R1a)

Exponential Distribution:

PDF: $P(x_i|\lambda) = \lambda e^{-\lambda x_i}, \lambda > 0$

$$L(\lambda) = P(x_1, \dots, x_N|\theta) \overset{i.i.d}{=} \prod_{i=1}^{N} P(x_i|\lambda) = \prod_{i=1}^{N} \lambda e^{-\lambda x_i} = \lambda^N \prod_{i=1}^{N} e^{-\lambda x_i} = \lambda^N e^{-\lambda \sum_{i=1}^{N} x_i}$$

$= \lambda^N e^{-\lambda \bar{x}}$ com $\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$ being the mean

$$\boxed{L(\lambda) = \lambda^N e^{-\lambda N \bar{x}}}$$

$$\ell = \log L(\lambda) = \ln(\lambda^N e^{-\lambda N \bar{x}}) = N[\ln(\lambda) - \lambda \bar{x}]$$

$$\boxed{\ell = N[\ln(\lambda) - \lambda \bar{x}]}$$

Estimator for λ, $\hat{\lambda}$

$$\frac{d\ell(\hat{\lambda})}{d\hat{\lambda}} = 0 \iff \frac{1}{\hat{\lambda}} = \bar{x} \iff \hat{\lambda} = \frac{1}{\frac{1}{N}\sum_{i=1}^{N} x_i} \iff \boxed{\hat{\lambda} = \frac{N}{\sum_{i=1}^{N} x_i}}$$

Rayleigh Distribution:

PDF: $P(x|\sigma) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}$

$$L(\sigma) = P(x_1, \dots, x_N|\theta) \overset{iid}{=} \prod_{i=1}^{N} P(x_i|\lambda) = \prod_{i=1}^{N} \frac{x_i}{\sigma^2} e^{-\frac{x_i^2}{2\sigma^2}}$$

$$\boxed{\ell(\sigma) = \ln(L(\sigma)) = \sum_{i=1}^{N} \left[\ln(x_i) - \ln(\sigma^2) - \frac{x_i^2}{2\sigma^2}\right]}$$

$\sigma^{-2} \to -2\sigma^{-3}$

Estimator for σ, $\hat{\sigma}$

$$\frac{\partial \ell(\sigma)}{\partial \sigma} = 0 \iff \sum_{i=1}^{N} \frac{-2}{\sigma} + 2\sum_{i=1}^{N} \frac{x_i^2}{2\sigma^3} = 0$$

$$\iff 2N = \frac{1}{\sigma^2} \sum_{i=1}^{N} x_i^2 \iff \sigma^2 = \frac{1}{2N} \sum_{i=1}^{N} x_i^2$$

$$\boxed{\Rightarrow \sigma = \sqrt{\frac{1}{2N} \sum_{i=1}^{N} x_i^2}}$$

Normal Distribution

$$P(x|\theta) = \frac{1}{\sqrt{2\pi\theta^2}} e^{\frac{-(x-\mu)^2}{2\theta^2}}$$

$$L(\theta) = P(x_1, \dots, x_N|\theta) = \prod_{i=1}^{N} P(x_i|\theta) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\theta^2}} e^{\frac{-(x_i-\mu)^2}{2\theta^2}}$$

$$\ell(\theta) = \ln(L(\theta)) = \sum_{i=1}^{N} -\frac{1}{2} \ln(2\pi\theta^2) - \sum_{i=1}^{N} \frac{(x_i-\mu)^2}{2\theta^2}$$

Estimator for θ, $\hat{\theta}$

$$\frac{d\ell(\hat{\theta})}{\partial\hat{\theta}} = 0 \iff \frac{N}{\hat{\theta}} = \frac{1}{\hat{\theta}^3} \sum_{i=1}^{N} (x_i-\mu)^2$$

$$\Rightarrow \boxed{\hat{\theta} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i-\mu)^2}}$$

Estimator for μ, $\hat{\mu}$

$$\frac{\partial\ell(\hat{\mu})}{\partial\hat{\mu}} = 0 \iff \frac{1}{2\theta^2} \sum_{i=1}^{N} 2(x_i-\hat{\mu}) = 0$$

$$\iff \sum_{i=1}^{N} x_i = N\hat{\mu} \Rightarrow \boxed{\hat{\mu} = \frac{1}{N} \sum_{i=1}^{N} x_i}$$