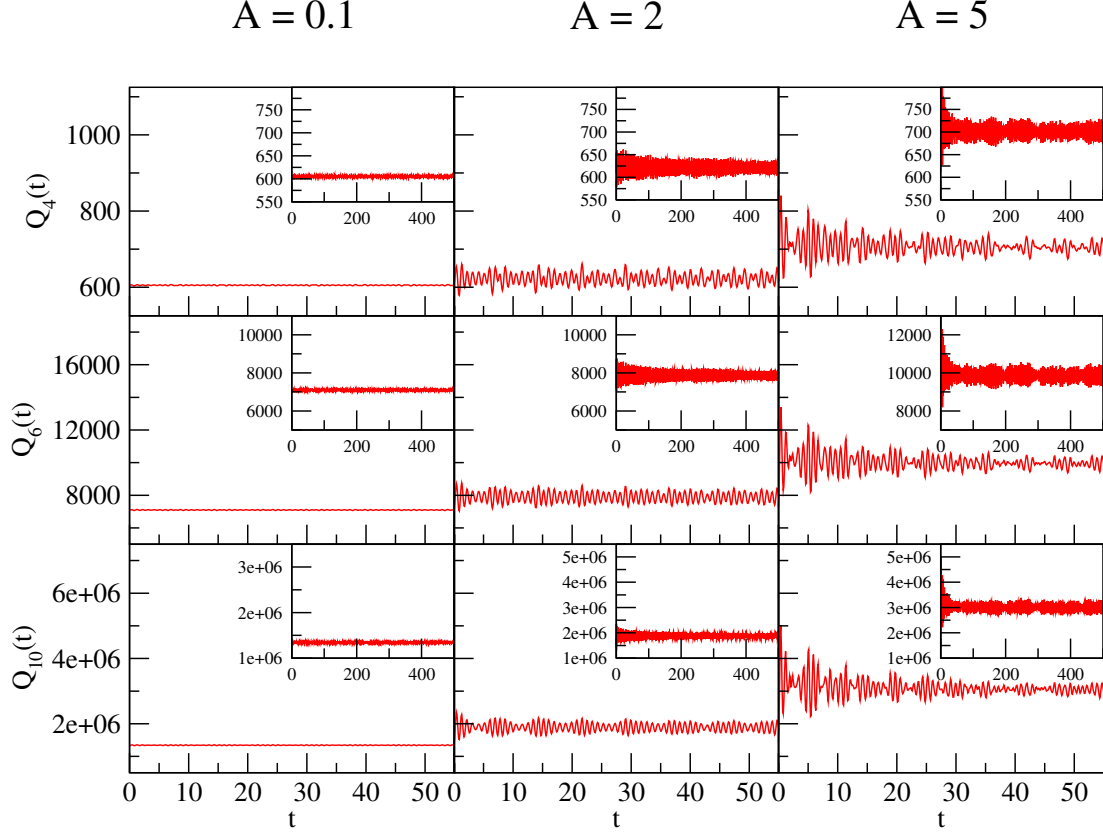


FIG. 4: The time evolution of the conserved charges, Q_n , of the Lieb-Liniger model post-quench. Shown are Q_4 , Q_6 , and Q_{10} for a release into a cosine potential of amplitude $A = 0.1, 2$, and 5 . The frequency of the cosine potential, ω , is set to $4\pi/L$.

III. POST-QUENCH DYNAMICS OF CONSERVED CHARGES

In this section we consider the time evolution of the Lieb-Liniger conserved quantities, Q_n . These quantities commute with the Lieb-Liniger Hamiltonian, $[Q_n, H_{LL}]$, but in the presence of the one-body cosine potential they become time dependent. To compute this time evolution, we first note the time evolution of our initial $t = 0$ state is expressible as

$$|\psi_{GS,para}(t)\rangle = \sum_{\alpha} c_{\alpha} |\psi_{\alpha,cos}\rangle e^{itE_{\alpha,cos}}. \quad (6)$$

To compute

$$Q_n(t) = \langle \psi_{GS,para}(t) | Q_n | \psi_{GS,para}(t) \rangle,$$

we then need to know $\langle \psi_{\beta,cos} | Q_n | \psi_{\alpha,cos} \rangle$. But because $|\psi_{\alpha,cos}\rangle$ is given in terms of Lieb-Liniger states (Eqn. 3), this amounts to knowing the action of the charges, Q_n , on such states. This

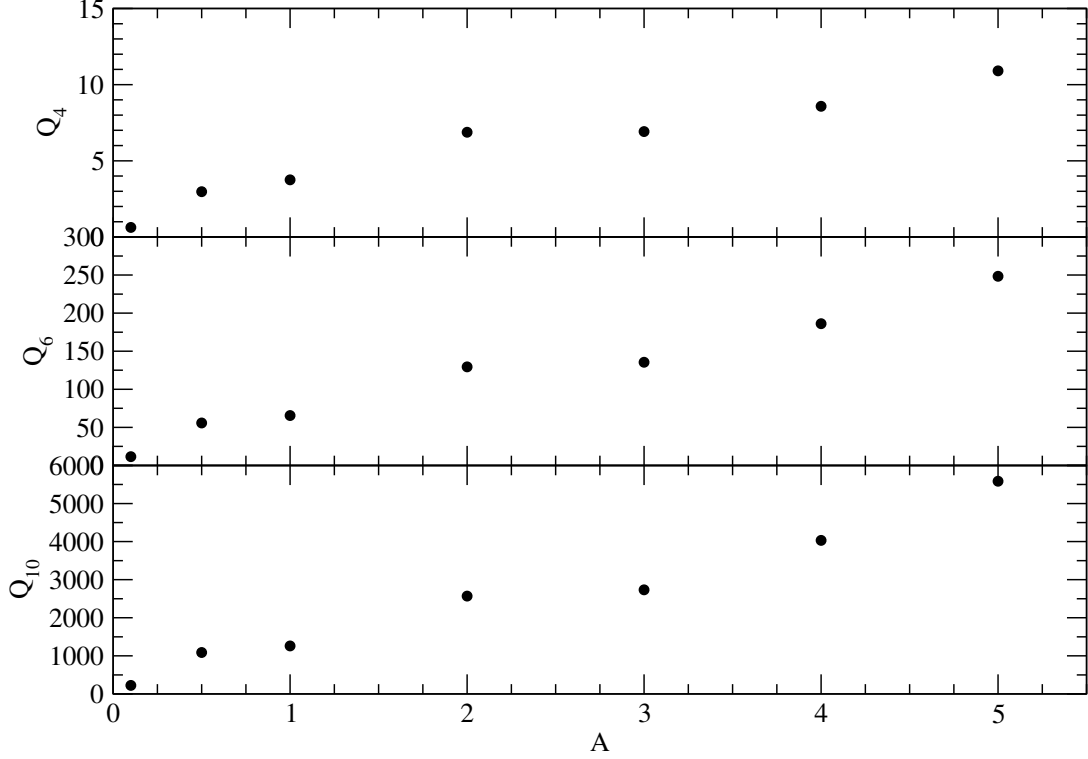


FIG. 5: Amplitudes of the oscillations of the conserved charges, Q_4 , Q_6 , and Q_{10} , as a function of the amplitude, A , of the post-quench cosine potential.

however is straightforward [3]:

$$Q_n|\psi_{LL,\alpha}\rangle = \sum_{i=1}^N \lambda_{\alpha,i}^n |\psi_{LL,\alpha}\rangle. \quad (7)$$

The first two charges in the sequence, Q_1 and Q_2 , give the momentum and energy respectively of the Lieb-Liniger state. Because of the system's parity symmetry, $x \rightarrow -x$, all of the odd charges evaluate to zero on $|\psi_{LL,\alpha}\rangle$. We thus will focus on the even charges, Q_{2n} , alone.

In Fig. 4 we present the time evolution of three charges Q_4 , Q_6 , and Q_{10} for quenching into cosine potentials with amplitudes $A = 0.1, 2$, and 5 . We see the charges oscillate in time with increasing amplitude as the amplitude of the integrability breaking cosine potential is increased. These oscillations occur about a well defined mean. This mean smoothly increases from its $A = 0$ value as we increase the strength of the cosine potential. We also see for small times, the expectation values of the charges are characterized by transients, but then settle into a steady state oscillatory behavior.

In Fig. 5 we plot the amplitudes of the oscillations of the charges as a function of the amplitude of the cosine potential. We see that these amplitudes of the oscillations are

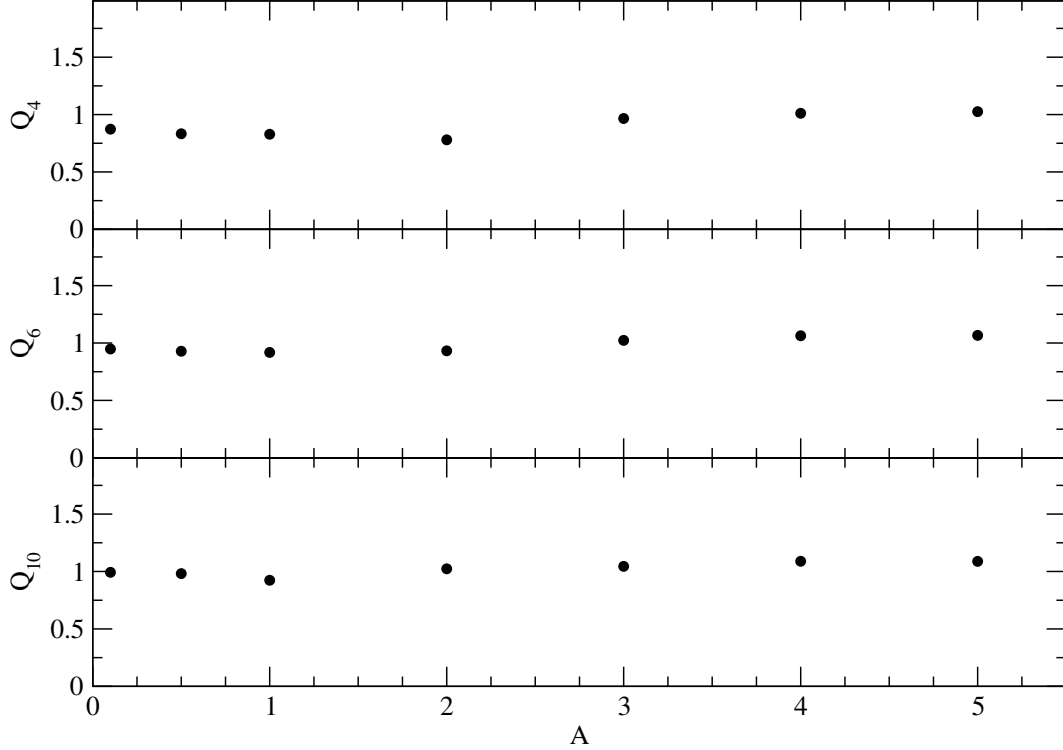


FIG. 6: Average frequency of the oscillations of the conserved charges, Q_4 , Q_6 , and Q_{10} as a function of the amplitude, A , of the post-quench cosine potential.

(roughly) linearly related to the amplitudes, A 's, of the cosine. In the limit that A vanishes, the charges become constants of motion, as expected. More interestingly, however, the frequencies at which the charges are oscillating, Fig. 6, are independent of the amplitude of the cosine potential. Instead these frequencies are roughly but uniformly equal to that of the cosine potential itself, $4\pi/L \sim 1$.

IV. DISCUSSION AND FUTURE DIRECTIONS

In this brief report we have considered post-quench dynamics of a cold atomic gas quenched from its ground state in a parabolic trap to a cosine shaped trap. To characterize these dynamics we have investigated the time dependency of the Lieb-Liniger conserved quantities induced by the presence of the post-quench integrability breaking cosine. In the presence of integrability breaking, the charges oscillate about a mean. This mean behaves smoothly as a function of the strength of integrability breaking, the amplitude, A , of the cosine. Similarly the amplitude of the oscillations go smoothly to zero as A is reduced to

zero. However for the frequency of the oscillations of the charges this is not the case. Instead this frequency is related directly to the cosine potential's own frequency, which in our quench protocol is kept constant.

Our findings for the behavior of the charges are then in accordance with Ref. [12–14], namely we find that the introduction of integrability breaking leads to a smooth interpolation of the expectations values of observables (here the charges) between that in the integrable limit and that in the fully chaotic limit where the standard thermodynamic ensembles govern dynamics. However we also find that the time scale, τ_{IB} , for integrability breaking is not necessarily associated with the strength of the integrability breaking. We find rather that the charges oscillate with a frequency independent of the amplitude of the one-body cosine potential. This suggests the intriguing possibility that even relatively large integrability breaking terms, if suitably low-frequency, might lead to long thermalization times before completely chaotic behavior is observed.

In future work we intend to explore these questions in terms of the momentum distribution function (MDF) of the gas. The long time behavior of the MDF is considerably more complicated to compute than the time dependency of the charges. In the presence of integrability breaking, it requires one to compute a large number of matrix elements of the form,

$$\langle \alpha, LL | \psi(x) \psi^\dagger(0) | \beta, LL \rangle,$$

where $\psi(x)$ is the Bose field operator. Each of these matrix elements is evaluated by inserting a resolution of the identity between the fields ψ , itself a computationally intensive task [17]. Having already carried out preliminary computations for systems sizes of $N = L = 14$, we nonetheless expect to be able to perform these calculations for system sizes of up to $N \sim 25 - 30$.

Acknowledgments

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